Research Article

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An extension of the Hermite-Hadamard inequality for a power of a convex function

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Abstract: In this article, we obtain an extension of the classical Hermite-Hadamard inequality for convex functions (concave functions) extending it to the power functions $[f(x)]^n$. Some related inequalities are also introduced. By applying those results in analysis, we obtain new upper and lower bounds for the error function.

Keywords: Hermite-Hadamard inequality, Jensen's inequality, convex function, error function, power function

MSC 2020: 52A40, 52A41, 26D15, 26D07

1 Introduction

Let f be a real-valued function on $I := [a, b] \subset \mathbb{R}$ for $a < b$. The function f is said to be a convex function on I if the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. Many inequalities have been established for convex functions, but the most famous is the Hermite-Hadamard inequality [1], due to its rich geometrical significance and applications, which is stated as follows:

Theorem 1.1. [1] (Hermite-Hadamard Inequality) *Let f be a real-valued and convex function on* $[a, b]$. *Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(b)+f(a)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave.

It is well known that the Hermite-Hadamard inequality plays an important role in the analysis and theory of convex functions. In recent years, there have been many generalizations of the Hermite-Hadamard inequality [2–9].

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The well-known Jensen's inequality for a convex function is given as follows.

Theorem 1.2. [10] (Jensen's inequality) *Let μ be a positive measure on a σ-algebra $\mathcal{F}$ in a set Ω such that $\mu(\Omega) = 1$. If f is a real function in $L^1(\mu)$, $a < f(x) < b$ for all $x \in \Omega$ and φ is convex on (a, b), then*

$$\varphi\left(\int_\Omega f \mathrm{d}\mu\right) \leq \int_\Omega \varphi(f)\mathrm{d}\mu.$$

2 Refinement of Hermite-Hadamard inequality

In this section, assume that f is a real-valued, nonnegative, and convex function on $[a, b]$ with $a < b$ and n is a positive integer and $f^n := f(x)^n := \underbrace{f(x) \times f(x) \times \cdots \times f(x)}_{n}$.

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a nonnegative convex function. Then f^n is convex and*

$$\left[f\left(\frac{a+b}{2}\right)\right]^n \leq \frac{1}{b-a}\int_a^b f(x)^n \mathrm{d}x \leq \frac{f(b)^n + f(a)^n}{2}. \tag{2.1}$$

Proof. Let $g(x) := x^n$. Since g and f are two convex functions,

$$[f(ta + (1-t)b]^n \leq [tf(a) + (1-t)f(b)]^n \leq tf(a)^n + (1-t)f(b)^n.$$

Hence, f^n is a convex function. The inequality (2.1) follows from Theorem 1.1. □

The following example shows that the nonnegativity in Lemma 2.1 is necessary.

Example 2.2. Set $f : [-1, 1] \to \mathbb{R}$: $f(x) = |x| - 1$. Then $f(x)$ is convex, but $[f(x)]^3$ is not convex (see Figure 1).

We now obtain an extension of Lemma 2.1.

Theorem 2.3. *Let f be a real-valued, nonnegative, and convex function on $[a, b]$ and n be a positive integer. Then*

$$\frac{\left[\int_a^b f(x)\mathrm{d}x\right]^n}{(b-a)^n} \leq \frac{1}{b-a}\int_a^b f(x)^n \mathrm{d}x \leq \frac{f(b)^{n+1} - f(a)^{n+1}}{(n+1)[f(b) - f(a)]}, \tag{2.2}$$

where we assume that if $f(a) = f(b)$, then $\frac{f(b)^{n+1} - f(a)^{n+1}}{f(b) - f(a)} := (n+1)f(a)^n$.

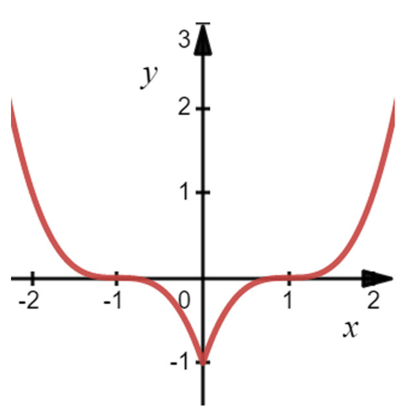

Figure 1: $y = (|x| - 1)^3$.

