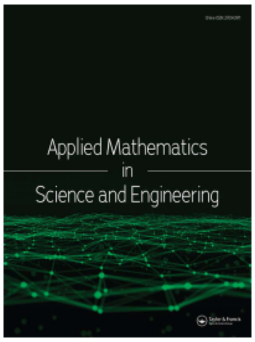

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A system of biadditive functional equations in Banach algebras

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ABSTRACT
In this paper, we obtain the general solution and the Hyers-Ulam stability of the system of biadditive functional equations

$$\begin{cases} 2f(x+y,z+w) - g(x,z) - g(x,w) = g(y,z) + g(y,w) \\ g(x+y,z+w) - 2f(x-y,z-w) = 4f(x,w) + 4f(y,z) \end{cases}$$

in complex Banach spaces. Furthermore, we prove the Hyers–Ulam stability of f-biderivations in complex Banach algebras.

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1. Introduction

In the fall of 1940, Ulam [1] raised the first stability problem. He proposed a question whether there exists an exact homomorphism near an approximate homomorphism. An answer to the problem was given by Hyers [2] in the setting of Banach spaces. Since then the stability problems have been extensively investigated for a variety of functional equations and spaces. In most cases, a functional equation is algebraic in nature whereas the stability is rather metrical. Hence, a normed linear space is a suitable choice to work with the stability of functional equations. We refer to [3–7] for results, references and examples.

Hyers was the first mathematician to present the concequence concerning the stability of functional equations. He answered the question of Ulam for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces (see [2]).

The method provided by Hyers [2] which produces the additive function will be called a direct method. This method is the most important and powerful tool to concerning the stability of different functional equations. That is, the exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the given approximate solution [6, 8]. The other significant method is fixed point theorem, that is, the exact solution of the functional equation is explicitly created as a fixed point of some certain map [9–12].

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Let $\mathcal{B}$ be a complex Banach algebra. A mapping $f : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is biadditive if f is additive in each variable. Furthermore, f is called a $\mathbb{C}$-bilinear mapping if f is $\mathbb{C}$-linear in each variable. Recently, the Hyers–Ulam stability and the hyperstability of biadditive functional equations were proved in [13]. Moreover, many mathematicians have studied the Hyers–Ulam stability of some derivations in algebras and rings (see [14]).

Lemma 1.1 ([15]): *Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras and $f : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ be a biadditive mapping such that $f(\lambda x, \mu y) = \lambda\mu f(x, y)$ for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y \in \mathcal{A}$, then f is $\mathbb{C}$-bilinear.*

Remark 1.1: In Lemma 1.1, $\mathbb{T}^1$ is too big. We can take a smaller set such as a part of the unit circle $\mathbb{T}^1$ or we can take a connected path to obtain the same result as in Lemma 1.1 (see [16]).

Definition 1.2 ([17, 18]): Let $\mathcal{B}$ be a ring. A biadditive mapping $g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is called a symmetric biderivation on $\mathcal{B}$ if g satisfies

$$\begin{cases} g(xy, z) = g(x, z)y + xg(y, z) \\ g(x, z) = g(z, x) \end{cases}$$

for all $x, y, z \in \mathcal{B}$.

Definition 1.3 ([19]): Let $\mathcal{B}$ be a complex Banach algebra. A $\mathbb{C}$-bilinear mapping $g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is called a biderivation on $\mathcal{B}$ if g satisfies

$$\begin{cases} g(xy, z) = g(x, z)y + xg(y, z) \\ g(x, zw) = g(x, z)w + zg(x, w) \end{cases}$$

for all $x, y, z, w \in \mathcal{B}$.

See [11, 20–22] for more information on biderivations in several spaces.

In this paper, we introduce f-biderivations in a Banach algebra.

Definition 1.4: Let $\mathcal{B}$ be a complex Banach algebra and $f : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be a $\mathbb{C}$-bilinear mapping. A $\mathbb{C}$-bilinear mapping $g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is called an f-biderivation on $\mathcal{B}$ if g satisfies

$$\begin{cases} g(xy, z^2) = g(x, z)f(y, z) + f(x, z)g(y, z) \\ g(x^2, zw) = g(x, z)f(x, w) + f(x, z)g(x, w) \end{cases}$$

for all $x, y, z, w \in \mathcal{B}$.

