Research Article

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A system of additive functional equations in complex Banach algebras

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Abstract: In this article, we solve the system of additive functional equations:

$$\begin{cases} 2f(x+y) - g(x) = g(y), \\ g(x+y) - 2f(y-x) = 4f(x) \end{cases}$$

and prove the Hyers-Ulam stability of the system of additive functional equations in complex Banach spaces. Furthermore, we prove the Hyers-Ulam stability of f-hom-ders in Banach algebras.

Keywords: additive mapping, f-hom-der, fixed point method, Hyers-Ulam stability, system of additive functional equations

MSC 2020: 47B47, 17B40, 39B72, 47H10

1 Introduction

Let $\mathcal{B}$ be a complex Banach algebra and $f : \mathcal{B} \to \mathcal{B}$ be a $\mathbb{C}$-linear mapping. Mirzavaziri and Moslehian [1] introduced the concept of f-derivation $g : \mathcal{B} \to \mathcal{B}$ as follows:

$$g(xy) = f(x)g(y) + g(x)f(y) \tag{1.1}$$

for all $x, y \in \mathcal{B}$.

Park et al. [2] introduced the concept of hom-derivation on $\mathcal{B}$, i.e., $g : \mathcal{B} \to \mathcal{B}$ is a homomorphism and f satisfies (1.1) for all $x, y \in \mathcal{B}$. Dehghanian et al. [3] introduced the concept of hom-der $g : \mathcal{B} \to \mathcal{B}$ as follows:

$$g(x)g(y) = xg(y) + g(x)y$$

for all $x, y \in \mathcal{B}$. Kheawborisuk et al. [4] defined and studied hom-ders in fuzzy Banach algebras.

Definition 1.1. Let $\mathcal{B}$ be a complex Banach algebra and $f : \mathcal{B} \to \mathcal{B}$ be a homomorphism. A $\mathbb{C}$-linear mapping $g : \mathcal{B} \to \mathcal{B}$ is called an f-hom-der if it satisfies

$$g(x)g(y) = f(x)g(y) + g(x)f(y)$$

for all $x, y \in \mathcal{B}$.

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Example 1.2. Let $\mathbf{M}_n$ be the collection of all $n \times n$ complex matrices and $g : \mathbf{M}_n \to \mathbf{M}_n$ defined by $g(X) = 2X$ and $f : \mathbf{M}_n \to \mathbf{M}_n$ defined by $f(X) = X$. Then f is a homomorphism, and g is an f-hom-der.

We say that an equation is stable if any function satisfying the equation approximately is near to an exact solution of the equation.

The stability analysis of functional equations emanated from a question of Ulam [5], which was raised in 1940 about the stability of group homomorphisms and then was extended by Hyers [6]. Recently, results on the so-called Hyers-Ulam stability have comfortable the stability conditions. Many mathematicians developed the Hyers results in various directions [7–14].

The method provided by Hyers [6], which produces the additive function, will be called a direct method. This method is the most significant and strong tool to concerning the stability of different functional equations. That is, the exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the given approximate solution [10,15]. The other significant method is fixed-point theorem, that is, the exact solutions of functional equations and differential equations are explicitly created as fixed points of some certain mappings [16–35].

We remember a fixed-point alternative theorem.

Theorem 1.3. [36] *If $(\mathcal{B}, d)$ is a complete generalized metric space and $J : \mathcal{B} \to \mathcal{B}$ is a strictly contractive mapping, that is,*

$$d(Ju, Jv) \le Ld(u, v)$$

for all $u, v \in \mathcal{B}$ and a Lipschitz constant $L < 1$. Then for each given element $u \in \mathcal{B}$, either

$$d(J^n u, J^{n+1}u) = +\infty, \quad \forall n \ge 0,$$

or

$$d(J^n u, J^{n+1}u) < +\infty, \quad \forall n \ge n_0,$$

for some positive integer n_0. Furthermore, if the second alternative holds, then:
- *(i) the sequence $(J^n u)$ is convergent to a fixed point v^* of J;*
- *(ii) v^* is the unique fixed point of J in the set $V := \{v \in \mathcal{B}, d(J^{n_0}u, v) < +\infty\}$;*
- *(iii) $d(v, v^*) \le \frac{1}{1-L} d(v, Jv)$ for all $u, v \in V$.*

In this article, we consider the following system of additive functional equations

$$\begin{cases} 2f(x+y) - g(x) = g(y), \\ g(x+y) - 2f(y-x) = 4f(x) \end{cases} \tag{1.2}$$

for all $x, y \in \mathcal{B}$. The aim of the present article is to solve the system of additive functional equations and prove the Hyers-Ulam stability of f-hom-ders in complex Banach algebras by using the fixed point method.

