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A holographic dual of hydrodynamics

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ABSTRACT: We consider a gravity dual description of time dependent, strongly interacting large- N_c $\mathcal{N} = 4$ SYM. We regard the gauge theory system as a fluid with shear viscosity. Our fluid is expanding in one direction following the Bjorken's picture that is relevant to RHIC experiments. We obtain the dual geometry at the late time that is consistent with dissipative hydrodynamics. We show that the integration constants that cannot be determined by hydrodynamics are given by looking at the horizon of the dual geometry. Relationship between time dependence of the energy density and bulk singularity is also discussed.

KEYWORDS: Gauge-gravity correspondence, Hadronic Colliders.

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1. Introduction

One of the attractive aspects of AdS/CFT [1] is applicability to the real systems after certain amount of deformations. In fact it has been suggested that the fireball in Relativistic Heavy Ion Collision (RHIC) can be explained from dual gravity point of view [2–5], since the quark-gluon systems created there are in the strong coupling region [6]. Although the YM theory described by the standard AdS/CFT is large- N_c $\mathcal{N} = 4$ SYM theory, there are many attempts to construct models closer to QCD [7]. SUSY is not very relevant in the finite temperature context because it is broken completely.

Since the RHIC fireball is expanding, we need to understand AdS/CFT in the time dependent situations. Recently, Janik and Peschanski [8, 9] discussed this problem in non-viscous cases. They use the conservation law and conformal invariance together with the holographic renormalization [10, 11] to express the bulk geometry with given boundary data. As a result, the bulk geometry reproduces the basic features of Bjorken theory [12]. It is also pointed out that inclusion of shear viscosity, although the value is small, is very important in the analyses of real RHIC physics since it plays an essential role in the elliptic flow (see for example, [13, 14]). In fact, the shear viscosity at the strong coupling limit was calculated for the $\mathcal{N} = 4$ SYM systems in ref. [15] using AdS/CFT. So it is natural to ask how the bulk geometry changes if we include the viscous effects in the boundary theory.

In this paper, we establish the dual geometry in the presence of shear viscosity by using the hydro-dynamics as the boundary data. Although our gauge theory is not QCD, we hope there is universal features in the character of strongly interacting gauge theory systems. In fact hydrodynamics, which is our input does not ask much about the details of the microscopic particles and the interactions once the equation of state is given. Therefore we have a chance to extract useful information on the macroscopic properties of the real quark-gluon fluid based on this universality.

If what we get is consistency with fluid dynamics, there would not be much point to consider AdS/CFT dual of it. In fact, the holographic dual of the hydrodynamics contains much more information than the hydrodynamics since AdS/CFT already contains essential information of microscopic gauge theory dynamics. For example, we will show that the holographic dual of the hydrodynamics gives integration constants in the hydrodynamic equations that cannot be determined by hydrodynamics alone. The dual geometry also gives a simple derivation of Stefan-Boltzmann's law in strongly coupled regime with precise Stefan-Boltzmann constant.

The organization of the paper is the following. In section 2, we analyze time dependence of the system in the framework of the relativistic hydrodynamics. We also review the basics of the dissipative relativistic hydrodynamics in order to clarify our setup. Section 3 gives the analysis in the gravity dual. We review the basic framework of the gravity dual and present some results for non-viscous cases obtained in ref. [8]. The main results of the present work will be given in section 3.3 where the late time dual geometry is proposed and consistency with the hydrodynamic analyses is checked. We will show that the holographic dual of the hydrodynamics contains more information than the hydrodynamics. We also make comments on the regularity of the bulk geometry in section 4. We conclude in the final section.

2. Relativistic hydrodynamics with shear viscosity

We begin with a short review of the relativistic hydrodynamics with dissipation. For the relevance to the RHIC fireball, we assume that it is described by the finite temperature theory of a variant of $\mathcal{N} = 4$ SYM. We also follow the Bjorken's picture [12]. In the Bjorken's model, the system undergoes one-dimensional expansion (Bjorken expansion) along the collision axis of the heavy ions, and the fluid of the quarks and gluons has boost symmetry in the so-called central rapidity region [12]. We shall consider only the late-time regime of the Bjorken expansion where the time evolution is slow enough to employ approximations.¹

The energy-momentum tensor in the framework of relativistic hydrodynamics is known to be²

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu} + \tau^{\mu\nu}, \quad (2.1)$$

¹Realistic model should contain three-dimensional expansion as tried in ref. [4]. There, it was suggested to use a dual of three-dimensional cosmic expansion.

²The convention of the signature of the metric is $(-, +, +, +)$ in this paper.

where ρ , P are the energy density and the pressure of the fluid, and $u^\mu = (\gamma, \gamma\vec{v})$ is the four-velocity field in terms of the local fluid velocity \vec{v} . $\tau^{\mu\nu}$ is the dissipative term. In a frame where the energy three-flux vanishes, $\tau^{\mu\nu}$ is given in terms of the bulk viscosity ξ and the shear viscosity η by

$$\tau^{\mu\nu} = -\eta(\Delta^{\mu\lambda}\nabla_\lambda u^\nu + \Delta^{\nu\lambda}\nabla_\lambda u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla_\lambda u^\lambda) - \xi\Delta^{\mu\nu}\nabla_\lambda u^\lambda, \quad (2.2)$$

under the assumption that $\tau^{\mu\nu}$ is of first order in gradients. We have defined the three-frame projector as $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$.