Example 1.5: Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $f(x, y) = \frac{1}{2}xy$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $g(x, y) = xy$ for all $x, y \in \mathbb{R}$. Then g is an f-biderivation.

In this paper, we consider the following system of functional equations

$$\begin{cases} 2f(x + y, z + w) - g(x, z) - g(x, w) = g(y, z) + g(y, w) \\ g(x + y, z + w) - 2f(x - y, z - w) = 4f(x, w) + 4f(y, z) \end{cases} \tag{1}$$

for all $x, y, z, w \in \mathcal{B}$.

The aim of the present paper is to solve the system of biadditive functional equations (1) and prove the Hyers–Ulam stability of f-biderivations in complex Banach algebras by using the fixed point method.

Throughout this paper, assume that $\mathcal{B}$ is a complex Banach algebra.

2. Stability of the system of biadditive functional equations (1)

We solve and investigate the system of biadditive functional equations (1) in complex Banach algebras.

Lemma 2.1: *Let $f, g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be mappings satisfying (1) for all $x, y, z, w \in \mathcal{B}$. Then the mappings $f, g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ are biadditive.*

Proof: Setting $x = y = z = w = 0$ in (1), we have

$$f(0,0) = g(0,0) = 0.$$

Putting $y = z = w = 0$ in (1), we have

$$\begin{cases} 2f(x,0) = 2g(x,0) \\ g(x,0) = 6f(x,0) \end{cases}$$

for all $x \in \mathcal{B}$. Hence

$$f(x,0) = g(x,0) = 0$$

for all $x \in \mathcal{B}$. Taking $x = y = w = 0$ in (1), we obtain

$$\begin{cases} 2f(0,z) = 2g(0,z) \\ g(0,z) = 6f(0,z) \end{cases}$$

and so

$$f(0,z) = g(0,z) = 0$$

for all $z \in \mathcal{B}$.

Letting $y = w = 0$ in (1), we have

$$g(x,z) = 2f(x,z) \tag{2}$$

for all $x, z \in \mathcal{B}$. So

$$g(x+y, z+w) = 2f(x+y, z+w) = g(x,z) + g(x,w) + g(y,z) + g(y,w)$$

for all $x, y, z, w \in \mathcal{B}$. Therefore the mapping $g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is biadditive and thus by (2) the mapping $f : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is biadditive. ∎

Using the fixed point method, we prove the Hyers–Ulam stability of the system of biadditive functional equations (1) in complex Banach algebras.

Theorem 2.2: *Suppose that $\Delta : \mathcal{B}^4 \to [0,\infty)$ is a function such that there exists an $L < 1$ with*

$$\Lambda\left(\frac{x}{2},\frac{y}{2},\frac{z}{2},\frac{w}{2}\right) \le \frac{L}{4}\Lambda(x,y,z,w) \tag{3}$$

for all $x,y,z,w \in \mathcal{B}$. Let $f,g : \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ be mappings satisfying

$$\begin{cases} \|2f(x+y,z+w) - g(x,z) - g(x,w) - g(y,z) - g(y,w)\| \le \Lambda(x,y,z,w) \\ \|g(x+y,z+w) - 2f(x-y,z-w) - 4f(x,w) - 4f(y,z)\| \le \Lambda(x,y,z,w) \end{cases} \tag{4}$$

for all $x,y,z,w \in \mathcal{B}$. Then there exist unique biadditive mappings $F,G : \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ such that

$$\begin{cases} \|F(x,z) - f(x,z)\| \le \frac{L+L^2}{8(1-L)}\Lambda(x,x,z,z) \\ \|G(x,z) - g(x,z)\| \le \frac{L+L^2}{4(1-L)}\Lambda(x,x,z,z) \end{cases} \tag{5}$$

for all $x,z \in \mathcal{B}$.

Proof: Letting $x = y = z = w = 0$ in (4), we obtain

$$\begin{cases} \|2f(0,0) - 4g(0,0)\| \le \Lambda(0,0,0,0) = 0 \\ \|g(0,0) - 10f(0,0)\| \le \Lambda(0,0,0,0) = 0 \end{cases}$$

and hence $f(0,0) = g(0,0) = 0$.

