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Numerical ranges and complex symmetric operators in semi-inner-product spaces

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Abstract

We introduce the numerical range of a bounded linear operator on a semi-inner-product space. We compute the numerical ranges of some operators on $\ell_2^p(\mathbb{C})$ ($1 \le p < \infty$) and show that the numerical range of the backward shift on an infinite-dimensional space ℓ^p is the open unit disc. We define a conjugation and a complex symmetric operator on a semi-inner-product space and discuss complex symmetry in the dual space. We prove some properties of a generalized adjoint of a complex symmetric operator. We also show that the numerical range of the complex conjugation on ℓ_n^p ($n \ge 2$) is the closed unit disc. Finally, we discuss the sequentially essential numerical ranges of operators on a semi-inner-product space.

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1 Introduction

For the study of operator theory in Banach spaces, Lumer [12] introduced a semi-inner-product, which is different from an inner product in that it is in general not conjugate symmetric. Thus a semi-inner-product is generally nonlinear with respect to its second variable. Giles [7] showed that in a fairly large class of Banach spaces, it is possible to construct a semi-inner-product with some desirable properties of the inner product. He proved that every normed space is a semi-inner-product space on which the semi-inner-product satisfies an extra homogeneity condition and gave fundamental properties extending Hilbert space type arguments to Banach spaces. Recently, semi-inner-products have been used as a useful tool in establishing the concept of reproducing kernel Banach spaces for machine learning [14].

On a separable complex Hilbert space $\mathcal{H}$, a conjugation is an isometric antilinear involution C from $\mathcal{H}$ to $\mathcal{H}$. A simple example of a conjugation on a Hilbert space is the pointwise complex conjugation on $L^2(\Omega,\mu)$, where (Ω,μ) is a measure space with a positive measure μ. Garcia et al. [6, Lemma 2.11] proved that there exists an orthonormal basis $\{e_n\}_{n\ge 1}$ in $\mathcal{H}$ such that $Ce_n = e_n$ for any positive integer n, which asserts that every conjugation is unitarily equivalent to the canonical conjugation on an ℓ^2-space of the appropriate dimension. Takagi [13] studied the antilinear eigenvalue problem $Tx = \lambda\overline{x}$ where T is an $n \times n$ symmetric complex matrix and $\overline{x}$ denotes the complex conjugation of the vector x in $\mathbb{C}^n$. The result of Godič and Lucenko [8] states that any unitary operator U on $\mathcal{H}$ can be factored as $U = CJ$ and is both C-symmetric and J-symmetric, where C and J are conjugations on $\mathcal{H}$. This generalizes the well-known fact that any planar rotation can be factored as the product of two reflections. Chō and Tanahashi [2] defined a conjugation on a complex Banach space and studied some spectral properties of complex symmetric operators.

The numerical range of $T \in \mathcal{L}(\mathcal{H})$ is the collection of complex numbers of the form $\langle T\xi, \xi\rangle$ with ξ ranging through the unit vectors in $\mathcal{H}$. The numerical range is very useful in studying operators and has many applications (see [9] for details), for example, numerical ranges are regarded as a rough estimate of eigenvalues, and generalizations of the numerical range are used to study quantum computing [3]. Recently, Hur and Lee [10] also studied the numerical ranges of conjugations and antilinear operators acting on a Hilbert space.

We now give a brief outline of the paper. In Sect. 2, we study the numerical range of a bounded linear operator on a semi-inner-product space. Using the standard semi-inner-product on $\ell_n^p(\mathbb{C})$, we compute numerical ranges of several operators, where $\ell_n^p(\mathbb{C})$ is the complex n-dimensional space with the ℓ^p-norm ($1 \leq p < \infty$). Particularly, we compute numerical ranges of some operators acting on $\ell_2^p(\mathbb{C})$ and show that the numerical range of the backward shift on $\ell^p(\mathbb{C})$ is the open unit disc, where $\ell^p(\mathbb{C})$ is an infinite-dimensional space. In Sect. 3, we introduce a conjugation and a complex symmetric operator on a semi-inner-product space and investigate their basic properties. We prove some properties of a generalized adjoint of a complex symmetric operator on a semi-inner-product space. Moreover, we show that the numerical range of the complex conjugation on $\ell_n^p(\mathbb{C})$ ($n \geq 2$) is the closed unit disc. Finally, we discuss the sequentially essential numerical ranges of operators multiplied by a conjugation in a semi-inner-product space.

2 Numerical ranges of semi-inner-product space operators

After introducing a semi-inner product space by Lumer [12], semi-inner-products have widely been applied to study bounded linear operators on Banach spaces [4]. Many properties of semi-inner-products were discovered by many authors, in particular, Giles [7]. We first recall the definitions of the semi-inner-product and the numerical range of a bounded operator on a semi-inner-product space and point out elementary properties of the numerical range.

Definition 2.1 Let $\mathcal{X}$ be a complex vector space. A semi-inner-product on $\mathcal{X}$ is a function $[\cdot,\cdot] : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ satisfying the following properties: for any $x, y, z \in \mathcal{X}$,

(1) $[x + y, z] = [x, z] + [y, z]$,
(2) $[\lambda x, y] = \lambda[x, y]$ for all $\lambda \in \mathbb{C}$,
(3) $[x, x] > 0$ for $x \neq 0$,
(4) $|[x, y]|^2 \leq [x, x][y, y]$.

We say that $\mathcal{X}$ equipped with a semi-inner-product is a semi-inner-product space.

Lumer [12] proved that a semi-inner-product space is a normed vector space with norm $\|x\| = [x, x]^{1/2}$ and every normed linear space can be made into a semi-inner-product space. We see that a semi-inner-product is an inner product if and only if the induced norm satisfies the parallelogram law. Giles [7] showed that every normed vector space can be represented as a semi-inner-product space with the homogeneity property, that is,

$$[x, \lambda y] = \overline{\lambda}[x, y] \quad \text{for all } x, y \in \mathcal{X} \text{ and } \lambda \in \mathbb{C}. \tag{1}$$

He proved the Riesz representation theorem for Hilbert space in the context of uniform semi-inner-product spaces, which says that if f is a continuous linear functional on $\mathcal{X}$, then there is a unique vector y in $\mathcal{X}$ such that $f(x) = [x, y]$ for all x in $\mathcal{X}$.

