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AN EQUALITY CONSTRAINED LEAST SQUARES APPROACH TO THE STRUCTURAL REANALYSIS

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ABSTRACT. An efficient method for reanalysis of a damaged structures is presented. Perturbation analysis for the equality constrained least squares problem is adapted to handle structural reanalysis, and related theoretical and numerical results are presented.

1. Introduction

In structural analysis, an important problem is the computation of the vector f of internal forces, given a finite element model of the structure and a set of applied loads. For notational purposes, let E denote the $p \times n$ *equilibrium matrix*, let d denote the p-vector of *nodal(applied) loads*, and let F denote the $n \times n$ element-level block diagonal *element flexibility matrix*. Here F represents the material properties of the structure and is generally symmetric, positive definite, and block diagonal, where the diagonal blocks correspond to the elements in the finite element model.

The internal forces vector f solves the quadratic programming problem [7]

$$\min \tfrac{1}{2} f^T F f \quad \text{subject to} \quad Ef = d. \tag{1}$$

It is assumed the flexbility matrix F is given in the decomposed form $F = G^T G$ [9], by the Cholesky algorithm, it follows that (1) is equivalent to the linear least squares problem subject to equality constraints (LSE problem)

$$\min \|Gf\| \quad \text{subject to} \quad Ef = d. \tag{2}$$

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We will assume that

$$(3) \qquad \operatorname{rank}(E) = p \quad \text{and} \quad \operatorname{null}(G) \cap \operatorname{null}(E) = \{0\}.$$

The assumption that E is of full rank ensures that d is in the range of E and the LSE problem has at least one solution. The second condition in (3) (equivalently, $\operatorname{rank}\binom{E}{G} = n$) guarantees that there is a unique solution. These assumptions are realistic for a wide range of important applications, including the structural analysis.

A standard method for solving the LSE problem is the nullspace method, so-called because it employs an orthogonal basis for the null-space of the constraint matrix E. In this paper we apply the nullspace method and the perturbatioin analysis of LSE problem to the reanalysis of a structure. Reanalysis refers to the analysis of a structure which has been slightly modified [4, 9]. Assume that the perturbed Cholesky factor $\overline{G}$ and the perturbed load vetor $\overline{d}$ are given by

$$(4) \qquad \overline{G} = G + \delta G \quad \text{and} \quad \overline{d} = d + \delta d,$$

where δG and δd are the changes in the Cholesky factor of flexibility matrix and in the load vector, respectively, due to the small scale modification in structure. Since the geometric layout of the element, for the static case, is assumed not to change in small scale modification case, the values of E remains the same.

The purpose of a reanalysis procedure is to consider the problem of finding the perturbed internal force vector $\overline{f}$ such that

$$(5) \qquad \min \|\overline{G}\,\overline{f}\| \quad \text{subject to} \quad E\overline{f} = \overline{d},$$

using, as much as possible, quantities calculated in the original LSE problem (2).

Various means to accomplish reanalysis of modified structures have been investigated in [2, 4, 9, 13]. In [12], Plemmons and White proposed reanalysis scheme by the force method work based on QR factorization for the case that only one element (that is, one small block of the matrix G) has been modified. In [8], Jang applied the perturbation analysis of LSE problem based on the Paige's formulation (in [10]) to handle reanalysis for the small scale damaged structure. However, the bounds in [12, 8] are rather complicated and it is more satisfactory to work directly form the easily derived argumented system, Kuhn -Tucker condition for the problem (2).

In this paper, the method that we devise will be based upon the Kuhn-Tucker conditions which are stated in the system of linear equations

$$\begin{bmatrix} 0 & 0 & E \\ 0 & I & G \\ E^T & G^T & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ r \\ f \end{bmatrix} = \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix}, \tag{6}$$

where λ is a vector of Lagrange multipliers and the residual vector r represents the nodal displacements.

2. The nullspace method

We describe the nullspace method for solving the LSE problem with a version based on the generalized QR factorization [11].

