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Analysis of the fractional polio model with the Mittag-Leffler kernels

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Abstract This article investigates the transmission of polio-virus disease in the human population. The classical model is considered for studying fatal disease. First of all, the model is converted into the fractal fractional epidemic model. Then, the existence of the solution for the said model is ensured with the help of the fixed point theory. Points of equilibria for the model are worked out. The basic reproduction number is described and its role in the disease communication and stability of the model is examined by some standard results. Simulated graphs are also plotted to support the pre-results and claims. Lastly, the findings of the study are presented.

1. Introduction

Polio is the simple or shortened name of poliomyelitis[1], it is known as a highly infectious or contagious disease. People of different ages have been suffering from polio for a long time and still are suffering as well [2]. An incident in Egyptian society of a young man was recovered, who had his leg defected by polio around 1400 BC[3]. Polio started hitting the mass in the early 1900s and started multiplying in the different countries of

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Europe and Asia and found uncountable cases of polio [4]. The Jonas Salk invented the vaccine to control the disease.

It spreads from person to person especially through fecal-oral transmission. The main hub of the virus is the mouth. First it goes in the throat after entering from the mouth, then starts increasing in quantity and finally it enter in the intestine, enters in the blood and reaches in the spinal cord and brain. As a result the person gets paralysis and in some cases death is possible.

Children under 5 years are the main victim of this virus, they are highly targeted. It is harmful because in 72 percent of infected individuals no symptoms appear, and the remaining infected have the common symptoms such as fever, nausea, headache, sore throat pain in body and stiffness in the neck muscles. The main source of transmission of Polio virus is uncleanliness such as poor sanitary condition, unclean or unhygienic food and contaminated water [5,6].

There are different symptoms of the infected individual ranging from normal to severe, and these conditions have different names, one is, Non-paralytic polio; it can stay 2 to 5 days but disappear gradually. Paralytic polio, 1 out of 200 suffers from it and causes paralysis which results in death. The severe symptoms due to spinal cord and brain, called post-polio syndrome, are stated as Paresthesia; feeling of pins and needles in the legs or arms. Meningitis, Infection of the coverings around the brain and medulla spinalis. Paralysis is the condition the movement of the legs, arms, and or breathing muscles is reduced [7,8].

Immunization is the only source to control the Polio. The leading and advanced countries and health organizations such as World Health Assembly, WHO, Rotary International, UNICEF, Malinda Gates Foundation. Gavi started a campaign against Polio and passed a resolution for the eradication of polio from the world in 1988, and the cases decreased by 99 percent since 1988 and there is not a single case in the USA during the 20 years because children are vaccinated at the ages of 2 months, 4 months and then twice before joining elementary school [9]. Afghanistan and Pakistan are the countries in the world which have cases of Polio. Two polio cases have been reported on 2 February, 2021 and both countries are considered as the exporters of the wild polio virus.

In Pakistan, there are some major complications in fighting against polio virus since 1994 because, most of the cases were reported in Karachi, North Waziristan, and FATA(KPK). The major barrier to eradicate the polio from Pakistan are militant groups such as Tehreek-e-Taliban and these militants mostly prevailed in FATA(KPK), North Waziristan, and some areas of big cities like Karachi. The reason behind the spreading of polio in these areas is the ban imposed by militant groups on polio vaccination campaigns in regions and these areas are under their control and very severe threats are against the campaigners. The misconception has been spread out by these militants that it causes infertility, Bats, on the other hand some religious scholar declared that this vaccination is Halal according to Sharia Law[10–12].

The campaign started on 11th January 2021 and 40 million children have been not only vaccinated by polio vaccination but also, have been administered a Vitamin A drops to build general immunity of children from 6 to 59 months across Pak0istan. The vaccination Dr. Jonas Salk used an Inactivated Polio Vaccine (IPV), Oral Polio Vaccine (OPV) developed by Dr. Albert Sabin and first time used in 1961[13]. Two different types of populations involved in the model are the responsible for the communication of the disease i.e. susceptible, exposed and infected. The immune system of people plays an important role to control the disease, and through vaccination immune system can be made strong to fight against viruses especially for a particular disease. Different research works have been considered for polio virus by many researchers such as Garfinkel (2003) and Tebbens (2005) [14]. They developed a model for stimulating the spread of poliovirus infection [15]. Various authors presented several mathematical models with integer and non-integer derivatives in many applications like biology, chemistry, engineering etc. [26–43,33].