Letting $y = x$ and $w = z$ in (4), we get

$$\begin{cases} \|2f(2x,2z) - 4g(x,z)\| \le \Lambda(x,x,z,z) \\ \|g(2x,2z) - 8f(x,z)\| \le \Lambda(x,x,z,z) \end{cases}$$

for all $x,z \in \mathcal{B}$. So

$$\begin{cases} \|f(x,z) - 16f\left(\frac{x}{4},\frac{z}{4}\right)\| \le \frac{1}{2}\Lambda\left(\frac{x}{2},\frac{x}{2},\frac{z}{2},\frac{z}{2}\right) + 2\Lambda(\frac{x}{4},\frac{x}{4},\frac{z}{4},\frac{z}{4}) \le \frac{L+L^2}{8}\Lambda(x,x,z,z) \\ \|g(x,z) - 16g\left(\frac{x}{4},\frac{z}{4}\right)\| \le \Lambda\left(\frac{x}{2},\frac{x}{2},\frac{z}{2},\frac{z}{2}\right) + 4\Lambda\left(\frac{x}{4},\frac{x}{4},\frac{z}{4},\frac{z}{4}\right) \le \frac{L+L^2}{4}\Lambda(x,x,z,z) \end{cases} \tag{6}$$

for all $x,z \in \mathcal{B}$.

We define

$$H = \{h : \mathcal{B}\times\mathcal{B} \to \mathcal{B} : h(0,0) = 0\}$$

and introduce the generalized metric on H as follows: $d : H \times H \to [0,\infty]$ by

$$d(\delta,h) = \inf\{k \in \mathbb{R}_+ : \|\delta(x,z) - h(x,z)\| \le k\Lambda(x,x,z,z), \forall x,z \in \mathcal{B}\}$$

and we consider $\inf \emptyset = +\infty$. Then d is a complete generalized metric on H (see [23]).

Now, we define the mapping $\mathcal{J} : (H,d) \to (H,d)$ such that

$$\mathcal{J}\delta(x,z) := 16\delta\left(\frac{x}{4},\frac{z}{4}\right)$$

for all $x,z \in \mathcal{B}$.

Assume that $\delta, h \in H$ such that $d(\delta, h) = k$. Then

$$\begin{aligned}\|\mathcal{J}\delta(x,z) - \mathcal{J}h(x,z)\| &= \left\|16\delta\left(\frac{x}{4},\frac{z}{4}\right) - 16h\left(\frac{x}{4},\frac{z}{4}\right)\right\| \\ &\leq 16k\Lambda\left(\frac{x}{4},\frac{x}{4},\frac{z}{4},\frac{z}{4}\right) \leq L^2k\Lambda(x,x,z,z)\end{aligned}$$

for all $x, z \in \mathcal{B}$. It follows that $d(\mathcal{J}\delta(x,z), \mathcal{J}h(x,z)) \leq L^2k$. Thus

$$d(\mathcal{J}\delta(x,z), \mathcal{J}h(x,z)) \leq L^2 d(\delta, h)$$

for all $x, z \in \mathcal{B}$ and all $\delta, h \in H$.

From (6), we have $d(f, \mathcal{J}f) \leq \frac{L+L^2}{8}$ and $d(g, \mathcal{J}g) \leq \frac{L+L^2}{4}$.

Using the fixed point alternative (see [24]), we deduce the existence of unique fixed points of $\mathcal{J}$, that is, the existence of mappings $F, G : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, respectively, such that

$$F(x,z) = 16F\left(\frac{x}{4},\frac{z}{4}\right), \quad G(x,z) = 16G\left(\frac{x}{4},\frac{z}{4}\right)$$

with the following property: there exist $k_1, k_2 \in (0, \infty)$ satisfying

$$\|f(x,z) - F(x,z)\| \leq k_1\Lambda(x,x,z,z), \quad \|g(x,z) - G(x,z)\| \leq k_2\Lambda(x,x,z,z)$$

for all $x, z \in \mathcal{B}$.

Since $\lim_{n\to\infty} d(\mathcal{J}^n f, F) = 0$ and $\lim_{n\to\infty} d(\mathcal{J}^n g, G) = 0$,

$$\lim_{n\to\infty} 4^{2n} f\left(\frac{x}{4^n},\frac{z}{4^n}\right) = F(x,z), \qquad \lim_{n\to\infty} 4^{2n} g\left(\frac{x}{4^n},\frac{z}{4^n}\right) = G(x,z)$$

for all $x, z \in \mathcal{B}$.