First compute an orthogonal matrix $Q \in \mathcal{R}^{n\times n}$ such that

$$Q^T E^T = \begin{bmatrix} S^T \\ 0 \end{bmatrix},$$

where $S \in \mathcal{R}^{p\times p}$ is lower triangular and nonsingular. The constraint $Ef = d$ may be written

$$Sy_1 = [S\ \ 0] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = d,$$

and

$$y = Q^T f.$$

Hence the constraint determines $y_1 \in \mathcal{R}^p$ as the solution of the triangular system $Sy_1 = d$ and leaves $y_2 \in \mathcal{R}^{n-p}$ arbitrary. Partition $Q = [Q_1\ Q_2]$ conformably with $[S\ 0]$. Then, clearly, null(E)=range(Q_2). Now form GQ and then construct orthogonal $U \in \mathcal{R}^{n\times n}$ to essentially produces

$$U^T GQ = U^T G[Q_1\ \ Q_2] = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}, \tag{7}$$

where $L_{11} \in \mathcal{R}^{p\times p}$ and $L_{22} \in \mathcal{R}^{(n-p)\times(n-p)}$ is lower triangular. We also can write

$$G[Q_1\ \ Q_2] = [U_1\ \ U_2] \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix},$$

so that $GQ_2 = U_2 L_{22}$. It follows that the second condition of (3) is equivalent to L_{22} being nonsingular. Since $\|Gf\|_2 = \|U^T GQy\|_2$, we see that we have to find

$$\min_{y_2}\left\|\begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22}\end{bmatrix}\begin{bmatrix} y_1 \\ y_2\end{bmatrix}\right\|_2 = \min_{y_2}\left\|\begin{bmatrix} L_{11}y \\ L_{21}y_1 - L_{22}y_2\end{bmatrix}\right\|_2.$$

Therefore y_2 is the solution to $L_{22}y_2 = L_{21}y_1$. Finally, f is recovered from $f = Qy$.

The computation in (7) does not take advantage of any special structure the matrix G have (G will be triangular if it is computed by the Cholesky factorization, and in our case G has block diagonal structures as well). We may use a Paige's formulation [10] in which the two orthogonal transformations U and Q simultaneously in a manner which retains the triangular structure of G throughout the computations. For implementation details, see [10].

3. Reanalysis with nullspace method

We here assume that the condition (3) holds for the perturbed data $\overline{G} = G + \delta G$, which will certainly be true if δG is sufficiently small [4], and will measure the perturbations by the smallest ϵ for which

$$\|\delta G\| \leq \epsilon\|G\| \quad \text{and} \quad \|\delta d\| \leq \epsilon\|d\|, \tag{8}$$

where $\|\cdot\|$ will always denote the 2-norm.

For the perturbed problem, the system (6) can be rewritten as

$$\begin{bmatrix} 0 & 0 & E \\ 0 & I & G+\delta G \\ E^T & G^T+\delta G^T & 0\end{bmatrix}\begin{bmatrix}\overline{\lambda} \\ \overline{r} \\ \overline{f}\end{bmatrix} = \begin{bmatrix} d+\delta d \\ 0 \\ 0\end{bmatrix}. \tag{9}$$

Perturbed data result in (4) lead to $\overline{\lambda} = \lambda + \delta\lambda$, $\overline{r} = r + \delta r$, $\overline{f} = f + \delta f$ of the perturbed problem (9). Subtracting (6) from (9) we obtain

$$\begin{bmatrix} 0 & 0 & E \\ 0 & I & G \\ E^T & G^T & 0\end{bmatrix}\begin{bmatrix}\delta\lambda \\ \delta r \\ \delta f\end{bmatrix} = \begin{bmatrix}\delta d \\ -(\delta G)\overline{f} \\ -(\delta G)^T\overline{r}\end{bmatrix}. \tag{10}$$

