Journal of Mathematical Physics

Isomorphisms between C^*-ternary algebras

Cite as: J. Math. Phys. **47**, 103512 (2006); https://doi.org/10.1063/1.2359576
Submitted: 09 March 2006 • Accepted: 12 September 2006 • Published Online: 31 October 2006

Choonkil Park

View Online Export Citation

ARTICLES YOU MAY BE INTERESTED IN

Inverted bulk-heterojunction organic photovoltaic device using a solution-derived ZnO underlayer
Applied Physics Letters **89**, 143517 (2006); https://doi.org/10.1063/1.2359579

Optical Society of America
Physics Today **14**, 108 (1961); https://doi.org/10.1063/1.3057320

Temperature Measurement and Control
Physics Today **14**, 108 (1961); https://doi.org/10.1063/1.3057322

AIP Publishing

© 2006 American Institute of Physics.

JOURNAL OF MATHEMATICAL PHYSICS **47**, 103512 (2006)

Isomorphisms between C^*-ternary algebras

Choonkil Park[a)]
Department of Mathematics, Hanyang University, Seoul, 133-791 Republic of Korea

(Received 9 March 2006; accepted 12 September 2006; published online 31 October 2006)

In this paper, we prove the Hyers-Ulam-Rassias stability of homomorphisms in C^*-ternary algebras and of derivations on C^*-ternary algebras for the following generalized Cauchy–Jensen additive mapping:

$$2f\left(\frac{\sum_{j=1}^{p} x_j}{2} + \sum_{j=1}^{d} y_j\right) = \sum_{j=1}^{p} f(x_j) + 2\sum_{j=1}^{d} f(y_j).$$

This is applied to investigate isomorphisms between C^*-ternary algebras. The concept of Hyers-Ulam-Rassias stability originated from the Rassias stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, see Proc. Amr. Math. Soc. **72**, 297–300 (1978). © *2006 American Institute of Physics.* [DOI: 10.1063/1.2359576]

I. INTRODUCTION AND PRELIMINARIES

Ternary algebraic operations were considered in the 19th century by several mathematicians and physicists such as Cayley,[5] who introduced the notion of a *cubic matrix*, which in turn was generalized by Kapranov *et al.*[15] The simplest example of such nontrivial ternary operation is given by the following composition rule:

$$\{a,b,c\}_{ijk} = \sum_{l,m,n} a_{nil} b_{ljm} c_{mkn} \quad (i,j,k,\ldots = 1,2,\ldots,N).$$

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their applications in physics. Some significant physical applications are as follows (see Refs. 16 and 17):

(1) The algebra of "*nonions*" generated by two matrices,

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix} \quad (\omega = e^{2\pi i/3}),$$

was introduced by Sylvester as a ternary analog of Hamilton's quaternions (cf. Ref. 1).

(2) A natural ternary composition of four-vectors in the four-dimensional Minkowskian space-time M_4 can be defined as an example of a ternary operation:

$$(X,Y,Z) \to U(X,Y,Z) \in M_4,$$

with the resulting four-vector U^μ defined via its components in a given coordinate system as follows:

[a)]Electronic address: baak@hanyang.ac.kr

 © 2006 American Institute of Physics

$$U^{\mu}(X,Y,Z) = g^{\mu\sigma}\eta_{\sigma\nu\lambda\rho}X^{\nu}Y^{\lambda}Z^{\rho}, \quad \mu,\nu,\ldots = 0,1,2,3,$$

where $g^{\mu\sigma}$ is the metric tensor and $\eta_{\sigma\nu\lambda\rho}$ is the canonical volume element of M_4 (see Ref. 17).

(3) The quark model inspired a particular brand of ternary algebraic systems. The so-called "*Nambu mechnics*" is based on such structures (see Ref. 7).

There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the nonstandard statistics, supersymmetric theory, and Yang-Baxter equation (cf. Refs. 1, 17, and 35).

Following the terminology of Ref. 2 a non-empty set G with a ternary operation $[\cdot,\cdot,\cdot]:G\times G\times G\to G$ is called a *ternary groupoid* and is denoted by $(G,[\cdot,\cdot,\cdot])$. The ternary groupoid $(G,[\cdot,\cdot,\cdot])$ is called *commutative* if $[x_1,x_2,x_3]=[x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}]$ for all $x_1,x_2,x_3\in G$ and all permutations σ of $\{1,2,3\}$.

If a binary operation $\circ$ is defined on G such that $[x,y,z]=(x\circ y)\circ z$ for all $x,y,z\in G$, then we say that $[\cdot,\cdot,\cdot]$ is derived from $\circ$. We say that $(G,[\cdot,\cdot,\cdot])$ is a *ternary semigroup* if the operation $[\cdot,\cdot,\cdot]$ is *associative*, i.e., if $[[x,y,z],u,v]=[x,[y,z,u],v]=[x,y,[z,u,v]]$ holds for all $x,y,z,u,v\in G$ (see Ref. 4).

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x,y,z)\mapsto[x,y,z]$ of A^3 into A, which is $\mathbb{C}$ linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x,y,[z,w,v]]=[x,[w,z,y],v]=[[x,y,z],w,v]$, and satisfies $\|[x,y,z]\|\le\|x\|\cdot\|y\|\cdot\|z\|$ and $\|[x,x,x]\|=\|x\|^3$ (see Refs. 2 and 36). Every left Hilbert C^* module is a C^*-ternary algebra via the ternary product $[x,y,z]:=\langle x,y\rangle z$.