It is well known that for $1 < p < \infty$, the space $\ell_n^p(\mathbb{C})$ has the semi-inner-product defined by

$$[x, y]_p = \frac{1}{\|y\|_p^{p-2}} \sum_{j=1}^{n} x_j \overline{y_j} |y_j|^{p-2} \quad \text{for } x, y \neq 0,$$

which is consistent with the ℓ^p-norm $\|\cdot\|_p$. For $p = 1$, the semi-inner-product is given by

$$[x, y]_1 = \|y\|_1 \sum_{j=1}^{n} x_j \operatorname{sgn}(\overline{y_j}),$$

where $\operatorname{sgn}(z)$ is $z/|z|$ if $z \in \mathbb{C} \setminus \{0\}$, and 0 if $z = 0$.

In a semi-inner-product space $\mathcal{X}$, the *numerical range* $W(T)$ of $T \in \mathcal{L}(\mathcal{X})$ was defined in [12] as the set of numbers

$$W(T) := \{[Tx, x] : [x, x] = 1, x \in \mathcal{X}\}.$$

This definition extends the classical one in a Hilbert space. It is well known that the numerical range of an operator in a Hilbert space is always convex; the proof can be done by reducing the problem to considering the numerical range of 2×2 matrices. However, the numerical range in a semi-inner-product space is not convex in general [12, Theorem 15].

Throughout this paper, $\mathcal{X}$ and $\mathcal{L}(\mathcal{X})$ denote a semi-inner-product space with semi-inner-product $[\cdot, \cdot]$ and the set of bounded linear operators on $\mathcal{X}$, respectively, unless specified otherwise. We always assume that every semi-inner-product space has this homogeneity property.

The following elementary properties were observed by Lumer [12]. Let $T, S \in \mathcal{L}(\mathcal{X})$ and $\alpha, \beta \in \mathbb{C}$. We denote by $\sigma_a(T)$ the approximate point spectrum and by $\partial\sigma(T)$ the boundary of the spectrum.

(i) $\frac{1}{4}\|T\| \leq w(T) \leq \|T\|$ for *the numerical radius* $w(T) = \sup\{|[Tx, x]| : x \in \mathcal{X}\}$,

(ii) $W(T) = \{\lambda\}$ if and only if $T = \lambda I$,

(iii) $W(T)$ contains all of the eigenvalues of T,

(iv) $W(\alpha T + \beta I) = \alpha W(T) + \beta$,

(v) $W(T + S) \subseteq W(T) + W(S)$,

(vi) $\sigma_a(T) \subseteq \mathrm{cl}[W(T)]$, i.e., $\partial\sigma(T) \subseteq \mathrm{cl}[W(T)]$ where $\mathrm{cl}[W(T)]$ denotes the closure of $W(T)$.

In this section, we explicitly compute the numerical range in a finite-dimensional semi-inner-product space. We denote by $\ell_n^p(\mathbb{C})$ $(p \geq 1)$ the complex n-dimensional space $\mathbb{C}^n$ equipped with the ℓ^p-norm. We first consider the numerical ranges of 2×2 matrices acting on the space $\ell_2^p(\mathbb{C})$.

Example 2.2

(1) If $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ acts on $\ell_2^p(\mathbb{C})$ $(1 \le p < \infty)$, then the numerical range $W(T)$ is the closed interval $[0,1]$ since $[Tu,u]_p = |x|^p$ for any unit vector $u = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$.

(2) For $T = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ acting on $\ell_2^1(\mathbb{C})$, the numerical range $W(T)$ is the open unit disc. Indeed, let $u = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ in $\ell_2^1(\mathbb{C})$ be any unit vector, so $|x| + |y| = 1$. Then we have that

$$[Tu,u]_1 = y\,\mathrm{sgn}(\overline{x}) = \begin{cases} 0 & \text{if } x = 0, \\ ye^{i\theta} & \text{if } x \neq 0, \end{cases}$$

where θ is a real number such that $e^{i\theta} = \frac{\overline{x}}{|x|}$. If $|[Tu,u]_1| = 1$ for nonzero x, then we have $1 = |[Tu,u]_1| = |y||\,\mathrm{sgn}(\overline{x})| = |y|$. However, we should have $|x| = 0$ because of $|x| + |y| = 1$, which contradicts to assumption. Thus the numerical range $W(T)$ is the open unit disc.

When $1 < p < \infty$, we now investigate the numerical range of $T = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ acting on $\ell_2^p(\mathbb{C})$.

Lemma 2.3 *For* $1 < p < \infty$, *the range of the function* $f(t) := (\tan t)^{\frac{2}{p}}(\cos t)^2$ *defined on* $[0,\pi/2)$ *is the interval* $[0, \frac{1}{p}(p-1)^{\frac{p-1}{p}}]$.

Proof We see that $f(0) = 0$ and $\lim_{t\to\frac{\pi}{2}^-} f(t) = 0$. Since

$$f'(t) = 2(\tan t)^{\frac{2}{p}} \cos t\left(\frac{1}{p\sin t} - \sin t\right),$$

f has the absolute maximum when $\sin t = 1/\sqrt{p}$. In this case, we have

$$\tan t = \frac{1}{\sqrt{p-1}} \quad \text{and} \quad \cos t = \sqrt{\frac{p-1}{p}},$$

so that the maximum value of f is equal to $\frac{1}{p}(p-1)^{\frac{p-1}{p}}$. By the intermediate value theorem the range of f is $[0, \frac{1}{p}(p-1)^{\frac{p-1}{p}}]$. □

Example 2.4 For $1 < p < \infty$, let D_p be the closed disc of radius $\frac{1}{p}(p-1)^{\frac{p-1}{p}}$ centered at the origin. For $T = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ acting on $\ell_2^p(\mathbb{C})$, we see that $W(T)$ is the closed disc D_p.

