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Isomorphisms between C^*-ternary algebras

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Abstract

In this paper, we prove the Hyers–Ulam–Rassias stability of homomorphisms in C^*-ternary algebras and of derivations on C^*-ternary algebras for the following Cauchy–Jensen additive mappings:

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)=f(x)+2f(z), \tag{0.1}$$

$$f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)=f(y), \tag{0.2}$$

$$2f\left(\frac{x+y}{2}+z\right)=f(x)+f(y)+2f(z). \tag{0.3}$$

These are applied to investigate isomorphisms between C^*-ternary algebras. The concept of Hyers–Ulam–Rassias stability originated from the Th.M. Rassias' stability theorem that appeared in his paper [Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978) 297–300].

Keywords: Cauchy–Jensen functional equation; C^*-ternary algebra isomorphism; Hyers–Ulam–Rassias stability; C^*-ternary derivation

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1. Introduction and preliminaries

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their applications in physics. Some significant physical applications are as follows (see [14,15]):

(1) The algebra of *'nonions'* generated by two matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix} \quad \left(\omega = e^{\frac{2\pi i}{3}}\right)$$

was introduced by Sylvester as a ternary analog of Hamilton's quaternions (cf. [1]).

(2) The quark model inspired a particular brand of ternary algebraic systems. The so-called *'Nambu mechanics'* is based on such structures (see [6]).

There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the nonstandard statistics, supersymmetric theory, and Yang–Baxter equation (cf. [1,15, 33]).

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \leqslant \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x, x, x]\| = \|x\|^3$ (see [2,34]). Every left Hilbert C^*-module is a C^*-ternary algebra via the ternary product $[x, y, z] := \langle x, y\rangle z$.

If a C^*-ternary algebra $(A, [\cdot, \cdot, \cdot])$ has an identity, i.e., an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [x, e, y]$ and $x^* := [e, x, e]$, is a unital C^*-algebra. Conversely, if $(A, \circ)$ is a unital C^*-algebra, then $[x, y, z] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra.

A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *C^*-ternary algebra homomorphism* if

$$H\big([x, y, z]\big) = \big[H(x), H(y), H(z)\big]$$

for all $x, y, z \in A$. If, in addition, the mapping H is bijective, then the mapping $H : A \to B$ is called a *C^*-ternary algebra isomorphism*. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *C^*-ternary derivation* if

$$\delta\big([x, y, z]\big) = \big[\delta(x), y, z\big] + \big[x, \delta(y), z\big] + \big[x, y, \delta(z)\big]$$

for all $x, y, z \in A$ (see [2,16]).

In 1940, S.M. Ulam [32] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms:

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

In 1941, D.H. Hyers [9] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\big\| f(x + y) - f(x) - f(y)\big\| \leqslant \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\| f(x) - L(x) \| \leqslant \epsilon.$$

In 1978, Th.M. Rassias [24] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1. (Th.M. Rassias) *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\| f(x+y) - f(x) - f(y) \| \leqslant \epsilon \left(\|x\|^p + \|y\|^p \right) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\| f(x) - L(x) \| \leqslant \frac{2\epsilon}{2 - 2^p} \|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) *holds for $x, y \neq 0$ and* (1.2) *for $x \neq 0$.*

In 1990, Th.M. Rassias [25] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geqslant 1$. In 1991, Z. Gajda [7] following the same approach as in Th.M. Rassias [24], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [7], as well as by Th.M. Rassias and P. Šemrl [30] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$. The counterexamples of Z. Gajda [7], as well as of Th.M. Rassias and P. Šemrl [30] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [8], S. Jung [13], who among others studied the Hyers–Ulam–Rassias stability of functional equations. The inequality (1.1) that was introduced for the first time by Th.M. Rassias [24] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept is known as *Hyers–Ulam–Rassias stability* of functional equations (cf. the books of P. Czerwik [5], D.H. Hyers, G. Isac and Th.M. Rassias [10]).

J.M. Rassias [22] following the spirit of the innovative approach of Th.M. Rassias [24] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (see also [23] for a number of other new results).