Next, $d(f, F) \leq \frac{1}{1-L} d(f, \mathcal{J}f)$ and $d(g, G) \leq \frac{1}{1-L} d(g, \mathcal{J}g)$ which imply

$$\begin{aligned}\|f(x,z) - F(x,z)\| &\leq \frac{L+L^2}{8(1-L)}\Lambda(x,x,z,z), \\ \|g(x,z) - G(x,z)\| &\leq \frac{L+L^2}{4(1-L)}\Lambda(x,x,z,z)\end{aligned}$$

for all $x, z \in \mathcal{B}$.

Using (3) and (4), we conclude that

$$\begin{aligned}&\|2F(x+y, z+w) - G(x,z) - G(x,w) - G(y,z) - G(y,w)\| \\ &= \lim_{n\to\infty} 4^{2n}\left\|2f\left(\frac{x+y}{4^n},\frac{z+w}{4^n}\right) - g\left(\frac{x}{4^n},\frac{z}{4^n}\right) - g\left(\frac{x}{4^n},\frac{w}{4^n}\right)\right. \\ &\quad \left. - g\left(\frac{y}{4^n},\frac{z}{4^n}\right) - g\left(\frac{y}{4^n},\frac{w}{4^n}\right)\right\| \\ &\leq \lim_{n\to\infty} 4^{2n}\Lambda\left(\frac{x}{4^n},\frac{y}{4^n},\frac{z}{4^n},\frac{w}{4^n}\right) \leq \lim_{n\to\infty} L^{2n}\Lambda(x,y,z,w) = 0\end{aligned}$$

and

$$\begin{aligned}&\|G(x+y,z+w)-2F(x-y,z-w)-4F(x,w)-4F(y,z)\|\\&=\lim_{n\to\infty}4^{2n}\left\|G\left(\frac{x+y}{4^n},\frac{z+w}{4^n}\right)-2f\left(\frac{x-y}{4^n},\frac{z-w}{4^n}\right)-4f\left(\frac{x}{4^n},\frac{w}{4^n}\right)-4f\left(\frac{y}{4^n},\frac{z}{4^n}\right)\right\|\\&\leq\lim_{n\to\infty}4^{2n}\Lambda\left(\frac{x}{4^n},\frac{y}{4^n},\frac{z}{4^n},\frac{w}{4^n}\right)\leq\lim_{n\to\infty}L^{2n}\Lambda(x,y,z,w)=0\end{aligned}$$

for all $x,y,z,w\in\mathcal{B}$, since $L<1$. Hence

$$\begin{cases}2F(x+y,z+w)-G(x,z)-G(x,w)=G(y,z)+G(y,w)\\G(x+y,z+w)-2F(x-y,z-w)=4F(x,w)+4F(y,z)\end{cases}$$

for all $x,y,z,w\in\mathcal{B}$. So by Lemma 2.1, the mappings $F,G:\mathcal{B}\to\mathcal{B}$ are biadditive. ■

Corollary 2.3: *Let θ,p be nonnegative real numbers with $p>2$ and $f,g:\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ be mappings satisfying*

$$\begin{cases}\|2f(x+y,z+w)-g(x,z)-g(x,w)-g(y,z)-g(y,w)\|\\\qquad\leq\theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)\\\|g(x+y,z+w)-2f(x-y,z-w)-4f(x,w)-4f(y,z)\|\\\qquad\leq\theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)\end{cases}$$

for all $x,y,z,w\in\mathcal{B}$. Then there exist unique biadditive mappings $F,G:\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ such that

$$\begin{cases}\|F(x,z)-f(x,z)\|\leq\dfrac{(2^p+8)\theta}{2^{p+1}(2^p-4)}(\|x\|^p+\|z\|^p)\\\|G(x,z)-g(x,z)\|\leq\dfrac{(2^p+8)\theta}{2^p(2^p-4)}(\|x\|^p+\|z\|^p)\end{cases}\tag{7}$$

for all $x,z\in\mathcal{B}$.