Proof. Since $\varphi(t) = t^n$ is a convex function on $(0, \infty)$, by the use of Theorem 1.2 with $\varphi(t) = t^n$, we have

$$\left[\int_0^1 f(ta + (1-t)b)\mathrm{d}t\right]^n \leq \int_0^1 f(ta + (1-t)b)^n \mathrm{d}t.$$

By substituting $x = ta + (1-t)b$, it is easy to observe that

$$\left[\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right]^n \leq \frac{1}{b-a}\int_a^b f(x)^n \mathrm{d}x.$$

On the other hand,

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)^n \mathrm{d}x &= \int_0^1 f(ta + (1-t)b)^n \mathrm{d}t\\ &\leq \int_0^1 [tf(a) + (1-t)f(b)]^n \mathrm{d}t\\ &= \int_0^1 \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} f(a)^k f(b)^{n-k}\mathrm{d}t\\ &= \sum_{k=0}^n \binom{n}{k} f(a)^k f(b)^{n-k}\int_0^1 t^k(1-t)^{n-k}\mathrm{d}t\\ &= \sum_{k=0}^n \binom{n}{k} f(a)^k f(b)^{n-k} B(k+1, n-k+1),\end{aligned}$$

where B is the beta function

$$B(m, n) = \int_0^1 t^{m-1}(1-t)^{n-1}\mathrm{d}t = \frac{(m-1)!(n-1)!}{(m+n-1)!},$$

for all $m, n \in \mathbb{N}$. Since

$$\binom{n}{k}\int_0^1 t^k(1-t)^{n-k}\mathrm{d}t = \binom{n}{k}B(k+1, n-k+1) = \frac{1}{n+1}$$

for all $n \in \mathbb{N}_0$ and $k \leq n$,

$$\frac{1}{b-a}\int_a^b f(x)^n \mathrm{d}x \leq \sum_{k=0}^n \frac{1}{n+1} f(a)^k f(b)^{n-k} = \frac{f(b)^{n+1} - f(a)^{n+1}}{(n+1)[f(b) - f(a)]}$$

for $f(a) \neq f(b)$, since $A^{n+1} - B^{n+1} = (A-B)(A^n + A^{n-1}B + \cdots + AB^{n-1} + B^n)$. For $f(a) = f(b)$, the right-hand side of the aforementioned inequality is $f(a)^n$. This completes the proof. □

Lemma 2.4. *Let a and b be positive real numbers with $a \neq b$. Then*

$$\frac{b^{n+1} - a^{n+1}}{(n+1)[b-a]} \leq \frac{b^n + a^n}{2}.$$

Proof. Without the loss of generality, suppose that $a < b$. By applying Theorem 1.1 with $f(x) = x^n$, we obtain

$$\left(\frac{a+b}{2}\right)^n \leq \frac{b^{n+1} - a^{n+1}}{(n+1)[b-a]} \leq \frac{b^n + a^n}{2}.$$

This completes the proof. □

Theorem 2.3 and Lemma 2.4 together yield the following theorem.

Theorem 2.5. *Let f be a real-valued, nonnegative, and convex function on $[a,b]$ and n be a positive integer. Then*

$$f\left(\frac{a+b}{2}\right)^n \le \left[\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right]^n \le \frac{1}{b-a}\int_a^b f(x)^n\mathrm{d}x \le \frac{f(b)^{n+1}-f(a)^{n+1}}{(n+1)[f(b)-f(a)]} \le \frac{f(b)^n+f(a)^n}{2}. \tag{2.3}$$

Proof. By using Lemma 2.4, we obtain

$$\frac{f(b)^{n+1}-f(a)^{n+1}}{(n+1)[f(b)-f(a)]} \le \frac{f(b)^n+f(a)^n}{2} \tag{2.4}$$

and by using Theorem 1.1, we have

$$f\left(\frac{a+b}{2}\right)^n \le \left[\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right]^n. \tag{2.5}$$