P. Găvruta [8] provided a further generalization of Th.M. Rassias' Theorem. In 1996, G. Isac and Th.M. Rassias [12] applied the Hyers–Ulam–Rassias stability theory to prove fixed point theorems and study some new applications in Nonlinear Analysis. In [11], D.H. Hyers, G. Isac and Th.M. Rassias studied the asymptoticity aspect of Hyers–Ulam stability of mappings. During the several papers have been published on various generalizations and applications of Hyers–Ulam stability and Hyers–Ulam–Rassias stability to a number of functional equations and mappings, for example: quadratic functional equation, invariant means, multiplicative mappings—superstability, bounded nth differences, convex functions, generalized orthogonality functional equation, Euler–Lagrange functional equation, Navier–Stokes equations. Several mathematician have contributed works on these subjects; we mention a few: C. Baak and M.S. Moslehian [4], C. Park [17–21], Th.M. Rassias [26–29], F. Skof [31].

In Section 2, we prove the Hyers–Ulam–Rassias stability of homomorphisms in C^*-ternary algebras for the Cauchy–Jensen additive mappings.

In Section 3, we investigate isomorphisms between unital C^*-ternary algebras, associated to the Cauchy–Jensen additive mappings.

In Section 4, we prove the Hyers–Ulam–Rassias stability of derivations on C^*-ternary algebras for the Cauchy–Jensen additive mappings.

2. Stability of homomorphisms in C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$ and that B is a C^*-ternary algebra with norm $\|\cdot\|_B$.

For a given mapping $f : A \to B$, we define

$$C_\mu f(x,y,z) := f\left(\frac{\mu x+\mu y}{2}+\mu z\right)+\mu f\left(\frac{x-y}{2}+z\right)-\mu f(x)-2\mu f(z),$$

$$D_\mu f(x,y,z) := f\left(\frac{\mu x+\mu y}{2}+\mu z\right)-\mu f\left(\frac{x-y}{2}+z\right)-\mu f(y),$$

$$E_\mu f(x,y,z) := 2f\left(\frac{\mu x+\mu y}{2}+\mu z\right)-\mu f(x)-\mu f(y)-2\mu f(z)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $x, y, z \in A$.

We prove the Hyers–Ulam–Rassias stability of homomorphisms in C^*-ternary algebras for the functional equation $C_\mu f(x,y,z) = 0$.

Theorem 2.1. *Let $r > 3$ and θ be positive real numbers, and let $f : A \to B$ be a mapping such that*

$$\|C_\mu f(x,y,z)\|_B \leqslant \theta\left(\|x\|_A^r+\|y\|_A^r+\|z\|_A^r\right), \tag{2.1}$$

$$\left\|f([x,y,z])-[f(x),f(y),f(z)]\right\|_B \leqslant \theta\left(\|x\|_A^r+\|y\|_A^r+\|z\|_A^r\right) \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ such that

$$\|f(x)-H(x)\|_B \leqslant \frac{3\theta}{2^r-2}\|x\|_A^r \tag{2.3}$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and $y = z = x$ in (2.1), we get

$$\|f(2x)-2f(x)\|_B \leqslant 3\theta\|x\|_A^r \tag{2.4}$$

for all $x \in A$. So

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|_B \leqslant \frac{3\theta}{2^r}\|x\|_A^r$$

for all $x \in A$. Hence

$$\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_B \leqslant \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_B$$
$$\leqslant \frac{3\theta}{2^r} \sum_{j=l}^{m-1} \frac{2^j}{2^{rj}} \|x\|_A^r \tag{2.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.5) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.5), we get (2.3).

It follows from (2.1) that

$$\left\|H\left(\frac{x+y}{2} + z\right) + H\left(\frac{x-y}{2} + z\right) - H(x) - 2H(z)\right\|_B$$
$$= \lim_{n\to\infty} 2^n \left\|f\left(\frac{x+y}{2^{n+1}} + \frac{z}{2^n}\right) + f\left(\frac{x-y}{2^{n+1}} + \frac{z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - 2f\left(\frac{z}{2^n}\right)\right\|_B$$
$$\leqslant \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}} \left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r\right) = 0$$

for all $x, y, z \in A$. So

$$H\left(\frac{x+y}{2} + z\right) + H\left(\frac{x-y}{2} + z\right) = H(x) + 2H(z)$$

for all $x, y, z \in A$. By Lemma 2.1 of [3], the mapping $H : A \to B$ is Cauchy additive.