If a C^*-ternary algebra $(A,[\cdot,\cdot,\cdot])$ has an identity, i.e., an element $e\in A$ such that $x=[x,e,e]=[e,e,x]$ for all $x\in A$, then it is routine to verify that A, endowed with $x\circ y:=[x,e,y]$ and $x^*:=[e,x,e]$, is a unital C^* algebra. Conversely, if $(A,\circ)$ is a unital C^* algebra, then $[x,y,z]:=x\circ y^*\circ z$ makes A into a C^*-ternary algebra.

A $\mathbb{C}$-linear mapping $H:A\to B$ is called a *C^*-ternary algebra homomorphism* if

$$H([x,y,z]) = [H(x),H(y),H(z)],$$

for all $x,y,z\in A$. If, in addition, the mapping H is bijective, then the mapping $H:A\to B$ is called a *C^*-ternary algebra isomorphism*. A $\mathbb{C}$-linear mapping $\delta:A\to A$ is called a *C^*-ternary derivation* if

$$\delta([x,y,z]) = [\delta(x),y,z] + [x,\delta(y),z] + [x,y,\delta(z)],$$

for all $x,y,z\in A$ (see Refs. 2 and 18).

In 1940, Ulam[34] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if $f:G\to G'$ satisfies $\rho(f(xy),f(x)f(y))<\delta$ for all $x,y\in G$, then a homomorphism $h:G\to G'$ exists with $\rho(f(x),h(x))<\epsilon$ for all $x\in G$?

In 1941, Hyers[10] considered the case of approximately additive mappings $f:E\to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*,

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon,$$

for all $x,y\in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

exists for all $x\in E$ and that $L:E\to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

In 1978, Rassias[26] provided a generalization of Hyers' Theorem that allows the *Cauchy difference to be unbounded*.

Theorem 1.1: (Rassias[26]) *Let $f:E\to E'$ be a mapping from a normed vector space E into a Banach space E', subject to the inequality*

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon(\|x\|^p+\|y\|^p), \tag{1.1}$$

for all $x,y\in E$, where ϵ and p are constants with $\epsilon>0$ and $p<1$. Then the limit

$$L(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n},$$

exists for all $x\in E$ and $L:E\to E'$ is the unique additive mapping that satisfies

$$\|f(x)-L(x)\| \le \frac{2\epsilon}{2-2^p}\|x\|^p, \tag{1.2}$$

for all $x\in E$. If $p<0$, then inequality (1.1) holds for $x,y\neq 0$ and (1.2) for $x\neq 0$.

In 1990, Rassias,[27] during the 27th International Symposium on Functional Equations, asked the question of whether such a theorem can also be proved for $p\ge 1$. In 1991, Gajda[8] following the same approach as in Rassias,[26] gave an affirmative solution to this question for $p>1$. It was shown by Gajda,[8] as well as by Rassias and Šemrl[32] that one cannot prove a Rassias' type theorem when $p=1$. The counterexamples of Gajda,[8] as well as of Rassias and Šemrl,[32] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings; cf. Găvruta,[9] Jung,[14] who, among others, studied the Hyers-Ulam-Rassias stability of functional equations. The inequality (1.1) that was introduced for the first time by Rassias[26] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as the *Hyers-Ulam-Rassias stability* of functional equations (cf. the books of Czerwik,[6] Hyers, Isac, and Rassias.[11])

Rassias,[24] following the spirit of the innovative approach of Rassias[26] for the unbounded Cauchy difference, proved a similar stability theorem in which he replaced the factor $\|x\|^p+\|y\|^p$ by $\|x\|^p\cdot\|y\|^q$ for $p,q\in\mathbb{R}$ with $p+q\neq 1$ (see also Ref. 25 for a number of other new results).

Găvruta[9] provided a further generalization of Rassias' Theorem. In 1996, Isac and Rassias[13] applied the Hyers-Ulam-Rassias stability theory to prove fixed point theorems and study some new applications in Nonlinear Analysis. In Ref. 12, Hyers, Isac, and Rassias studied the asymptoticity aspect of Hyers-Ulam stability of mappings. During the past few years several mathematicians have published on various generalizations and applications of Hyers-Ulam stability and Hyers-Ulam-Rassias stability to a number of functional equations and mappings, for example, quadratic functional equation, invariant means, multiplicative mappings—superstability, bounded nth differences, convex functions, generalized orthogonality functional equation, Euler-Lagrange functional equation, and Navier-Stokes equations. Several mathematicians have contributed works on these subjects; we mention a few: Baak and Moslehian,[3] Park,[19–23] Rassias,[28–31] and Skof.[33]

Throughout this paper, assume that p, d are non-negative integers with $p+d\ge 3$.

In Sec. II, we prove the Hyers-Ulam-Rassias stability of homomorphisms in C^*-ternary algebras for the generalized Cauchy-Jensen additive mapping.

In Sec. III, we investigate isomorphisms between unital C^*-ternary algebras associated with the generalized Cauchy-Jensen additive mapping.

In Sec. IV, we prove the Hyers-Ulam-Rassias stability of derivations on C^*-ternary algebras for the generalized Cauchy-Jensen additive mapping.

II. STABILITY OF HOMOMORPHISMS IN C^*-TERNARY ALGEBRAS

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$ and that B is a C^*-ternary algebra with norm $\|\cdot\|_B$.

For a given mapping $f:A\to B$, we define

$$C_\mu f(x_1, \dots, x_p, y_1, \dots, y_d) \coloneqq 2f\left(\frac{\sum_{j=1}^{p} \mu x_j}{2} + \sum_{j=1}^{d} \mu y_j\right) - \sum_{j=1}^{p} \mu f(x_j) - 2\sum_{j=1}^{d} \mu f(y_j),$$

for all $\mu \in \mathbb{T}^1 \coloneqq \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$ and all $x_1, \dots, x_p, y_1, \dots, y_d \in A$.

