Generic Bell Inequalities for Multipartite Arbitrary Dimensional Systems

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We present generic Bell inequalities for multipartite arbitrary dimensional systems. The inequalities that any local realistic theory must obey are violated by quantum mechanics for even dimensional systems. A large set of variants are shown to naturally emerge from the generic Bell inequalities. We discuss particular variants of Bell inequalities that are violated for all the systems including odd dimensional systems.

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Quantum nonlocality is the most significant evidence of physical observations that cannot be explained by theories based upon local realism. Local realism is rooted in the classical view of measurement, namely, that an observation on one of a pair of subsystems cannot affect the other system faster than the speed of light. In fact, since the advent of quantum mechanics, the nonclassical implications have given rise to fundamental questions on the nature of the act of measurement. In quantum mechanics, a measurement does not provide a preexisting value of the system, rather it is a manifestation of the state of the probed system and the probing apparatus, as advocated by Mermin [1].

As early as 1964, Bell [2] proved that local realism implies constraints on a correlation of measurements between two separate systems. These constraints are incompatible with the quantitative predictions by quantum theory in the case of two coupled spin-$1/2$ particles. These constraints, expressed as so-called Bell inequalities, are of paramount importance in the conceptual foundations of quantum mechanics. But these are idealized experiments with archetypal nonclassical system. A key question remains of whether a complex system of high-dimensional quantum subsystems could eventually simulate a pseudoclassical system that does not contradict local realism.

Since this startling discovery [2], investigating Bell theorem for a general system has been regarded as one of the most important challenges in quantum mechanics and quantum information science (QIS) [3–15]. The motivation is obvious from a scientific and technological viewpoint. First, proving the Bell theorem for a general system would show that quantum physics would apply to macroscopic complex systems. Second, for QIS to outperform the acquisition, manipulation, and transmission of information over its classical counterpart, the property of nonlocality is closely related to its extraordinary power. In fact, the manipulation of a complex quantum system rather than a simple one has practical advantages. Gaining access to the states is easier and more efficient (for example, efficient cluster state QIS [16] and increased security in high-dimensional quantum cryptography [17]). It follows that a nonlocality test for such a complex system is highly desirable. Third, the controllability of quantum operations depends on the nature of the system. Certain logical operations are relatively easy for one system but impossible or difficult for another. It is then essential to understand the nonlocal properties of different systems in order to couple them together, to arrange interfaces between the systems.

Suppose that measurements are performed locally on N subsystems. On each subsystem, one out of *two* observables is measured, bearing d outcomes each, in order to consider a Bell inequality in the composite system of (N, d) [3]. Garg and Mermin formulated marginal probabilities predicted by quantum mechanics for $(2, d)$ and investigated if they can be derived from higher-order joint probabilities [4]. This investigation suggested loss of incompatibility between quantum mechanics and local realism in the limit of $d \to \infty$. Kaszlikowski *et al.* [5] argued that loss of incompatibility was due to restrictions in the type of measurement. They suggested the full use of the d-dimensional Hilbert space to address this issue. Later, Collins *et al.* derived Bell inequalities for high d values, which can exhibit incompatibility for $d \to \infty$ [6]. These works together with an information-theoretic approach of the Bell theorem [18] are all for a composite system of $(2, d)$. For a multipartite spin-$1/2$ system of $(N, 2)$, Mermin [7], Ardehali [8], and Belinskii and Klyshko [9] formulated Bell inequalities based on the statistical properties of the Greenberger, Horne and Zeilinger (GHZ) nonlocality [10]. In summary, we have seen proofs of the Bell theorem for systems of $(2, d)$ and $(N, 2)$ for any N and d. However, note that there are no proofs for arbitrary (N, d) despite its prime importance, as this will complete the proof to rule out any local realistic view which has been driven into the corner to stand for measurements having a continuum of values for a continuum of particles. Very recently, Lee *et al.* showed Bell theorem without inequalities for multipartite multidimensional systems in generalized GHZ states [11]. In this Letter, we generalize the Mermin-Ardehali Bell in-

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equality for a system of arbitrary (N, d). However, we neither suggest to find all the Bell inequalities nor their nonlocality condition. We do not consider many measurement settings other than two, either. This is because our interest is the currently important issue, a proof of the Bell theorem for arbitrary (N, d) system, which has been a long-awaited problem.