By the same reasoning as in the proof of Theorem 2.1 of [19], the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.2) that

$$\left\|H([x, y, z]) - [H(x), H(y), H(z)]\right\|_B$$
$$= \lim_{n\to\infty} 8^n \left\|f\left(\frac{[x, y, z]}{2^n \cdot 2^n \cdot 2^n}\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)\right]\right\|_B$$
$$\leqslant \lim_{n\to\infty} \frac{8^n\theta}{2^{nr}} \left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r\right) = 0$$

for all $x, y, z \in A$. So

$$H([x, y, z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in A$.

Now, let $T : A \to B$ be another Cauchy–Jensen additive mapping satisfying (2.3). Then we have

$$\left\|H(x) - T(x)\right\|_B = 2^n \left\|H\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|_B$$
$$\leqslant 2^n \left(\left\|H\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_B + \left\|T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_B\right)$$
$$\leqslant \frac{6 \cdot 2^n\theta}{(2^r - 2)2^{nr}} \|x\|_A^r,$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $H(x) = T(x)$ for all $x \in A$. This proves the uniqueness of H. Thus the mapping $H : A \to B$ is a unique C^*-ternary algebra homomorphism satisfying (2.3). □

Theorem 2.2. *Let $r < 1$ and θ be positive real numbers, and let $f : A \to B$ be a mapping satisfying* (2.1) *and* (2.2)*. Then there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ such that*

$$\left\| f(x) - H(x) \right\|_B \leqslant \frac{3\theta}{2 - 2^r} \|x\|_A^r \tag{2.6}$$

for all $x \in A$.

Proof. It follows from (2.4) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\|_B \leqslant \frac{3\theta}{2} \|x\|_A^r$$

for all $x \in A$. So

$$\begin{aligned} \left\| \frac{1}{2^l} f\left(2^l x\right) - \frac{1}{2^m} f\left(2^m x\right) \right\|_B &\leqslant \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\|_B \\ &\leqslant \frac{3\theta}{2} \sum_{j=l}^{m-1} \frac{2^{rj}}{2^j} \|x\|_A^r \end{aligned} \tag{2.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.7) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n \to \infty} \frac{1}{2^n} f\left(2^n x\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.7), we get (2.6).

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.3. *Let $r > 1$ and θ be positive real numbers, and let $f : A \to B$ be a mapping such that*

$$\left\| C_\mu f(x, y, z) \right\|_B \leqslant \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \cdot \|z\|_A^r, \tag{2.8}$$

$$\left\| f\left([x, y, z]\right) - \left[f(x), f(y), f(z) \right] \right\|_B \leqslant \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \cdot \|z\|_A^r \tag{2.9}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ such that

$$\left\| f(x) - H(x) \right\|_B \leqslant \frac{\theta}{8^r - 2} \|x\|_A^{3r} \tag{2.10}$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and $y = z = x$ in (2.8), we get

$$\left\| f(2x) - 2f(x) \right\|_B \leqslant \theta \|x\|_A^{3r} \tag{2.11}$$

for all $x \in A$. So

$$\left\| f(x) - 2f\left(\frac{x}{2}\right)\right\|_B \leqslant \frac{\theta}{8^r}\|x\|_A^{3r}$$

for all $x \in A$. Hence

$$\begin{aligned}\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_B &\leqslant \sum_{j=l}^{m-1}\left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_B \\ &\leqslant \frac{\theta}{8^r}\sum_{j=l}^{m-1}\frac{2^j}{8^{rj}}\|x\|_A^{3r} \qquad (2.12)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.12) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.12), we get (2.10).

