Hyers–Ulam–Rassias stability of a generalized Euler–Lagrange type additive mapping and isomorphisms between C^*-algebras*

Chun-Gil Park

Abstract

Let X, Y be Banach modules over a C^*-algebra and let $r_1, \cdots, r_n \in (0, \infty)$ be given. We prove the Hyers–Ulam–Rassias stability of the following functional equation in Banach modules over a unital C^*-algebra:

$$\sum_{i=1}^{n} r_i f\left(\sum_{j=1}^{n} r_j(x_i - x_j)\right) + \left(\sum_{i=1}^{n} r_i\right) f\left(\sum_{i=1}^{n} r_i x_i\right) = \left(\sum_{i=1}^{n} r_i\right) \sum_{i=1}^{n} r_i f(x_i). (0.1)$$

We show that if $r_1 = \cdots = r_n = r$ and an odd mapping $f : X \to Y$ satisfies the functional equation (0.1) then the odd mapping $f : X \to Y$ is Cauchy additive. As an application, we show that every almost linear bijection $h : A \to B$ of a unital C^*-algebra A onto a unital C^*-algebra B is a C^*-algebra isomorphism when $h((nr)^d uy) = h((nr)^d u)h(y)$ for all unitaries $u \in A$, all $y \in A$, and all $d \in \mathbf{Z}$.

1 Introduction

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbf{R}$ for each fixed

*Supported by Korea Research Foundation Grant KRF-2005-070-C00009

Received by the editors April 2005.

Communicated by F. Bastin.

1991 *Mathematics Subject Classification* : Primary 39B52, 46L05, 47B48.

Key words and phrases : Hyers–Ulam–Rassias stability, generalized Euler–Lagrange type additive mapping, isomorphism between C^*-algebras.

$x \in X$. Assume that there exist constants $\theta \geq 0$ and $p \in [0,1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th.M. Rassias [24] showed that there exists a unique **R**-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$. A number of mathematicians were attracted to this result of Th.M. Rassias and stimulated to investigate the stability problems of functional equations. The stability phenomenon that was introduced and proved by Th.M. Rassias is called the *Hyers–Ulam–Rassias stability*. Găvruta [2] generalized the Rassias' result: Let G be an abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in G$. C. Park [10] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. Several functional equations have been investigated in [4]–[6], [10]–[12], [14]–[31]. Many authors have studied the structure of C^*-algebras (see [9], [13]).

In [1], S. Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation. J.M. Rassias [22, 23] solved the stability problem of Ulam for the Euler–Lagrange type quadratic functional equation

$$f(rx + sy) + f(sx - ry) = (r^2 + s^2)[f(x) + f(y)].$$

Recently, Jun and Kim [7] solved the stability problem of Ulam for another Euler–Lagrange type quadratic functional equation. Jun and Kim [8] introduced and investigated the following quadratic functional equation of Euler–Lagrange type

$$\begin{aligned}\sum_{i=1}^{n} r_i Q\left(\sum_{j=1}^{n} r_j(x_i - x_j)\right) + \left(\sum_{i=1}^{n} r_i\right) Q\left(\sum_{i=1}^{n} r_i x_i\right)& \\ = \left(\sum_{i=1}^{n} r_i\right)^2 \sum_{i=1}^{n} r_i Q(x_i),&\end{aligned}$$

whose solution is said to be a generalized quadratic mapping of Euler–Lagrange type.

In this paper, we introduce the following functional equation

$$\sum_{i=1}^{n} r_i L\left(\sum_{j=1}^{n} r_j(x_i - x_j)\right) + \left(\sum_{i=1}^{n} r_i\right) L\left(\sum_{i=1}^{n} r_i x_i\right) = \left(\sum_{i=1}^{n} r_i\right) \sum_{i=1}^{n} r_i L(x_i), \qquad r_i \in (0,\infty) \tag{1.1}$$

whose solution is called a *generalized Euler–Lagrange type additive mapping.* We investigate the Hyers–Ulam–Rassias stability of a generalized Euler–Lagrange type additive mapping in Banach modules over a C^*-algebra. These results are applied to investigate C^*-algebra isomorphisms between unital C^*-algebras.