Most of the Poliovirus models are classical i.e., integer order derivations. But in this paper, fractional-order AB derivatives are considered for studying the transmission of the disease. Fixed point theory is applied to guarantee the solution.

The integer-order models involve the classical derivatives and these derivatives are local in nature. So, they can measure the change over an interval rather than a point.

Organization of the paper is as follows, a mathematical model for the Polio virus [39] is considered and converted into a fractional order model. Fixed point reduction is also presented in this Section. Section 3 is meant for the existence of the solution. Equilibrium points and stability analysis is worked out in Section 4. Analysis of the model is made with the help of Mittag- leffer kernel in Section 5. Numerical simulations are demonstrated in the Section number 6. Conclusion of the work is drawn in the last section.

2. Mathematical model/Fixed point reduction

For the fixed point reduction, following system of equations are considered.

$$\frac{dS}{dt} = A - \beta SI - r\beta SE - (\mu + v)S \tag{2.1}$$

$$\frac{dE}{dt} = \beta SI + r\beta SE - (b + \mu + v_1)E \tag{2.2}$$

$$\frac{dI}{dt} = (b + v_1)E - (\mu + \alpha)I \tag{2.3}$$

$$\frac{dV}{dt} = vS - \mu V \tag{2.4}$$

We use AB-fractional derivatives and obtain the following system of equations.

$${}^{AB}D_t^v S(t) = A - \beta SI - r\beta SE - (\mu + v)S \tag{2.5}$$

$${}^{AB}D_t^v E(t) = \beta SI + r\beta SE - (b + \mu + v_1)E \tag{2.6}$$

$${}^{AB}D_t^v I(t) = (b + v_1)E - (\mu + \alpha)I \tag{2.7}$$

$${}^{AB}D_t^v V(t) = vS - \mu V \tag{2.8}$$

Subject to the following initial conditions

$S = S_0 \geqslant 0, E = E_0 \geqslant 0, I = I_0 \geqslant = 0, V = V_0 \geqslant = 0.$ with $v \in (0, 1]$ The inclusion of fractional order parameters in the mathematics model increases the degree of freedom, which improves the fitting of the curve. This property of the fractional-order differential operator is helpful in decreasing the disease dynamics and predicting the future perspectives.

Our goal in this article is to find the solutions of the system (2.5)–(2.8). For simplicity, we define, the system as follows,

$$K_1(t,S,E,I,V) = A - \beta SI - r\beta SE - (\mu+v)S$$
$$K_2(t,S,E,I,V) = \beta SI + r\beta SE - (b+\mu+v_1)E$$
$$K_3(t,S,E,I,V) = (b+v_1)E - (\mu+\alpha)I$$
$$K_4(t,S,E,I,V) = vS - \mu V$$

Then, by applying the AB integral [17], we have the subsequent set of equations.

$$S(t)-S(0) = \frac{1-v}{\mathscr{AB}(v)}K_1(t,S,E,I,V) + \frac{v}{\mathscr{AB}(v)\Gamma(v)} \times \int_0^t (t-\zeta)^{v-1}K_1(\zeta,S,E,I,V)d\zeta \tag{2.9}$$

$$E(t)-E(0) = \frac{1-v}{\mathscr{AB}(v)}K_2(t,S,E,I,V) + \frac{v}{\mathscr{AB}(v)\Gamma(v)} \times \int_0^t (t-\zeta)^{v-1}K_2(\zeta,S,E,I,V)d\zeta \tag{2.10}$$

$$I(t)-I(0) = \frac{1-v}{\mathscr{AB}(v)}K_3(t,S,E,I,V) + \frac{v}{\mathscr{AB}(v)\Gamma(v)} \times \int_0^t (t-\zeta)^{v-1}K_3(\zeta,S,E,I,V)d\zeta \tag{2.11}$$

$$V(t)-V(0) = \frac{1-v}{\mathscr{AB}(v)}K_4(t,S,E,I,V) + \frac{v}{\mathscr{AB}(v)\Gamma(v)} \times \int_0^t (t-\zeta)^{v-1}K_4(\zeta,S,E,I,V)d\zeta \tag{2.12}$$

Next we construct the closed bolls for the above operators. Let C be the space of all continuous functions and we consider four closed ball with the radious r and center S_0, E_0, I_0, V_0 in the space of continuous functions

$$B_r(S_0) = [S, S \in C[0,\rho]; ||S - S_0|| \leqslant r]$$

where $||\cdot||$ represents the supremum norm defined on the domain.