The desired result follows from (2.4), (2.5), and Theorem 2.3. □

Corollary 2.6. *Let n be a positive integer and $\sqrt[n]{f}$ be a real-valued, nonnegative, and convex function on $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \le \left[\frac{1}{b-a}\int_a^b \sqrt[n]{f(x)}\,\mathrm{d}x\right]^n \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(b)\sqrt[n]{f(b)}-f(a)\sqrt[n]{f(a)}}{(n+1)[\sqrt[n]{f(b)}-\sqrt[n]{f(a)}]} \le \frac{f(b)+f(a)}{2}, \tag{2.6}$$

where we assume that if $f(b)=f(a)$, then

$$\frac{f(b)\sqrt[n]{f(b)}-f(a)\sqrt[n]{f(a)}}{\sqrt[n]{f(b)}-\sqrt[n]{f(a)}} := (n+1)f(a).$$

Proof. If $f(x)$ is replaced by $\sqrt[n]{f(x)}$, (2.6) follows from (2.3). □

Remark 2.7. With the notations in Corollary 2.6, if $n=1$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(b)+f(a)}{2},$$

which is analogous to inequality (1.1).

Now, we provide an example, which is the importance and superiority of our result compared to the existing results.

Example 2.8. Let $n=3$ and $f:[0,2]\to\mathbb{R}$ be defined by $f(x)=e^{3x^2}$. Then by Corollary 2.6,

$$e^3 \le \left(\frac{1}{2}\int_0^2 \sqrt[3]{e^{3x^2}}\,\mathrm{d}x\right)^3 \le \frac{1}{2}\int_0^2 e^{3x^2}\mathrm{d}x \le \frac{e^{16}-1}{4(e^4-1)} \le \frac{e^{12}+1}{2}.$$

We have the following:

$$e^3 \simeq 20.1 \text{ is a lower bound of the Hermite-Hadamard inequality},$$
$$\left(\frac{1}{2}\int_0^2 \sqrt[3]{e^{3x^2}}\,\mathrm{d}x\right)^3 \simeq 556.7 \text{ is a lower bound of the inequality in Corollary 2.6},$$
$$\frac{e^{16}-1}{4(e^4-1)} \simeq 41447.8 \text{ is an upper bound of the inequality in Corollary 2.6},$$
$$\frac{e^{12}+1}{2} \simeq 81377.9 \text{ is an upper bound of Hermite-Hadamard inequality}.$$

So

$$20.1 \leq 556.7 \leq \frac{1}{2}\int_0^2 e^{3x^2}\mathrm{d}x \leq 41447.8 \leq 81377.9.$$

Corollary 2.9. *Let $f \geq 0$ and $\sqrt{f}$ be a convex function on $[a, b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \left[\frac{1}{b-a}\int_a^b \sqrt{f(x)}\,\mathrm{d}x\right]^2 \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)+\sqrt{f(a)f(b)}}{3} \leq \frac{f(b)+f(a)}{2}.$$

Example 2.10. *Let $f(x) = e^{-x^2}$. Note that $f(x)$ is not convex on $[0, 1)$. But $\sqrt{f(x)}$ is convex on $[1, b]$ for all $b > 1$. Thus,*

$$e^{-\frac{1+b}{2}} \leq \left[\frac{1}{b-1}\int_1^b e^{\frac{-x^2}{2}}\mathrm{d}x\right]^2 \leq \frac{1}{b-1}\int_1^b e^{-x^2}\mathrm{d}x \leq \frac{e^{-1}+e^{-b^2}+e^{-\frac{1+b^2}{2}}}{3} \leq \frac{e^{-1}+e^{-b^2}}{2}$$

for all $b > 1$.