Then the inverse of the matrix on the left-hand side of (10) is (see, e.g., [6])

$$\begin{bmatrix} (GE_G^+)^T GE_G^+ & -(GE_G^+)^T & E_G^{+^T} \\ -GE_G^+ & I-(GP)(GP)^+ & (P(GP)^+)^T \\ E_G^+ & (GP)^+ & -P((GP)^T GP)^+ P^T\end{bmatrix}, \tag{11}$$

$$E^+ = Q\begin{bmatrix} S^{-1} \\ 0\end{bmatrix},$$

$$P = I_n - E^+E,$$
$$(GP)^+ = Q\begin{bmatrix} 0 & 0 \\ 0 & L_{22}^{-1} \end{bmatrix} U^T,$$
$$E_G^+ = (I_n - (GP)^+G)E^+.$$

Perturbation theory for the LSE problem is given in several earlier references [1, 3, 6]. In [5], Cox and Higham give a full analysis in order to investigate the sharpness of the bounds. Applying the perturbation analysis discussed in [5] to our case, we get the following results which could be viewed as an important special case of Cox and Higham's work.

THEOREM 1. *Consider the LSE problem (2). Let* $\overline{G} = G + \delta G$ *and* $\overline{d} = d + \delta d$. *Suppose that the conditions in* (3) *for* $\overline{G}$ *and* (8) *hold. Then the LSE solution* $\overline{f}$ *of* (5) *is given by* $\overline{f} = f + \delta f$, *where*

$$\delta f = E_G^+(\delta d) - [(GP)^+(\delta G) + ((GP)^TGP)^+(\delta G)^TG]f + O(\epsilon^2) \tag{12}$$

and

$$\|\delta f\| \le \epsilon\left[\|E_G^+\|\|d\| + (1+\kappa_E(G))\kappa_E(G)\|f\|\right] + O(\epsilon^2) \equiv \beta_f. \tag{13}$$

Here

$$\kappa_E(G) = \|G\|\|(GP)^+\|$$

and P, $(GP)^+$, *and* E_G^+ *are as in* (11).

Proof. Using the factorization (7) and (11) it is straightforward to show that

$$(P(GP)^+)^T = (GP)^{+^T},$$
$$P((GP)^TGP)^+P^T = ((GP)^TGP)^+,$$

which yields the following expression for the inverse

$$\begin{bmatrix} (GE_G^+)^TGE_G^+ & -(GE_G^+)^T & E_G^{+^T} \\ -GE_G^+ & I - (GP)(GP)^+ & (GP)^{+^T} \\ E_G^+ & (GP)^+ & -((GP)^TGP)^+ \end{bmatrix}. \tag{14}$$

Using (14) and from (10) we obtain

$$\delta f = E_G^+(\delta d) - (GP)^+(\delta G)\overline{f} + ((GP)^TGP)^+(\delta G)^T\overline{r}.$$

Since δf and δr are all of order ϵ, we can replace $\overline{f}$ and $\overline{r}$ by their unperturbed counterparts to obtain first order expressions. And since $r = -Gf$, we have

$$\delta f = E_G^+(\delta d) - [(GP)^+(\delta G) + ((GP)^T GP)^+(\delta G)^T G]f + O(\epsilon^2).$$

Taking 2-norms we obtain

$$\|\delta f\| \leq \|E_G^+\|\|\delta d\| + \left[\|(GP)^+\| + \|((GP)^T GP)^+\|\|G\|\right] \|\delta G\|\|f\| + O(\epsilon^2).$$

Using (8), and the fact that $\|((GP)^T GP)^+\| = \|(GP)^+\|^2$, we have

$$\|\delta f\| \leq \epsilon \left[\|E_G^+\|\|d\| + \left(1 + \|(GP)^+\|\|G\|\right) \|(GP)^+\|\|G\|\|f\|\right] + O(\epsilon^2).$$

Define $\kappa_E(G) = \|G\|\|(GP)^+\|$, the bound can be rewritten as

$$\|\delta f\| \leq \epsilon \left[\|E_G^+\|\|d\| + (1 + \kappa_E(G))\kappa_E(G)\|f\|\right] + O(\epsilon^2).$$

□

We see that the sensitivity is governed by $\|E_G^+\|$ and $\kappa_E(G)$, and a sufficient condition for the LSE problem to be well conditioned is that E and GP are both well conditioned. To obtain a sharp bound we must combine the two δG terms before taking norms. However, the bound obtained such a way (see [5]) is much more difficult to interpret than (13).