Throughout this article, assume that $\mathcal{B}$ is a complex Banach algebra.

2 Stability of system of additive functional equations

We solve and investigate the system of additive functional equations (1.2) in complex Banach algebras.

Lemma 2.1. *Let $f, g : \mathcal{B} \to \mathcal{B}$ be mappings satisfying* (1.2) *for all $x, y \in \mathcal{B}$. Then the mappings $f, g : \mathcal{B} \to \mathcal{B}$ are additive.*

Proof. By substituting $x = y = 0$ in (1.2), we obtain

$$f(0) = g(0) = 0.$$

By substituting $y = 0$ in (1.2), we have

$$\begin{aligned} 2f(x) &= g(x), \\ f(-x) &= -f(x) \end{aligned} \tag{2.1}$$

for all $x \in \mathcal{B}$. So

$$g(x + y) = 2f(x + y) = g(x) + g(y)$$

for all $x, y \in \mathcal{B}$. Hence, the mapping $g : \mathcal{B} \to \mathcal{B}$ is additive, and thus by (2.1), the mapping $f : \mathcal{B} \to \mathcal{B}$ is additive. □

By using the fixed-point technique, we prove the Hyers-Ulam stability of the system of additive functional equations (1.2) in complex Banach algebras.

Theorem 2.2. *Suppose that $\Delta : \mathcal{B}^2 \to [0, \infty)$ is a function such that there exists an $L < 1$ with*

$$\Delta\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{2}\Delta(x, y) \tag{2.2}$$

for all $x, y \in \mathcal{B}$. Let $f, g : \mathcal{B} \to \mathcal{B}$ be mappings satisfying

$$\begin{cases} \|2f(x + y) - g(x) - g(y)\| \le \Delta(x, y), \\ \|g(x + y) - 2f(y - x) - 4f(x)\| \le \Delta(x, y) \end{cases} \tag{2.3}$$

for all $x, y \in \mathcal{B}$. Then there exist unique additive mappings $F, G : \mathcal{B} \to \mathcal{B}$ such that

$$\|F(x) - f(x)\| \le \frac{L^2 + L}{4(1 - L)}\Delta(x, x), \tag{2.4}$$

$$\|G(x) - g(x)\| \le \frac{L^2 + L}{2(1 - L)}\Delta(x, x) \tag{2.5}$$

for all $x \in \mathcal{B}$.

Proof. By substituting $x = y = 0$ in (2.3), we obtain

$$\begin{cases} \|2f(0) - 2g(0)\| \le \Delta(0, 0) = 0, \\ \|g(0) - 6f(0)\| \le \Delta(0, 0) = 0 \end{cases}$$

and so $f(0) = g(0) = 0$.

By substituting $y = x$ in (2.3), we obtain

$$\begin{cases} \|2f(2x) - 2g(x)\| \le \Delta(x, x), \\ \|g(2x) - 4f(x)\| \le \Delta(x, x) \end{cases}$$

and so

$$\begin{cases} \left\| g(x) - 4g\left(\frac{x}{4}\right) \right\| \le 2\Delta\left(\frac{x}{4}, \frac{x}{4}\right) + \Delta\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{L^2 + L}{2}\Delta(x, x), \\ \left\| f(x) - 4f\left(\frac{x}{4}\right) \right\| \le \Delta\left(\frac{x}{4}, \frac{x}{4}\right) + \frac{1}{2}\Delta\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{L^2 + L}{4}\Delta(x, x) \end{cases} \tag{2.6}$$

for all $x \in \mathcal{B}$.

Let $\Gamma = \{\gamma : \mathcal{B} \to \mathcal{B} : \gamma(0) = 0\}$. We define a generalized metric on Γ as follows: $d : \Gamma \times \Gamma \longrightarrow [0, \infty]$ by

$$d(\delta, \gamma) = \inf\{\mu \in \mathbb{R}_+ : \|\delta(x) - \gamma(x)\| \le \mu\Delta(x, x), \quad \forall x \in \mathcal{B}\},$$

and we consider $\inf\varnothing = +\infty$. Then d is a complete generalized metric on Γ (see [37]).