2 Hyers–Ulam–Rassias stability of a generalized Euler–Lagrange type additive mapping in Banach modules over a C^*-algebra

Throughout this section, assume that A is a unital C^*-algebra with norm $|\cdot|$ and unitary group $U(A)$, and that X and Y are left Banach modules over a unital C^*-algebra A with norms $||\cdot||$ and $\|\cdot\|$, respectively. We set $N := \sum_{i=1}^{n} r_i$.

For a given mapping $f : X \to Y$ and a given $u \in U(A)$, we define $D_u f : X^n \to Y$ by

$$D_u f(x_1, \cdots, x_n) := \sum_{i=1}^{n} r_i f\left(\sum_{j=1}^{n} r_j(ux_i - ux_j)\right) + \left(\sum_{i=1}^{n} r_i\right) f\left(\sum_{i=1}^{n} r_i ux_i\right) - \left(\sum_{i=1}^{n} r_i\right) \sum_{i=1}^{n} r_i uf(x_i)$$

for all $x_1, \cdots, x_n \in X$.

Lemma 2.1. *Assume that a mapping $L : X \to Y$ satisfies the functional equation* (1.1) *and that $L(0) = 0$. Then we have*

$$L(N^k x) = N^k L(x) \tag{2.1}$$

for all $x \in X$ and all $k \in \mathbf{Z}$.

Proof. Putting $x_1 = \cdots = x_n = x$ in (1.1), we get $NL(Nx) = N^2 L(x)$ for all $x \in X$. So we get

$$L(N^k x) = N^k L(x) \tag{2.2}$$

for all $x \in X$ by induction on $k \in \mathbf{N}$.

It follows from (2.2) that

$$L(\frac{x}{N^k}) = \frac{1}{N^k} L(x)$$

for all $x \in X$ and all $k \in \mathbf{N}$. So we get the equality (2.1). ∎

We investigate the Hyers–Ulam–Rassias stability of a generalized Euler–Lagrange type additive mapping in Banach spaces.

Theorem 2.2. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^n \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} \frac{1}{N^j} \varphi(N^j x_1, \cdots, N^j x_n) < \infty, \tag{2.3}$$

$$\|D_1 f(x_1, \cdots, x_n)\| \le \varphi(x_1, \cdots, x_n) \tag{2.4}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique generalized Euler–Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{1}{N^2} \widetilde{\varphi}(\underbrace{x, \cdots, x}_{n\ times}) \tag{2.5}$$

for all $x \in X$.

Note that if $N = 1$ in (2.3), then φ is identically zero. So $f = L$ is itself a generalized Euler–Lagrange type additive mapping. Thus we assume that $N \neq 1$.

Proof. Letting $x_1 = \cdots = x_n = x$ in (2.4), we get the following inequality

$$\left\| N f(Nx) - N^2 f(x) \right\| \le \varphi(\underbrace{x, \cdots, x}_{n \text{ times}}) \tag{2.6}$$

for all $x \in X$. It follows from (2.6) that

$$\left\| f(x) - \frac{f(Nx)}{N} \right\| \le \frac{1}{N^2} \varphi(\underbrace{x, \cdots, x}_{n \text{ times}}) \tag{2.7}$$

for all $x \in X$. Now applying a standard procedure of direct method [3, 24] to the inequality (2.7), we obtain that for all nonnegative integers k, l with $k > l$

$$\left\| \frac{f(N^l x)}{N^l} - \frac{f(N^k x)}{N^k} \right\| \le \frac{1}{N^2} \sum_{j=l}^{k-1} \frac{1}{N^j} \varphi(\underbrace{N^j x, \cdots, N^j x}_{n \text{ times}}) \tag{2.8}$$