Proof: The proof follows from Theorem 2.2 by taking $L=2^{2-p}$ and $\Lambda(x,y,z,w)=\theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p)$ for all $x,y,z,w\in\mathcal{B}$. ■

Corollary 2.4: *Let p, q, r, s be nonnegative real numbers with $p+q+r+s>4$ and $f,g:\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ be mappings satisfying*

$$\begin{cases}\|2f(x+y,z+w)-g(x,z)-g(x,w)-g(y,z)-g(y,w)\|\\\qquad\leq\|x\|^p\|y\|^q\|z\|^r\|w\|^s\\\|g(x+y,z+w)-2f(x-y,z-w)-4f(x,w)-4f(y,z)\|\\\qquad\leq\|x\|^p\|y\|^q\|z\|^r\|w\|^s\end{cases}$$

for all $x, y, z, w \in \mathcal{B}$. Then there exist unique biadditive mappings $F, G : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ such that

$$\begin{cases} \|F(x,z) - f(x,z)\| \le \dfrac{2(2^p + 16)}{2^p(2^p - 16}\|x\|^{p+q+r+s} \\ \|G(x,z) - g(x,z)\| \le \dfrac{2^p + 16}{2^p(2^p - 16}\|x\|^{p+q+r+s} \end{cases} \tag{8}$$

for all $x, z \in \mathcal{B}$.

Proof: The proof follows from Theorem 2.2 by taking $L = 2^{4-(p+q+r+s)}$ and $\Lambda(x, y, z, w) = \|x\|^p\|y\|^q\|z\|^r\|w\|^s$ for all $x, y, z, w \in \mathcal{B}$. ■

3. Stability of *f*-biderivations in Banach algebras

In this section, by using the fixed point technique, we prove the Hyers–Ulam stability of f-biderivations in complex Banach algebras.

Lemma 3.1: *Let $f, g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be mappings satisfying*

$$\begin{cases} 2f(\lambda(x+y), \mu(z+w)) - \lambda\mu g(x,z) - \lambda\mu g(x,w) = \lambda\mu g(y,z) + \lambda\mu g(y,w) \\ g(\lambda(x+y), \mu(z+w)) - 2f(\lambda(x-y), \mu(z-w)) = 4\lambda\mu f(x,w) + 4\lambda\mu f(y,z) \end{cases} \tag{9}$$

for all $x, y, z, w \in \mathcal{B}$ and all $\lambda, \mu \in \mathbb{T}^1$. Then the mappings $f, g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ are $\mathbb{C}$-bilinear.

Proof: If we put $\lambda = \mu = 1$ in (9), then f and g are biadditive and

$$g(x,z) = 2f(x,z) \tag{10}$$

for all $x, z \in \mathcal{B}$ by Lemma 2.1.

Now, taking $y = x$ and $w = z$ in (9), we have

$$\begin{cases} 8f(\lambda x, \mu z) = 4\lambda\mu g(x,z) \\ 4g(\lambda x, \mu y) = 8\lambda\mu f(x,z) \end{cases} \tag{11}$$

for all $x, z \in \mathcal{B}$ and all $\lambda, \mu \in \mathbb{T}^1$. From (10) and (11), we obtain

$$f(\lambda x, \mu z) = \lambda\mu f(x,z), \quad g(\lambda x, \mu z) = \lambda\mu g(x,z)$$

for all $x, z \in \mathcal{B}$ and all $\lambda, \mu \in \mathbb{T}^1$. Thus by Lemma 1.1 the mappings f and g are $\mathbb{C}$-bilinear. ■

Theorem 3.2: *Suppose that* $\Delta : \mathcal{B}^4 \to [0,\infty)$ *is a function such that there exists an* $L<1$ *with*

$$\Lambda(\frac{x}{2},\frac{y}{2},\frac{z}{2},\frac{w}{2}) \le \frac{L}{16}\Lambda(x,y,z,w) \tag{12}$$

for all $x,y,z,w \in \mathcal{B}$. *Let* $f,g : \mathcal{B}\times\mathcal{B}\to\mathcal{B}$ *be mappings satisfying*

$$\begin{cases} \|2f(\lambda(x+y),\mu(z+w)) - \lambda\mu g(x,z) - \lambda\mu g(x,w) - \lambda\mu g(y,z) - \lambda\mu g(y,w)\| \\ \quad \le \Lambda(x,y,z,w) \\ \|g(\lambda(x+y),\mu(z+w)) - 2f(\lambda(x-y),\mu(z-w)) - 4\lambda\mu f(x,w) - 4\lambda\mu f(y,z)\| \\ \quad \le \Lambda(x,y,z,w) \end{cases} \tag{13}$$