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.4. *Let $r < \frac{1}{3}$ and θ be positive real numbers, and let $f : A \to B$ be a mapping satisfying* (2.8) *and* (2.9). *Then there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ such that*

$$\left\| f(x) - H(x)\right\|_B \leqslant \frac{\theta}{2 - 8^r}\|x\|_A^{3r} \qquad (2.13)$$

for all $x \in A$.

Proof. It follows from (2.11) that

$$\left\| f(x) - \frac{1}{2}f(2x)\right\|_B \leqslant \frac{\theta}{2}\|x\|_A^{3r}$$

for all $x \in A$. So

$$\begin{aligned}\left\| \frac{1}{2^l} f\left(2^l x\right) - \frac{1}{2^m} f\left(2^m x\right)\right\|_B &\leqslant \sum_{j=l}^{m-1}\left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right)\right\|_B \\ &\leqslant \frac{\theta}{2}\sum_{j=l}^{m-1}\frac{8^{rj}}{2^j}\|x\|_A^{3r} \qquad (2.14)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.14) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.14), we get (2.13).

The rest of the proof is similar to the proof of Theorem 2.1. □

One can obtain similar results for the functional equations $D_\mu f(x,y,z)=0$ and $E_\mu f(x,y,z)=0$.

3. Isomorphisms between C^*-ternary algebras

Throughout this section, assume that A is a unital C^*-ternary algebra with norm $\|\cdot\|_A$ and unit e, and that B is a unital C^*-ternary algebra with norm $\|\cdot\|_B$ and unit e'.

We investigate isomorphisms between C^*-ternary algebras, associated to the functional equation $D_\mu f(x,y,z)=0$.

Theorem 3.1. *Let $r>1$ and θ be positive real numbers, and let $f:A\to B$ be a bijective mapping such that*

$$\left\|D_\mu f(x,y,z)\right\|_B \leqslant \theta\left(\|x\|_A^r+\|y\|_A^r+\|z\|_A^r\right), \tag{3.1}$$

$$f\left([x,y,z]\right)=\left[f(x),f(y),f(z)\right] \tag{3.2}$$

for all $\mu\in\mathbb{T}^1$ and all $x,y,z\in A$. If $\lim_{n\to\infty}2^n f(\frac{e}{2^n})=e'$, then the mapping $f:A\to B$ is a C^*-ternary algebra isomorphism.

Proof. Letting $\mu=1$ and $y=z=x$ in (3.1), we get

$$\left\|f(2x)-2f(x)\right\|_B \leqslant 3\theta\|x\|_A^r \tag{3.3}$$

for all $x\in A$. So

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|_B \leqslant \frac{3\theta}{2^r}\|x\|_A^r$$

for all $x\in A$. Hence

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right\|_B &\leqslant \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right)-2^{j+1}f\left(\frac{x}{2^{j+1}}\right)\right\|_B\\ &\leqslant \frac{3\theta}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\|x\|_A^r \end{aligned}\tag{3.4}$$

for all nonnegative integers m and l with $m>l$ and all $x\in A$. It follows from (3.4) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x\in A$. Since B is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $H:A\to B$ by

$$H(x):=\lim_{n\to\infty}2^n f\left(\frac{x}{2^n}\right)$$

for all $x\in A$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (3.4), we get

$$\left\|f(x)-H(x)\right\|_B \leqslant \frac{3\theta}{2^r-2}\|x\|_A^r$$

for all $x\in A$.

It follows from (3.1) that

$$\begin{aligned}&\left\|H\left(\frac{x+y}{2}+z\right)-H\left(\frac{x-y}{2}+z\right)-H(y)\right\|_B\\&\quad=\lim_{n\to\infty}2^n\left\|f\left(\frac{x+y}{2^{n+1}}+\frac{z}{2^n}\right)-f\left(\frac{x-y}{2^{n+1}}+\frac{z}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right\|_B\\&\quad\leqslant\lim_{n\to\infty}\frac{2^n\theta}{2^{nr}}\left(\|x\|_A^r+\|y\|_A^r+\|z\|_A^r\right)=0\end{aligned}$$

for all $x, y, z \in A$. So

$$H\left(\frac{x+y}{2}+z\right)-H\left(\frac{x-y}{2}+z\right)=H(y)$$

for all $x, y, z \in A$. By Lemma 2.1 of [3], the mapping $H : A \to B$ is Cauchy additive.