for all $x \in X$. Since the right hand side of (2.8) tends to zero as $l \to \infty$, the sequence $\{\frac{f(N^k x)}{N^k}\}$ is a Cauchy sequence for all $x \in X$, and thus converges by the completeness of Y. Thus we can define a mapping $L : X \to Y$ by

$$L(x) = \lim_{k \to \infty} \frac{f(N^k x)}{N^k}$$

for all $x \in X$. Letting $l = 0$ in (2.8), we obtain

$$\left\| f(x) - \frac{f(N^k x)}{N^k} \right\| \leq \frac{1}{N^2} \sum_{j=0}^{k-1} \frac{1}{N^j} \varphi(\underbrace{N^j x, \cdots, N^j x}_{n \text{ times}}) \tag{2.9}$$

for all $x \in X$ and all $k \in \mathbf{N}$. Taking the limit as $k \to \infty$ in (2.9), we obtain the desired inequality (2.5).

It follows from (2.3) and (2.4) that

$$\begin{aligned} \|D_1 L(x_1, \cdots, x_n)\| &= \lim_{k \to \infty} \frac{1}{N^k} \|D_1 f(N^k x_1, \cdots, N^k x_n)\| \\ &\leq \lim_{k \to \infty} \frac{1}{N^k} \varphi(N^k x_1, \cdots, N^k x_n) = 0. \end{aligned} \tag{2.10}$$

Therefore, the mapping $L : X \to Y$ satisfies the equation (1.1) and hence L is a generalized Euler–Lagrange type additive mapping.

To prove the uniqueness, let L' be another generalized Euler–Lagrange type additive mapping satisfying (2.5). By Lemma 2.1, we get $L'(N^k x) = N^k L'(x)$ for all $x \in X$ and all $k \in \mathbf{N}$. Thus we have, for any positive integer k,

$$\begin{aligned} \|L(x) - L'(x)\| &\leq \frac{1}{N^k} \left\{ \left\| L(N^k x) - f(N^k x) \right\| + \left\| f(N^k x) - L'(N^k x) \right\| \right\} \\ &\leq \frac{2}{N^2} \sum_{j=0}^{\infty} \frac{1}{N^{k+j}} \varphi(\underbrace{N^{k+j} x, \cdots, N^{k+j} x}_{n \text{ times}}). \end{aligned}$$

Taking the limit as $k \to \infty$, we conclude that $L(x) = L'(x)$ for all $x \in X$. ∎

Corollary 2.3. *Let $\epsilon \geq 0$ and let p be a real number with $0 < p < 1$ if $N > 1$ and with $p > 1$ if $N < 1$. Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$\|D_1 f(x_1, \cdots, x_n)\| \leq \epsilon \sum_{j=1}^{n} ||x_j||^p$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique generalized Euler–Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{n\epsilon}{N^2 - N^{p+1}} ||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \cdots, x_n) := \epsilon \sum_{j=1}^{n} ||x_j||^p$, and apply Theorem 2.2. ∎

Theorem 2.4. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^n \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_n) := \sum_{j=1}^{\infty} N^j \varphi(\frac{x_1}{N^j}, \cdots, \frac{x_n}{N^j}) < \infty, \tag{2.11}$$

$$\|D_1 f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n) \tag{2.12}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique generalized Euler–Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{1}{N^2}\widetilde{\varphi}(\underbrace{x, \cdots, x}_{n\ times}) \tag{2.13}$$

for all $x \in X$.