In this paper, we consider pure $\mathcal{N} = 4$ SYM theory whose energy-momentum tensor is traceless. Now, the trace of the energy-momentum tensor is given by

$$T^\mu_\mu = -\rho + 3P - 3\xi\nabla_\lambda u^\lambda. \quad (2.3)$$

Demanding $T^\mu_\mu = 0$ for all the possible frames where (2.2) is valid, we obtain

$$\xi = 0, \quad \text{and} \quad \rho = 3P. \quad (2.4)$$

Notice that the bulk viscosity in the realistic RHIC setup might also be negligible. (See for example, ref. [14].)

We assume that our fluid system is boost invariant following Bjorken [12] since it is actually supported by experiments. We want to take a ‘‘co-moving frame’’ where each point of the fluid labels the coordinate, a concept called Lagrangian frame in fluid dynamics. In this frame all the fluid points are at rest by definition, hence all the fluid points share the same proper time. We can use the rapidity of each fluid-point as a spatial coordinate and the common proper time of each fluid-point as a time coordinate. Therefore a local rest frame (LRF) of the fluid can be given by proper time(τ)-rapidity(y), whose relationship with the cartesian coordinate is $(x^0, x^1, x^2, x^3) = (\tau \cosh y, \tau \sinh y, x^2, x^3)$. We have chosen the collision axis to be in the x^1 direction.

The Minkowski metric in this coordinate has the form of

$$ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_\perp^2, \quad (2.5)$$

where $dx_\perp^2 = (dx^2)^2 + (dx^3)^2$. We assume that the collision happened at $\tau = 0$ and we consider only $\tau \geq 0$ region. Note that $|y| \sim \infty$ corresponds to the fronts of the expanding fluid. Therefore, the whole region on the y-coordinate axis is occupied by the fluid. We also assume that the fluid is extended in the x^2, x^3 directions homogeneously. Since the real fireball produced by RHIC experiment is localized, the set up we use is an idealized one. Nevertheless, the present setup is proper since we are interested in the central rapidity region.

The four-velocity of the fluid at any point in the LRF is $u^\mu = (1, 0, 0, 0)$, and this makes the energy-momentum tensor to be diagonal:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{1}{\tau^2} \left(P - \frac{4\eta}{3\tau} \right) & 0 & 0 \\ 0 & 0 & P + \frac{2\eta}{3\tau} & 0 \\ 0 & 0 & 0 & P + \frac{2\eta}{3\tau} \end{pmatrix}. \quad (2.6)$$

We have three independent quantities, ρ , P and η in (2.6). However, the energy-momentum conservation, $\nabla_\mu T^{\mu\nu} = 0$, together with the equation of state $\rho = 3P$, reduces the number of the independent quantities to be one. One finds that the energy-momentum tensor is written by using only ρ in the following way:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{1}{\tau^2}(-\rho - \tau\dot{\rho}) & 0 & 0 \\ 0 & 0 & \rho + \frac{1}{2}\tau\dot{\rho} & 0 \\ 0 & 0 & 0 & \rho + \frac{1}{2}\tau\dot{\rho} \end{pmatrix}, \quad (2.7)$$

where $\dot{\rho} \equiv \frac{d\rho}{d\tau}$. By identifying (2.6) with (2.7), we obtain the following differential equation that connects η and ρ :³

$$\frac{d\rho}{d\tau} = -\frac{4}{3}\frac{\rho}{\tau} + \frac{4}{3}\frac{\eta}{\tau^2}. \quad (2.8)$$

Note that both of ρ and η depend on the proper time τ in general. Let's assume that the shear viscosity evolves by

$$\eta = \frac{\eta_0}{\tau^\beta}, \quad (2.9)$$

where η_0 is a positive constant. The solution of (2.8) is then given by

$$\rho(\tau) = \frac{\rho_0}{\tau^{4/3}} + \frac{4\eta_0}{1-3\beta} \frac{1}{\tau^{1+\beta}} \quad (\text{for } \beta \neq 1/3), \quad (2.10)$$

$$\rho(\tau) = \frac{\rho_0}{\tau^{4/3}} + \frac{4\eta_0}{3} \frac{\ln(\tau)}{\tau^{4/3}} \quad (\text{for } \beta = 1/3), \quad (2.11)$$

where ρ_0 is a positive constant. For $\beta \leq 1/3$ case, the viscous corrections in the hydrodynamic quantities become dominant in the late time, which invalidates the hydrodynamic description. If $\beta > \frac{1}{3}$, the shear viscosity term is sub-leading in the late time behavior as we expect. Therefore we will consider only $\beta > \frac{1}{3}$ case from now on.

The proper time dependence of the temperature T can be read off by assuming the Stefan-Boltzmann's law $\rho \propto T^4$:

$$T = T_0 \left(\frac{1}{\tau^{1/3}} + \frac{\eta_0}{\rho_0} \frac{1}{1-3\beta} \frac{1}{\tau^\beta} + \dots \right). \quad (2.12)$$

In the *static* finite temperature system of strongly coupled $\mathcal{N} = 4$ SYM theory, it is known that $\eta \propto T^3$ [15]. Let us assume that the same is true in the slowly varying non-static cases. Then we set $\beta = 1$:

$$\eta = \frac{\eta_0}{\tau}. \quad (2.13)$$

³Equation (2.8) turns out to be the same as the one appearing in so-called first order (or standard) dissipative relativistic hydrodynamics. (See for example, ref. [16] and the cited therein.) It is known that the first order formalism has a problem of acausal signal propagation. However, it gives good enough results for our purposes. The details of the causal dissipative relativistic hydrodynamics and consistency of our analysis are shown in appendix B.