To show that $W(T) = D_p$, we take any unit vector $u = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ in $\ell_2^p(\mathbb{C})$ with $|x|^p + |y|^p = 1$. Since $[Tu,u]_p = y\overline{x}|x|^{p-2}$ and $|y|^p = 1 - |x|^p$, we have

$$\big|[Tu,u]_p\big|^p = \big(1 - |x|^p\big)|x|^{p(p-1)}.$$

Since the function $g(t) = (1-t^p)t^{p(p-1)}$ has the maximum value $\frac{1}{p^p}(p-1)^{p-1}$ at $t = (\frac{p-1}{p})^{1/p}$, $|[Tu,u]_p|$ has the maximum value $\frac{1}{p}(p-1)^{\frac{p-1}{p}}$ when $|x|^p = \frac{p-1}{p}$. Thus we see that $W(T)$ is contained in the closed disc D_p.

To show the reverse inclusion, let λ be any complex number in D_p. We can write $\lambda = re^{i\theta}$ for some $0 \le r \le \frac{1}{p}(p-1)^{\frac{p-1}{p}}$ and $\theta \in \mathbb{R}$. We take a unit vector

$$u = \big(|\cos\alpha|^{\frac{2}{p}}, e^{i\theta}|\sin\alpha|^{\frac{2}{p}}\big)^t \in \ell_2^p(\mathbb{C})$$

for any real number α. We note that the arctangent function $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is 1–1 and onto. By Lemma 2.3 there exists a unique real number $\alpha \in [0, \arctan((p-1)^{\frac{p-1}{2}} p^{-\frac{p}{2}})]$ such that $r = |\tan\alpha|^{\frac{2}{p}} |\cos\alpha|^2$. Consequently, for such a unit vector u, we have

$$[Tu, u]_p = e^{i\theta} |\sin\alpha|^{\frac{2}{p}} |\cos\alpha|^{\frac{2p-2}{p}} = e^{i\theta} |\tan\alpha|^{\frac{2}{p}} |\cos\alpha|^2 = re^{i\theta} = \lambda,$$

which completes the proof.

Proposition 2.5 *Let $T = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$ act on $\ell_2^1(\mathbb{C})$ where $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{C}$. Then*

$$W(T) = \begin{cases} \{x + iy \in \mathbb{C} : (x+a)^2 + y^2 \leq 4a^2\} & \textit{if } |b| = 2|a|, \\ \{x + iy \in \mathbb{C} : y^2 \leq \frac{|b|^2}{|4a^2 - |b|^2|}(x-a)^2\} & \textit{if } |b| \neq 2|a|. \end{cases}$$

Proof For any unit vector $u = (e^{i\alpha} \cos^2\theta, e^{i\beta} \sin^2\theta)^t \in \ell_2^1(\mathbb{C})$ with $0 \leq \alpha, \beta < 2\pi$ and $0 \leq \theta < \pi$, we have

$$\begin{aligned} [Tu, u]_1 &= a\cos^2\theta + be^{i(\beta-\alpha)} \sin^2\theta - a\sin^2\theta \\ &= a\cos 2\theta + \frac{b}{2}(1 - \cos 2\theta)e^{i(\beta-\alpha)}. \end{aligned}$$

Letting $[Tu, u]_1 =: x + iy$ with $x, y \in \mathbb{R}$, we have the equation

$$(x - a\cos 2\theta)^2 + y^2 = \frac{|b|^2}{4}(1 - \cos 2\theta)^2,$$

so that the following quadratic equation for $\cos 2\theta$ holds;

$$\left(4a^2 - |b|^2\right)\cos^2 2\theta - 2\left(4ax - |b|^2\right)\cos 2\theta + 4x^2 + 4y^2 - |b|^2 = 0. \tag{2}$$

In the case of $|b| = 2|a|$, we have the equation $2a(x-a)\cos 2\theta = x^2 + y^2 - a^2$. If $x = a$, then we have $y = 0$. Assumet that $x \neq a$. Then it follows that

$$|\cos 2\theta| = \left| \frac{x^2 + y^2 - a^2}{2a(x-a)} \right| \leq 1.$$

In the case of $a(x-a) > 0$, we have $W(T) = \emptyset$. On the other hand, in the case of $a(x-a) < 0$, we get the inequality $(x+a)^2 + y^2 \leq 4a^2$, so that

$$W(T) = \left\{x + iy \in \mathbb{C} : (x+a)^2 + y^2 \leq 4a^2\right\} \setminus \left\{(a, 0)\right\}.$$

By combining these cases we have $W(T) = \{x + iy \in \mathbb{C} : (x+a)^2 + y^2 \leq 4a^2\}$.

In the case of $|b| < 2|a|$, since $\cos 2\theta$ must be real, the discriminant of equation (2) gives the inequality

$$\left(4ax - |b|^2\right)^2 - \left(4a^2 - |b|^2\right)\left(4x^2 + 4y^2 - |b|^2\right) \geq 0,$$

so that

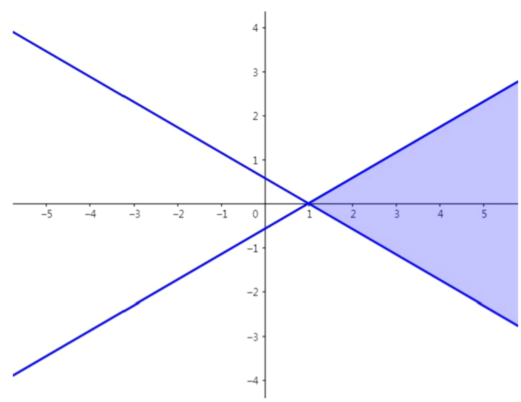
$$y^2 \le \frac{|b|^2}{4a^2 - |b|^2}(x-a)^2.$$

In the case of $|b| > 2|a|$, we similarly get the inequality

$$y^2 \le \frac{|b|^2}{|b|^2 - 4a^2}(x-a)^2.$$

These complete the proof. □

Example 2.6 If $T = \left(\begin{smallmatrix} 1 & i \\ 0 & -1 \end{smallmatrix}\right)$ acts on $\ell_2^1(\mathbb{C})$, then it follows form Proposition 2.5 that the numerical range $W(T)$ is the region satisfying $-\frac{1}{\sqrt{3}}(x-1) \le y \le \frac{1}{\sqrt{3}}(x-1)$ as follows.