Proposition 2.11. *Let $a < b$. Then*

$$(b-a)e^{\frac{a+b}{2}} \leq \int_a^b e^{x^2}\mathrm{d}x \leq \frac{e^{b^2}-e^{a^2}}{b+a} \leq \frac{(b-a)(e^{b^2}+e^{a^2})}{2}.$$

Proof. Let $a < b$. Define $f(x) = e^{x^2}$ on $[a, b]$. Since $\sqrt[n]{e^{x^2}} = e^{\frac{x^2}{n}}$ is convex on $[a, b]$ for all $n \in \mathbb{N}$, and by Corollary 2.6, we have

$$e^{\frac{a+b}{2}} \leq \frac{1}{b-a}\int_a^b e^{x^2}\mathrm{d}x \leq \frac{e^{\left(1+\frac{1}{n}\right)b^2} - e^{\left(1+\frac{1}{n}\right)a^2}}{(n+1)(e^{\frac{1}{n}b^2} - e^{\frac{1}{n}a^2})} \leq \frac{e^{b^2}+e^{a^2}}{2}$$

for all $n \in \mathbb{N}$. By taking the limit as $n \to \infty$, we have that

$$e^{\frac{a+b}{2}} \leq \frac{1}{b-a}\int_a^b e^{x^2}\mathrm{d}x \leq \lim_{n\to\infty} \frac{e^{\left(1+\frac{1}{n}\right)b^2} - e^{\left(1+\frac{1}{n}\right)a^2}}{(n+1)(e^{\frac{1}{n}b^2} - e^{\frac{1}{n}a^2})} \leq \frac{e^{b^2}+e^{a^2}}{2}.$$

Since

$$\lim_{n\to\infty} \frac{e^{\left(1+\frac{1}{n}\right)b^2} - e^{\left(1+\frac{1}{n}\right)a^2}}{(n+1)(e^{\frac{1}{n}b^2} - e^{\frac{1}{n}a^2})} = \lim_{x\to 0} \frac{x(e^{(1+x)b^2} - e^{(1+x)a^2})}{(x+1)(e^{xb^2} - e^{xa^2})} = \frac{e^{b^2}-e^{a^2}}{b^2-a^2},$$

$$e^{\frac{a+b}{2}} \leq \frac{1}{b-a}\int_a^b e^{x^2}\mathrm{d}x \leq \frac{e^{b^2}-e^{a^2}}{b^2-a^2} \leq \frac{e^{b^2}+e^{a^2}}{2}. \tag{2.7}$$

Let $b^2 = a^2 + y$. Then $y > 0$, and the last inequality in (2.7) becomes $e^{a^2 \frac{e^y - 1}{y}} \leq e^{a^2 \frac{e^y + 1}{2}}$. Let $g(y) = ye^y + y - 2e^y + 2$. Then $g'(y) = e^y(y - 1) + 1$, $g''(y) = ye^y \geq 0$, and $g'(0) = g(0) = 0$ and so $g'(y) \geq 0$ and $g(y) \geq 0$ for all $y \geq 0$. Since $e^{a^2} > 0$, the last inequality in (2.7) holds true. Thus,

$$(b - a)e^{\frac{a+b}{2}} \leq \int_a^b e^{x^2}\mathrm{d}x \leq \frac{e^{b^2} - e^{a^2}}{b + a} \leq \frac{(b - a)(e^{b^2} + e^{a^2})}{2}.$$

This completes the proof. □

Corollary 2.12. *Let $a \neq b$ be two positive real numbers. Then*

$$\left(\frac{a + b}{2}\right)^{kn} \leq \frac{b^{kn+1} - a^{kn+1}}{(b - a)(kn + 1)} \leq \frac{b^{kn+k} - a^{kn+k}}{(b - a)(n + 1)} \leq \frac{b^{kn} + a^{kn}}{2},$$

for all $k, n \in \mathbb{N}$.