Now, we define the mapping $\mathcal{J} : (\Gamma, d) \to (\Gamma, d)$ such that

$$\mathcal{J}\delta(x) := 4\delta\left(\frac{x}{4}\right)$$

for all $x \in \mathcal{B}$.

Actually, let $\delta, \gamma \in (\Gamma, d)$ be given such that $d(\delta, \gamma) = \mu$. Then

$$\|\delta(x) - \gamma(x)\| \le \mu\Delta(x, x)$$

for all $x \in \mathcal{B}$. Hence,

$$\|\mathcal{J}\delta(x) - \mathcal{J}\gamma(x)\| = \left\| 4\delta\left(\frac{x}{4}\right) - 4\gamma\left(\frac{x}{4}\right) \right\| \le 4\mu\Delta\left(\frac{x}{4}, \frac{x}{4}\right) \le L^2\mu\Delta(x, x)$$

for all $x \in \mathcal{B}$. It follows that $d(\mathcal{J}\delta(x), \mathcal{J}\gamma(x)) \le L^2\mu$. So

$$d(\mathcal{J}\delta(x), \mathcal{J}\gamma(x)) \le L^2 d(\delta, \gamma)$$

for all $x \in \mathcal{B}$ and all $\delta, \gamma \in \Gamma$.

It follows from (2.6) that $d(f, \mathcal{J}f) \le \frac{L^2+L}{4}$ and $d(g, \mathcal{J}g) \le \frac{L^2+L}{2}$.

By using the fixed-point alternative we deduce the existence of a unique fixed point of $\mathcal{J}$ and a unique fixed point of $\mathcal{J}$, that is, the existence of mappings $F, G : \mathcal{B} \to \mathcal{B}$, respectively, such that

$$F(x) = 4F\left(\frac{x}{4}\right), \quad G(x) = 4G\left(\frac{x}{4}\right)$$

with the following property: there exist $\mu, \eta \in (0, \infty)$ satisfying

$$\|f(x) - F(x)\| \le \mu\Delta(x, x), \quad \|g(x) - G(x)\| \le \eta\Delta(x, x)$$

for all $x \in \mathcal{B}$.

Since $\lim_{n\to\infty} d(\mathcal{J}^n f, F) = 0$ and $\lim_{n\to\infty} d(\mathcal{J}^n g, G) = 0$,

$$\lim_{n\to\infty} 4^n f\left(\frac{x}{4^n}\right) = F(x), \quad \lim_{n\to\infty} 4^n g\left(\frac{x}{4^n}\right) = G(x)$$

for all $x \in \mathcal{B}$.

Next, $d(f, F) \le \frac{1}{1-L} d(f, \mathcal{J}f)$ and $d(g, G) \le \frac{1}{1-L} d(g, \mathcal{J}g)$, which imply

$$\|f(x) - F(x)\| \le \frac{L^2 + L}{4(1 - L)}\Delta(x, x), \quad \|g(x) - G(x)\| \le \frac{L^2 + L}{2(1 - L)}\Delta(x, x)$$

for all $x \in \mathcal{B}$.

By using (2.2) and (2.3), we conclude that

$$\begin{aligned}\|2F(x + y) - G(x) - G(y)\| &= \lim_{n\to\infty} 4^n \left\| 2f\left(\frac{x + y}{4^n}\right) + g\left(\frac{x}{4^n}\right) - g\left(\frac{y}{4^n}\right) \right\| \le \lim_{n\to\infty} 4^n\Delta\left(\frac{x}{4^n}, \frac{y}{4^n}\right) \\ &\le \lim_{n\to\infty} L^{2n}\Delta(x, y) = 0\end{aligned}$$

and

$$\begin{aligned}\|G(x + y) - 2F(y - x) - 4F(x)\| &= \lim_{n\to\infty} 4^n \left\| g\left(\frac{x + y}{4^n}\right) - 2f\left(\frac{y - x}{4^n}\right) - 4f\left(\frac{x}{4^n}\right) \right\| \\ &\le \lim_{n\to\infty} 4^n\Delta\left(\frac{x}{4^n}, \frac{y}{4^n}\right) \le \lim_{n\to\infty} L^{2n}\Delta(x, y) = 0\end{aligned}$$

for all $x, y \in \mathcal{B}$, since $L < 1$. Hence,

$$\begin{cases} 2F(x+y) - G(x) = G(y), \\ G(x+y) - 2F(y-x) = 4F(x) \end{cases}$$

for all $x, y \in \mathcal{B}$. Therefore, by Lemma 2.1, the mappings $F, G : \mathcal{B} \to \mathcal{B}$ are additive. □