We know $\rho \sim T^4$ and $\eta \sim T^3$ cannot be consistent without an additional term in (2.13), but the correction term is negligible in our case. One can check the consistency of our approach in appendix C. The temperature behavior is then given by

$$T = T_0 \left(\frac{1}{\tau^{1/3}} - \frac{\eta_0}{2\rho_0} \frac{1}{\tau} + \dots \right). \quad (2.14)$$

We can evaluate the entropy change in the presence of shear viscosity by using hydrodynamics. The conservation of energy-momentum tensor can be rewritten as

$$\frac{d(\tau\rho)}{d\tau} + P = \frac{4\eta}{3\tau}. \quad (2.15)$$

Using the nature of the one-dimensional expansion, the above can be rephrased as

$$T \frac{d(\tau s)}{d\tau} = \frac{4\eta}{3\tau}, \quad (2.16)$$

where s denotes the entropy density and $\tau s \equiv S$ is the entropy per unit rapidity and unit transverse area.⁴ Notice that in the absence of viscosity, S is constant. Now, the entropy per unit rapidity and unit transverse area has time dependence,

$$\begin{aligned} S(\tau) &= \int d\tau \frac{4\eta}{3\tau T} \\ &= S_\infty - 2 \frac{\eta_0}{T_0} \tau^{-2/3} + O(\tau^{-4/3}), \end{aligned} \quad (2.17)$$

due to creation of entropy by dissipation. However, the creation rate of the entropy slows down with time. S_∞ , that is the entropy per unit rapidity and unit transverse area at $\tau = \infty$, is an integration constant which we cannot determine in the framework of hydrodynamics. We will show that its precise value is given by using the gravity dual in section 3.

3. Holographic Dual of Hydrodynamics

In this section, we will find a five-dimensional metric which is dual to the hydrodynamic description of the YM fluid in the previous section. The basic strategy is to use the Einstein's equation together with the boundary condition given by the energy-momentum tensor at the boundary [10, 11, 8]. We consider general asymptotically AdS metrics in the Fefferman-Graham coordinate:

$$ds^2 = r_0^2 \frac{g_{\mu\nu} dx^\mu dx^\nu + dz^2}{z^2}, \quad (3.1)$$

where $x^\mu = (\tau, y, x^2, x^3)$ in our case. $r_0 \equiv (4\pi g_s N_c \alpha'^2)^{1/4}$ is the length scale given by the string coupling g_s and the number of the colors N_c . The four-dimensional metric $g_{\mu\nu}$ is expanded with respect to z in the following form [10, 11]:

$$g_{\mu\nu}(\tau, z) = g_{\mu\nu}^{(0)}(\tau) + z^2 g_{\mu\nu}^{(2)}(\tau) + z^4 g_{\mu\nu}^{(4)}(\tau) + z^6 g_{\mu\nu}^{(6)}(\tau) + \dots \quad (3.2)$$

⁴A precise definition of S is the entropy within a unit 3d region on the (y, x^2, x^3) coordinate. The volume of this region is τ and it is expanding with time in the x^1 direction.

$g_{\mu\nu}^{(0)}$ is the physical four-dimensional metric for the gauge theory on the boundary, that is given by (2.5) in the present case. The $g_{\mu\nu}^{(n)}$'s depend only on τ because of the translational symmetry in the x^2, x^3 directions and the boost symmetry in the y direction in our setup. $g_{\mu\nu}^{(2)}$ is found to be zero. We can identify the first non-trivial data in (3.2), $g_{\mu\nu}^{(4)}$, with the energy-momentum tensor at the boundary [10]:

$$g_{\mu\nu}^{(4)} = \frac{4\pi G_5}{r_0^3} \langle T_{\mu\nu} \rangle, \tag{3.3}$$

where G_5 is the 5d Newton's constant given by $G_5 = 8\pi^3 \alpha'^4 g_s^2 / r_0^5$ in our notation. For the time being, we set $4\pi G_5 = 1$ and $r_0 = 1$. The higher-order terms in (3.2) are determined by solving the Einstein's equation with negative cosmological constant $\Lambda = -6$ [10, 8]:

$$R_{MN} - \frac{1}{2} G_{MN} R - 6G_{MN} = 0, \tag{3.4}$$

where the metric and the curvature tensor are for the five-dimensional ones of (3.1). $g_{\mu\nu}^{(2n)}$ is described by $g_{\mu\nu}^{(2n-2)}, g_{\mu\nu}^{(2n-4)}, \dots, g_{\mu\nu}^{(0)}$ through solving the Einstein's equation. In other words, we can obtain the higher-order terms in (3.2) recursively by starting with the initial data $g_{\mu\nu}^{(0)}$ (\sim Minkowski) and $g_{\mu\nu}^{(4)}$ ($\sim T_{\mu\nu}$).