$$\Rightarrow$$
$$||S|| \leqslant (r + S_0)$$
$$B_r(E_0) = [E, E \in C[0,\rho]; ||E - E_0|| \leqslant r]$$
$$\Rightarrow$$
$$|E|| \leqslant (r + E_0)$$
$$B_r(I_0) = [I, I \in C[0,\rho]; ||I - I_0|| \leqslant r]$$
$$\Rightarrow ||I|| \leqslant (r + I_0)$$
$$B_r(V_0) = [V, V \in C[0,\rho]; ||V - V_0|| \leqslant r]$$
$$\Rightarrow$$
$$||V|| \leqslant (r + V_0)$$

3. Existence of the Solution

In this section, we are to find the conditions for the existence of the solution for the above model fractional differential equations[20] (2.5) to (2.8) one by one by fixed point theory, using Banach fixed point theorem, Schauder's fixed point theorem [18]. We find Lipschitz condition, self-mapping and relative compactness.

3.1. Mapping properties of fixed-point operators:

Each operator maps the ball B into itself. For Eq. (2.9)

$$S(t) = S(0) + \frac{1-v}{\mathscr{AB}(v)}K_1(t,S,E,I,V) + \frac{v}{\mathscr{AB}(v)\Gamma(v)} \times \int_0^t (t-\zeta)^{v-1}K_1(\zeta,S,E,I,V)d\zeta$$

by putting

$$K_1(t,S,E,I,V) = A - \beta SI - r\beta SE - (\mu+v)S$$

$$S(t) = S(0) + \tfrac{1-v}{\mathscr{AB}(v)}[A - \beta SI - r\beta SE - (\mu+v)S] + \tfrac{v}{\mathscr{AB}(v)\Gamma(v)}\int_0^t (t-\zeta)^{v-1}[A - \beta SI - r\beta SE - (\mu+v)S]d\zeta$$

applying norm on both sides we have

$$||S(t)-S(0)|| \leqslant \left|\tfrac{1-v}{\mathscr{AB}(v)}\right|||A - \beta SI - r\beta SE - (\mu+v)S|| + \left|\tfrac{v}{\mathscr{AB}(v)\Gamma(v)}\right|\int_0^t |t-\zeta|^{v-1}||A - \beta SI - r\beta SE - (\mu+v)S||d\zeta$$

$$||S(t)-S(0)|| \leqslant \left|\tfrac{1-v}{\mathscr{AB}(v)}\right|[||A|| + |\beta|||S||||I|| + |r||\beta|||S||||E|| + |\mu+v|||S||] + \left|\tfrac{v}{\mathscr{AB}(v)\Gamma(v)}\right|\int_0^t |t-\tau|^{v-1}[||A|| + |\beta|||S||||I|| + |r||\beta|||S||||E|| + |\mu+v|||S||]d\zeta$$

Consider for $K_1(t,S,E,I,V) = A - \beta SI - r\beta SE - (\mu+v)S$we have

$$||S(t)-S(0)|| \leqslant \left|\frac{1-v}{\mathscr{AB}(v)}\right|U^*(r) + \left|\frac{v}{\mathscr{AB}(v)\Gamma(v)}\right|U^*(r)\int_0^t |t-\zeta|^{v-1}d\zeta$$

where $A + (|\beta|(r+C) + |r||\beta|(r+C) + |\mu+v|)(r+C) = U^*(r)$

$\because$In both the cases $t > \zeta$ and $\zeta > t$ the integral $\int_0^t |t-\zeta|^{v-1}d\zeta = \frac{\rho^v}{v}$. for self mapping we have

$$||S(t)-S(0)|| \leqslant r$$

$$\left|\frac{1-v}{\mathscr{AB}(v)}\right|U^*(r) + \left|\frac{v}{\mathscr{AB}(v)\Gamma(v)}\right|U^*(r)\frac{\rho^v}{v} \leqslant r$$

$$\left|\frac{1-v}{\mathscr{AB}(v)}\right| + \left|\frac{v}{\mathscr{AB}(v)\Gamma(v)}\right|\frac{\rho^v}{v} \leqslant \frac{r}{U^*(r)}$$

$$\rho \leqslant \left[|\mathscr{AB}(v)\Gamma(v)|\left(\frac{r}{U^*(r)} - \left|\frac{1-v}{\mathscr{AB}(v)}\right|\right)\right]^{\frac{1}{v}} \tag{3.1}$$

Which is the condition for self-mapping.