Corollary 2.3. *Let $f, g : \mathcal{B} \to \mathcal{B}$ be mappings satisfying*

$$\begin{cases} \|2f(x+y) - g(x) - g(y)\| \le \|xy\|, \\ \|g(x+y) - 2f(y-x) - 4f(x)\| \le \|xy\| \end{cases}$$

for all $x, y \in \mathcal{B}$. Then there exist unique additive mappings $F, G : \mathcal{B} \to \mathcal{B}$ such that

$$\begin{cases} \|F(x) - f(x)\| \le \frac{3}{8}\|x\|^2, \\ \|G(x) - g(x)\| \le \frac{3}{4}\|x\|^2 \end{cases}$$

for all $x \in \mathcal{B}$.

Proof. The proof follows from Theorem 2.2 by taking $L = \frac{1}{2}$ and $\Delta(x, y) = \|xy\|$ for all $x, y \in \mathcal{B}$. □

Corollary 2.4. *Let p and θ be nonnegative real numbers with $p > 1$ and $f, g : \mathcal{B} \to \mathcal{B}$ be mappings satisfying*

$$\begin{cases} \|2f(x+y) - g(x) - g(y)\| \le \theta(\|x\|^p + \|y\|^p), \\ \|g(x+y) - 2f(y-x) - 4f(x)\| \le \theta(\|x\|^p + \|y\|^p) \end{cases}$$

for all $x, y \in \mathcal{B}$. Then there exist unique additive mappings $F, G : \mathcal{B} \to \mathcal{B}$ such that

$$\begin{cases} \|F(x) - f(x)\| \le \frac{2^p + 2}{2^p(2^p - 2)}\theta\|x\|^p, \\ \|G(x) - g(x)\| \le \frac{2(2^p + 2)}{2^p(2^p - 2)}\theta\|x\|^p \end{cases}$$

for all $x \in \mathcal{B}$.

Proof. The proof follows from Theorem 2.2 by taking $L = \frac{2}{2^p}$ and $\Delta(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in \mathcal{B}$. □

3 Stability of F-hom-ders in Banach algebras

In this section, by using the fixed-point technique, we prove the Hyers-Ulam stability of F-hom-ders in complex Banach algebras.

Lemma 3.1. [38] *Let $\mathcal{B}$ be a complex Banach algebra and $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ be an additive mapping such that $\mathcal{F}(\alpha x) = \alpha\mathcal{F}(x)$ for all $\alpha \in \mathbb{T}^1 := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ and all $x \in \mathcal{B}$. Then $\mathcal{F}$ is $\mathbb{C}$-linear.*

Lemma 3.2. *Let $f, g : \mathcal{B} \to \mathcal{B}$ be mappings satisfying*

$$\begin{cases} 2f(\lambda(x+y)) - g(\lambda x) = \lambda g(y), \\ g(\lambda(x+y)) - 2f(\lambda(y-x)) = 4\lambda f(x) \end{cases} \tag{3.1}$$

for all $x, y \in \mathcal{B}$ and all $\lambda \in \mathbb{T}^1$. Then the mappings $f, g : \mathcal{B} \to \mathcal{B}$ are $\mathbb{C}$-linear.

Proof. If we substitute $\lambda = 1$ in (3.1), then f and g are additive by Lemma 2.1.

By substituting $y = 0$ in (3.1), we have

$$\begin{cases} 2f(\lambda x) = \lambda g(x), \\ g(\lambda x) = 2\lambda f(x) \end{cases} \tag{3.2}$$

for all $x \in \mathcal{B}$ and all $\lambda \in \mathbb{T}^1$, since the mappings f and g are additive. By substituting $\lambda = 1$ in (3.2), we obtain

$$\begin{cases} 2f(x) = g(x), \\ g(x) = 2f(x), \end{cases}$$

and so

$$\begin{cases} 2f(\lambda x) = 2\lambda f(x), \\ g(\lambda x) = \lambda g(x) \end{cases}$$

for all $x \in \mathcal{B}$ and all $\lambda \in \mathbb{T}^1$. So by Lemma 3.2, the mappings f and g are $\mathbb{C}$-linear. □