Proof. It follows from Corollary 2.6 by choosing $f(x) = x^{kn}$ and appropriate elementary calculations. □

Remark 2.13. Under the notation of Corollary 2.12, if $k = 1$, then

$$\left(\frac{a + b}{2}\right)^{n} \leq \frac{b^{n+1} - a^{n+1}}{(b - a)(n + 1)} \leq \frac{b^{n} + a^{n}}{2},$$

which is a generalization of the following Haber inequality [11]

$$\left(\frac{a + b}{2}\right)^{n} \leq \frac{1}{n + 1}\sum_{i=0}^{n} a^i b^{n-i}.$$

3 Extension of Hermite-Hadamard inequality for the concave functions

In this section, we extend the Hermite-Hadamard inequality for the class of concave functions.

Theorem 3.1. *Let f be a real-valued, nonnegative, and concave function on $[a, b]$, and let n be a positive integer. Then*

$$\frac{[f(b)]^{n+1} - [f(a)]^{n+1}}{(n + 1)[f(b) - f(a)]} \leq \frac{1}{b - a}\int_a^b [f(x)]^n \mathrm{d}x \leq 2^{n-1}[f\left(\frac{a + b}{2}\right)]^n, \tag{3.1}$$

where we assume that if $f(a) = f(b)$, then $\frac{[f(b)]^{n+1} - [f(a)]^{n+1}}{f(b) - f(a)} := (n + 1)[f(a)]^n$.

Theorem 3.2. *Let $\sqrt[n]{f}$ be a real-valued, nonnegative, and concave function on $[a, b]$, and let n be a positive integer. Then*

$$\frac{f(b)\sqrt[n]{f(b)} - f(a)\sqrt[n]{f(a)}}{(n + 1)[\sqrt[n]{f(b)} - \sqrt[n]{f(a)}]} \leq \frac{1}{b - a}\int_a^b f(x)\mathrm{d}x \leq 2^{n-1}f\left(\frac{a + b}{2}\right), \tag{3.2}$$

where we assume that if $f(b) = f(a)$, then

$$\frac{f(b)\sqrt[n]{f(b)} - f(a)\sqrt[n]{f(a)}}{\sqrt[n]{f(b)} - \sqrt[n]{f(a)}} := (n + 1)f(a).$$

Proof. If $f(x)$ is replaced by $\sqrt[n]{f(x)}$, (3.2) follows from (3.1). □

Proposition 3.3. *Let* $0 \le a < b$. *Then*

$$\frac{e^{-a^2} - e^{-b^2}}{a + b} \le \int_a^b e^{-x^2} \mathrm{d}x \le (b - a)2^{m-1} e^{-\left(\frac{a+b}{2}\right)^2},$$

where m *is the smallest integer greater than or equal to* $2b^2$.

Proof. Let $0 \le a < b$. Define $f(x) = e^{-x^2}$ on $[a, b]$. Since $\sqrt[n]{e^{-x^2}} = e^{\frac{-x^2}{n}}$ is concave on $\left[0, \sqrt{\frac{n}{2}}\right]$ for all $n \in \mathbb{N}$, $\sqrt[n]{e^{-x^2}} = e^{\frac{-x^2}{n}}$ is concave on $[a, b] \subseteq \left[0, \sqrt{\frac{n}{2}}\right]$ for all $n \ge 2b^2$. By Theorem 3.2, we have

$$\frac{e^{-b^2\left(1+\frac{1}{n}\right)} - e^{-a^2\left(1+\frac{1}{n}\right)}}{(n + 1)\left[e^{-\frac{b^2}{n}} - e^{-\frac{a^2}{n}}\right]} \le \frac{1}{b - a}\int_a^b e^{-x^2} \mathrm{d}x \le 2^{n-1} e^{-\left(\frac{a+b}{2}\right)^2}, \tag{3.3}$$

for all $n \ge 2b^2$. Hence,

$$\frac{1}{b - a}\int_a^b e^{-x^2} \mathrm{d}x \le 2^{m-1} e^{-\left(\frac{a+b}{2}\right)^2},$$

where m is the smallest integer greater than or equal to $2b^2$.