and

$$\begin{cases} g(xy,z^2) - g(x,z)f(y,z) - f(x,z)g(y,z) \le \Lambda(x,y,z,z) \\ g(x^2,zw) - g(x,z)f(x,w) - f(x,z)g(x,w) \le \Lambda(x,x,z,w) \end{cases} \tag{14}$$

for all $x,y,z,w \in \mathcal{B}$ *and all* $\lambda,\mu \in \mathbb{T}^1$. *Then there exist unique* $\mathbb{C}$*-bilinear mappings* $F,G : \mathcal{B}\times\mathcal{B}\to\mathcal{B}$ *such that G is an F-biderivation and satisfying* (5).

Proof: Let $\lambda=\mu=1$ in (13). By the same reasoning as in the proof of Theorem 2.2, there exist unique biadditive mappings $F,G : \mathcal{B}\times\mathcal{B}\to\mathcal{B}$ satisfying (5), which are given by

$$F(x,z) = \lim_{n\to\infty} 4^{2n} f\left(\frac{x}{4^n},\frac{z}{4^n}\right), \qquad G(x,z) = \lim_{n\to\infty} 4^{2n} g\left(\frac{x}{4^n},\frac{z}{4^n}\right)$$

for all $x,z \in \mathcal{B}$.

From (12) and (13), we have

$$\begin{aligned} &\|2F(\lambda(x+y),\mu(z+w)) - \lambda\mu G(x,z) - \lambda\mu G(x,w) - \lambda\mu G(y,z) - \lambda\mu G(y,w)\| \\ &= \lim_{n\to\infty} 4^{2n}\left\| 2f\left(\frac{\lambda(x+y)}{4^n},\frac{\mu(z+w)}{4^n}\right) - \lambda\mu g\left(\frac{x}{4^n},\frac{z}{4^n}\right) - \lambda\mu g\left(\frac{x}{4^n},\frac{w}{4^n}\right) \right. \\ &\qquad \left. - \lambda\mu g\left(\frac{y}{4^n},\frac{z}{4^n}\right) - \lambda\mu g\left(\frac{y}{4^n},\frac{w}{4^n}\right) \right\| \\ &\le \lim_{n\to\infty} 4^{2n}\Lambda\left(\frac{x}{4^n},\frac{y}{4^n},\frac{z}{4^n},\frac{w}{4^n}\right) \le \lim_{n\to\infty} \frac{L^{2n}}{4^{2n}}\Lambda(x,y,z,w) = 0 \end{aligned}$$

for all $x,y,z,w \in \mathcal{B}$ and all $\lambda,\mu \in \mathbb{T}^1$. Similarly, one can show that

$$\begin{aligned} &\|G(\lambda(x+y),\mu(z+w)) - F(\lambda(x-y),\mu(z-w)) - 4\lambda\mu F(x,w) - 4\lambda\mu F(y,z)\| \\ &\le \lim_{n\to\infty} 4^{2n}\Lambda\left(\frac{x}{4^n},\frac{y}{4^n},\frac{z}{4^n},\frac{w}{4^n}\right) \le \lim_{n\to\infty} \frac{L^{2n}}{4^{2n}}\Lambda(x,y,z,w) = 0 \end{aligned}$$

for all $x,y,z,w \in \mathcal{B}$ and all $\lambda,\mu \in \mathbb{T}^1$. So

$$\begin{cases} 2F(\lambda(x+y),\mu(z+w)) - \lambda\mu G(x,z) - \lambda\mu G(x,w) = \lambda\mu G(y,z) + \lambda\mu G(y,w) \\ G(\lambda(x+y),\mu(z+w)) - F(\lambda(x-y),\mu(z-w)) = 4\lambda\mu F(x,w) + 4\lambda\mu F(y,z) \end{cases}$$

for all $x,y,z,w \in \mathcal{B}$ and all $\lambda,\mu \in \mathbb{T}^1$. Therefore, by Lemma 3.1, the mappings $F,G : \mathcal{B}\times\mathcal{B}\to\mathcal{B}$ are $\mathbb{C}$-bilinear.