By the same reasoning as in the proof of Theorem 2.1 of [19], the mapping $H : A \to B$ is $\mathbb{C}$-linear.

Since $f([x, y, z]) = [f(x), f(y), f(z)]$ for all $x, y, z \in A$,

$$\begin{aligned}H\big([x,y,z]\big)&=\lim_{n\to\infty}8^n f\left(\left[\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right]\right)=\lim_{n\to\infty}\left[2^n f\left(\frac{x}{2^n}\right),2^n f\left(\frac{y}{2^n}\right),2^n f\left(\frac{z}{2^n}\right)\right]\\&=\big[H(x),H(y),H(z)\big]\end{aligned}$$

for all $x, y, z \in A$. So the mapping $H : A \to B$ is a C^*-ternary algebra homomorphism.

It follows from (3.2) that

$$\begin{aligned}H(x)&=H\big([e,e,x]\big)=\lim_{n\to\infty}4^n f\left(\frac{1}{4^n}[e,e,x]\right)=\lim_{n\to\infty}4^n f\left(\left[\frac{e}{2^n},\frac{e}{2^n},x\right]\right)\\&=\lim_{n\to\infty}\left[2^n f\left(\frac{e}{2^n}\right),2^n f\left(\frac{e}{2^n}\right),f(x)\right]=\big[e',e',f(x)\big]=f(x)\end{aligned}$$

for all $x \in A$. Hence the bijective mapping $f : A \to B$ is a C^*-ternary algebra isomorphism. □

Theorem 3.2. *Let $r < 1$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (3.1) *and* (3.2). *If $\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e'$, then the mapping $f : A \to B$ is a C^*-ternary algebra isomorphism.*

Proof. It follows from (3.3) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\|_B\leqslant\frac{3\theta}{2}\|x\|_A^r$$

for all $x \in A$. So

$$\begin{aligned}\left\|\frac{1}{2^l}f\big(2^l x\big)-\frac{1}{2^m}f\big(2^m x\big)\right\|_B&\leqslant\sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\big(2^j x\big)-\frac{1}{2^{j+1}}f\big(2^{j+1}x\big)\right\|_B\\&\leqslant\frac{3\theta}{2}\sum_{j=l}^{m-1}\frac{2^{rj}}{2^j}\|x\|_A^r\end{aligned}\tag{3.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (3.5) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.5), we get

$$\left\| f(x) - H(x) \right\|_B \leqslant \frac{3\theta}{2 - 2^r} \|x\|_A^r$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.3. *Let $r > \frac{1}{3}$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (3.2) *such that*

$$\left\| D_\mu f(x, y, z) \right\|_B \leqslant \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \cdot \|z\|_A^r \tag{3.6}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. If $\lim_{n\to\infty} 2^n f(\frac{e}{2^n}) = e'$, *then the mapping $f : A \to B$ is a C^*-ternary algebra isomorphism.*

Proof. Letting $\mu = 1$ and $y = z = x$ in (3.6), we get

$$\left\| f(2x) - 2f(x) \right\|_B \leqslant \theta \|x\|_A^{3r} \tag{3.7}$$

for all $x \in A$. So

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\|_B \leqslant \frac{\theta}{8^r} \|x\|_A^{3r}$$

for all $x \in A$. Hence

$$\begin{aligned} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\|_B &\leqslant \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\|_B \\ &\leqslant \frac{\theta}{8^r} \sum_{j=l}^{m-1} \frac{2^j}{8^{rj}} \|x\|_A^{3r} \end{aligned} \tag{3.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (3.8) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get

$$\left\| f(x) - H(x) \right\|_B \leqslant \frac{\theta}{8^r - 2} \|x\|_A^{3r}$$

for all $x \in A$.