3.1 Static cases

In order to demonstrate the use of the above procedure, we first workout the static case using cartesian coordinate $x^\mu = (t, x^1, x^2, x^3)$ instead of the Bjorken's coordinate. The energy-momentum tensor in this case is given by

$$T_{\mu\nu} = \text{diag}(\rho, \rho/3, \rho/3, \rho/3). \tag{3.5}$$

The result of the above procedure gives the solution of the Einstein's equation in Fefferman-Graham co-ordinate:

$$ds^2 = \frac{1}{z^2} \left\{ -\frac{(1 - \frac{\rho}{3}z^4)^2}{1 + \frac{\rho}{3}z^4} dt^2 + \left(1 + \frac{\rho}{3}z^4\right) (dx_1^2 + dx_2^2 + dx_3^2) \right\} + \frac{dz^2}{z^2}, \tag{3.6}$$

which is equivalent to the AdS-Schwarzschild Black hole.⁵ The Hawking temperature is given by

$$T_H = \sqrt{2}/(z_0\pi), \tag{3.7}$$

where $z_0(\tau) = [3/\rho]^{1/4}$ is the position of the horizon. By restoring $4\pi G_5$ and $r_0 \equiv (4\pi g_s N_c \alpha'^2)^{1/4}$, and by identifying T_H with gauge theory temperature T , we obtain the Stefan-Boltzmann's law⁶

$$\rho = \frac{r_0^3}{4\pi G_5} \frac{3\pi^4}{4} T_H^4 = \frac{3}{8} \pi^2 N_c^2 T^4. \tag{3.8}$$

This result agrees with ref. [17].

⁵Notice that the metric is mapped to the standard form of the AdS-Schwarzschild metric through the coordinate transformation $\tilde{z} = z/\sqrt{1 + z^4/z_0^4}$.

⁶Actually Stefan Boltzmann's law is about intensity $I = c\rho/4$ in terms of temperature. But we use the terminology abusively to name (3.8).

3.2 Non-viscous time dependent cases

Coming back to Bjorken's setup, Janik and Peschanski obtained the late time bulk metric [8] by the above procedure. Here we briefly review their work in our language. By using (2.7) and (2.10), the energy-momentum tensor for non-viscous case is explicitly written as

$$T_{\mu\nu} = \begin{pmatrix} \frac{\rho_0}{\tau^{4/3}} & 0 & 0 & 0 \\ 0 & \tau^2 \frac{\rho_0}{3\tau^{4/3}} & 0 & 0 \\ 0 & 0 & \frac{\rho_0}{3\tau^{4/3}} & 0 \\ 0 & 0 & 0 & \frac{\rho_0}{3\tau^{4/3}} \end{pmatrix}. \quad (3.9)$$

Then the metric is given by

$$\begin{aligned} g_{\tau\tau} &= -1 + \frac{\rho_0}{\tau^{4/3}} z^4 + O(z^6), \\ \frac{g_{yy}}{\tau^2} &= 1 + \frac{\rho_0}{3\tau^{4/3}} z^4 + O(z^6), \\ g_{xx} &= 1 + \frac{\rho_0}{3\tau^{4/3}} z^4 + O(z^6), \end{aligned} \quad (3.10)$$

where $g_{xx} = g_{22} = g_{33}$. Notice that our Minkowski metric is given by (2.5). Let's focus on the late time behaviour of the metric, since $T_{\mu\nu}$ is given only for the late time.⁷ If we take $\tau \rightarrow \infty$ limit naively, what we obtain is just the Minkowski metric (2.5). To extract a non-trivial result, the authors of ref. [8] take the limit such that $g^{(4)} z^4$ does not go to zero nor infinity as $\tau \rightarrow \infty$:

$$\tau \rightarrow \infty \quad \text{with} \quad \frac{z}{\tau^{1/3}} \equiv v \text{ fixed.} \quad (3.11)$$

Then by solving the Einstein's equation recursively up to certain order of z , we find $g_{\tau\tau}$, g_{yy}/τ^2 and g_{xx} have the following structure:

$$f^{(1)}(v) + f^{(2)}(v)/\tau^{4/3} + \dots \quad (3.12)$$

By neglecting the $O(\tau^{-4/3})$ quantities, they obtained an analytic expression of the late time metric:

$$ds^2 = \frac{1}{z^2} \left\{ -\frac{\left(1 - \frac{\rho_0}{3} \frac{z^4}{\tau^{4/3}}\right)^2}{1 + \frac{\rho_0}{3} \frac{z^4}{\tau^{4/3}}} d\tau^2 + \left(1 + \frac{\rho_0}{3} \frac{z^4}{\tau^{4/3}}\right) (\tau^2 dy^2 + dx_{\perp}^2) \right\} + \frac{dz^2}{z^2}. \quad (3.13)$$

We can explicitly check that the above metric satisfies

$$G^{LM}(R_{MN} - \frac{1}{2}G_{MN}R - 6G_{MN}) \sim O(1/\tau^{4/3}). \quad (3.14)$$

Notice that (3.13) is a black hole in AdS space with time-dependent horizon. The time dependence of the entropy and the Hawking temperature from the metric (3.13) reproduces the Bjorken's results [12] $S \sim \text{constant}$ as well as $T \sim \tau^{-1/3}$. Interestingly, they observed that the regularity of the geometry at the horizon uniquely selects the power of time

⁷The slow time evolution is necessary to justify the hydrodynamic treatment of the fluid.

evolution of the energy density $\rho \sim \tau^{-4/3}$, which is a consequence of hydrodynamics. (See section 4 for the details.)

If we replace $\rho = \frac{\rho_0}{3} \frac{1}{\tau^{4/3}}$ with $\rho = \frac{\rho_0 \log \tau}{3 \tau^{4/3}}$, we find that the left-hand-side of (3.14) is at the order of $1/(\tau^{4/3} \log \tau)$. In this sense, this replacement makes another late time solution of (3.4). We will use this solution in section 4.