Theorem 3.3. *Suppose that $\Delta : \mathcal{B}^2 \to [0, \infty)$ is a function such that there exists an $L < 1$ with*

$$\Delta(x, y) \le \frac{L}{4}\Delta(2x, 2y) \le \frac{L}{2}\Delta(2x, 2y) \tag{3.3}$$

for all $x, y \in \mathcal{B}$. Let $f, g : \mathcal{B} \to \mathcal{B}$ be mappings satisfying

$$\begin{cases} \|2f(\lambda(x + y)) - g(\lambda x) - \lambda g(y)\| \le \Delta(x, y), \\ \|g(\lambda(x + y)) - 2f(\lambda(y - x)) - 4\lambda f(x)\| \le \Delta(x, y), \end{cases} \tag{3.4}$$

$$\|f(xy) - f(x)f(y)\| \le \Delta(x, y), \tag{3.5}$$

$$\|g(x)g(y) - f(x)g(y) - g(x)f(y)\| \le \Delta(x, y) \tag{3.6}$$

for all $x, y \in \mathcal{B}$, and all $\lambda \in \mathbb{T}^1$. Then there exist unique mappings $F, G : \mathcal{B} \to \mathcal{B}$ satisfying (2.4) and (2.5) and F is a homomorphism and G is an F-hom-der.

Proof. Let $\lambda = 1$ in (3.4). Since

$$\Delta(x, y) \le \frac{L}{4}\Delta(2x, 2y) \le \frac{L}{2}\Delta(2x, 2y)$$

for all $x, y \in \mathcal{B}$, by Theorem 2.2, there exist unique mappings $F, G : \mathcal{B} \to \mathcal{B}$ satisfying (2.4) and (2.5), which are given by

$$\lim_{n\to\infty} 4^n f\left(\frac{x}{4^n}\right) = F(x), \quad \lim_{n\to\infty} 4^n g\left(\frac{x}{4^n}\right) = G(x)$$

for all $x \in \mathcal{B}$.

By using (3.3), we conclude that

$$\begin{aligned} \|2F(\lambda(x + y)) - G(\lambda x) - \lambda G(y)\| &= \lim_{n\to\infty} 4^n \left\| 2f\left(\frac{\lambda(x + y)}{4^n}\right) - g\left(\frac{\lambda x}{4^n}\right) - \lambda g\left(\frac{y}{4^n}\right) \right\| \\ &\le \lim_{n\to\infty} 4^n \Delta\left(\frac{x}{4^n}, \frac{y}{4^n}\right) \le \lim_{n\to\infty} L^{2n}\Delta(x, y) = 0 \end{aligned}$$

and

$$\begin{aligned} \|G(\lambda(x + y)) - 2F(\lambda(y - x)) - 4\lambda F(x)\| &= \lim_{n\to\infty} 4^n \left\| g\left(\frac{\lambda(x + y)}{4^n}\right) - 2f\left(\frac{\lambda(y - x)}{4^n}\right) - 4\lambda f\left(\frac{x}{4^n}\right) \right\| \\ &\le \lim_{n\to\infty} 4^n \Delta\left(\frac{x}{4^n}, \frac{y}{4^n}\right) \le \lim_{n\to\infty} L^{2n}\Delta(x, y) = 0 \end{aligned}$$

for all $x, y \in \mathcal{B}$, since $L < 1$. Hence,

$$\begin{cases} 2F(\lambda(x+y)) - G(\lambda x) = \lambda G(y), \\ G(\lambda(x+y)) - 2F(\lambda(y-x)) = 4\lambda F(x) \end{cases}$$

for all $x, y \in \mathcal{B}$ and all $\lambda \in \mathbb{T}^1$. Therefore, by Lemma 3.2, the mappings $F, G : \mathcal{B} \to \mathcal{B}$ are $\mathbb{C}$-linear.

It follows from (3.5) that

$$\begin{aligned} \|F(xy) - F(x)F(y)\| &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{4^n}\right) - f\left(\frac{x}{2^n}\right) f\left(\frac{y}{2^n}\right) \right\| \\ &\le \lim_{n\to\infty} 4^n \Delta\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \le \lim_{n\to\infty} L^n \Delta(x,y) = 0 \end{aligned}$$

for all $x, y \in \mathcal{B}$. So

$$F(xy) = F(x)F(y)$$

for all $x, y \in \mathcal{B}$. Thus, F is a homomorphism.