Proof. It follows from (2.6) that

$$\left\|f(x) - Nf(\frac{1}{N}x)\right\| \le \frac{1}{N}\varphi(\underbrace{\frac{x}{N}, \cdots, \frac{x}{N}}_{n \text{ times}}) \tag{2.14}$$

for all $x \in X$. Now applying a standard procedure of direct method [3, 24] to the inequality (2.14), we obtain that for all nonnegative integers k, l with $k > l$

$$\left\|N^l f(\frac{x}{N^l}) - N^k f(\frac{x}{N^k})\right\| \le \frac{1}{N^2}\sum_{j=l+1}^{k} N^j \varphi(\underbrace{\frac{x}{N^j}, \cdots, \frac{x}{N^j}}_{n \text{ times}}) \tag{2.15}$$

for all $x \in X$. Since the right hand side of (2.15) tends to zero as $l \to \infty$, the sequence $\{N^k f(\frac{x}{N^k})\}$ is a Cauchy sequence for all $x \in X$, and thus converges by the completeness of Y. Thus we can define a mapping $L : X \to Y$ by

$$L(x) = \lim_{k\to\infty} N^k f(\frac{x}{N^k})$$

for all $x \in X$. Letting $l = 0$ in (2.15), we obtain

$$\left\|f(x) - N^k f(\frac{x}{N^k})\right\| \le \frac{1}{N^2}\sum_{j=1}^{k} N^j \varphi(\underbrace{\frac{x}{N^j}, \cdots, \frac{x}{N^j}}_{n \text{ times}}) \tag{2.16}$$

for all $x \in X$ and all $k \in \mathbf{N}$. Taking the limit as $k \to \infty$ in (2.16), we obtain the desired inequality (2.13).

The rest of the proof is similar to the proof of Theorem 2.2. ■

Corollary 2.5. *Let $\epsilon \ge 0$ and let p be a real number with $0 < p < 1$ if $N < 1$ and with $p > 1$ if $N > 1$. Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$\|D_1 f(x_1, \cdots, x_n)\| \le \epsilon \sum_{j=1}^{n} ||x_j||^p$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique generalized Euler–Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{n\epsilon}{N^{p+1} - N^2}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \cdots, x_n) := \epsilon \sum_{j=1}^{n} ||x_j||^p$, and apply Theorem 2.4. ■

3 Hyers–Ulam–Rassias stability of linear mappings in Banach modules over a C^*-algebra

Throughout this section, assume that $r_1 = \cdots = r_n = r \in (0, \infty)$. Let A be a unital C^*-algebra with norm $|\cdot|$ and unitary group $U(A)$, and let X and Y be left Banach modules over a unital C^*-algebra A with norms $||\cdot||$ and $\|\cdot\|$, respectively.

Lemma 3.1. *If an odd mapping $L : X \to Y$ satisfies* (1.1) *for all $x_1, x_2, \cdots, x_n \in X$, then L is Cauchy additive.*

Proof. Assume that $L : X \to Y$ satisfies (1.1) for all $x_1, x_2, \cdots, x_n \in X$.

Note that $L(0) = 0$ and $L(-x) = -L(x)$ for all $x \in X$ since L is an odd mapping. Putting $x_1 = x, x_2 = y$ and $x_3 = \cdots = x_n = 0$ in (1.1), we get

$$NL(rx + ry) = Nr\Big(L(x) + L(y)\Big) \tag{3.1}$$

for all $x, y \in X$. Letting $y = 0$ in (3.1), $NL(rx) = NrL(x)$ for all $x \in X$. So

$$NrL(x + y) = Nr\Big(L(x) + L(y)\Big)$$

for all $x, y \in X$. Thus L is Cauchy additive. ■

Theorem 3.2. *Let $f : X \to Y$ be an odd mapping for which there is a function $\varphi : X^n \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} \frac{1}{(nr)^j} \varphi((nr)^j x_1, \cdots, (nr)^j x_n) < \infty, \tag{3.2}$$

$$\|D_u f(x_1, \cdots, x_n)\| \le \varphi(x_1, \cdots, x_n) \tag{3.3}$$

for all $u \in U(A)$ and all $x_1, \cdots, x_n \in X$. Then there exists a unique A-linear generalized Euler–Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \frac{1}{(nr)^2} \widetilde{\varphi}(\underbrace{x, \cdots, x}_{n \text{ times}}) \tag{3.4}$$

for all $x \in X$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd mapping. Let $u = 1 \in U(A)$. By Theorem 2.2, there exists a unique generalized Euler–Lagrange type additive mapping $L : X \to Y$ satisfying (3.4).