By taking the limit as $n \to \infty$ in (3.3), we deduce that

$$\lim_{n\to\infty} \frac{e^{-b^2\left(1+\frac{1}{n}\right)} - e^{-a^2\left(1+\frac{1}{n}\right)}}{(n + 1)\left[e^{-\frac{b^2}{n}} - e^{-\frac{a^2}{n}}\right]} \le \frac{1}{b - a}\int_a^b e^{-x^2} \mathrm{d}x.$$

Since

$$\lim_{n\to\infty} \frac{e^{-b^2\left(1+\frac{1}{n}\right)} - e^{-a^2\left(1+\frac{1}{n}\right)}}{(n + 1)\left[e^{-\frac{b^2}{n}} - e^{-\frac{a^2}{n}}\right]} = \lim_{x\to 0} \frac{x(e^{-b^2(1+x)} - e^{-a^2(1+x)})}{(1 + x)[e^{-xb^2} - e^{-xa^2}]} = \frac{e^{-a^2} - e^{-b^2}}{b^2 - a^2},$$

$$\frac{e^{-a^2} - e^{-b^2}}{b^2 - a^2} \le \frac{1}{b - a}\int_a^b e^{-x^2} \mathrm{d}x.$$

Therefore,

$$\frac{e^{-a^2} - e^{-b^2}}{a + b} \le \int_a^b e^{-x^2} \mathrm{d}x \le (b - a)2^{m-1} e^{-\left(\frac{a+b}{2}\right)^2}.$$

This completes the proof. □

Remark 3.4. Note that $m \le 2b^2 + 1$. Thus,

$$\frac{e^{-a^2} - e^{-b^2}}{a + b} \le \int_a^b e^{-x^2} \mathrm{d}x \le (b - a)4^{b^2} e^{-\left(\frac{a+b}{2}\right)^2},$$

for all $b > a \ge 0$.

The error function erf(x) and the complementary error function erfc(x) are defined as follows:

$$\operatorname{erf}(x) = \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}} dt, \quad \operatorname{erfc}(x) = \int_x^\infty \frac{2e^{-t^2}}{\sqrt{\pi}} dt.$$

Proposition 3.5. *Let $x > 0$ and m be the smallest integer greater than or equal to $2x^2$. Then*
(1) $\frac{2(1-e^{-x^2})}{\sqrt{\pi}x} \le \operatorname{erf}(x) \le \frac{2^m}{\sqrt{\pi}} x e^{-\frac{1}{4}x^2} \le \frac{2x4^{x^2}}{\sqrt{\pi}} e^{-\frac{1}{4}x^2}$, *for all $x > 0$.*
(2) $1 - \frac{2x4^{x^2}}{\sqrt{\pi}} e^{-\frac{1}{4}x^2} \le \operatorname{erfc}(x) \le \frac{\sqrt{\pi}x - 2(1-e^{-x^2})}{\sqrt{\pi}x}$.

Proof.
(1) Applying Proposition 3.3 and Remark 3.4 with $a = 0$, $b = x$, after some calculations the desired assertion follows.
(2) Since erfc(x) = 1 − erf(x), the result follows from (1).

The proof is completed. □

4 Hermite-Hadamard inequality for the power of a convex function

Let f be a convex function on $[a, b]$. We define

$$f_0 \equiv \min\{\min\{f(x) : a \le x \le b\}, 0\},$$

and the convex positive part of a real-valued function f is defined by the formula:

$$f_c(x) := f(x) - f_0.$$

The function f can be expressed in terms of f_c and f_0 as $f(x) = f_c(x) + f_0$.

Example 4.1. Set $f : [\frac{3}{2}, 4] \longrightarrow \mathbb{R} : f(x) = x - \ln(x-1) - 3$. Then $f_0 = -1$ and $f_c(x) = x - \ln(x-1) - 2$ (see Figure 2).

Lemma 4.2. *Let f be a real-valued and convex function on $[a, b]$. Then $f_c = f_c(x)$ is a nonnegative convex function on $[a, b]$ and f_0 is nonpositive.*

Proof. It is easy. □

Denote $E_n := \{2k : 2k \le n, k = 0, 1, 2, \ldots\}$ and $O_n := \{2k+1 : 2k+1 \le n, k = 0, 1, 2, \ldots\}$.