Clearly for ρ to be positive we must ensure that

$$r > \frac{(1-v)U^*(r)}{\mathscr{AB}(v)} \tag{3.2}$$

Similarly on the same line, for $E, I$and Vwe get the following apriori estimation.

$$\rho \leqslant \left[|\mathscr{AB}(v)\Gamma(v)|\left(\frac{r}{M^*(r)} - \left|\frac{1-v}{\mathscr{AB}(v)}\right|\right)\right]^{\frac{1}{v}} \tag{3.3}$$

$$r > \frac{(1-v)M^*(r)}{\mathscr{AB}(v)} \tag{3.4}$$

$$\rho \leqslant \left[|\mathscr{AB}(v)\Gamma(v)|\left(\frac{r}{N^*(r)} - \left|\frac{1-v}{\mathscr{AB}(v)}\right|\right)\right]^{\frac{1}{v}} \tag{3.5}$$

$$r > \frac{(1-v)N^*(r)}{\mathscr{A}\mathscr{B}(v)} \tag{3.6}$$

$$\rho \leqslant \left[|\mathscr{A}\mathscr{B}(v)\Gamma(v)|\left(\frac{r}{R^*(r)} - \left|\frac{1-v}{\mathscr{A}\mathscr{B}(v)}\right|\right)\right]^{\frac{1}{v}} \tag{3.7}$$

$$r > \frac{(1-v)R^*(r)}{AB(v)} \tag{3.8}$$

Theorem 3.1. If the inequalities (3.1, 3.3,3.5, 3.7) and (3.2, 3.4,3.6, 3.8) are satisfied then the operator vectors (S, E, I, V) maps the closed set $B \to B$.

3.2. Relative Compact Mapping:

We have

$$S(t) = S(0) + \frac{1-v}{\mathscr{A}\mathscr{B}(v)}K_1(t,S,E,I,V) + \frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)} \times \int_0^t (t-\zeta)^{v-1}K_1(\tau,S,E,I,V)d\zeta$$

We take the family of functions

$$S_i(t) = S(0) + \tfrac{1-v}{\mathscr{A}\mathscr{B}(v)}K_1(t,S_i,E,I,V) + \tfrac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\int_0^t (t-\zeta)^{v-1}K_1(\zeta,S_i,E,I,V)d\zeta$$

$$S_i(t^*) = S(0) + \tfrac{1-v}{\mathscr{A}\mathscr{B}(v)}K_1(t^*,S_i,E,I,V) + \tfrac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\int_0^{t^*} (t^*-\zeta)^{v-1}K_1(\zeta,S_i,E,I,V)d\zeta$$

subtracting the above equations, we get

$$S_i(t) - S_i(t^*) = \frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\left[\int_0^t (t-\zeta)^{v-1}d\zeta - \int_0^{t^*}(t^*-\zeta)^{v-1}d\zeta\right]K_1(\zeta,S_i,E,I,V)$$

$$S_i(t) - S_i(t^*) = \frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\left[\int_0^t (t-\zeta)^{v-1}d\zeta - \int_0^{t^*}(t^*-\zeta)^{v-1}d\zeta\right](A - \beta SI - r\beta SE - (\mu+v)S)$$

applying norm on both sides

$$||S_i(t) - S_i(t^*)|| = \left|\left|\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\left[\int_0^t (t-\zeta)^{v-1}d\zeta - \int_0^{t^*}(t^*-\zeta)^{v-1}d\zeta\right](A - \beta SI - r\beta SE - (\mu+v)S)\right|\right|$$

$$||S_i(t) - S_i(t^*)|| \leqslant \left|\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\right|\left|\int_0^t (t-\zeta)^{v-1}d\zeta - \int_0^{t^*}(t^*-\zeta)^{v-1}d\zeta\right| \times(|A| + |\beta||S||I|| + |r||\beta||S||E|| + |\mu+v||S|)$$

In self mapping we see that $[A + |\beta|(r+C) + |r||\beta|(r+C) + |\mu+v|(r+C)] = U^*(r)$by putting in above equation

$$||S_i(t) - S_i(t^*)|| \leqslant \left|\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\right|\left|\int_0^t (t-\zeta)^{v-1}d\zeta - \int_0^{t^*}(t^*-\zeta)^{v-1}d\zeta\right|U^*(r) \tag{3.9}$$