By the assumption, for each $u \in U(A)$, we get

$$\begin{aligned} \|D_u L(x, \underbrace{0, \cdots, 0}_{n-1 \text{ times}})\| &= \lim_{d\to\infty} \frac{1}{(nr)^d} \|D_u f((nr)^d x, \underbrace{0, \cdots, 0}_{n-1 \text{ times}})\| \\ &\le \lim_{d\to\infty} \frac{1}{(nr)^d} \varphi((nr)^d x, \underbrace{0, \cdots, 0}_{n-1 \text{ times}}) = 0 \end{aligned}$$

for all $x \in X$. So

$$nrL(rux) = nr \cdot ruL(x)$$

for all $u \in U(A)$ and all $x \in X$. By (3.1),

$$L(ux) = \frac{1}{r}L(rux) = uL(x) \tag{3.5}$$

for all $u \in U(A)$ and all $x \in X$.

By the same reasoning as in the proofs of [16] and [19],

$$L(ax + by) = L(ax) + L(by) = aL(x) + bL(y)$$

for all $a, b \in A(a, b \neq 0)$ and all $x, y \in X$. And $L(0x) = 0 = 0L(x)$ for all $x \in X$. So the unique generalized Euler–Lagrange type additive mapping $L : X \to Y$ is an A-linear mapping. ■

Corollary 3.3. *Let $\epsilon \geq 0$ and let p be a real number with $0 < p < 1$ if $nr > 1$ and with $p > 1$ if $nr < 1$. Let $f : X \to Y$ be an odd mapping such that*

$$\|D_u f(x_1, \cdots, x_n)\| \leq \epsilon \sum_{j=1}^{n} ||x_j||^p$$

for all $u \in U(A)$ and all $x_1, \cdots, x_n \in X$. Then there exists a unique A-linear generalized Euler–Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{n\epsilon}{(nr)^2 - (nr)^{p+1}}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \cdots, x_n) = \epsilon \sum_{j=1}^{n} ||x_j||^p$, and apply Theorem 3.2. ■

Theorem 3.4. *Let $f : X \to Y$ be an odd mapping for which there is a function $\varphi : X^n \to [0, \infty)$ satisfying (3.3) such that*

$$\widetilde{\varphi}(x_1, \cdots, x_n) := \sum_{j=1}^{\infty}(nr)^j \varphi(\frac{1}{(nr)^j}x_1, \cdots, \frac{1}{(nr)^j}x_n) < \infty$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique A-linear generalized Euler–Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{1}{(nr)^2}\widetilde{\varphi}(\underbrace{x, \cdots, x}_{n \text{ times}}) \tag{3.6}$$

for all $x \in X$.

Proof. The proof is similar to the proofs of Theorems 2.4 and 3.2. ■

Corollary 3.5. *Let $\epsilon \geq 0$ and let p be a real number with $0 < p < 1$ if $nr < 1$ and with $p > 1$ if $nr > 1$. Let $f : X \to Y$ be an odd mapping such that*

$$\|D_u f(x_1, \cdots, x_n)\| \leq \epsilon \sum_{j=1}^{n} ||x_j||^p$$

for all $u \in U(A)$ and all $x_1, \cdots, x_n \in X$. Then there exists a unique A-linear generalized Euler–Lagrange type additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \frac{n\epsilon}{(nr)^{p+1} - (nr)^2} ||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \cdots, x_n) = \epsilon \sum_{j=1}^{n} ||x_j||^p$, and apply Theorem 3.2. ∎

4 Isomorphisms between unital C^*-algebras

Throughout this section, assume that $r_1 = \cdots = r_n = r \in \mathbf{Q} \cap (0, \infty)$. Assume that A is a unital C^*-algebra with norm $|| \cdot ||$ and unit e, and that B is a unital C^*-algebra with norm $\| \cdot \|$. Let $U(A)$ be the set of unitary elements in A.