Proposition 2.7 *Let $T = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ act on ℓ_2^1, where $a, c \in \mathbb{R} \setminus \{0\}$, $a + c \ne 0$, and $b \in \mathbb{C}$. For any unit vector $u = (x, y)^t \in \ell_2^1$, we have*

$$[Tu, u]_1 = \begin{cases} a & \text{if } |x| = 1 \text{ and } |y| = 0, \\ c & \text{if } |x| = 0 \text{ and } |y| = 1, \\ a|x| + c(1-|x|) + \frac{b\bar{x}y}{|x|} & \text{if } |x| \ne 0 \text{ and } |y| = 1 - |x|. \end{cases}$$

Proof The proof can be obtained by simple computations, so we omit it. □

Remark 2.8 In Proposition 2.7, we suppose that b, x, y are pure imaginary numbers. Let $b = \alpha i$ for a nonzero real number α. If $|x| = r$ for $0 < r < 1$, then

$$[Tu, u]_1 = \{ra + (1-r)c\} + (1-r)\alpha i.$$

So, $[Tu, u]_1$ converges to a as $r \to 1$. On the other hand, $[Tu, u]_1$ goes to $c \pm \alpha i$ as $r \to 0$.

Corollary 2.9 *Let T be as in Proposition 2.7, and let $u = (x, y)^t \in \ell_2^1$ be a unit vector. If, in addition, b is a nonzero real number, then for $x \ne 0$ and y with $|y| = 1 - |x|$,*

$$W(T) \subset \left\{\lambda \in \mathbb{C} : \left|\mathrm{Re}(\lambda)\right| \le \frac{|a+b+c|}{2} \text{ and } \left|\mathrm{Im}(\lambda)\right| \le \frac{|b|}{2}\right\}.$$

Proof If $x = x_1 + ix_2$ and $y = y_1 + iy_2$, where x_j and y_j are real numbers ($j = 1, 2$), then we obtain from Proposition 2.7 that

$$[Tu, u]_1 = \left\{(a-c)|x| + c + \frac{b}{|x|}(x_1y_1 + x_2y_2)\right\} + i\frac{b}{|x|}(x_1y_2 - x_2y_1).$$

By the Cauchy–Schwarz inequality, we get that for $t := \sqrt{x_1^2 + x_2^2}$,

$$\begin{aligned}(x_1y_1 + x_2y_2)^2 &\le \left(x_1^2 + x_2^2\right)\left(y_1^2 + y_2^2\right) \\ &= \left(x_1^2 + x_2^2\right)\left(1 - \sqrt{x_1^2 + x_2^2}\right)^2 \\ &= t^2(1-t)^2 =: f(t).\end{aligned}$$

On the interval $(0, 1)$, f has the maximum $\frac{1}{16}$ at $t = \frac{1}{2}$, so that we get $|x_1y_1 + x_2y_2| \le \frac{1}{4}$, and this gives the inequality

$$\left|\mathrm{Re}\left([Tu, u]_1\right)\right| \le \frac{|a+b+c|}{2}.$$

In this case, we also get the inequality $|\mathrm{Im}([Tu, u]_1)| = |2b(x_1y_2 - x_2y_1)| \le \frac{|b|}{2}$ by a similar method. This completes the proof. □

Theorem 2.10 *Let T be the backward shift on an infinite dimensional Banach space $\ell^p(\mathbb{C})$ for $1 \le p < \infty$. Then the numerical range $W(T)$ is the open unit disc.*

Proof Let $x = (x_1, x_2, x_3, \dots)$ be any unit vector in $\ell^p(\mathbb{C})$, and let $k = \min\{i \ge 1 : x_i \ne 0\}$. Then we have

$$\begin{aligned}\left|[Tx, x]_p\right| &\le \sum_{j=k}^{\infty} |x_{j+1}||x_j|^{p-1} \\ &\le \frac{1}{p}\sum_{j=k}^{\infty}\left\{|x_{j+1}|^p + (p-1)|x_j|^p\right\} \\ &= \frac{1}{p}\left\{(p-1)|x_k|^p + p\sum_{j=k+1}^{\infty}|x_j|^p\right\} \\ &= \frac{1}{p}\left\{p\sum_{j=k}^{\infty}|x_j|^p - |x_k|^p\right\} = 1 - \frac{|x_k|^p}{p} < 1,\end{aligned}$$

where the second inequality follows from the inequality of arithmetic and geometric means. Hence we obtain that $|[Tx, x]_p| < 1$ for any unit vector $x \in \ell^p(\mathbb{C})$, which implies that $W(T)$ is contained in the open unit disc.

To show the reverse inclusion, let $\lambda = re^{i\theta}$ be any vector in the open unit disc with $0 \le r < 1$. We take the vector $x \in \ell^p(\mathbb{C})$ given by

$$x = \left(\left(1-r^p\right)^{\frac{1}{p}}, r\left(1-r^p\right)^{\frac{1}{p}}e^{i\theta}, r^2\left(1-r^p\right)^{\frac{1}{p}}e^{2i\theta}, r^3\left(1-r^p\right)^{\frac{1}{p}}e^{3i\theta}, \dots\right).$$

Then we see that $\|x\|_p = 1$, so that $[x,x]_p = \|x\|_p^2 = 1$. Moreover, we get that

$$[Tx,x]_p = re^{i\theta}\left(1 - r^p\right)\sum_{k=1}^{\infty} r^{(k-1)p} = re^{i\theta} = \lambda,$$

which implies that $W(T)$ contains the open unit disc. This completes the proof. □

3 Conjugations and complex symmetric operators

A conjugation C defined on a complex Hilbert space $\mathcal{H}$ is an antilinear operator that is involutive ($C^2 = I_{\mathcal{H}}$) and isometric, meaning that the following equality holds;

$$\langle C\xi, C\eta\rangle = \langle \eta, \xi\rangle \quad \text{for all } \xi, \eta \in \mathcal{H}. \tag{3}$$

Thus it follows from (3) that $\langle C\xi, C\eta\rangle = \overline{\langle \xi, \eta\rangle}$. Chō and Tanahashi [2] introduced a conjugation C on a complex Banach space $\mathcal{B}$ as the operator satisfying the following relations;

$$C^2 = I_{\mathcal{B}}, \qquad \|C\| \le 1, \qquad C(x+y) = Cx + Cy \quad \text{and} \quad C(\lambda x) = \overline{\lambda} Cx \tag{4}$$

for all $x, y \in \mathcal{B}$ and $\lambda \in \mathbb{C}$.