Generic Bell inequalities.—Before investigating general cases, in order to understand the principal ideas, we consider a tripartite arbitrary dimensional system which is already significant. Consider three observers and allow each to independently choose one of two variables. The variables are denoted by A_j and B_j for the jth observer. Each variable takes, as its value, an element in the set $S = \{1, \omega, \ldots, \omega^{d-1}\}$ where the elements of S are the dth roots of unity over the complex field. With these variables and their powers, A_j^n and B_j^n, we propose a generic Bell function, $\mathcal{B}$:

$$\mathcal{B} = \frac{1}{2^3} \sum_{n=1}^{d-1} \left\langle \prod_{j=1}^{3} (A_j^n + \omega^{n/2} B_j^n) \right\rangle + \text{c.c.}, \qquad (1)$$

where c.c. stands for complex conjugate. The symbol $\langle \cdot \rangle$ is introduced to denote the statistical average over many runs of the experiment. It is remarkable that the higher-order correlation functions appear in our Bell function. If $d = 2$, our generic Bell function is reduced to Mermin's [7]. In the classical view of the statistical average, the local realism implies that the values for the variables are predetermined, before measurement, by local hidden variable λ: $A_j(\lambda)$ and $B_j(\lambda)$. The correlation among the variables is the statistical average over λ, i.e.,

$$\int d\lambda \rho(\lambda) \prod_{j=1}^{3} V_j(\lambda),$$

where $V_j \in \{A_j, B_j\}$, and $\rho(\lambda)$ is the statistical distribution of λ with $\rho(\lambda) \geq 0$ and $\int d\lambda \rho(\lambda) = 1$. The classical upper bound of the Bell function $\mathcal{B}$ will be obtained by noting the following facts. First, by definition the values of A_j and B_j are ω^{α_j} and ω^{β_j}, where α_j and β_j are integers. Second, for integer α, we have two identities: (a) $\sum_{n=0}^{d-1} \omega^{\alpha n} = d\delta_d(\alpha)$, where $\delta_d(\alpha) = 1$ if $\alpha \equiv 0 \bmod d$ and $\delta_d(\alpha) = 0$ otherwise, and (b) $\sum_{n=1}^{d-1} \omega^{(\alpha+1/2)n} + \text{c.c.} = 0$. Third, the average value of a function $f(\lambda)$ must be less than its maximum: $\int d\lambda \rho(\lambda) f(\lambda) \leq \sup_\lambda f(\lambda)$. We then obtain

$$\mathcal{B} \leq \frac{d}{4}[\delta_d(\alpha_1 + \alpha_2 + \alpha_3) + \delta_d(\alpha_1 + \beta_2 + \beta_3 + 1) + \delta_d(\beta_1 + \alpha_2 + \beta_3 + 1) + \delta_d(\beta_1 + \beta_2 + \alpha_3 + 1)] - 1.$$

If d is even, the four arguments in the δ_d functions cannot all be 0 mod d because $\alpha_1 + \alpha_2 + \alpha_3$ would be even and then the sum of the last three arguments would be odd. Hence

$$\mathcal{B} \leq \frac{3d}{4} - 1, \quad \text{if } d \text{ is even.} \qquad (2)$$

For *odd d*, on the other hand, the Bell function has a larger classical upper bound, $B \leq d - 1$. We will show that quantum mechanics violates the generic Bell inequalities in even dimensions while it does not in odd dimensions.