One can easily show that a mapping $f: A \to B$ satisfies $C_1 f(x_1, \dots, x_p, y_1, \dots, y_d) = 0$ if and only if f is Cauchy additive, and that if a mapping $f: A \to B$ satisfies $C_1 f(x_1, \dots, x_p, y_1, \dots, y_d) = 0$ then $f(0) = 0$.

We prove the Hyers-Ulam-Rassias stability of homomorphisms in C^*-ternary algebras for the functional equation $C_\mu f(x_1, \dots, x_p, y_1, \dots, y_d) = 0$.

Theorem 2.1: *Let $r > 3$ and θ be non-negative real numbers, and let $f: A \to B$ be a mapping such that*

$$\|C_\mu f(x_1, \dots, x_p, y_1, \dots, y_d)\|_B \le \theta\left(\sum_{j=1}^{p} \|x_j\|_A^r + \sum_{j=1}^{d} \|y_j\|_A^r\right), \tag{2.1}$$

$$\|f([x,y,z]) - [f(x), f(y), f(z)]\|_B \le \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r), \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \dots, x_p, y_1, \dots, y_d \in A$. Then there exists a unique C^*-ternary algebra homomorphism $H: A \to B$, such that

$$\|f(x) - H(x)\|_B \le \frac{p+d}{2(p+2d)^r - (p+2d)2^r}\theta\|x\|_A^r, \tag{2.3}$$

for all $x \in A$.

Proof: Let us assume $\mu = 1$ and $x_1 = \dots = x_p = y_1 = \dots = y_d = x$ in (2.1). Then we get

$$\left\|2f\left(\frac{p+2d}{2}x\right) - (p+2d)f(x)\right\|_B \le (p+d)\theta\|x\|_A^r, \tag{2.4}$$

for all $x \in A$. So

$$\left\|f(x) - \frac{p+2d}{2}f\left(\frac{2}{p+2d}x\right)\right\|_B \le \frac{p+d}{2(p+2d)^r}\theta\|x\|_A^r,$$

for all $x \in A$. Hence

$$\begin{aligned}
&\left\|\frac{(p+2d)^l}{2^l}f\left(\frac{2^l}{(p+2d)^l}x\right) - \frac{(p+2d)^m}{2^m}f\left(\frac{2^m}{(p+2d)^m}x\right)\right\|_B \\
&\le \sum_{j=l}^{m-1}\left\|\frac{(p+2d)^j}{2^j}f\left(\frac{2^j}{(p+2d)^j}x\right) - \frac{(p+2d)^{j+1}}{2^{j+1}}f\left(\frac{2^{j+1}}{(p+2d)^{j+1}}x\right)\right\|_B \\
&\le \frac{(p+d)}{2(p+2d)^r}\sum_{j=l}^{m-1}\frac{2^{rj}(p+2d)^j}{2^j(p+2d)^{rj}}\theta\|x\|_A^r,
\end{aligned} \tag{2.5}$$

for all non-negative integers m and l with $m > l$ and all $x \in A$. From this it follows that the sequence $\{[(p+2d)^n/2^n]f([2^n/(p+2d)^n]x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{[(p+2d)^n/2^n]f([2^n/(p+2d)^n]x)\}$ converges. Thus one can define the mapping $H: A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{(p+2d)^n}{2^n} f\left(\frac{2^n}{(p+2d)^n} x\right),$$

for all $x \in A$. Moreover, letting $l=0$ and passing the limit $m \to \infty$ in (2.5), we get (2.3).

It follows from (2.1) that

$$\begin{aligned}
&\left\| 2H\left(\frac{\sum_{j=1}^{p} x_j}{2} + \sum_{j=1}^{d} y_j\right) - \sum_{j=1}^{p} H(x_j) - 2\sum_{j=1}^{d} H(y_j)\right\|_B \\
&= \lim_{n\to\infty} \frac{(p+2d)^n}{2^n} \left\| 2f\left(\frac{2^n}{(p+2d)^n}\frac{\sum_{j=1}^{p} x_j}{2} + \frac{2^n}{(p+2d)^n}\sum_{j=1}^{d} y_j\right)\right. \\
&\qquad \left. - \sum_{j=1}^{p} f\left(\frac{2^n}{(p+2d)^n} x_j\right) - 2\sum_{j=1}^{d} f\left(\frac{2^n}{(p+2d)^n} y_j\right)\right\|_B \\
&\le \lim_{n\to\infty} \frac{2^{nr}(p+2d)^n}{2^n (p+2d)^{nr}} \theta\left(\sum_{j=1}^{p} \|x_j\|_A^r + \sum_{j=1}^{d} \|y_j\|_A^r\right) = 0,
\end{aligned}$$

for all $x_1, \ldots, x_p, y_1, \ldots, y_d \in A$. Hence

$$2H\left(\frac{\sum_{j=1}^{p} x_j}{2} + \sum_{j=1}^{d} y_j\right) = \sum_{j=1}^{p} H(x_j) + 2\sum_{j=1}^{d} H(y_j),$$

for all $x_1, \ldots, x_p, y_1, \ldots, y_d \in A$. So the mapping $H: A \to B$ is Cauchy additive.

By the same reasoning as in the proof of Theorem 2.1 of Ref. 21, the mapping $H: A \to B$ is $\mathbb{C}$-linear.