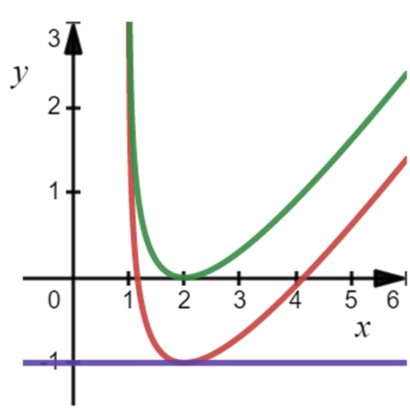

Figure 2: f, f_0, and f_c.

Theorem 4.3. *Let f be a real-valued and convex function on $[a, b]$, and let n be a positive integer. Then*

$$\begin{aligned}
&\sum_{k\in O_n}\binom{n}{k}(f_0)^k\frac{f_c(b)^{n-k}+f_c(a)^{n-k}}{2}+\sum_{k\in E_n}\binom{n}{k}(f_0)^k\left[f_c\left(\frac{a+b}{2}\right)\right]^{n-k}\\
&\le \sum_{k\in O_n}\binom{n}{k}(f_0)^k\frac{f_c(b)^{n-k+1}-f_c(a)^{n-k+1}}{(n-k+1)(f_c(b)-f_c(a))}+\sum_{k\in E_n}\binom{n}{k}(f_0)^k\left[f_c\left(\frac{a+b}{2}\right)\right]^{n-k}\\
&\le \frac{1}{b-a}\int_a^b f(x)^n\mathrm{d}x\\
&\le \sum_{k\in E_n}\binom{n}{k}(f_0)^k\frac{f_c(b)^{n-k+1}-f_c(a)^{n-k+1}}{(n-k+1)(f_c(b)-f_c(a))}+\sum_{k\in O_n}\binom{n}{k}(f_0)^k\left[f_c\left(\frac{a+b}{2}\right)\right]^{n-k}\\
&\le \sum_{k\in E_n}\binom{n}{k}(f_0)^k\frac{f_c(b)^{n-k}+f_c(a)^{n-k}}{2}+\sum_{k\in O_n}\binom{n}{k}(f_0)^k\left[f_c\left(\frac{a+b}{2}\right)\right]^{n-k},
\end{aligned}$$

where $\frac{f_c(b)^{n+1}-f_c(a)^{n+1}}{f_c(b)-f_c(a)} := (n+1)f_c(a)^n$ *for* $f_c(b)=f_c(a)$.

Proof. Let f be a convex function. By the use of Lemma 4.2, the function f_c is a nonnegative function and f_0 is nonpositive. Since

$$f(x)^n = [f_c(x)+f_0]^n = \sum_{k=0}^{n}\binom{n}{k}f_0^k f_c(x)^{n-k},$$

by Theorem 2.3, we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)^n\mathrm{d}x &= \frac{1}{b-a}\int_a^b [f_c(x)+f_0]^n\mathrm{d}x\\
&= \frac{1}{b-a}\left[\sum_{k\in E_n}\binom{n}{k}f_0^k\int_a^b f_c(x)^{n-k}\mathrm{d}x+\sum_{k\in O_n}\binom{n}{k}f_0^k\int_a^b f_c(x)^{n-k}\mathrm{d}x\right]\\
&\le \sum_{k\in E_n}\binom{n}{k}f_0^k\frac{f_c(b)^{n-k+1}-f_c(a)^{n-k+1}}{(n-k+1)(f_c(b)-f_c(a))}+\sum_{k\in O_n}\binom{n}{k}f_0^k\left[f_c\left(\frac{a+b}{2}\right)\right]^{n-k}.
\end{aligned}$$