Like in a Hilbert space, we will define a conjugation on a semi-inner-product space using a semi-inner-product. Throughout this section, $\mathcal{X}$ denotes a semi-inner-product space with a semi-inner-product $[\cdot,\cdot]$, unless specified otherwise.

Definition 3.1 An operator $C : \mathcal{X} \to \mathcal{X}$ is a *conjugation* if it is involutive ($C^2 = I_{\mathcal{X}}$) and

$$[Cx, Cy] = \overline{[x,y]} \quad \text{for all } x, y \in \mathcal{X}. \tag{5}$$

Proposition 3.2 *If C is a conjugation on $\mathcal{X}$, then relation* (4) *holds for all $x, y \in \mathcal{X}$ and $\lambda \in \mathbb{C}$.*

Proof By the Cauchy–Schwarz inequality for a semi-inner-product, we have that $\|Cx\|^2 = [Cx, Cx] = \overline{[x,x]} \le \|x\|^2$ for every $x \in \mathcal{X}$, which implies that $\|C\| \le 1$. Since a semi-inner-product is linear in the first variable, we have

$$\begin{aligned}\left[C(x+y), Cz\right] &= \overline{[x+y,z]} = \overline{[x,z]} + \overline{[y,z]} \\ &= [Cx, Cz] + [Cy, Cz] = [Cx + Cy, Cz]\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Since the operator C is surjective, we can take $z \in \mathcal{X}$ such that

$$Cz := C(x+y) - Cx - Cy.$$

Then we get that $0 = [C(x+y) - Cx - Cy, Cz] = [Cz, Cz]$, so that $Cz = 0$, that is, $C(x+y) = Cx + Cy$. To show that $C(\lambda x) = \overline{\lambda} Cx$ for any $x \in \mathcal{X}$ and $\lambda \in \mathbb{C}$, take any element $y \in \mathcal{X}$. Then we have

$$\left[C(\lambda x), y\right] = \overline{[\lambda x, Cy]} = \overline{\lambda}\overline{[x, Cy]}$$

$$= \overline{\lambda}[Cx, y] = [\overline{\lambda}Cx, y],$$

which means that $C(\lambda x) = \overline{\lambda}Cx$. Therefore C satisfies relation (4). □

Let C be a conjugation on a complex Hilbert space $\mathcal{H}$. A bounded linear operator T on $\mathcal{H}$ is C-symmetric if $T = CT^*C$, where T^* is a Hilbert space adjoint of T, which is equivalent to $\langle Tx, y\rangle = \langle x, CTCy\rangle$ for all $x, y \in \mathcal{H}$. Chō et al. [1] have extend the notion of C-symmetric operators to Banach space operators via linear functionals in its dual space. However, we would like to extend the notion of the complex symmetry to semi-inner-product space operators without using linear functionals. Even though semi-inner-products in general are not additive in the second variables, we will use a semi-inner-product to define the C-symmetric operator on a semi-inner-product space.

Definition 3.3 Let C be a conjugation on $\mathcal{X}$. We say that $T \in \mathcal{L}(\mathcal{X})$ is *C-symmetric* if

$$[Tx, y] = [x, CTCy] \quad \text{for all } x, y \in \mathcal{X}. \tag{6}$$

Remark 3.4 In Definition 3.3, equation (6) is equivalent to

$$[x, Ty] = [CTCx, y] \quad \text{for all } x, y \in \mathcal{X}. \tag{7}$$

Indeed, by putting Cx, Cy into (6) instead of x, y we obtain that $[TCx, Cy] = [Cx, CTy]$. It follows from the definition of a conjugation C that $[CTCx, y] = \overline{[Cx, CTy]} = [x, Ty]$.

Proposition 3.5 *Let C be a conjugation on $\mathcal{X}$, and let $T \in \mathcal{L}(\mathcal{X})$ be a C-symmetric operator.*

(i) *λT is C-symmetric for any complex number λ.*
(ii) *If T is invertible, then T^{-1} is also C-symmetric.*
(iii) *If $S \in \mathcal{L}(\mathcal{X})$ is C-symmetric and commutes with T, then so is TS.*

Proof (i) For any complex number λ, we have

$$\begin{aligned}\bigl[(\lambda T)x, y\bigr] &= \lambda[Tx, y] = \lambda[x, CTCy] \\ &= [x, \overline{\lambda}CTCy] = \bigl[x, C(\lambda T)Cy\bigr],\end{aligned}$$

so that λT is C-symmetric.

(ii) For any $y \in \mathcal{X}$, there exists $z \in \mathcal{X}$ such that $y = CTCz$. Indeed, since T is invertible and C is a conjugation, $CT^{-1}C$ is also invertible. Putting $z := CT^{-1}Cy$, we get $y = CTCz$. For any $x, y \in \mathcal{X}$, we have

$$\bigl[T^{-1}x, y\bigr] = \bigl[T^{-1}x, CTCz\bigr] = \bigl[TT^{-1}x, z\bigr] = [x, z] = \bigl[x, CT^{-1}Cy\bigr],$$

where the second equality follows from the C-symmetry of T. Thus T^{-1} is C-symmetric, which completes the proof.