Violation by quantum mechanics.—For quantum mechanical description, we introduce an operator $\hat{V}_j$ to represent the measurement for a variable $V_j \in \{A_j, B_j\}$ of the jth observer. An *orthogonal* measurement of a given variable V is described by a complete set of orthonormal basis vectors $\{|\alpha\rangle_V\}$. Distinguishing the measurement outcomes can be indicated by a set of values, called eigenvalues. As the variable $V \in \{A, B\}$ takes a value $\omega^\alpha \in S$, let the eigenvalues be the elements in S so that the operator is represented by $\hat{V} = \sum_{\alpha=0}^{d-1} \omega^\alpha |\alpha\rangle_{VV}\langle\alpha|$. In this representation the "observable" operator $\hat{V}$ is unitary [11,12,19]. Each measurement described is nondegenerate with all distinct eigenvalues, called a maximal test [20].

The statistical average over the local hidden variable λ in Eq. (1) is replaced by a quantum average to derive the quantum mechanical Bell function. For the purpose we obtain the nth order quantum correlation function

$$E^n_{V_1V_2V_3} = \langle\psi|\hat{V}_1^n \otimes \hat{V}_2^n \otimes \hat{V}_3^n|\psi\rangle,$$

where $\hat{V}_j^n$ is the nth power of $\hat{V}_j$ and $|\psi\rangle$ is a quantum state of the system. It is obvious that $|E^n| \leq 1$ as the operator $\hat{V}_j$ and its powers are all unitary. After replacing the local hidden-variable averages for A_j and B_j with the quantum averages for $\hat{A}_j$ and $\hat{B}_j$ in Eq. (1), we derive the quantum mechanical Bell function in a useful form of

$$\mathcal{B}_q = \frac{1}{2^2} \sum_{n=1}^{d-1} (E^n_{A_1A_2A_3} + \omega^n E^n_{A_1B_2B_3} + \omega^n E^n_{B_1A_2B_3} + \omega^n E^n_{B_1B_2A_3}). \qquad (3)$$

By noting $|E^n| \leq 1$, the generalized triangle inequality implies that $\mathcal{B}_q$ is bounded from above,

$$|\mathcal{B}_q| \leq d - 1. \qquad (4)$$

The quantum upper bound $(d - 1)$ is larger than the classical upper bound $(3d/4 - 1)$ for an even dimension, as shown in (2), while they are equal for an odd dimension. It is not clear yet whether $\mathcal{B}_q$ actually takes the quantum upper bound as its maximum. If so, it implies that quantum mechanics violates the generic Bell inequalities (2) for even d's that any local realistic theories must satisfy. We will show that this is indeed the case if the system is prepared in a generalized GHZ state and the observable operators $\hat{A}_j$ and $\hat{B}_j$ are given by ones which are employed in showing a generalized GHZ nonlocality [11].

A generalized GHZ state for a tripartite d dimensional system is defined as

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{\alpha=0}^{d-1} |\alpha, \alpha, \alpha\rangle, \quad (5)$$

where $\{|\alpha\rangle\}$ is an orthonormal basis set. We consider the two observable operators $\hat{A}$ and $\hat{B}$ that are introduced in Ref. [11]. For a given eigenvalue ω^{α}, the eigenvector of $\hat{A}$ is given by applying quantum Fourier transformation $\hat{F}$ on the basis vector $|\alpha\rangle$ in Eq. (5):

$$|\alpha\rangle_A = \hat{F}|\alpha\rangle = \frac{1}{\sqrt{d}} \sum_{\beta=0}^{d-1} \omega^{-\alpha\beta}|\beta\rangle, \quad (6)$$

where the subscript A stands for the observable $\hat{A}$. Similarly, the eigenvector of $\hat{B}$ is given by

$$|\alpha\rangle_B = \hat{P}_{1/2}\hat{F}|\alpha\rangle = \frac{1}{\sqrt{d}} \sum_{\beta=0}^{d-1} \omega^{-(\alpha+1/2)\beta}|\beta\rangle, \quad (7)$$