It follows from (3.6) that

$$\begin{aligned} \|G(x)G(y) - F(x)G(y) - G(x)F(y)\| &= \lim_{n\to\infty} 4^n \left\| g\left(\frac{x}{2^n}\right) g\left(\frac{y}{2^n}\right) - f\left(\frac{x}{2^n}\right) g\left(\frac{y}{2^n}\right) - g\left(\frac{x}{2^n}\right) f\left(\frac{y}{2^n}\right) \right\| \\ &\le \lim_{n\to\infty} 4^n \Delta\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \le \lim_{n\to\infty} L^n \Delta(x,y) = 0 \end{aligned}$$

for all $x, y \in \mathcal{B}$. So

$$G(x)G(y) = F(x)G(y) + G(x)F(y)$$

for all $x, y \in \mathcal{B}$. Thus, the $\mathbb{C}$-linear mapping G is an F-hom-der. □

Corollary 3.4. *Let p and q be nonnegative real numbers with $p + q > 2$ and $f, g : \mathcal{B} \to \mathcal{B}$ be mappings satisfying*

$$\begin{cases} \|2f(\lambda(x+y)) - g(\lambda x) - \lambda g(y)\| \le \|x\|^p \|y\|^q, \\ \|g(\lambda(x+y)) - 2f(\lambda(y-x)) - 4\lambda f(x)\| \le \|x\|^p \|y\|^q \end{cases}$$

and

$$\|f(xy) - f(x)f(y)\| \le \|x\|^p \|y\|^q,$$

$$\|g(xy) - f(x)g(y) - g(x)f(y)\| \le \|x\|^p \|y\|^q$$

for all $x, y \in \mathcal{B}$ and all $\lambda \in \mathbb{T}^1$. Then there exist unique mappings $F, G : \mathcal{B} \to \mathcal{B}$ such that F is a homomorphism and G is an F-hom-der and

$$\begin{cases} \|F(x) - f(x)\| \le \dfrac{2^{p+1} + 4}{2^{p+q}(2^{p+q} - 4)} \|x\|^{p+q}, \\ \|G(x) - g(x)\| \le \dfrac{2(2^{p+1} + 4)}{2^{p+1}(2^{p+q} - 4)} \|x\|^{p+q} \end{cases}$$

for all $x \in \mathcal{B}$.

Proof. The proof follows from Theorem 3.3 by taking $\Delta(x, y) = \|x\|^p \|y\|^q$ for all $x, y \in \mathcal{B}$. By choosing $L = 2^{2-p-q}$, we obtain the desired result. □

Corollary 3.5. *Let p and θ be nonnegative real numbers with $p > 2$ and $f, g : \mathcal{B} \to \mathcal{B}$ be mappings satisfying*

$$\begin{cases} \|2f(\lambda(x+y)) - g(\lambda x) - \lambda g(y)\| \le \theta(\|x\|^p + \|y\|^p), \\ \|g(\lambda(x+y)) - 2f(\lambda(y-x)) - 4\lambda f(x)\| \le \theta(\|x\|^p + \|y\|^p) \end{cases}$$

and

$$\|f(xy) - f(x)f(y)\| \le \theta(\|x\|^p + \|y\|^p),$$

$$\|g(xy) - f(x)g(y) - g(x)f(y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in \mathcal{B}$ *and all* $\lambda \in \mathbb{T}^1$. *Then there exist unique mappings* $F, G : \mathcal{B} \to \mathcal{B}$ *such that F is a homomorphism and G is an F-hom-der and*

$$\begin{cases} \|F(x) - f(x)\| \le \dfrac{2(2^p + 4)}{2^p(2^p - 4)}\theta\|x\|^p, \\ \|G(x) - g(x)\| \le \dfrac{4(2^p + 4)}{2^p(2^p - 4)}\theta\|x\|^p \end{cases}$$

for all $x \in \mathcal{B}$.

Proof. The proof follows from Theorem 3.3 by taking $\Delta(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in \mathcal{B}$. By choosing $L = 2^{2-p}$, we obtain the desired result. □

4 Conclusion

In this article, we introduced a new system of additive functional equations in complex Banach spaces and the concept of f-hom-der in complex Banach algebras and proved the Hyers-Ulam stability of the new system of additive functional equations in complex Banach spaces and the Hyers-Ulam stability of f-hom-ders in complex Banach algebras.