Let $I = \int_0^t (t-\zeta)^{v-1}d\zeta - \int_0^{t^*}(t^*-\zeta)^{v-1}d\zeta$. In the first case when t^* lies between $(0,t)$

$$I_1 = \int_0^t (t-\zeta)^{v-1}d\zeta - \int_0^{t^*}(t^*-\zeta)^{v-1}d\zeta$$

$$= \int_0^t (t-\zeta)^{v-1}d\zeta - \int_0^{t^*}(t^*-\zeta)^{v-1}d\zeta + \int_t^{t^*}(t^*-\zeta)^{v-1}d\zeta - \int_t^{t^*}(t^*-\zeta)^{v-1}d\zeta$$

$$= \int_0^t (t-\zeta)^{v-1}d\zeta - \int_0^{t^*}(t^*-\zeta)^{v-1}d\zeta - \int_{t^*}^t (t-\zeta)^{v-1}d\tau - \int_t^{t^*}(t^*-\zeta)^{v-1}d\zeta$$

$$= \int_0^t (t-\zeta)^{v-1}d\zeta - \int_0^t (t^*-\zeta)^{v-1}d\zeta - \int_t^{t^*}(t^*-\zeta)^{v-1}d\zeta$$

$$= -\left[\int_0^t (t^*-\zeta)^{v-1} - (t-\zeta)^{v-1}d\zeta\right] - \int_t^{t^*}(t^*-\zeta)^{v-1}d\zeta$$

$$= -\left[\int_0^t (t^*-\zeta)^{v-1} - (t-\zeta)^{v-1}d\zeta + \int_t^{t^*}(t^*-\zeta)^{v-1}d\zeta\right]$$

by putting the above value in Eq. (3.9), we get

$$||S_i(t) - S_i(t^*)|| \leqslant \left|\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\right|\left|-\left[\int_0^t (t^*-\zeta)^{v-1} - (t-\zeta)^{v-1}d\zeta + \int_t^{t^*}(t^*-\zeta)^{v-1}d\zeta\right]\right|U^*(r)$$

$$||S_i(t) - S_i(t^*)|| \leqslant \left|\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\right|\int_0^t \left|(t^*-\zeta)^{v-1} - (t-\zeta)^{v-1}\right|d\zeta + \int_t^{t^*}\left|(t^*-\zeta)^{v-1}\right|d\zeta U^*(r) \tag{3.10}$$

clearly $f(t) = (t-\zeta)^{v-1}$ satisfying mean value theorem

$f(t) = (t-\zeta)^{v-1}$

is continuous in $[t^*, t]$ and

$f(t) = (t-\zeta)^{v-1}$

is differentiable (t^*, t) then $\exists\ d \in (t^*, t)$

$\Rightarrow$

$$\frac{(t-\zeta)^{v-1} - (t^*-\zeta)^{v-1}}{(t-t^*)} = f'(d)$$

$$(t-\zeta)^{v-1} - (t^*-\zeta)^{v-1} = (t-t^*)(v-1)(t-d)^{v-2}$$

applying norm on both sides

$$\left|\left|(t-\zeta)^{v-1} - (t^*-\zeta)^{v-1}\right|\right| \leqslant ||(t^*-t)||\left|(v-1)(t-d)^{v-2}\right|$$

let $L = (v-1)(t-d)^{v-2}$ and

$$\left|\left|(t^*-\zeta)^{v-1} - (t-\zeta)^{v-1}\right|\right| \leqslant |(t^*-t)|L$$

By putting value in Eq. (3.10), we get

$$||S_i(t) - S_i(t^*)|| \leqslant \left|\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\right|\left[\int_0^t L|(t^*-t)|d\zeta + \int_t^{t^*}|(t^*-\zeta)|^{v-1}d\zeta\right]U^*(r)$$

$$||S_i(t) - S_i(t^*)|| \leqslant \left|\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\right|\left[L|(t^*-t)|\int_0^t d\zeta + \int_t^{t^*}|(t^*-\zeta)|^{v-1}d\zeta\right]U^*(r)$$

$$||S_i(t) - S_i(t^*)|| \leqslant \left|\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\right|\left[L|(t^*-t)|t + \frac{|t^*-t|^v}{v}\right]U^*(r)$$