where $\hat{P}_\nu$ is a phase shift operator such that $\hat{P}_\nu|\alpha\rangle = \omega^{-\nu\alpha}|\alpha\rangle$. If $d = 2$, the two observable operators $\hat{A}$ and $\hat{B}$ reduce to Pauli operators $\hat{\sigma}_x$ and $\hat{\sigma}_y$, respectively. It is convenient to introduce a raising operator $\hat{J}$ such that $\hat{J}|d-1\rangle = 0$ and $\hat{J}|\alpha\rangle = |\alpha+1\rangle$ for $0 \le \alpha < d-1$. In particular, the raising operator can be written in terms of $\hat{A}$ and $\hat{B}$ as $\hat{J} = (\hat{A} + \omega^{1/2}\hat{B})/2$. Its Hermitian conjugate (H.c.), $\hat{J}^\dagger$, is a lowering operator. The operator $\hat{J}^n$ ($\hat{J}^{n\dagger}$) implies raising (lowering) by n levels. It is remarkable that the n-level raising operator can be expressed by the nth powers of $\hat{A}$ and $\hat{B}$, that is,

$$\hat{J}^n = \frac{1}{2}\left(\hat{A}^n + \omega^{n/2}\hat{B}^n\right). \quad (8)$$

When $\mathcal{B}_q$ is written in the form similar to Eq. (1), the generic Bell operator is now given by

$$\hat{\mathcal{B}}_q = \sum_{n=1}^{d-1} \bigotimes_{j=1}^{3} \hat{J}_j^n + \text{H.c.} \quad (9)$$

The generalized GHZ state $|\psi\rangle$ is an eigenstate of $\hat{\mathcal{B}}_q$ with the eigenvalue $(d-1)$, i.e., $\mathcal{B}_q = \langle\psi|\hat{\mathcal{B}}_q|\psi\rangle = d-1$. It implies that the quantum expectation $\mathcal{B}_q$ takes as its maximum the quantum upper bound $(d-1)$, derived in Eq. (4), if the tripartite system is in the generalized GHZ state $|\psi\rangle$ and the measurements are chosen with their bases in Eqs. (6) and (7). For the tripartite even dimensional system, we have thus shown the Bell theorem that quantum mechanics conflicts with any local realistic description.

In showing the Bell theorem, the maximal tests of measurements enable the quantum expectation $\mathcal{B}_q$ to take the quantum upper bound in Eq. (4). As the violations are being exhibited in even dimensions, nevertheless, one might apprehend that our Bell inequalities would be simple extensions of or equivalent to the ones with dichotomic observables, for instance, Mermin's inequality [7]. This is not the case: No dichotomic observables can achieve the quantum upper bound (4). Further, if the observables are simultaneously decomposable into the direct sum of dichotomic observables, the inequality is equivalent to a two-dimensional one [11–13]. However, our generic Bell inequalities are genuinely d dimensional in the sense that the observables are not simultaneously decomposable into any subdimensional observables [11].

Generalization to multipartite systems.—The Bell theorem for a tripartite even dimensional system has been proven by showing the incompatibility of the quantum expectation with the generic Bell inequality (2). Our formulation can be generalized to *arbitrary* multipartite *even* dimensional systems simply by increasing the number of parties, $N \ge 3$. Then, the Bell inequality becomes $\mathcal{B} \le d(2^{-N/2} + 2^{-1}) - 1$ if N is even, and $\mathcal{B} \le d(2^{-(N+1)/2} + 2^{-1}) - 1$ otherwise. The quantum expectation $\mathcal{B}_q$ is independent of N and it takes the maximum of $(d-1)$, which is clearly larger than the upper bounds of the Bell inequalities. For qubits of $d = 2$, our generic Bell inequalities reduce to Mermin's [7].

For a bipartite and/or odd dimensional system, on the other hand, quantum mechanics does not violate the generic Bell inequalities as the classical upper bounds are equal to the quantum. For such a system, however, one may consider variants from the generic Bell inequalities and show that some of them are violated by quantum mechanics, as done for qubits [8].