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References

[1] M. Mirzavaziri and M. S. Moslehian, *Automatic continuity of σ-derivations on C^*-algebras*, Proc. Am. Math. Soc. **134** (2006), no. 11, 3319–3327.

[2] C. Park, J. Lee, and X. Zhang, *Additive s-functional inequality and hom-derivations in Banach algebras*, J. Fixed Point Theory Appl. **21** (2019), Paper No. 18, DOI: https://doi.org/10.1007/s11784-018-0652-0.
[3] M. Dehghanian, C. Park, and Y. Sayyari, *On the stability of hom-der on Banach algebras* (preprint).
[4] A. Kheawborisuk, S. Paokanta, and J. Senasukh, *Ulam stability of hom-ders in fuzzy Banach algebars, C. Park*, AIMS Math. **7** (2022), no. 9, 16556–16568, DOI: https://doi.org/10.3934/math.2022907.
[5] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publications, New York, 1960.
[6] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
[7] M. Dehghanian and S. M. S. Modarres, *Ternary γ-homomorphisms and ternary γ-derivations on ternary semigroups*, J. Inequal. Appl. **2012** (2012), Paper No. 34, DOI: https://doi.org/10.1186/1029-242X-2012-34.
[8] M. Dehghanian, S. M. S. Modarres, C. Park, and D. Shin, *C^*-Ternary 3-derivations on C^*-ternary algebras*, J. Inequal. Appl. **2013** (2013), Paper No. 124, DOI: https://doi.org/10.1186/1029-242X-2013-124.
[9] M. Dehghanian and C. Park, *C^*-Ternary 3-homomorphisms on C^*-ternary algebras*, Results Math. **66** (2014), 385–404, DOI: https://doi.org/10.1007/s00025-014-0383-5.
[10] M. Dehghanian, Y. Sayyari, and C. Park, *Hadamard homomorphisms and Hadamard derivations on Banach algebras*, Miskolc Math. Notes (in press).
[11] C. Park, J. M. Rassias, A. Bodaghi, and S. Kim, *Approximate homomorphisms from ternary semigroups to modular spaces*, Rev. R. Acad. Cienc. Exactas Fiiis. Nat. Ser. A Mat. RACSAM **113** (2019), no. 3, 2175–2188, DOI: https://doi.org/10.1007/s13398-018-0608-7.
[12] C. Park, K. Tamilvanan, G. Balasubramanian, B. Noori, and A. Najati, *On a functional equation that has the quadratic-multiplicative property*, Open Math. **18** (2020), 837–845, DOI: https://doi.org/10.1515/math-2020-0032.
[13] A. Thanyacharoen and W. Sintunavarat, *The new investigation of the stability of mixed type additive-quartic functional equations in non-Archimdean spaces*, Demonstr. Math. **53** (2020), 174–192, DOI: https://doi.org/10.1515/dema-2020-0009.
[14] A. Thanyacharoen and W. Sintunavarat, *On new stability results for composite functional equations in quasi-β-normed spaces*, Demonstr. Math. **54** (2021), 68–84, DOI: https://doi.org/10.1515/dema-2021-0002.
[15] Y. Sayyari, M. Dehghanian, C. Park, and J. Lee, *Stability of hyper homomorphisms and hyper derivations in complex Banach algebras*, AIMS Math. **7** (2022), no. 6, 10700–10710, DOI: https://doi.org/10.3934/math.2022597.
[16] I. Hwang and C. Park, *Bihom derivations in Banach algebras*, J. Fixed Point Theory Appl. **21** (2019), no. 3, Paper No. 81, DOI: https://doi.org/10.1007/s11784-019-0722-x.
[17] G. Lu and C. Park, *Hyers-Ulam stability of general Jensen-type mappings in Banach algebras*, Results Math. **66** (2014), 87–98, DOI: https://doi.org/10.1007/s00025-014-0365-7.
[18] C. Park, *An additive (α, β)-functional equation and linear mappings in Banach spaces*, J. Fixed Point Theory Appl. **18** (2016), 495–504, DOI: https://doi.org/10.1007/s11784-016-0283-2.
[19] C. Park, *The stability of an additive (ρ_1, ρ_2)-functional inequality in Banach spaces*, J. Math. Inequal. **13** (2019), 95–104, DOI: https://doi.org/10.7153/jmi-2019-13-07.
[20] J. Brzdęk, L. Cădariu, and K. Ciepliński, *Fixed point theory and the Ulam stability*, J. Funct. Spaces **2014** (2014), Article ID 829419, DOI: https://doi.org/10.1155/2014/829419.
[21] J. Brzdęk, E. Karapinar, and A. Petruşel, *A fixed point theorem and the Ulam stability in generalized d_q-metric spaces*, J. Math. Anal. Appl. **467** (2018), no. 1, 501–520, DOI: https://doi.org/10.1016/j.jmaa.2018.07.022.
[22] D. Popa, G. Pugna, and I. Rasa, *On Ulam stability of the second order linear differential equation*, Adv. Theory Nonlinear Anal. Appl. **2** (2018), no. 2, 106–112.
[23] A. Salim, M. Benchohra, E. Karapinar, and J. E. Lazreg, *Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations*, Adv. Difference Equations **2020** (2020), Paper No. 601, DOI: https://doi.org/10.1186/s13662-020-03063-4.
[24] A. Lachouri and A. Ardjouni, *The existence and Ulam-Hyers stability results for generalized Hilfer fractional integro-differential equations with nonlocal integral boundary conditions*, Adv. Theory Nonlinear Anal. Appl. **6** (2022), no. 1, 101–117.
[25] H. Mohamed, *Sequential fractional pantograph differential equations with nonlocal boundary conditions: Uniqueness and Ulam-Hyers-Rassias stability*, Results Nonlinear Anal. **5** (2022), no. 1, 29–41, DOI: https://doi.org/10.53006/rna.928654.
[26] R. Atmania, *Existence and stability results for a nonlinear implicit fractional differential equation with a discrete delay*, Adv. Theory Nonlinear Anal. Appl. **6** (2022), no. 2, 246–257.
[27] E. Karapinar, H. D. Binh, N. H. Luc, and N. H. Can, *On continuity of the fractional derivative of the time-fractional semilinear pseudo-parabolic systems*, Adv. Difference Equations **2021** (2021), Paper No. 70, DOI: https://doi.org/10.1186/s13662-021-03232-z.
[28] J. E. Lazreg, S. Abbas, M. Benchohra, and E. Karapinar, *Impulsive Caputo-Fabrizio fractional differential equations in b-metric spaces*, Open Math. **19** (2021), 363–372, DOI: https://doi.org/10.1515/math-2021-0040.
[29] R. S. Adigüzel, U. Aksoy, E. Karapinar, and I. M. Erhan, *On the solution of a boundary value problem associated with a fractional differential equation*, Math. Methods Appl. Sci. (in press), DOI: https://doi.org/10.1002/mma.6652.