3.3 Viscous cases

Let us come back to our main interest to obtain the bulk geometry in the presence of shear viscosity. The energy-momentum tensor for $\beta = 1$ is written by using (2.7) and (2.10) as

$$T_{\mu\nu} = \begin{pmatrix} \frac{\rho_0}{\tau^{4/3}} - \frac{2\eta_0}{\tau^2} & 0 & 0 & 0 \\ 0 & \tau^2 \left(\frac{\rho_0}{3\tau^{4/3}} - \frac{2\eta_0}{\tau^2} \right) & 0 & 0 \\ 0 & 0 & \frac{\rho_0}{3\tau^{4/3}} & 0 \\ 0 & 0 & 0 & \frac{\rho_0}{3\tau^{4/3}} \end{pmatrix}. \quad (3.15)$$

We find that the metric components, $g_{\tau\tau}$, g_{yy}/τ^2 , g_{xx} have the following structure by solving the Einstein's equation recursively:

$$f^{(1)}(v) + \eta_0 h^{(1)}(v)/\tau^{2/3} + \tilde{f}^{(2)}(v)/\tau^{4/3} + \dots, \quad (3.16)$$

Note that *the viscosity dependent terms exist at the order of $\tau^{-2/3}$ and these are more important than the higher-order terms neglected in (3.13)*. We are considering the late time region $\tau \gg 1$. But to see the effects of viscosity, we need to keep the terms at least to the order of $\tau^{-2/3}$. In this paper we consider the viscosity effects to the minimal order.

Now we solve the Einstein's equation recursively. The power series that appear in the solution can be re-summed to give a compact form of the metric. For the detail, see appendix A. The late time 5d bulk geometry is given by

$$ds^2 = \frac{1}{z^2} \left\{ -\frac{(1 - \frac{\rho z^4}{3})^2}{1 + \frac{\rho z^4}{3}} d^2\tau + \left(1 + \frac{\rho z^4}{3}\right) \left(\frac{1 + \frac{\rho z^4}{3}}{1 - \frac{\rho z^4}{3}}\right)^{-2\gamma} \tau^2 d^2y + \left(1 + \frac{\rho z^4}{3}\right) \left(\frac{1 + \frac{\rho z^4}{3}}{1 - \frac{\rho z^4}{3}}\right)^\gamma d^2x_\perp \right\} + \frac{dz^2}{z^2}, \quad (3.17)$$

where

$$\gamma \equiv \frac{\eta_0}{\rho_0 \tau^{2/3}} \quad \text{and} \quad \rho = \frac{\rho_0}{\tau^{4/3}} - \frac{2\eta_0}{\tau^2}. \quad (3.18)$$

Notice that the energy momentum tensor (3.15) can NOT be written in terms of whole $\rho(\tau)$. It is truly amazing that the final metric nevertheless can be written in terms of $\rho(\tau)$ (apart from the powers) in the compact way. This implies that the position of the horizon can be determined solely by the energy density.⁸

⁸One should keep in mind that we are looking for the late time geometry; the metric (3.17) is correct only to the order of γ and the $O(\gamma^2)$ contributions are not unambiguously determined. The representation of (3.17) is chosen since it makes the volume of the horizon finite.

The Hawking temperature in the adiabatic approximation is given by $T(\tau) = \sqrt{2}/\pi(z_0(\tau))$, where $z_0(\tau) = [3/\rho(\tau)]^{1/4}$ is the time dependent position of the horizon. Just as the static case, we obtain

$$\rho = \frac{3}{8}\pi^2 N_c^2 T^4(\tau), \tag{3.19}$$

by restoring $4\pi G_5$ and r_0 . The entropy per unit rapidity and unit transverse area is given by

$$\begin{aligned} S &= \frac{1}{4G_5} \frac{2\sqrt{2}\tau r_0^3}{z_0^3(\tau)} \\ &= \left(\frac{N_c^2}{2\pi}\right)^{1/4} \left(\frac{\pi}{3}\right)^{3/4} 2\sqrt{2}\rho_0^{3/4} \left(1 - \frac{3}{2}\frac{\eta_0}{\rho_0\tau^{2/3}} + O(\tau^{-4/3})\right). \end{aligned} \tag{3.20}$$

One remarkable thing is that the value of S at $\tau = \infty$, that cannot be determined by hydrodynamics alone, is precisely determined to be

$$S_\infty = \left(\frac{N_c^2}{2\pi}\right)^{1/4} \left(\frac{\pi}{3}\right)^{3/4} 2\sqrt{2}\rho_0^{3/4}, \tag{3.21}$$

in terms of the initial condition ρ_0 .

Let us check consistency of (3.20) and (2.17). The normalized entropy-creation rate is given by

$$\frac{1}{S} \frac{dS}{d\tau} = \frac{\eta_0}{\rho_0\tau^{5/3}} + O(\tau^{-7/3}) \tag{3.22}$$

from the gravity dual and

$$\frac{1}{S} \frac{dS}{d\tau} = \frac{4}{3} \frac{\eta_0}{T_0 S_\infty \tau^{5/3}} + O(\tau^{-7/3}) \tag{3.23}$$

from (2.17) of the hydrodynamics. Comparing (3.22) and (3.23), we obtain

$$S_\infty = \frac{4}{3} \frac{\rho_0}{T_0} = \frac{4}{3} \frac{\rho\tau}{T} \Big|_{\tau=\infty}. \tag{3.24}$$

This is nothing but the relationship among the entropy, the energy (per unit rapidity and unit transverse area) and the temperature obtained by thermodynamics at $\tau = \infty$ where the system reaches thermal equilibrium.