(iii) If $S \in \mathcal{L}(\mathcal{X})$ commutes with T and is C-symmetric, then it follows that

$$\big[(TS)x, y\big] = [Sx, CTCy] = \big[x, CSC(CTCy)\big] = \big[x, C(ST)Cy\big] = \big[x, C(TS)Cy\big].$$

Hence TS is C-symmetric. □

Let $T \in \mathcal{L}(\mathcal{X})$ and $y \in \mathcal{X}$. By the Riesz representation theorem in a semi-inner-product space [7], there is a unique vector $T^\dagger y$ such that $[Tx, y] = [x, T^\dagger y]$ for all $x \in \mathcal{X}$, where $T^\dagger$ is a generalized adjoint, which is not usually linear [11]. On the other hand, if C is a conjugation on $\mathcal{X}$ and if $T \in \mathcal{L}(\mathcal{X})$ is C-symmetric, then we obtain that

$$\big[x, T^\dagger y\big] = [Tx, y] = [x, CTCy] \quad \text{for all } x, y \in \mathcal{X},$$

so that $T^\dagger = CTC$. Since CTC is linear, $T^\dagger$ becomes a linear operator on $\mathcal{X}$. It follows from (7) that $[x, Ty] = [CTCx, y] = [T^\dagger x, y]$ for all $x, y \in \mathcal{X}$. Furthermore, $T^\dagger$ is also C-symmetric. Indeed, for all $x, y \in \mathcal{X}$,

$$\big[T^\dagger x, y\big] = [CTCx, y] = \overline{[TCx, Cy]} = \overline{\big[Cx, T^\dagger Cy\big]} = \big[x, CT^\dagger Cy\big].$$

A *uniform* semi-inner-product space means a uniformly continuous semi-inner-product space where the induced normed vector space is complete and uniformly convex. Here the (uniform) continuity implies that

$$\operatorname{Re}\big\{[y, x + ty]\big\} \to \operatorname{Re}\big\{[y, x]\big\} \quad \text{(uniformly) as } t \in \mathbb{R} \to 0.$$

Giles [7, Theorem 7] proved that for a uniform semi-inner-product space $\mathcal{X}$, the dual space $\mathcal{X}^\star$ is also a uniform complex semi-inner-product space with respect to the semi-inner-product defined by $[x^\star, y^\star]_\star = [y, x]$. Moreover, he proved that for every continuous linear functional $x^\star$ in a dual space $\mathcal{X}^\star$, there exists a unique vector $x \in \mathcal{X}$ such that

$$x^\star(z) = [z, x] \quad \text{for all } z \in \mathcal{X},$$

so that the map $x \mapsto x^\star = [\cdot, x]$ is a one-to-one mapping from $\mathcal{X}$ onto $\mathcal{X}^\star$. For any $T \in \mathcal{L}(\mathcal{X})$, the dual operator $T^\star \in \mathcal{L}(\mathcal{X}^\star)$ is given by $T^\star y^\star(z) = y^\star(Tz)$ for all $y^\star \in \mathcal{X}^\star$ and $z \in \mathcal{X}$.

If C is a conjugation on a uniform semi-inner-product space $\mathcal{X}$, then we define the dual operator $C^\star : \mathcal{X}^\star \to \mathcal{X}^\star$ by

$$\big(C^\star\big(x^\star\big)\big)(z) := \overline{x^\star(Cz)} \quad \text{for all } z \in \mathcal{X}. \tag{8}$$

We have that $(C^\star(x^\star))(z) = \overline{x^\star(Cz)} = \overline{[Cz, x]} = [z, Cx] = (Cx)^\star(z)$, so that $C^\star(x^\star) = (Cx)^\star$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\;C\;} & \mathcal{X} \\ \big\downarrow & & \big\downarrow \\ \mathcal{X}^\star & \xrightarrow[C^\star]{} & \mathcal{X}^\star \end{array}$$

Moreover, the dual operator $C^\star$ is a conjugation on $\mathcal{X}^\star$. Indeed, for any $x^\star, y^\star \in \mathcal{X}^\star$, there exist unique vectors x and y in $\mathcal{X}$ such that

$$[C^\star x^\star, C^\star y^\star]_\star = [(Cx)^\star, (Cy)^\star]_\star = [Cy, Cx] = \overline{[y,x]} = \overline{[x^\star, y^\star]_\star}.$$

Since $(Cx)^\star = C^\star(x^\star)$ for all $x \in \mathcal{X}$, we observe that relation (4) implies equation (5).

Proposition 3.6 *Let C be a conjugation on a uniform semi-inner-product space $\mathcal{X}$.*

(i) *If $T \in \mathcal{L}(\mathcal{X})$ is C-symmetric, then $T^\star \in \mathcal{L}(\mathcal{X}^\star)$ is also $C^\star$-symmetric.*

(ii) *If $T \in \mathcal{L}(\mathcal{X})$ is C-symmetric, then $(T^\dagger)^\star = (T^\star)^\dagger$.*

(iii) *If $\{T_n\}$ is a sequence of C-symmetric operators such that $T_n \to S$ in the strong topology, then S is C-symmetric.*

Proof (i) Suppose that T is a C-symmetric operator on $\mathcal{X}$. Let f and g be arbitrary elements in the dual space $\mathcal{X}^\star$. Since $\mathcal{X}$ is a uniform semi-inner-product space, there exist unique vectors x and y in $\mathcal{X}$ such that $x^\star = f$ and $y^\star = g$. First, we observe that $T^\star x^\star = (T^\dagger x)^\star$. Indeed, for any $z \in \mathcal{X}$, we have

$$(T^\star x^\star)(z) = x^\star(Tz) = [Tz, x] = [z, T^\dagger x] = (T^\dagger x)^\star(z).$$

Moreover, for any $z \in \mathcal{X}$ and $y^\star \in \mathcal{X}^\star$, we see that

$$(C^\star T^\star C^\star)y^\star(z) = y^\star(CTCz) = (CTC)^\star y^\star(z),$$

so $C^\star T^\star C^\star = (CTC)^\star$. Thus we have

$$\begin{aligned}[T^\star x^\star, y^\star]_\star &= [y, T^\dagger x] = [CT^\dagger Cy, x] \\ &= [x^\star, (CT^\dagger Cy)^\star]_\star = [x^\star, (CTC)^\star y^\star]_\star \\ &= [x^\star, C^\star T^\star C^\star y^\star]_\star,\end{aligned}$$

which means that $T^\star$ is $C^\star$-symmetric.