We investigate C^*-algebra isomorphisms between unital C^*-algebras.

Theorem 4.1. *Let $h : A \to B$ be an odd bijective mapping satisfying $h((nr)^d uy) = h((nr)^d u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $d \in \mathbf{Z}$, for which there exists a function $\varphi : A^n \to [0, \infty)$ such that*

$$\sum_{j=0}^{\infty} \frac{1}{(nr)^j} \varphi((nr)^j x_1, \cdots, (nr)^j x_n) < \infty, \tag{4.1}$$

$$\|D_\mu h(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n),$$

$$\|h((nr)^d u^*) - h((nr)^d u)^*\| \leq \varphi(\underbrace{(nr)^d u, \cdots, (nr)^d u}_{n \text{ times}}) \tag{4.2}$$

for all $\mu \in S^1 := \{\lambda \in \mathbf{C} \mid |\lambda| = 1\}$, all $u \in U(A)$, all $d \in \mathbf{Z}$ and all $x_1, \cdots, x_n \in A$. Assume that

(4.i) $\displaystyle\lim_{d\to\infty} \frac{1}{(nr)^d} h((nr)^d e)$ *is invertible.*

Then the odd bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. Consider the C^*-algebras A and B as left Banach modules over the unital C^*-algebra $\mathbf{C}$. By Theorem 3.2, there exists a unique $\mathbf{C}$-linear generalized Euler–Lagrange type additive mapping $H : A \to B$ such that

$$\|h(x) - H(x)\| \leq \frac{1}{(nr)^2} \widetilde{\varphi}(\underbrace{x, \cdots, x}_{n \text{ times}}) \tag{4.3}$$

for all $x \in A$. The generalized Euler–Lagrange type additive mapping $H : A \to B$ is given by

$$H(x) = \lim_{d\to\infty} \frac{1}{(nr)^d} h((nr)^d x)$$

for all $x \in A$.

By (4.1) and (4.2), we get

$$\begin{aligned} H(u^*) &= \lim_{d\to\infty} \frac{1}{(nr)^d} h((nr)^d u^*) = \lim_{d\to\infty} \frac{1}{(nr)^d} h((nr)^d u)^* \\ &= (\lim_{d\to\infty} \frac{1}{(nr)^d} h((nr)^d u))^* = H(u)^* \end{aligned}$$

for all $u \in U(A)$. Since H is $\mathbf{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements (see [9]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbf{C}, u_j \in U(A))$,

$$\begin{aligned} H(x^*) &= H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j)^* = (\sum_{j=1}^m \lambda_j H(u_j))^* \\ &= H(\sum_{j=1}^m \lambda_j u_j)^* = H(x)^* \end{aligned}$$

for all $x \in A$.

Since $h((nr)^d uy) = h((nr)^d u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $d \in \mathbf{Z}$,

$$H(uy) = \lim_{d\to\infty} \frac{1}{(nr)^d} h((nr)^d uy) = \lim_{d\to\infty} \frac{1}{(nr)^d} h((nr)^d u)h(y) = H(u)h(y) \tag{4.4}$$

for all $u \in U(A)$ and all $y \in A$. By the additivity of H and (4.4),

$$(nr)^d H(uy) = H((nr)^d uy) = H(u((nr)^d y)) = H(u)h((nr)^d y)$$

for all $u \in U(A)$ and all $y \in A$. Hence

$$H(uy) = \frac{1}{(nr)^d} H(u)h((nr)^d y) = H(u)\frac{1}{(nr)^d} h((nr)^d y) \tag{4.5}$$

for all $u \in U(A)$ and all $y \in A$. Taking the limit in (4.5) as $d \to \infty$, we obtain