It follows from (2.2) that

$$\begin{aligned}
\|H([x,y,z]) - [H(x),H(y),H(z)]\|_B &= \lim_{n\to\infty} \frac{(p+2d)^{3n}}{8^n} \left\| f\left(\frac{8^n [x,y,z]}{(p+2d)^{3n}}\right)\right. \\
&\qquad \left. - \left[f\left(\frac{2^n x}{(p+2d)^n}\right), f\left(\frac{2^n y}{(p+2d)^n}\right), f\left(\frac{2^n z}{(p+2d)^n}\right)\right]\right\|_B \\
&\le \lim_{n\to\infty} \frac{2^{nr}(p+2d)^{3n}}{8^n (p+2d)^{nr}} \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) = 0,
\end{aligned}$$

for all $x, y, z \in A$. Thus

$$H([x,y,z]) = [H(x),H(y),H(z)],$$

for all $x, y, z \in A$.

Now, let $T: A \to B$ be another generalized Cauchy-Jensen additive mapping satisfying (2.3). Then we have

$$\begin{aligned}\|H(x)-T(x)\|_B &= \frac{(p+2d)^n}{2^n}\left\|H\left(\frac{2^n x}{(p+2d)^n}\right)-T\left(\frac{2^n x}{(p+2d)^n}\right)\right\|_B \\ &\le \frac{(p+2d)^n}{2^n}\left(\left\|H\left(\frac{2^n x}{(p+2d)^n}\right)-f\left(\frac{2^n x}{(p+2d)^n}\right)\right\|_B \right.\\ &\quad \left.+\left\|T\left(\frac{2^n x}{(p+2d)^n}\right)-f\left(\frac{2^n x}{(p+2d)^n}\right)\right\|_B\right) \\ &\le \frac{p+d}{2(p+2d)^r-(p+2d)2^r}\cdot\frac{2^{nr+1}(p+2d)^n}{2^n(p+2d)^{nr}}\theta\|x\|_A^r,\end{aligned}$$

which tends to zero as $n\to\infty$ for all $x\in A$. So we can conclude that $H(x)=T(x)$ for all $x\in A$. This proves the uniqueness of H. Thus, the mapping $H:A\to B$ is a unique C^*-ternary algebra homomorphism satisfying (2.3).

Theorem 2.2: *Let $r<1$ and θ be non-negative real numbers, and let $f:A\to B$ be a mapping satisfying* (2.1) and (2.2). *Then there exists a unique C^*-ternary algebra homomorphism $H:A\to B$ such that*

$$\|f(x)-H(x)\|_B \le \frac{2^r(p+d)}{2^r(p+2d)-2(p+2d)^r}\theta\|x\|_A^r, \tag{2.6}$$

for all $x\in A$.

Proof: It follows from (2.4) that

$$\left\|f(x)-\frac{2}{p+2d}f\left(\frac{p+2d}{2}x\right)\right\|_B \le \frac{p+d}{p+2d}\theta\|x\|_A^r,$$

for all $x\in A$. So

$$\begin{aligned}&\left\|\frac{2^l}{(p+2d)^l}f\left(\frac{(p+2d)^l}{2^l}x\right)-\frac{2^m}{(p+2d)^m}f\left(\frac{(p+2d)^m}{2^m}x\right)\right\|_B \\ &\quad\le \sum_{j=l}^{m-1}\left\|\frac{2^j}{(p+2d)^j}f\left(\frac{(p+2d)^j}{2^j}x\right)-\frac{2^{j+1}}{(p+2d)^{j+1}}f\left(\frac{(p+2d)^{j+1}}{2^{j+1}}x\right)\right\|_B \\ &\quad\le \frac{p+d}{p+2d}\sum_{j=l}^{m-1}\frac{2^j(p+2d)^{jr}}{2^{jr}(p+2d)^j}\theta\|x\|_A^r,\end{aligned} \tag{2.7}$$

for all non-negative integers m and l with $m>l$ and all $x\in A$. From this it follows that the sequence $\{[2^n/(p+2d)^n]f([(p+2d)^n/2^n]x)\}$ is a Cauchy sequence for all $x\in A$. Since B is complete, the sequence $\{[2^n/(p+2d)^n]f([(p+2d)^n/2^n]x)\}$ converges. So one can define the mapping $H:A\to B$ by

$$H(x) := \lim_{n\to\infty}\frac{2^n}{(p+2d)^n}f\left(\frac{(p+2d)^n}{2^n}x\right),$$

for all $x\in A$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (2.7), we get (2.6).

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.3: *Let $r>1$ and θ be non-negative real numbers, and let $f:A\to B$ be a mapping such that*

$$\|C_\mu f(x_1,\ldots,x_p,y_1,\ldots,y_d)\|_B \le \theta\prod_{j=1}^{p}\|x_j\|_A^r\cdot\prod_{j=1}^{d}\|y_j\|_A^r, \tag{2.8}$$

$$\|f([x,y,z])-[f(x),f(y),f(z)]\|_B \le \theta\cdot\|x\|_A^r\cdot\|y\|_A^r\cdot\|z\|_A^r, \tag{2.9}$$

for all $\mu\in\mathbb{T}^1$ and all $x,y,z,x_1,\ldots,x_p,y_1,\ldots,y_d\in A$. Then there exists a unique C^*-ternary algebra homomorphism $H:A\to B$ such that

$$\|f(x)-H(x)\|_B \le \frac{2^{(p+d)r}}{2(p+2d)^{(p+d)r}-2^{(p+d)r}(p+2d)}\theta\|x\|_A^{(p+d)r}, \tag{2.10}$$

for all $x\in A$.