[30] R. S. Adigüzel, U. Aksoy, E. Karapinar, and I. M. Erhan, *Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115** (2021), Paper No. 155, DOI: https://doi.org/10.1007/s13398-021-01095-3.

[31] R. S. Adigüzel, U. Aksoy, E. Karapinar, and I. M. Erhan, *On the solutions of fractional differential equations via Geraghty type hybrid contractions*, Appl. Comput. Math. **20** (2021), 313–333.

[32] H. Afshari and E. Karapinar, *A solution of the fractional differential equations in the setting of b-metric space*, Carpathian Math. Publ. **13** (2021), 764–774, DOI: https://doi.org/10.15330/cmp.13.3.764-774.

[33] H. Afshari, H. Shojaat, and M. S. Moradi, *Existence of the positive solutions for a tripled system of fractional differential equations via integral boundary conditions*, Results Nonlinear Anal. **4** (2021), Article ID 186193, DOI: https://doi.org/10.53006/rna.938851.

[34] M. Asadi, B. Moeini, A. Mukheimer, and H. Aydi, *Complex valued M-metric spaces and related fixed point results via complex C-class functions*, J. Inequal. Spec. Funct. **10** (2019), no. 1, 101–110.

[35] B. Moeini, M. Asadi, H. Aydi, and M. S. Noorani, *C^*-Algebra-valued M-metric spaces and some related fixed point results*, Ital. J. Pure Appl. Math. **41** (2019), 708–723.

[36] J. B. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Am. Math. Soc. **74** (1968), 305–309.

[37] D. Mihet and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), no. 1, 567–572, DOI: https://doi.org/10.1016/j.jmaa.2008.01.100.

[38] C. Park, *Homomorphisms between Poisson JC^*-algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97, DOI: https://doi.org/10.1007/s00574-005-0029-z.