(ii) For any $z \in \mathcal{X}$ and $y^\star \in \mathcal{X}^\star$, we obtain that

$$(T^\dagger)^\star y^\star(z) = (CTC)^\star y^\star(z) = (C^\star T^\star C^\star)y^\star(z).$$

On the other hand, it follows from (i) that $T^\star$ is $C^\star$-symmetric. Hence, for all $x^\star \in \mathcal{X}^\star$,

$$[x^\star, (T^\star)^\dagger y^\star]_\star = [T^\star x^\star, y^\star]_\star = [x^\star, (C^\star T^\star C^\star)y^\star]_\star.$$

This means that $(T^\dagger)^\star = (T^\star)^\dagger$.

(iii) Since $\|(S - T_n)x\| \to 0$ for all $x \in \mathcal{X}$, for all $x, y \in \mathcal{X}$, we have

$$[CSCx, y] = \lim_{n\to\infty}[CT_nCx, y] = \lim_{n\to\infty}[x, T_n y] = [x, Sy],$$

where the third equality follows from uniform continuity. Thus S is a C-symmetric operator. □

Now we compute the numerical range of a conjugation on $\ell_n^p(\mathbb{C})$.

Example 3.7 Let C be a complex conjugation on $\ell_n^p(\mathbb{C})$ $(1 \le p < \infty)$ given by $Cx = \overline{x} = (\overline{x}_1, \dots, \overline{x}_n)$ for $x \in \ell_n^p(\mathbb{C})$. Then we have:

(1) $W(C) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ for $n = 1$,

(2) $W(C) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$ for $n \ge 2$.

It is easy to prove (1). Indeed, for any $x \in \ell_1^p(\mathbb{C})$ with $|x| = 1$, we write $x = e^{i\theta}$ for some real number θ. Obviously, we have $[Cx, x]_p = [\overline{x}, x]_p = e^{-2i\theta}$, and so $W(C) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

To show the second statement, let $x \in \ell_n^p(\mathbb{C})$ be any unit vector, i.e., $\|x\|_p^2 = [x, x]_p = 1$. By the Cauchy–Schwarz inequality we have

$$\big|[Cx, x]_p\big|^2 \le [Cx, Cx]_p[x, x]_p = \overline{[x, x]_p}[x, x]_p = 1,$$

which implies that $W(C) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$.

For the reverse inclusion, let λ be any complex number with $|\lambda| \le 1$. We write a polar form $\lambda = |\lambda|e^{i\theta}$ for some real number θ. Now we take a unit vector $x \in \ell_n^p(\mathbb{C})$ given by

$$x = \left(\left(\frac{1 + |\lambda|}{2}\right)^{\frac{1}{p}} e^{-\frac{i\theta}{2}}, \left(\frac{1 - |\lambda|}{2}\right)^{\frac{1}{p}} ie^{-\frac{i\theta}{2}}, 0, \dots, 0\right).$$

Then we have

$$[Cx, x]_p = [\overline{x}, x]_p = \left(\frac{1 + |\lambda|}{2} - \frac{1 - |\lambda|}{2}\right)e^{i\theta} = |\lambda|e^{i\theta} = \lambda,$$

which implies that $W(C)$ contains the closed unit disc. Therefore the numerical range $W(C)$ is the closed unit disc.

Let C be the usual complex conjugation given in Example 3.7. Then we see that

$$w(C) = \sup\left\{\big|[Cx, x]_p\big| : [x, x] = 1, x \in \ell_n^p\right\} = 1 \quad \text{for all } n \ge 1,$$

where $w(C)$ is the numerical radius of C. Moreover, we can find infinitely many unit vectors x that attain the numerical radius of the complex conjugation C on $\ell_n^1(\mathbb{C})$ $(n \ge 1)$, that is, vectors x with $|[Cx, x]_1| = 1$. We explicitly construct vectors attaining the numerical radius $w(C)$ in the following example.

Example 3.8 Let $n \ge 2$. For any $\lambda \in \mathbb{C}$ with $0 < |\lambda| \le \frac{1}{n-1}$, we take the vector $x = (x_1, x_2, \dots, x_n)^t \in \ell_n^1(\mathbb{C})$ given by

$$x_j = \begin{cases} \overline{\lambda} & \text{if } 1 \le j \le n-1, \\ (\frac{1}{|\lambda|} - n + 1)\overline{\lambda} & \text{if } j = n. \end{cases} \tag{9}$$

Then we have that

$$[x, x]_1 = 1 \quad \text{and} \quad \big|[Cx, x]_1\big| = \left|\left(\frac{\lambda}{|\lambda|}\right)\right|^2 = 1.$$

For any complex number λ with $|\lambda| \geq n-1$, we put $x = (x_1, x_2, \dots, x_n) \in \ell_n^1(\mathbb{C})$, where

$$x_j = \begin{cases} \overline{\lambda}^{-1} & \text{if } 1 \leq j \leq n-1, \\ \overline{\lambda}^{-1}(|\lambda| - n + 1) & \text{if } j = n. \end{cases} \tag{10}$$

Then it follows that

$$[x,x]_1 = 1 \quad \text{and} \quad \big|[Cx,x]_1\big| = \left|\left(\frac{|\lambda|}{\lambda}\right)\right|^2 = 1.$$

Similarly, we also have infinitely many numerical radius attaining vectors in the infinite-dimensional space $\ell^1(\mathbb{N})$ in the same way as (9) and (10) except for jth terms with 0 ($j > n$).