Variants from the generic Bell inequalities.—A large set of variants can emerge from the generic Bell operators. Such a variant Bell operator is written in the form of

$$\hat{\mathcal{B}}_\nu = \frac{1}{2^N} \sum_{n=1}^{d-1} \omega^{\nu n} \bigotimes_{j=1}^{N} (\hat{A}_j^n + \omega^{n/2}\hat{B}_j^n) + \text{H.c.}, \quad (10)$$

where ν is a rational number. We obtain the variant Bell operators from a generic one by some local unitary transformation; for instance, $\hat{\mathcal{B}}_\nu$ is obtained by applying the phase shift operation on the Nth subsystem: $\hat{P}_\nu^\dagger \frac{1}{2}(\hat{A}_N^n + \omega^{n/2}\hat{B}_N^n)\hat{P}_\nu = \hat{P}_\nu^\dagger \hat{J}_N^n \hat{P}_\nu = \omega^{\nu n}\hat{J}_N^n$, where $\hat{P}_\nu$ is defined after Eq. (7). These variants have the same quantum maximum of $(d-1)$ for a generalized GHZ state since they are obtained by local unitary transformation.

All variants of Bell inequalities are not violated by quantum mechanics. For instance, if ν is an integer, the variant becomes equivalent to the generic Bell inequality, which is not violated in a bipartite system. For any N-partite d dimensional system, consider the variant Bell functions, $\mathcal{B}_{1/4}^o$ for an odd N and $\mathcal{B}_{1/4}^e$ for an even N, that satisfy the inequalities,

$$\mathcal{B}_{1/4}^o \le \frac{1}{2^{N-1}} \sum_{k=0}^{(N-1)/2} b_{N,k} - 1, \quad (11)$$

$$\mathcal{B}^{e}_{1/4} \leq \frac{1}{2^N}\left(\sum_{k=0}^{(N-2)/2} b_{N+1,k} + b_{N+1,N/2}\right) - 1, \qquad (12)$$

where

$$b_{n,k} = (-1)^k \binom{n}{(n-1-2k)/2} \cot[\pi(2k+1)/4d]$$

with a binomial coefficient $\binom{a}{b}$. The classical upper bounds in the inequalities (11) and (12) are smaller than the quantum maximum, $(d-1)$. Thus, the Bell theorem holds for all multipartite arbitrary dimensional systems. For qubits, the variant inequalities are equivalent to those derived in Ref. [8]. For a bipartite system of $N = 2$, in particular, the inequality (12) becomes

$$\mathcal{B}^{e}_{1/4} \leq \frac{1}{4}\left(3\cot\frac{\pi}{4d} - \cot\frac{3\pi}{4d}\right) - 1. \qquad (13)$$

Further, a two-qubit system has the classical and quantum upper bounds of $1/\sqrt{2}$ and unity, respectively, and thus their ratio is equal to $\sqrt{2}$ as is in the Clauser-Horne-Shimony-Holt inequality [21].

Remarks.—In our Bell inequalities, the quantum to classical upper bound ratios (QCRs) are always larger than unity, which is a clear proof of the Bell theorem for all the systems. The QCRs in our Bell inequalities decrease as the dimensionality grows, which should be compared with the opposite behavior of the Bell inequality that Collins *et al.* proposed for a bipartite arbitrary dimensional system [6]. Acin *et al.* [15] also suggested an inequality to show a similar trend for a tripartite three dimensional system. It has been shown [22] that the composite measurements of Collins *et al.* are classically correlated. On the other hand, those in this Letter are composed of mutually independent local measurements. The potential inconsistency is in fact not a problem as Bell inequalities are not uniquely defined in any way. However, one important issue to point out is that whereas the inequalities by the others are maximally violated for the *partially* entangled state [23], our Bell inequalities show the maximum violation for the *maximally* entangled state. It is an open question as to whether one can construct other variants of Bell inequalities which show the increase of the QCR for the increase of dimensions.

In summary, we proposed the Bell inequalities for all multipartite arbitrary dimensional systems. They were constructed based on the generalized GHZ nonlocality for multipartite multidimensional systems [11]. Quantum mechanics violates the generic Bell inequalities for even dimensional N-partite systems for $N \geq 3$. The generic Bell inequalities were shown to be genuinely multidimensional in the sense that the observables are not simultaneously decomposable into any subdimensional ones. It was found that the large set of variants of Bell inequalities naturally emerge from the generic ones and a particular variant is violated by quantum mechanics for every N-partite d dimensional system.

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