The rest of the proof is similar to the proofs of Theorems 2.3 and 3.1. □

Theorem 3.4. *Let $r < \frac{1}{3}$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (3.2) *and* (3.6)*. If $\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e'$, then the mapping $f : A \to B$ is a C^*-ternary algebra isomorphism.*

Proof. It follows from (3.7) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\|_B \leqslant \frac{\theta}{2} \|x\|_A^{3r}$$

for all $x \in A$. So

$$\begin{aligned}\left\| \frac{1}{2^l} f\left(2^l x\right) - \frac{1}{2^m} f\left(2^m x\right) \right\|_B &\leqslant \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\|_B \\ &\leqslant \frac{\theta}{2} \sum_{j=l}^{m-1} \frac{8^{rj}}{2^j} \|x\|_A^{3r} \end{aligned} \tag{3.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (3.9) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n} f\left(2^n x\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get

$$\left\| f(x) - H(x) \right\|_B \leqslant \frac{\theta}{2 - 8^r} \|x\|_A^{3r}$$

for all $x \in A$.

The rest of the proof is similar to the proofs of Theorems 2.4 and 3.1. □

One can obtain similar results for the functional equations $C_\mu f(x, y, z) = 0$ and $E_\mu f(x, y, z) = 0$.

4. Stability of derivations on C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$.

We prove the Hyers–Ulam–Rassias stability of derivations on C^*-ternary algebras for the functional equation $E_\mu f(x, y, z) = 0$.

Theorem 4.1. *Let $r > 3$ and θ be positive real numbers, and let $f : A \to A$ be a mapping such that*

$$\left\| E_\mu f(x, y, z) \right\|_A \leqslant \theta\left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r\right), \tag{4.1}$$

$$\begin{aligned}&\left\| f\left([x, y, z]\right) - \left[f(x), y, z\right] - \left[x, f(y), z\right] - \left[x, y, f(z)\right] \right\|_A \\ &\quad \leqslant \theta\left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r\right)\end{aligned} \tag{4.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique C^*-ternary derivation $\delta : A \to A$ such that

$$\left\| f(x) - \delta(x) \right\|_A \leqslant \frac{3\theta}{2^{r+1} - 4} \|x\|_A^r \tag{4.3}$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and all $y = z = x$ in (4.1), we get

$$\left\|2f(2x) - 4f(x)\right\|_A \leqslant 3\theta\|x\|_A^r \tag{4.4}$$

for all $x \in A$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_A \leqslant \frac{3\theta}{2^{r+1}}\|x\|_A^r$$

for all $x \in A$. Hence

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_A &\leqslant \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_A \\ &\leqslant \frac{3\theta}{2\cdot 2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\|x\|_A^r \end{aligned} \tag{4.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (4.5) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $\delta : A \to A$ by

$$\delta(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.5), we get (4.3).

It follows from (4.1) that

$$\begin{aligned}&\left\|2\delta\left(\frac{x+y}{2} + z\right) - \delta(x) - \delta(y) - 2\delta(z)\right\|_A \\ &\quad = \lim_{n\to\infty} 2^n\left\|2f\left(\frac{x+y}{2^{n+1}} + \frac{z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - 2f\left(\frac{z}{2^n}\right)\right\|_A \\ &\quad \leqslant \lim_{n\to\infty}\frac{2^n\theta}{2^{nr}}\left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r\right) = 0\end{aligned}$$

for all $x, y, z \in A$. So

$$\delta\left(\frac{x+y}{2} + z\right) = \delta(x) + \delta(y) + 2\delta(z)$$

for all $x, y, z \in A$. By Lemma 2.1 of [3], the mapping $\delta : A \to A$ is Cauchy additive.

By the same reasoning as in the proof of Theorem 2.1 of [19], the mapping $\delta : A \to A$ is $\mathbb{C}$-linear.