Before closing this section, let us give a technical remark to clarify the meaning of the late time limit (3.11). The readers might want to skip this paragraph at first reading. One finds that the higher order terms we have neglected in (3.12) and (3.16) contain the terms proportional to $v^6\tau^{-4/3}$ for example. On the other hand, the leading order terms in $g_{\tau\tau}$ contains arbitrary higher power of v . Therefore, neglect of the $O(\tau^{-4/3})$ terms is justified only when v is larger than $O(1)$. In fact, we can justify the limit (3.11) when we extract the thermodynamic quantities of the system. Such quantities are associated with the horizon of the black hole and the value of v at the horizon is indeed $O(1)$ constant at the late time. Now, we understand why the naive $\tau \rightarrow \infty$ limit on the z -coordinate (the

limit with fixing z to be constant) is not good for our purpose. The position of the horizon grows with time: $z_0 \sim \tau^{1/3}$. If we treat z to be a constant in the $\tau \rightarrow \infty$ limit, the region we describe becomes infinitely far from the horizon. In other words, neglect of the terms of $(z/\tau^{1/3})^n$ ($n > 0$) is not justified around the horizon. This means, a suitable coordinate is the v -coordinate rather than the z -coordinate to describe near the horizon at $\tau \rightarrow \infty$.

4. Conditions on energy density and bulk singularity

In ref. [8], singularity analysis was used to select a physical metric. Namely, starting with energy density (without viscosity)⁹

$$\rho = \frac{\rho_0}{\tau^l}, \tag{4.1}$$

it is found that the late time bulk geometry is singular except for a special value of l . More precisely, $(R_{MNKL})^2$ at the order of $(\tau)^0$ has singularity at the horizon except for $l = \frac{4}{3}$, the value for the perfect fluid. In fluid dynamics, this value of l is determined by the conservation law and the equation of state. However the above means the bulk metric knows the correct form of the energy density independently [8]. Therefore it is interesting to see whether requiring the regularity of the metric at the horizon gives further control over the behavior of the viscous term as well. The late time bulk geometry with generic value of β can be obtained through similar calculations starting with the energy-momentum tensor (2.7) with ρ given in (2.10) or (2.11).

If $\beta < 1/3$, the viscous correction in ρ is dominant at the late time and $\rho \sim 1/\tau^{1+\beta}$ from (2.10). This leads to the singular geometry since the proper time dependence of the dominant term in ρ , $1/\tau^{1+\beta}$, is not $1/\tau^{4/3}$.

If $\beta > 1/3$, the viscous correction is sub-leading and we should consider $(R_{MNKL})^2$ to the sub-leading order. The late time geometry is in the same form of (3.17) except the following replacement

$$\gamma \rightarrow \gamma' \equiv \eta_0/(\rho_0\tau^{\beta-1/3}), \tag{4.2}$$

and with ρ given by (2.10). We find that $(R_{MNKL})^2$ has the following structure:

$$(R_{MNKL})^2 = \frac{8(5w^{16} + 20w^{12} + 174w^8 + 20w^4 + 5)}{(1 + w^4)^4} + O(\tau^{-4/3}), \tag{4.3}$$

where

$$w \equiv z \left(\frac{\rho}{3}\right)^{1/4} = \frac{z}{\tau^{1/3}} \left(\frac{\rho_0}{3}\right)^{1/4} \left(1 + \frac{4\gamma'}{1 - 3\beta}\right)^{1/4}. \tag{4.4}$$

In the absence of viscosity, the above result is reduced to that of ref. [8]. The first term in the right-hand side of (4.3), that contains the viscous sub-leading corrections, is finite. So the consideration of singularity does not give any further restriction to the viscosity

⁹The value of l is restricted to be $0 < l < 4$ by the positive energy condition for (2.7) [8].

term. If $\beta > 5/3$ the corrections due to the viscosity-dependence give smaller effects in the metric than the non-viscous $O(\tau^{-4/3})$ corrections which are already discarded in the late time geometry. So up to our approximation, the viscous effect is not visible in this case.

For $\beta = 1/3$ case, the viscous corrections are leading order. We find that the metric (3.13), where $\frac{\rho_0}{3} \frac{1}{\tau^{4/3}}$ is replaced with the second term of ρ in (2.11), gives the late time geometry¹⁰ as mentioned in section 3.2. $(R_{MNKL})^2$ at the leading order is given in the same form as that of (4.3) where the ρ in $w = z(\rho/3)^{1/4}$ has to be taken from the second term of (2.11). There is no divergence at the leading order, although the value of β is not consistent with the hydrodynamics because of the dominance of the viscous corrections.

Altogether, our geometry is regular and the viscosity effect is meaningful in the region of

$$1/3 \leq \beta < 5/3. \tag{4.5}$$

Indeed, $\beta = 1$ is within this region. This means that in the late time geometry (or the late time fireball dynamics), the viscous term gives visible contribution to the dynamics of the fireball.

5. Conclusions

We considered the gravity dual of large- N_c $\mathcal{N} = 4$ SYM fluid undergoing one-dimensional expansion with account of shear viscosity. We obtained the late time bulk geometry to the minimal order of the viscous corrections in the analytic form. We found that our viscous corrections do not break the regularity within our approximation. We also found that the time evolution of the thermodynamic quantities given by the late time geometry is consistent with the hydrodynamic analyses.