The bound (13) requires computation of the quantities $\|E_G^+\|$ and $\|(GP)^+\|$. Using (7) and (11) these quantities can be expressed as

$$\|E_G^+\| = \|L_{22}^{-1} L_{21} S^{-1}\|$$

and

$$\|(GP)^+\| = \|L_{22}^{-1}\|.$$

The computational saving to estimate the norms of these matrices is discussed in [5].

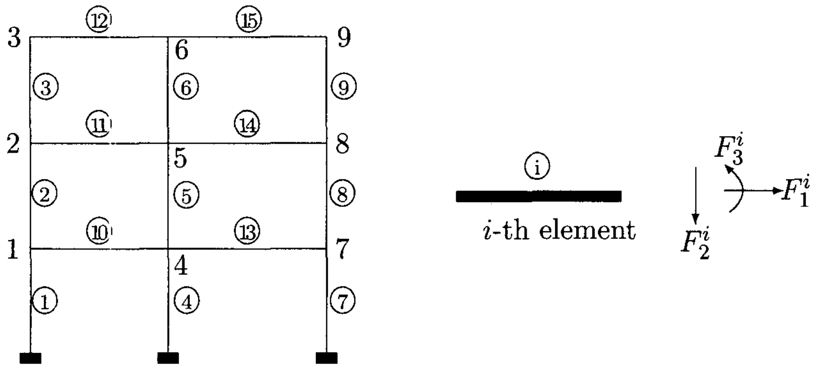

FIGURE 1. Two-dimensional frame with element and node numbering

TABLE 1. 1 element(G_i, for i=12) has been modified

d_K^{12}	0.99999	0.99	0.9	0.09	0.0009	0.000009
$\kappa_2(\overline{G})$	2.95e+04	9.32e+02	2.95e+02	9.77e+01	9.33e+01	9.32e+01
$\|\delta f\|_2$	4.89e+01	4.81e+01	4.11e+01	2.69e+00	2.59e−02	2.58e−04
β_f	2.16e+05	6.17e+03	1.49e+03	3.31e+01	3.08e−01	3.09e−03

TABLE 2. 5 elements(G_i, for i=1,2,3,13,14) have been modified

$d_K^{1,2,3}$	0.99999	0.99	0.9	0.09	0.0009	0.000009
$d_K^{13,14}$	0.000009	0.0009	0.09	0.9	0.99	0.99999
$\kappa_2(\overline{G})$	5.17e+03	1.64e+02	9.32e+01	1.11e+02	3.51e+02	1.11e+04
$\|\delta f\|_2$	6.86e+02	6.57e+02	4.72e+02	2.31e+02	3.30e+02	3.49e+02
β_f	1.36e+05	3.87e+03	9.20e+02	1.86e+03	7.76e+03	2.72e+05

4. Numerical experiments

To illustrate numerical results for reanalysis of a small scale damaged structure, consider the two dimensional frame with 15 elements and 9 nodes shown in Figure 1. In this case the equilibrium matrix E is 27×45 and the element flexibility matrix F is 45×45 matrix. For the element and node numbering of this structure example, we used the method in [12], which we call the substructuring method with proper partitions.

Damage will be measured on an element-by-element basis, and the effect upon the ith block (corresponds to the ith element) of element flexibility is

$$[F_i]_{dF} = [F_i]/(1 - d_K^i),$$

where dF refers to values in the damaged state and d_K^i is 'stiffness damage' to the ith element. The number d_K^i lies between 0 and 1 inclusive, and represent a fractional decrease in load capacity. Small scale damage is defined as less-than-total stiffness damage, i.e., when $d_K^i < 1$.

Some numerical results obtained by using MATLAB(for various d_K^i and corresponding condition number $\kappa_2(\overline{G})$) are listed in Table 1 and 2. We see from the tables that our bound for δf is suitable for practical computation.

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