Also, by Theorem 2.3, we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)^n\mathrm{d}x &= \frac{1}{b-a}\int_a^b [f_c(x)+f_0]^n\mathrm{d}x\\
&= \frac{1}{b-a}\left[\sum_{k\in O_n}\binom{n}{k}f_0^k\int_a^b f_c(x)^{n-k}\mathrm{d}x+\sum_{k\in E_n}\binom{n}{k}f_0^k\int_a^b f_c(x)^{n-k}\mathrm{d}x\right]\\
&\ge \sum_{k\in O_n}\binom{n}{k}f_0^k\frac{f_c(b)^{n-k+1}-f_c(a)^{n-k+1}}{(n-k+1)(f_c(b)-f_c(a))}+\sum_{k\in E_n}\binom{n}{k}f_0^k\left[f_c\left(\frac{a+b}{2}\right)\right]^{n-k},
\end{aligned}$$

which completes the proof. □

Theorem 4.4. *With the notations in* Theorem 4.3, *if* $n = 1$, *then*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\le\frac{f(b)+f(a)}{2},$$

which is analogous to inequality (1.1).

Corollary 4.5. *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)^2 + 2f_0\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right] \\ & \le f\left(\frac{a+b}{2}\right)^2 + 2f_0\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right] + f_0^2 \\ & \le \frac{1}{b-a}\int_a^b f(x)^2 \mathrm{d}x \\ & \le \frac{f(a)^2 + f(a)f(b) + f(b)^2}{3} + 2f_0\left[f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{2}\right] \\ & \le \frac{f(b)^2 + f(a)^2}{2} + 2f_0\left[f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{2}\right]. \end{aligned}$$

Proof. We apply Theorem 4.3 with $n = 2$ to obtain

$$\begin{aligned} f_0(f_c(b) + f_c(a)) + f_c\left(\frac{a+b}{2}\right)^2 + f_0^2 & \le f_0\frac{f_c(b)^2 - f_c(a)^2}{f_c(b) - f_c(a)} + f_c\left(\frac{a+b}{2}\right)^2 + 2f_0^2 \\ & \le \frac{1}{b-a}\int_a^b f(x)^n \mathrm{d}x \\ & \le \frac{f_c(b)^3 - f_c(a)^3}{3(f_c(b) - f_c(a))} + f_0^2 + 2f_0 f_c\left(\frac{a+b}{2}\right) \\ & \le \frac{f_c(b)^2 + f_c(a)^2}{2} + f_0^2 + 2f_0 f_c\left(\frac{a+b}{2}\right). \end{aligned}$$

Thus, by replacing $f_c(a)$ by $f(a) - f_0$, $f_c(b)$ by $f(b) - f_0$ and $f_c\left(\frac{a+b}{2}\right)$ by $f\left(\frac{a+b}{2}\right) - f_0$, we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)^2 + 2f_0\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right] \\ & \le f\left(\frac{a+b}{2}\right)^2 + 2f_0\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right] + f_0^2 \\ & \le \frac{1}{b-a}\int_a^b f(x)^2 \mathrm{d}x \\ & \le \frac{f(a)^2 + f(a)f(b) + f(b)^2}{3} + 2f_0\left[f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{2}\right] \\ & \le \frac{f(b)^2 + f(a)^2}{2} + 2f_0\left[\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{2}\right]. \end{aligned}$$

This completes the proof. □

Remark 4.6. The superiority of the results obtained in this article compared to the existing results for the Hermite-Hadamard's inequality (1.1) is as follows:

(1) In Theorem 2.5, we have generalized the Hermite-Hadamard bound for functions whose powers of n are convex.
(2) In Corollary 2.6, we have improved the Hermite-Hadamard bound for functions, where $f^{\frac{1}{n}}$ is convex, especially, if $n = 2$, then Corollary 2.9 is obtained.
(3) In Theorem 4.3, we have obtained the Hermite-Hadamard inequality for the power n of a convex function, especially, if $n = 2$, then Corollary 4.5 is obtained.
(4) Examples 2.8 and 2.10 and Corollary 2.12 show the superiority of the work compared to the existing results.

5 Conclusion

We obtained an extension of the classical Hermite-Hadamard inequality for convex functions (concave functions) extending it to the power functions $[f(x)]^n$. Some related inequalities were also introduced. By applying those results in the analysis, we obtained new upper and lower bounds for the error function.

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