The essential numerical range for a bounded linear operator on a Hilbert space is defined as the closure of the numerical range of the image in the Calkin algebra, and many equivalent conditions are known [5]. We now introduce the *sequentially essential numerical range* of T on a semi-inner-product space $\mathcal{X}$ by

$$W_e(T) = \Big\{ z \in \mathbb{C} : \lim_n [Tx_n, x_n] = z \text{ for some } \{x_n\} \subset \mathcal{X} \text{ with } [x_n, x_n] = 1, x_n \xrightarrow{w} 0 \Big\}.$$

Theorem 3.9 *Let $T \in \mathcal{L}(\mathcal{X})$, and let C be a conjugation on $\mathcal{X}$. Then we have*

$$W(CTC) = \overline{W(T)} \quad \textit{and} \quad W_e(T) = \overline{W_e(CTC)},$$

where $\overline{S}$ denotes the complex conjugation of S.

Proof If $z \in W(CTC)$, then there exists a vector $x \in \mathcal{X}$ with $[x,x] = 1$ such that

$$z = [CTCx, x] = \overline{[TCx, Cx]} \in \overline{W(T)}.$$

This means that $W(CTC) \subset \overline{W(T)}$. Since $W(T) = W(C^2TC^2) \subset \overline{W(CTC)}$, we get the reverse inclusion. Therefore we have $W(CTC) = \overline{W(T)}$.

If $z \in W_e(CTC)$, then there exists a sequence $\{x_n\} \subset \mathcal{X}$ with $[x_n, x_n] = 1$ and $x_n \xrightarrow{w} 0$. Since $\lim_n x_n = 0$ in the weak sense, we obtain that $\lim_n f(x_n) = 0$ for all $f \in \mathcal{X}^\star$. Since $C^\star f \in \mathcal{X}^\star$ for all $f \in \mathcal{X}^\star$, we have $\lim_n f(Cx_n) = \lim_n C^\star f(x_n) = 0$, which implies that $Cx_n \xrightarrow{w} 0$. Thus we have

$$z = \lim_n [CTCx_n, x_n] = \lim_n \overline{[TCx_n, Cx_n]} \in \overline{W_e(T)}.$$

This implies that $W_e(CTC) \subset \overline{W_e(T)}$. The reverse inclusion follows from

$$W_e(T) = W_e\big(C^2TC^2\big) \subset \overline{W_e(CTC)},$$

which completes the proof. □

Corollary 3.10 *Let C be a conjugation on $\mathcal{X}$, and let $T \in \mathcal{L}(\mathcal{X})$ be C-symmetric.*

(i) $W(T^\dagger) = \overline{W(T)}$ *and* $W_e(T^\dagger) = \overline{W_e(T)}$.
(ii) $W(T) = \{\overline{[x, Tx]} : [x,x] = 1, x \in \mathcal{X}\}$.
(iii) *If, in addition,* $\mathcal{X}$ *is a uniform semi-inner-product space, then*

$$W(C^\star T^\star C^\star) = \overline{W(T^\star)} = \{[x^\star, T^\star x^\star]_\star : [x^\star, x^\star]_\star = 1, x^\star \in \mathcal{X}^\star\}.$$

Proof It immediately follows from Proposition 3.6 and Theorem 3.9. □

We say that $T \in \mathcal{L}(\mathcal{X})$ is an *isometry* if $[Tx, Ty] = [x,y]$ for $x, y \in \mathcal{X}$, a *unitary* if it is isometric and surjective, and a *Hermitian operator* if $W(T) \subset \mathbb{R}$. For a conjugation C on $\mathcal{X}$, we have that T is an isometry (a unitary or a Hermitian operator, respectively) if and only if CTC is an isometry (a unitary or a Hermitian operator, respectively). Indeed, if T is an isometry, then for $x, y \in \mathcal{X}$,

$$[CTCx, CTCy] = \overline{[TCx, TCy]} = \overline{[Cx, Cy]} = [x, y],$$

which implies that CTC is an isometry. Conversely, if CTC is an isometry, then for $x, y \in \mathcal{X}$,

$$[Tx, Ty] = [TCz, TCw] = \overline{[CTCz, CTCw]} = \overline{[z, w]} = [Cz, Cw] = [x, y],$$

where $z = Cx$ and $w = Cy$. Similarly, we can see that T is a unitary if and only if CTC is a unitary. It follows from Theorem 3.9 that T is Hermitian if and only if CTC is also Hermitian.

In [8, Lemma 3.1] and [6, Theorem 3.1], it has been proved that any unitary operator on a Hilbert space can be constructed by gluing together two copies of essentially the same antilinear operator. The following proposition provides a perspective on the structure of unitary operators in a semi-inner-product space.

Proposition 3.11 *If* C *and* G *are conjugations on* $\mathcal{X}$*, then* $U = CG$ *is a unitary and is both* C*-symmetric and* G*-symmetric.*

Proof For any $x, y \in \mathcal{X}$, we have

$$[Ux, Uy] = [CGx, CGy] = \overline{[Gx, Gy]} = [x, y],$$

which means that U is isometric. Since C and G are conjugations on $\mathcal{X}$, it is obvious that U is surjective, so that it is a unitary. Moreover, we have

$$[CUCx, y] = [GCx, y] = \overline{[Cx, Gy]} = [x, CGy] = [x, Uy]$$

and

$$[GUGx, y] = [GCx, y] = \overline{[Cx, Gy]} = [x, CGy] = [x, Uy].$$

Thus U is both C-symmetric and G-symmetric. □

Remark 3.12 (i) In Proposition 3.11, if $U^\dagger$ is a generalized adjoint of a unitary $U = CG$, then we get from C-symmetry of U that $U^\dagger = CUC = GC$. Hence we have $UU^\dagger = U^\dagger U = I_{\mathcal{X}}$. This means that $U^\dagger = U^{-1}$.

(ii) Suppose that $\mathcal{X}$ in Proposition 3.11 is a uniform semi-inner-product space. Let $C^\star$ and $G^\star$ be conjugations on $\mathcal{X}^\star$ corresponding to C and G, which are given by (8). By Propositions 3.6 and 3.11, $U^\star = C^\star G^\star$ is a unitary on $\mathcal{X}^\star$ and is both $C^\star$-symmetric and $G^\star$-symmetric. It also follows from $C^\star$-symmetry of $U^\star$ that $(U^\star)^\dagger = G^\star C^\star$, so that $(U^\star)^\dagger = (U^\star)^{-1}$.

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