Proof: Let us assume $\mu=1$ and $x_1=\cdots=x_p=y_1=\cdots=y_d=x$ in (2.8). Then we get

$$\left\|2f\left(\frac{p+2d}{2}x\right)-(p+2d)f(x)\right\|_B \le \theta\|x\|_A^{(p+d)r}, \tag{2.11}$$

for all $x\in A$.So

$$\left\|f(x)-\frac{p+2d}{2}f\left(\frac{2}{p+2d}x\right)\right\|_B \le \frac{2^{(p+d)r}}{2(p+2d)^{(p+d)r}}\theta\|x\|_A^{(p+d)r},$$

for all $x\in A$. Hence

$$\begin{aligned}&\left\|\frac{(p+2d)^l}{2^l}f\left(\frac{2^l}{(p+2d)^l}x\right)-\frac{(p+2d)^m}{2^m}f\left(\frac{2^m}{(p+2d)^m}x\right)\right\|_B\\ &\le\sum_{j=l}^{m-1}\left\|\frac{(p+2d)^j}{2^j}f\left(\frac{2^j}{(p+2d)^j}x\right)-\frac{(p+2d)^{j+1}}{2^{j+1}}f\left(\frac{2^{j+1}}{(p+2d)^{j+1}}x\right)\right\|_B\\ &\le\frac{2^{(p+d)r}}{2(p+2d)^{(p+d)r}}\sum_{j=l}^{m-1}\frac{2^{(p+d)rj}(p+2d)^j}{2^j(p+2d)^{(p+d)rj}}\theta\|x\|_A^{(p+d)r},\end{aligned} \tag{2.12}$$

for all non-negative integers m and l with $m>l$ and all $x\in A$. From this it follows that the sequence $\{[(p+2d)^n/2^n]f([2^n/(p+2d)^n]x)\}$ is a Cauchy sequence for all $x\in A$. Since B is complete, the sequence $\{[(p+2d)^n/2^n]f([2^n/(p+2d)^n]x)\}$ converges. Thus one can define the mapping $H:A\to B$ by

$$H(x):=\lim_{n\to\infty}\frac{(p+2d)^n}{2^n}f\left(\frac{2^n}{(p+2d)^n}x\right),$$

for all $x\in A$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (2.12), we get (2.10).

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.4: *Let $r<1/(p+d)$ and θ be non-negative real numbers, and let $f:A\to B$ be a mapping satisfying* (2.8) and (2.9). *Then there exists a unique C^*-ternary algebra homomorphism $H:A\to B$ such that*

$$\|f(x)-H(x)\|_B \le \frac{2^{(p+d)r}}{2^{(p+d)r}(p+2d)-2(p+2d)^{(p+d)r}}\theta\|x\|_A^{(p+d)r}, \tag{2.13}$$

for all $x\in A$.

Proof: It follows from (2.11) that

$$\left\|f(x)-\frac{2}{p+2d}f\left(\frac{p+2d}{2}x\right)\right\|_B \le \frac{\theta}{p+2d}\|x\|_A^{(p+d)r},$$

for all $x\in A$. So

$$\left\|\frac{2^l}{(p+2d)^l}f\left(\frac{(p+2d)^l}{2^l}x\right)-\frac{2^m}{(p+2d)^m}f\left(\frac{(p+2d)^m}{2^m}x\right)\right\|_B$$
$$\leq\sum_{j=l}^{m-1}\left\|\frac{2^j}{(p+2d)^j}f\left(\frac{(p+2d)^j}{2^j}x\right)-\frac{2^{j+1}}{(p+2d)^{j+1}}f\left(\frac{(p+2d)^{j+1}}{2^{j+1}}x\right)\right\|_B$$
$$\leq\frac{\theta}{p+2d}\sum_{j=l}^{m-1}\frac{2^j(p+2d)^{j(p+d)r}}{2^{j(p+d)r}(p+2d)^j}\|x\|_A^{(p+d)r}, \tag{2.14}$$

for all non-negative integers m and l with $m>l$ and all $x\in A$. From this it follows that the sequence $\{[2^n/(p+2d)^n]f([(p+2d)^n/2^n]x)\}$ is a Cauchy sequence for all $x\in A$. Since B is complete, the sequence $\{[2^n/(p+2d)^n]f([(p+2d)^n/2^n]x)\}$ converges. So one can define the mapping $H:A\to B$ by

$$H(x):=\lim_{n\to\infty}\frac{2^n}{(p+2d)^n}f\left(\frac{(p+2d)^n}{2^n}x\right),$$

for all $x\in A$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (2.14), we get (2.13).

The rest of the proof is similar to the proof of Theorem 2.1. □

III. ISOMORPHISMS BETWEEN C^*-TERNARY ALGEBRAS

Throughout this section, assume that A is a unital C^*-ternary algebra with norm $\|\cdot\|_A$ and unit e, and that B is a unital C^*-ternary algebra with norm $\|\cdot\|_B$ and unit e'.

We investigate isomorphisms between C^*-ternary algebras associated with the functional equation $C_\mu f(x_1,\ldots,x_p,y_1,\ldots,y_d)=0$.

Theorem 3.1: *Let $r>1$ and θ be non-negative real numbers, and let $f:A\to B$ be a bijective mapping satisfying* (2.1), *such that*

$$f([x,y,z])=[f(x),f(y),f(z)], \tag{3.1}$$

for all $x,y,z\in A$. If $\lim_{n\to\infty}[(p+2d)^n/2^n]f(2^ne/(p+2d)^n)=e'$, then the mapping $f:A\to B$ is a C^*-ternary algebra isomorphism.