We saw that the holographic dual of the hydrodynamics contains much more information than the hydrodynamics, since AdS/CFT already contains essential information of microscopic gauge theory dynamics. For example, the holographic dual of the hydrodynamics gave a simple derivation of Stefan-Boltzmann's law in the strongly coupled region with precise Stefan-Boltzmann constant. The integration constant in the hydrodynamic equation was also given by looking at the horizon of the dual geometry.

We believe that by probing the resulting geometry, one can extract many of information on strongly interacting systems in principle. It is important to compute various physical quantities based on the obtained geometry [19]. We can also consider various extensions of the present work. Inclusion of the higher-order corrections, generalization to the systems with chemical potential, consideration of the systems with three-dimensional expansion are possible directions. One can also consider the effects of the bulk viscosity whose presence violates the conformal invariance. In the real QCD, conformal invariance must be broken and including it might be relevant for more realistic account of RHIC fireball. We hope that the present work will shed light on AdS/CFT for non-static non-equilibrium systems and holographic description of RHIC physics.

¹⁰In this case, the late time limit should be taken by fixing $v = z(\log \tau)^{1/4}/\tau^{1/3}$.

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A. Derivation of metric (3.17)

First, we find the following expressions by solving the Einstein's equation recursively to the order of v^{28} :

$$\begin{aligned}
 g_{\tau\tau}(\tau, v) &= -1 + 3a - 4a^2 + 4a^3 - 4a^4 + 4a^5 - 4a^6 + 4a^7 + \dots \\
 &\quad - \frac{2\eta_0 v^4}{\tau^{2/3}} \left(1 + \frac{4a}{3} (-2 + 3a - 4a^2 + 5a^3 - 6a^4 + 7a^5 + \dots) \right) + O(\tau^{-4/3}), \\
 \frac{g_{yy}(\tau, v)}{\tau^2} &= 1 + a - \frac{2\eta_0 v^4}{\tau^{2/3}} \left(1 + \frac{2a}{3} \left(1 + \frac{a}{3} + \frac{a^2}{3} + \frac{a^3}{5} + \frac{a^4}{5} + \frac{a^5}{7} + \dots \right) \right) + O(\tau^{-4/3}), \\
 g_{xx}(\tau, v) &= 1 + a + \frac{2\eta_0 v^4}{\tau^{2/3}} \frac{a}{3} \left(1 + \frac{a}{3} + \frac{a^2}{3} + \frac{a^3}{5} + \frac{a^4}{5} + \frac{a^5}{7} + \dots \right) + O(\tau^{-4/3}), \quad (\text{A.1})
 \end{aligned}$$

where $a = \rho_0 v^4/3$.

The power series in the right-hand sides are re-summed to give the analytic form of the metric:

$$\begin{aligned}
 g_{\tau\tau}(\tau, v) &= -\frac{(1 - \frac{\rho_0 v^4}{3})^2}{1 + \frac{\rho_0 v^4}{3}} - \frac{2\eta_0 v^4}{3\tau^{2/3}} \frac{(1 - \frac{\rho_0 v^4}{3})(3 + \frac{\rho_0 v^4}{3})}{(1 + \frac{\rho_0 v^4}{3})^2} + O(\tau^{-4/3}), \\
 \frac{g_{yy}(\tau, v)}{\tau^2} &= 1 + \frac{\rho_0 v^4}{3} - \frac{2\eta_0 v^4}{3\tau^{2/3}} \left(1 + \frac{1 + \frac{\rho_0 v^4}{3}}{\frac{\rho_0 v^4}{3}} \log \left(\frac{1 + \frac{\rho_0 v^4}{3}}{1 - \frac{\rho_0 v^4}{3}} \right) \right) + O(\tau^{-4/3}), \\
 g_{xx}(\tau, v) &= 1 + \frac{\rho_0 v^4}{3} - \frac{2\eta_0 v^4}{3\tau^{2/3}} \left(1 - \frac{1}{2} \frac{1 + \frac{\rho_0 v^4}{3}}{\frac{\rho_0 v^4}{3}} \log \left(\frac{1 + \frac{\rho_0 v^4}{3}}{1 - \frac{\rho_0 v^4}{3}} \right) \right) + O(\tau^{-4/3}), \quad (\text{A.2})
 \end{aligned}$$

We can check explicitly that (A.2) is indeed the solution of the Einstein's equation that is accurate to the order of $\tau^{-2/3}$. The important feature of (A.2) is that all the terms at the order of $\tau^{-2/3}$ are proportional to η_0 . up to $O(\gamma^2)$ terms:

$$\begin{aligned}
 g_{\tau\tau}(\tau, v) &= -\frac{(1 - \frac{\rho_0(1-2\gamma)v^4}{3})^2}{1 + \frac{\rho_0(1-2\gamma)v^4}{3}} + O(\tau^{-4/3}), \\
 \frac{g_{yy}(\tau, v)}{\tau^2} &= \left(1 + \frac{\rho_0(1-2\gamma)v^4}{3} \right) \left(1 - 2\gamma \log \left(\frac{1 + \frac{\rho_0(1-2\gamma)v^4}{3}}{1 - \frac{\rho_0(1-2\gamma)v^4}{3}} \right) \right) + O(\tau^{-4/3}), \\
 g_{xx}(\tau, v) &= \left(1 + \frac{\rho_0(1-2\gamma)v^4}{3} \right) \left(1 + \gamma \log \left(\frac{1 + \frac{\rho_0(1-2\gamma)v^4}{3}}{1 - \frac{\rho_0(1-2\gamma)v^4}{3}} \right) \right) + O(\tau^{-4/3}), \quad (\text{A.3})
 \end{aligned}$$