$$H(uy) = H(u)H(y) \tag{4.6}$$

for all $u \in U(A)$ and all $y \in A$. Since H is $\mathbf{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbf{C}, u_j \in U(A))$, it follows from (4.6) that

$$\begin{aligned} H(xy) &= H(\sum_{j=1}^m \lambda_j u_j y) = \sum_{j=1}^m \lambda_j H(u_j y) = \sum_{j=1}^m \lambda_j H(u_j)H(y) = H(\sum_{j=1}^m \lambda_j u_j)H(y) \\ &= H(x)H(y) \end{aligned}$$

for all $x, y \in A$.

By (4.4) and (4.6),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in A$. Since $\lim_{d\to\infty} \frac{1}{(nr)^d} h((nr)^d e) = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in A$.

Therefore, the odd bijective mapping $h : A \to B$ is a C^*-algebra isomorphism. ∎

Corollary 4.2. *Let $\epsilon \geq 0$ and let p be a real number with $0 < p < 1$ if $nr > 1$ and with $p > 1$ if $nr < 1$. Let $h : A \to B$ be an odd bijective mapping satisfying $h((nr)^d uy) = h((nr)^d u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $d \in \mathbf{Z}$, such that*

$$\|D_\mu h(x_1, \cdots, x_n)\| \leq \epsilon \sum_{j=1}^{n} ||x_j||^p,$$
$$\|h((nr)^d u^*) - h((nr)^d u)^*\| \leq n\ (nr)^{dp}\epsilon$$

for all $\mu \in S^1$, all $u \in U(A)$, all $d \in \mathbf{Z}$, and all $x_1, \cdots, x_n \in A$. Assume that $\lim_{d\to\infty} \frac{1}{(nr)^d} h((nr)^d e)$ is invertible. Then the odd bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. Define $\varphi(x_1, \cdots, x_n) = \epsilon \sum_{j=1}^{n} ||x_j||^p$, and apply Theorem 4.1. ∎

Theorem 4.3. *Let $h : A \to B$ be an odd bijective mapping satisfying $h((nr)^d uy) = h((nr)^d u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $d \in \mathbf{Z}$, for which there exists a function $\varphi : A^n \to [0, \infty)$ satisfying (4.1), (4.2), and (4.i) such that*

$$\|D_\mu h(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n) \tag{4.7}$$

for $\mu = 1, i$, and all $x_1, \cdots, x_n \in A$. If $h(tx)$ is continuous in $t \in \mathbf{R}$ for each fixed $x \in A$, then the odd bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. Put $\mu = 1$ in (4.7). By the same reasoning as in the proof of Theorem 2.2, there exists a unique generalized Euler–Lagrange type additive mapping $H : A \to B$ satisfying (4.3). By the same reasoning as in the proof of [24], the generalized Euler–Lagrange type additive mapping $H : A \to B$ is $\mathbf{R}$-linear.

Put $\mu = i$ in (4.7). By the same method as in the proof of Theorem 3.2, one can obtain that

$$H(ix) = \lim_{d\to\infty} \frac{1}{(nr)^d} h((nr)^d ix) = \lim_{d\to\infty} \frac{i}{(nr)^d} h((nr)^d x) = iH(x)$$

for all $x \in A$.

For each element $\lambda \in \mathbf{C}$, $\lambda = s + it$, where $s, t \in \mathbf{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) = (s + it)H(x) \\ &= \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbf{C}$ and all $x \in A$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbf{C}$, and all $x, y \in A$. Hence the generalized Euler–Lagrange type additive mapping $H : A \to B$ is $\mathbf{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 4.1. ∎

References

[1] S. Czerwik, *The stability of the quadratic functional equation*, in ‘Stability of Mappings of Hyers–Ulam Type’ (edited by Th. M. Rassias and J. Tabor), Hadronic Press, Florida, 1994, pp. 81–91.