Next, from (12) and (14) it follows that

$$\begin{aligned}
&\|G(xy,z^2) - G(x,z)F(y,z) - F(x,z)G(y,z)\| \\
&= \lim_{n\to\infty} 4^{4n}\left\| g\left(\frac{xy}{4^{2n}}, \frac{z^2}{4^{2n}}\right) - g\left(\frac{x}{4^n}, \frac{z}{4^n}\right) f\left(\frac{y}{4^n}, \frac{z}{4^n}\right) - f\left(\frac{x}{4^n}, \frac{z}{4^n}\right) g\left(\frac{y}{4^n}, \frac{z}{4^n}\right)\right\| \\
&\le \lim_{n\to\infty} 4^{4n}\Lambda\left(\frac{x}{4^n}, \frac{y}{4^n}, \frac{z}{4^n}, \frac{z}{4^n}\right) \le \lim_{n\to\infty} L^{2n}\Lambda(x,y,z,z) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\|G(x^2,zw) - G(x,z)F(x,w) - F(x,z)G(x,w)\| \\
&= \lim_{n\to\infty} 4^{4n}\left\| g\left(\frac{x^2}{4^{2n}}, \frac{zw}{4^{2n}}\right) - g\left(\frac{x}{4^n}, \frac{z}{4^n}\right) f\left(\frac{x}{4^n}, \frac{w}{4^n}\right) - f\left(\frac{x}{4^n}, \frac{z}{4^n}\right) g\left(\frac{x}{4^n}, \frac{w}{4^n}\right)\right\| \\
&\le \lim_{n\to\infty} 4^{4n}\Lambda\left(\frac{x}{4^n}, \frac{x}{4^n}, \frac{z}{4^n}, \frac{w}{4^n}\right) \le \lim_{n\to\infty} L^{2n}\Lambda(x,x,z,w) = 0
\end{aligned}$$

for all $x, y, z, w \in \mathcal{B}$ and all $\lambda, \mu \in \mathbb{T}^1$, since $L < 1$. Thus

$$\begin{cases} G(xy,z^2) = G(x,z)F(y,z) + F(x,z)G(y,z) \\ G(x^2,zw) = G(x,z)F(x,w) + F(x,z)G(x,w) \end{cases}$$

for all $x, y, z, w \in \mathcal{B}$. Hence the mapping $G : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is an F-biderivation. ■

Corollary 3.3: *Let θ, p be nonnegative real numbers with $p > 4$ and $f, g : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be mappings satisfying*

$$\begin{cases} \|2f(\lambda(x+y),\mu(z+w)) - \lambda\mu g(x,z) - \lambda\mu g(x,w) - \lambda\mu g(y,z) - \lambda\mu g(y,w)\| \\ \qquad \le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) \\ \|g(\lambda(x+y),\mu(z+w)) - 2f(\lambda(x-y),\mu(z-w)) - 4\lambda\mu f(x,w) - 4\lambda\mu f(y,z)\| \\ \qquad \le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) \end{cases}$$

for all $x, y, z, w \in \mathcal{B}$ and all $\lambda, \mu \in \mathbb{T}^1$. Then there exist unique $\mathbb{C}$-bilinear mappings $F, G : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ such that G is an F-biderivation and satisfying (7).

Proof: The proof follows from Theorem 3.2 and Corollary 2.3. ■

Corollary 3.4: *Let p, q, r, s be nonnegative real numbers with $p + q + r + s > 16$ and $f, g :$ $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be mappings satisfying*

$$\begin{cases} \|2f(\lambda(x+y),\mu(z+w)) - \lambda\mu g(x,z) - \lambda\mu g(x,w) - \lambda\mu g(y,z) - \lambda\mu g(y,w)\| \\ \qquad \le \|x\|^p\|y\|^q\|z\|^r\|w\|^s \\ \|g(\lambda(x+y),\mu(z+w)) - 2f(\lambda(x-y),\mu(z-w)) - 4\lambda\mu f(x,w) - 4\lambda\mu f(y,z)\| \\ \qquad \le \|x\|^p\|y\|^q\|z\|^r\|w\|^s \end{cases}$$

for all $x, y, z, w \in \mathcal{B}$. Then there exist unique $\mathbb{C}$-bilinear mappings $F, G : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ such that G is an F-biderivation and satisfying (8).

***Proof*:** The proof follows from Theorem 3.2 and Corollary 3.4. ■

Remark 3.1: We can obtain some results on asymptotically generalized Lie bi-derivations corresponding to the results given in [25].

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Declarations

Human and animal rights: We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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