Proof: By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $H:A\to B$ such that

$$\|f(x)-H(x)\|_B\leq\frac{p+d}{2(p+2d)^r-(p+2d)2^r}\theta\|x\|_A^r,$$

for all $x\in A$. The mapping $H:A\to B$ is defined by

$$H(x):=\lim_{n\to\infty}\frac{(p+2d)^n}{2^n}f\left(\frac{2^n}{(p+2d)^n}x\right),$$

for all $x\in A$.

Since $f([x,y,z])=[f(x),f(y),f(z)]$ for all $x,y,z\in A$,

$$H([x,y,z])=\lim_{n\to\infty}\frac{(p+2d)^{3n}}{8^n}f\left(\left[\frac{2^nx}{(p+2d)^n},\frac{2^ny}{(p+2d)^n},\frac{2^nz}{(p+2d)^n}\right]\right)$$
$$=\lim_{n\to\infty}\left[\frac{(p+2d)^n}{2^n}f\left(\frac{2^nx}{(p+2d)^n}\right),\frac{(p+2d)^n}{2^n}f\left(\frac{2^ny}{(p+2d)^n}\right),\frac{(p+2d)^n}{2^n}f\left(\frac{2^nz}{(p+2d)^n}\right)\right]$$
$$=[H(x),H(y),H(z)],$$

for all $x,y,z\in A$. So the mapping $H:A\to B$ is a C^*-ternary algebra homomorphism.

It follows from (3.1) that

$$H(x) = H([e,e,x]) = \lim_{n\to\infty} \frac{(p+2d)^{2n}}{4^n} f\left(\frac{4^n}{(p+2d)^{2n}}[e,e,x]\right)$$
$$= \lim_{n\to\infty} \frac{(p+2d)^{2n}}{4^n} f\left(\left[\frac{2^n e}{(p+2d)^n}, \frac{2^n e}{(p+2d)^n}, x\right]\right)$$
$$= \lim_{n\to\infty} \left[\frac{(p+2d)^n}{2^n} f\left(\frac{2^n e}{(p+2d)^n}\right), \frac{(p+2d)^n}{2^n} f\left(\frac{2^n e}{(p+2d)^n}\right), f(x)\right] = [e', e', f(x)] = f(x),$$

for all $x \in A$. Hence the bijective mapping $f:A \to B$ is a C^*-ternary algebra isomorphism. □

Theorem 3.2: *Let $r<1$ and θ be non-negative real numbers, and let $f:A\to B$ be a bijective mapping satisfying* (2.1) and (3.1). *If* $\lim_{n\to\infty}[2^n/(p+2d)^n]f([(p+2d)^n/2^n]e)=e'$, *then the mapping $f:A\to B$ is a C^*-ternary algebra isomorphism.*

Proof: By the same reasoning as in the proofs of Theorems 2.1 and 2.2, there exists a unique $\mathbb{C}$-linear mapping $H:A\to B$ such that

$$\|f(x)-H(x)\|_B \le \frac{2^r(p+d)}{2^r(p+2d)-2(p+2d)^r}\theta\|x\|_A^r,$$

for all $x \in A$. The mapping $H:A\to B$ is defined by

$$H(x) := \lim_{n\to\infty} \frac{2^n}{(p+2d)^n} f\left(\frac{(p+2d)^n}{2^n}x\right),$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.3: *Let $r>1/(p+d)$ and θ be non-negative real numbers, and let $f:A\to B$ be a bijective mapping satisfying* (2.8) and (3.1). *If $\lim_{n\to\infty}[(p+2d)^n/2^n]f(2^n e/(p+2d)^n)=e'$, then the mapping $f:A\to B$ is a C^*-ternary algebra isomorphism.*

Proof: By the same reasoning as in the proofs of Theorems 2.1 and 2.3, there exists a unique $\mathbb{C}$-linear mapping $H:A\to B$ such that

$$\|f(x)-H(x)\|_B \le \frac{2^{(p+d)r}}{2(p+2d)^{(p+d)r}-2^{(p+d)r}(p+2d)}\theta\|x\|_A^{(p+d)r},$$

for all $x \in A$. The mapping $H:A\to B$ is defined by

$$H(x) := \lim_{n\to\infty} \frac{(p+2d)^n}{2^n} f\left(\frac{2^n}{(p+2d)^n}x\right),$$

for all $x \in A$.

The rest of the proof is similar to the proofs of Theorems 2.3 and 3.1. □

Theorem 3.4: *Let $r<1/(p+d)$ and θ be non-negative real numbers, and let $f:A\to B$ be a bijective mapping satisfying* (2.8) and (3.1). *If* $\lim_{n\to\infty}[2^n/(p+2d)^n]f([(p+2d)^n/2^n]e)=e'$, *then the mapping $f:A\to B$ is a C^*-ternary algebra isomorphism.*

Proof: By the same reasoning as in the proofs of Theorems 2.1 and 2.4, there exists a unique $\mathbb{C}$-linear mapping $H:A\to B$ such that

$$\|f(x)-H(x)\|_B \le \frac{2^{(p+d)r}}{2^{(p+d)r}(p+2d)-2(p+2d)^{(p+d)r}}\theta\|x\|_A^{(p+d)r},$$

for all $x \in A$. The mapping $H:A\to B$ is defined by

$$H(x) := \lim_{n\to\infty} \frac{2^n}{(p+2d)^n} f\left(\frac{(p+2d)^n}{2^n}x\right),$$

for all $x \in A$.