[2] P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[3] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. **27** (1941), 222–224.

[4] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[5] D.H. Hyers, G. Isac and Th.M. Rassias, *On the asymptoticity aspect of Hyers–Ulam stability of mappings*, Proc. Amer. Math. Soc. **126** (1998), 425–430.

[6] K. Jun and H. Kim, *On the stability of Appolonius’ equation*, Bull. Belg. Math. Soc. Simon Stevin **11** (2004), 615–624.

[7] K. Jun and H. Kim, *Ulam stability problem for quadratic mappings of Euler–Lagrange*, Nonlinear Anal.–TMA **61** (2005), 1093–1104.

[8] K. Jun and H. Kim, *On the generalized A-quadratic mappings associated with the variance of a discrete type distribution* Nonlinear Anal.–TMA **62** (2005), 975–987.

[9] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Academic Press, New York, 1983.

[10] C. Park, *On the stability of the linear mapping in Banach modules*, J. Math. Anal. Appl. **275** (2002), 711–720.

[11] C. Park, *Linear functional equations in Banach modules over a C^*-algebra*, Acta Appl. Math. **77** (2003), 125–161.

[12] C. Park, *Modified Trif's functional equations in Banach modules over a C^*-algebra and approximate algebra homomorphisms*, J. Math. Anal. Appl. **278** (2003), 93–108.

[13] C. Park, *The bundle structure of noncommutative tori over UHF-algebras*, Bull. Belg. Math. Soc. Simon Stevin **10** (2003), 321–328.

[14] C. Park, *Universal Jensen's equations in Banach modules over a C^*-algebra and its unitary group*, Acta Math. Sinica **20** (2004), 1047–1056.

[15] C. Park, *Lie $*$-homomorphisms between Lie C^*-algebras and Lie $*$-derivations on Lie C^*-algebras*, J. Math. Anal. Appl. **293** (2004), 419–434.

[16] C. Park, *Homomorphisms between Lie JC^*-algebras and Cauchy–Rassias stability of Lie JC^*-algebra derivations*, J. Lie Theory **15** (2005), 393–414.

[17] C. Park, *Approximate homomorphisms on JB^*-triples*, J. Math. Anal. Appl. **306** (2005), 375–381.

[18] C. Park, *Generalized Hyers–Ulam–Rassias stability of n-sesquilinear-quadratic mappings on Banach modules over C^*-algebras*, J. Comput. Appl. Math. **180** (2005), 51–63.

[19] C. Park, *Homomorphisms between Poisson JC^*-algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97.

[20] C. Park and J. Hou, *Homomorphisms between C^*-algebras associated with the Trif functional equation and linear derivations on C^*-algebras*, J. Korean Math. Soc. **41** (2004), 461–477.

[21] C. Park, J. Hou and S. Oh, *Homomorphisms between JC^*-algebras and between Lie C^*-algebras*, Acta Math. Sinica **21**, 1391–1398.

[22] J.M. Rassias, *On the stability of the Euler–Lagrange functional equation*, Chinese J. Math. **20** (1992), 185–190.

[23] J.M. Rassias, *Solution of the Ulam stability problem for Euler–Lagrange quadratic mappings*, J. Math. Anal. Appl. **220** (1998), 613–639.

[24] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[25] Th.M. Rassias, *On the stability of the quadratic functional equation and its applications*, Studia Univ. Babes-Bolyai **XLIII** (1998), 89–124.

[26] Th.M. Rassias, *The problem of S.M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000), 352–378.

[27] Th.M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.

[28] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.

[29] Th.M. Rassias and P. Šemrl, *On the Hyers–Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993), 325–338.

[30] Th.M. Rassias and K. Shibata, *Variational problem of some quadratic functionals in complex analysis*, J. Math. Anal. Appl. **228** (1998), 234–253.

[31] S.M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

Department of Mathematics
Hanyang University
Seoul 133-791
Republic of Korea
baak@hanyang.ac.kr