or more simply,

$$\begin{aligned}
 g_{\tau\tau}(\tau, v) &= -\frac{\left(1 - \frac{\rho_0(1-2\gamma)v^4}{3}\right)^2}{1 + \frac{\rho_0(1-2\gamma)v^4}{3}} + O(\tau^{-4/3}), \\
 \frac{g_{yy}(\tau, v)}{\tau^2} &= \left(1 + \frac{\rho_0(1-2\gamma)v^4}{3}\right) \left(\frac{1 + \frac{\rho_0(1-2\gamma)v^4}{3}}{1 - \frac{\rho_0(1-2\gamma)v^4}{3}}\right)^{-2\gamma} + O(\tau^{-4/3}), \\
 g_{xx}(\tau, v) &= \left(1 + \frac{\rho_0(1-2\gamma)v^4}{3}\right) \left(\frac{1 + \frac{\rho_0(1-2\gamma)v^4}{3}}{1 - \frac{\rho_0(1-2\gamma)v^4}{3}}\right)^\gamma + O(\tau^{-4/3}). \tag{A.4}
 \end{aligned}$$

The differences among (A.2), (A.3) and (A.4) are at the order of $O(\tau^{-4/3})$ (that is the same order of $O(\gamma^2)$) and they share the same terms to the order of γ .

B. Second order formalism of dissipative relativistic hydrodynamics

In this appendix, we briefly introduce the causal dissipative relativistic hydrodynamics that is also referred to the second order dissipative relativistic hydrodynamics. We shall show that the first order formalism we have employed gives a good approximation of the second order formalism at the late time in our setup.

It is known that the first order formalism of the dissipative relativistic hydrodynamics has a problem; the viscous and the thermal signal propagates instantaneously and causality is broken. (See for example, [16].) In the second order formalism [18], the relaxation time of the fluid is introduced to maintain the causality. The energy density evolution equation in our setting is [16]:

$$\frac{d\rho}{d\tau} = -\frac{4}{3}\frac{\rho}{\tau} + \frac{\Phi}{\tau}, \tag{B.1}$$

where

$$\Phi = \frac{4}{3}\frac{\eta}{\tau} \quad (\text{the first order formalism}), \tag{B.2}$$

$$\tau_\pi \frac{d\Phi}{d\tau} = -\Phi + \frac{4}{3}\frac{\eta}{\tau} \quad (\text{the second order formalism}), \tag{B.3}$$

and τ_π is the relaxation time of the system. Note that the $\tau_\pi \rightarrow 0$ limit gives the first order formalism.

Let us evaluate the difference between the first order formalism and the second order formalism for our case. We begin with the assumption that the proper time dependence of the shear viscosity is given by (2.9). Substituting (2.9) to (B.3), we find

$$\begin{aligned}
 \Phi &= \frac{4}{3}\frac{\eta}{\tau} \left\{ 1 + (1 + \beta)\frac{\tau_\pi}{\tau} + (1 + \beta)(2 + \beta)\left(\frac{\tau_\pi}{\tau}\right)^2 + \dots \right\} \\
 &+ \text{constant} \times e^{-\tau/\tau_\pi}. \tag{B.4}
 \end{aligned}$$

This means that the second order formalism approaches to the first order formalism when

$$\frac{\tau_\pi}{\tau} \ll 1. \tag{B.5}$$

Therefore, our analyses based on the first order formalism (2.8) with assumption (2.9) and the late time approximation are self-consistent. The condition (B.5) also agrees with our basic assumption that the microscopic time scale τ_π is short enough comparing to the macroscopic time scale so that the hydrodynamic description is valid. Small value of τ_π also matches the fact that our fluid consists of strongly interacting particles.

C. Consistency of $\eta = \eta_0/\tau$ with $\eta \sim T^3$

Let us check self-consistency of our assumption (2.13). Starting with $\eta = \eta_0/\tau$, the energy density is given as

$$\rho(\tau) = \frac{\rho_0}{\tau^{4/3}} (1 - 2\gamma), \tag{C.1}$$

where

$$\gamma \equiv \frac{\eta_0}{\rho_0 \tau^{2/3}}. \tag{C.2}$$

The relationship $\eta \propto T^3 \propto \rho^{3/4}$ makes further corrections to the shear viscosity like

$$\eta = \frac{\eta_0}{\tau} \{1 + O(\gamma)\}, \tag{C.3}$$

and the $O(\gamma\tau^{-1})$ correction in (C.3) makes further corrections to the energy density recursively. However, all such corrections are at the higher order of γ and we can neglect them if $\gamma \ll 1$. Let us define our approximation precisely:

- We consider only the region of $\gamma \ll 1$.
- We consider $\eta\tau$ to the order of 1 and $\rho\tau^{4/3}$ to the order of γ . In other words, we consider only to the order of η_0 .

The above makes our framework to be self-consistent.¹¹ Note that the positive energy condition for (C.1) is also guaranteed in the above approximation.

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¹¹The physical meaning of $\gamma \ll 1$ is that the contribution of the shear viscosity to the energy density is small. This is also a necessary condition to justify the hydrodynamic treatment of the system.

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