The rest of the proof is similar to the proofs of Theorems 2.4 and 3.1. □

IV. STABILITY OF DERIVATIONS ON C^*-TERNARY ALGEBRAS

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$.

We prove the Hyers-Ulam-Rassias stability of derivations on C^*-ternary algebras for the functional equation $C_\mu f(x_1, \ldots, x_p, y_1, \ldots, y_d) = 0$.

Theorem 4.1: *Let $r > 3$ and θ be non-negative real numbers, and let $f: A \to A$ be a mapping satisfying (2.1) such that*

$$\|f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)]\|_A \le \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r), \tag{4.1}$$

for all $x, y, z \in A$. Then there exists a unique C^*-ternary derivation $\delta: A \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{p+d}{2(p+2d)^r - (p+2d)2^r} \theta \|x\|_A^r, \tag{4.2}$$

for all $x \in A$.

Proof: By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $\delta: A \to A$ satisfying (4.2). The mapping $\delta: A \to A$ is defined by

$$\delta(x) := \lim_{n\to\infty} \frac{(p+2d)^n}{2^n} f\left(\frac{2^n}{(p+2d)^n}x\right),$$

for all $x \in A$.

It follows from (4.1) that

$$\begin{aligned}
&\|\delta([x,y,z]) - [\delta(x),y,z] - [x,\delta(y),z] - [x,y,\delta(z)]\|_A \\
&= \lim_{n\to\infty} \frac{(p+2d)^{3n}}{8^n} \left\| f\left(\frac{8^n}{(p+2d)^{3n}}[x,y,z]\right) - \left[f\left(\frac{2^n x}{(p+2d)^n}\right), \frac{2^n y}{(p+2d)^n}, \frac{2^n z}{(p+2d)^n}\right] \right. \\
&\quad \left. - \left[\frac{2^n x}{(p+2d)^n}, f\left(\frac{2^n y}{(p+2d)^n}\right), \frac{2^n z}{(p+2d)^n}\right] - \left[\frac{2^n x}{(p+2d)^n}, \frac{2^n y}{(p+2d)^n}, f\left(\frac{2^n z}{(p+2d)^n}\right)\right] \right\|_A \\
&\le \lim_{n\to\infty} \frac{2^{nr}(p+2d)^{3n}}{8^n (p+2d)^{nr}} \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) = 0,
\end{aligned}$$

for all $x, y, z \in A$. Hence

$$\delta([x,y,z]) = [\delta(x),y,z] + [x,\delta(y),z] + [x,y,\delta(z)],$$

for all $x, y, z \in A$. Thus the mapping $\delta: A \to A$ is a unique C^*-ternary derivation satisfying (4.2). □

Theorem 4.2: *Let $r < 1$ and θ be nonnegative real numbers, and let $f: A \to A$ be a mapping satisfying (2.1) and (4.1). Then there exists a unique C^*-ternary derivation $\delta: A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \le \frac{2^r(p+d)}{2^r(p+2d) - 2(p+2d)^r} \theta \|x\|_A^r, \tag{4.3}$$

for all $x \in A$.

Proof: By the same reasoning as in the proofs of Theorems 2.1 and 2.2, there exists a unique $\mathbb{C}$-linear mapping $\delta: A \to A$ satisfying (4.3). The mapping $\delta: A \to A$ is defined by

$$\delta(x) := \lim_{n\to\infty} \frac{2^n}{(p+2d)^n} f\left(\frac{(p+2d)^n}{2^n}x\right),$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 4.1. □

Theorem 4.3: *Let $r>1$ and θ be non-negative real numbers, and let $f:A\to A$ be a mapping satisfying (2.8) such that*

$$\|f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)]\|_A \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \cdot \|z\|_A^r, \tag{4.4}$$

for all $x,y,z \in A$. Then there exists a unique C^*-ternary derivation $\delta:A\to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{2^{(p+d)r}}{2(p+2d)^{(p+d)r} - 2^{(p+d)r}(p+2d)} \theta \|x\|_A^{(p+d)r}, \tag{4.5}$$

for all $x \in A$.

Proof: By the same reasoning as in the proofs of Theorems 2.1 and 2.3, there exists a unique $\mathbb{C}$-linear mapping $\delta:A\to A$ satisfying (4.5). The mapping $\delta:A\to A$ is defined by

$$\delta(x) := \lim_{n\to\infty} \frac{(p+2d)^n}{2^n} f\left(\frac{2^n}{(p+2d)^n}x\right),$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 4.1. □

Theorem 4.4: *Let $r<1/(p+d)$ and θ be non-negative real numbers, and let $f:A\to A$ be a mapping satisfying (2.8)* and *(4.4). Then there exists a unique C^*-ternary derivation $\delta:A\to A$ such that*

$$\|f(x) - \delta(x)\|_A \le \frac{2^{(p+d)r}}{2^{(p+d)r}(p+2d) - 2(p+2d)^{(p+d)r}} \theta \|x\|_A^{(p+d)r}, \tag{4.6}$$

for all $x \in A$.

Proof: By the same reasoning as in the proofs of Theorems 2.1 and 2.4, there exists a unique $\mathbb{C}$-linear mapping $\delta:A\to A$ satisfying (4.6). The mapping $\delta:A\to A$ is defined by

$$\delta(x) := \lim_{n\to\infty} \frac{2^n}{(p+2d)^n} f\left(\frac{(p+2d)^n}{2^n}x\right),$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 4.1. □

[1] Abramov, V., Kerner, R., and Le Roy, B., "Hypersymmetry: a $\mathbb{Z}_3$-graded generalization of supersymmetry," J. Math. Phys. **38**, 1650–1669 (1997).