It follows from (4.2) that

$$\begin{aligned}&\left\|\delta([x,y,z]) - [\delta(x), y, z] - [x, \delta(y), z] - [x, y, \delta(z)]\right\|_A \\ &\quad = \lim_{n\to\infty} 8^n\left\|f\left(\frac{[x,y,z]}{8^n}\right) - \left[f\left(\frac{x}{2^n}\right), \frac{y}{2^n}, \frac{z}{2^n}\right]\right. \\ &\qquad \left. - \left[\frac{x}{2^n}, f\left(\frac{y}{2^n}\right), \frac{z}{2^n}\right] - \left[\frac{x}{2^n}, \frac{y}{2^n}, f\left(\frac{z}{2^n}\right)\right]\right\|_A \\ &\quad \leqslant \lim_{n\to\infty}\frac{8^n\theta}{2^{nr}}\left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r\right) = 0\end{aligned}$$

for all $x, y, z \in A$. So

$$\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in A$.

Now, let $T : A \to A$ be another Cauchy–Jensen additive mapping satisfying (4.3). Then we have

$$\begin{aligned}
\|\delta(x) - T(x)\|_A &= 2^n \left\| \delta\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\|_A \\
&\leqslant 2^n \left(\left\| \delta\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\|_A + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\|_A \right) \\
&\leqslant \frac{3 \cdot 2^n \theta}{(2^r - 2)2^{nr}} \|x\|_A^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $\delta(x) = T(x)$ for all $x \in A$. This proves the uniqueness of δ. Thus the mapping $\delta : A \to A$ is a unique C^*-ternary derivation satisfying (4.3). □

Theorem 4.2. *Let $r < 1$ and θ be positive real numbers, and let $f : A \to A$ be a mapping satisfying* (4.1) *and* (4.2)*. Then there exists a unique C^*-ternary derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \leqslant \frac{3\theta}{4 - 2^{r+1}} \|x\|_A^r \tag{4.6}$$

for all $x \in A$.

Proof. It follows from (4.4) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\|_A \leqslant \frac{3\theta}{4} \|x\|_A^r$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 4.1. □

Theorem 4.3. *Let $r > 1$ and θ be positive real numbers, and let $f : A \to A$ be a mapping such that*

$$\|E_\mu f(x, y, z)\|_A \leqslant \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \cdot \|z\|_A^r, \tag{4.7}$$

$$\begin{aligned}
&\|f([x, y, z]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)]\|_A \\
&\quad \leqslant \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \cdot \|z\|_A^r
\end{aligned} \tag{4.8}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique C^*-ternary derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leqslant \frac{\theta}{2(8^r - 2)} \|x\|_A^{3r} \tag{4.9}$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and $y = z = x$ in (4.7), we get

$$\|2f(2x) - 4f(x)\|_A \leqslant \theta \|x\|_A^{3r} \tag{4.10}$$

for all $x \in A$. So

$$\left\| f(x) - 2f\left(\frac{x}{2}\right)\right\|_A \leqslant \frac{\theta}{2\cdot 8^r}\|x\|_A^{3r}$$

for all $x \in A$. Hence

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_A &\leqslant \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_A \\ &\leqslant \frac{\theta}{2\cdot 8^r}\sum_{j=l}^{m-1}\frac{2^j}{8^{rj}}\|x\|_A^{3r} \qquad (4.11)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (4.11) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $\delta : A \to A$ by

$$\delta(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.11), we get (4.9).

The rest of the proof is similar to the proof of Theorem 4.1. □

Theorem 4.4. *Let $r < \frac{1}{3}$ and θ be positive real numbers, and let $f : A \to A$ be a mapping satisfying* (4.7) *and* (4.8). *Then there exists a unique C^*-ternary derivation $\delta : A \to A$ such that*

$$\| f(x) - \delta(x)\|_A \leqslant \frac{\theta}{2(2-8^r)}\|x\|_A^{3r} \qquad (4.12)$$

for all $x \in A$.

Proof. It follows from (4.10) that

$$\left\| f(x) - \frac{1}{2} f(2x)\right\|_A \leqslant \frac{\theta}{4}\|x\|_A^{3r}$$

for all $x \in A$.

The rest of the proof is similar to the proofs of Theorems 4.1 and 4.3. □

One can obtain similar results for the functional equations $C_\mu f(x, y, z) = 0$ and $D_\mu f(x, y, z) = 0$.

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