[2] Amyari, M., and Moslehian, M. S., "Approximately ternary semigroup homomorphisms," Lett. Math. Phys. **77**, 1–9 (2006).

[3] Baak, C., and Moslehian, M. S., "On the stability of J^*-homomorphisms," Nonlinear Anal. Theory, Methods Appl. **63**, 42–48 (2005).

[4] Bazunova, N., Borowiec, A., and Kerner, R., "Universal differential calculus on ternary algebras," Lett. Math. Phys. **67**, 195–206 (2004).

[5] Cayley, A., "On the 34 concomitants of the ternary cubic," Am. J. Math. **4**, 1–15 (1881).

[6] Czerwik, P., *Functional Equations and Inequalities in Several Variables* (World Scientific, Singapore, 2002).

[7] Daletskii, Y. L., and Takhtajan, L., "Leibniz and Lie algebra structures for Nambu algebras," Lett. Math. Phys. **39**, 127–141 (1997).

[8] Gajda, Z., "On stability of additive mappings," Int. J. Math. Math. Sci. **14**, 431–434 (1991).

[9] Găvruta, P., "A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings," J. Math. Anal. Appl. **184**, 431–436 (1994).

[10] Hyers, D. H., "On the stability of the linear functional equation," Proc. Natl. Acad. Sci. U.S.A. **27**, 222–224 (1941).

[11] Hyers, D. H., Isac, G., and Rassias, Th. M., *Stability of Functional Equations in Several Variables* (Birkhäuser, Basel, 1998).

[12] Hyers, D. H., Isac, G., and Rassias, Th. M., "On the asymptoticity aspect of Hyers–Ulam stability of mappings," Proc. Am. Math. Soc. **126**, 425–430 (1998).
[13] Isac, G., and Rassias, Th. M., "Stability of ψ-additive mappings: Applications to nonlinear analysis," Int. J. Math. Math. Sci. **19**, 219–228 (1996).
[14] Jung, S., "On the Hyers–Ulam–Rassias stability of approximately additive mappings," J. Math. Anal. Appl. **204**, 221–226 (1996).
[15] Kapranov, M., Gelfand, I. M., and Zelevinskii, A., *Discriminants, Resultants and Multidimensional Determinants* (Birkhäuser, Berlin, 1994).
[16] Kerner, R., "The cubic chessboard: Geometry and physics," Class. Quantum Grav. **14**, A203–A225 (1997).
[17] Kerner, R., "Ternary algebraic structures and their applications in physics" (preprint).
[18] Moslehian, M. S., "Almost derivations on C^*-ternary rings" (preprint).
[19] Park, C., "Lie *-homomorphisms between Lie C^*-algebras and Lie *-derivations on Lie C^*-algebras," J. Math. Anal. Appl. **293**, 419–434 (2004).
[20] Park, C., "Homomorphisms between Lie JC^*-algebras and Cauchy–Rassias stability of Lie JC^*-algebra derivations," J. Lie Theory **15**, 393–414 (2005).
[21] Park, C., "Homomorphisms between Poisson JC^*-algebras," Bull. Braz. Math. Soc. N. S. **36**, 79–97 (2005).
[22] Park, C., "Hyers-Ulam-Rassias stability of a generalized Euler–Lagrange type additive mapping and isomorphisms between C^*-algebras," Bull. Belgian Math. Soc.–Simon Stevin (to appear).
[23] Park, C., "Hyers-Ulam-Rassias stability of a generalized Apollonius–Jensen type additive mapping and isomorphisms between C^*-algebras," Math. Nachr. (to appear).
[24] Rassias, J. M., "On approximation of approximately linear mappings by linear mappings," Bull. Sci. Math. **108**, 445–446 (1984).
[25] Rassias, J. M., "Solution of a problem of Ulam," J. Approx. Theory **57**, 268–273 (1989).
[26] Rassias, Th. M., "On the stability of the linear mapping in Banach spaces," Proc. Am. Math. Soc. **72**, 297–300 (1978).
[27] Rassias, Th. M., "Problem 16; 2, Report of the 27th International Symposium on Functional Equations," Aequ. Math. **39**, 292–293; 309 (1990).
[28] Rassias, Th. M., "The problem of S.M. Ulam for approximately multiplicative mappings," J. Math. Anal. Appl. **246**, 352–378 (2000).
[29] Rassias, Th. M., "On the stability of functional equations in Banach spaces," J. Math. Anal. Appl. **251**, 264–284 (2000).
[30] Rassias, Th. M., "On the stability of functional equations and a problem of Ulam," Acta Appl. Math. **62**, 23–130 (2000).
[31] Rassias, Th. M., *Functional Equations, Inequalities and Applications* (Kluwer Academic, Dordrecht, 2003).
[32] Rassias, Th. M., and Šemrl, P., "On the behaviour of mappings which do not satisfy Hyers–Ulam stability," Proc. Am. Math. Soc. **114**, 989–993 (1992).
[33] Skof, F., "Proprietà locali e approssimazione di operatori," Rend. Semin. Mat. Fis. Milano **53**, 113–129 (1983).
[34] Ulam, S. M., *A Collection of the Mathematical Problems* (Interscience Publ. New York, 1960).
[35] Vainerman, L., and Kerner, R., "On special classes of n-algebras," J. Math. Phys. **37**, 2553–2565 (1996).
[36] Zettl, H., "A characterization of ternary rings of operators," Adv. Math. **48**, 117–143 (1983).