$$||S_i(t) - S_i(t^*)|| \leqslant \left[\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}U^*(r)Lt + \frac{|t^*-t|^{v-1}}{AB(v)\Gamma(v)}U^*(r)\right]|(t^*-t)|$$

clearly as

$|t^* - t| \to 0$ *then* $||S_i(t) - S_i(t^*)|| \to 0$

so,if $|t^* - t| \leqslant \delta$ $||S_i(t) - S_i(t^*)|| \leqslant D\delta = \epsilon$ where $D\delta = \epsilon$, then

$$||S_i(t) - S_i(t^*)|| < \epsilon \tag{3.11}$$

where

$$D = \left[\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)} U^*(r)Lt + \frac{|t^* - t|^{v-1}}{AB(v)\Gamma(v)} U^*(r)\right]$$

secondly we consider the case when t between $(0, t^*)$

$$I_2 = \int_0^t (t-\zeta)^{v-1}d\zeta - \int_0^{t^*} (t^*-\zeta)^{v-1}d\zeta$$

$$= \int_0^t (t-\zeta)^{v-1}d\zeta - \int_0^{t^*} (t^*-\zeta)^{v-1}d\zeta + \int_{t^*}^t (t-\zeta)^{v-1}d\zeta - \int_{t^*}^t (t-\zeta)^{v-1}d\zeta$$

$$= \int_0^t (t-\zeta)^{v-1}d\zeta - \int_0^{t^*} (t^*-\zeta)^{v-1}d\zeta + \int_{t^*}^t (t-\zeta)^{v-1}d\zeta + \int_t^{t^*} (t-\zeta)^{v-1}d\zeta$$

$$= \int_0^{t^*} (t-\zeta)^{v-1}d\zeta - \int_0^{t^*} (t^*-\zeta)^{v-1}d\zeta + \int_{t^*}^t (t-\zeta)^{v-1}d\zeta$$

$$= \int_0^{t^*} \left[(t-\zeta)^{v-1}d\zeta - (t^*-\zeta)^{v-1}\right]d\zeta + \int_{t^*}^t (t-zeta)^{v-1}d\zeta$$

by putting the above value in the Eq. (3.9)we get

$$||S_i(t) - S_i(t^*)|| \leqslant \left|\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\right|\left|\left[\int_0^{t^*} (t-\zeta)^{v-1} - (t^*-\zeta)^{v-1}d\zeta + \int_{t^*}^t (t-\zeta)^{v-1}d\zeta\right]\right| U^*(r)$$

$$||S_i(t) - S_i(t^*)|| \leqslant \left|\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\right| \int_0^{t^*} \left|(t-\zeta)^{v-1} - (t^*-\zeta)^{v-1}\right|d\zeta + \int_{t^*}^t \left|(t-\zeta)^{v-1}\right|d\zeta U^*(r) \tag{3.12}$$

clearly $f(t) = (t-\zeta)^{v-1}$ satisfying mean value theorem

$f(t) = (t-\zeta)^{v-1}$

is continuous in $[t, t^*]$ and

$f(t) = (t-\zeta)^{v-1}$

is differentiable in (t, t^*) then $\exists\, d \in (t, t^*)$

$\Rightarrow$

$$\frac{(t^*-\zeta)^{v-1} - (t-\zeta)^{v-1}}{(t^*-t)} = f'(d)$$

$$(t-\zeta)^{v-1} - (t^*-\zeta)^{v-1} = (t^*-t)(v-1)(t-d)^{v-2}$$

applying norm on both sides

$$\left|\left|(t-\tau)^{v-1} - (t^*-\zeta)^{v-1}\right|\right| \leqslant ||(t-t^*)||\left|(v-1)(t-d)^{v-2}\right|$$

let $L = (v-1)(t-d)^{v-2}$ Now

$$\left|\left|(t-\zeta)^{v-1} - (t^*-\zeta)^{v-1}\right|\right| \leqslant |t^*-t|L$$

now by putting value in Eq. (3.12), we get

$$||S_i(t) - S_i(t^*)|| \leqslant \left|\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\right|\left[\int_0^{t^*} L|(t^*-t|d\zeta + \int_{t^*}^t |(t-\tau)|^{v-1}d\zeta\right] U^*(r)$$

$$||S_i(t) - S_i(t^*)|| \leqslant \left|\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\right|\left[L|t^*-t|\int_0^{t^*} d\zeta + \int_{t^*}^t |(t-\tau)|^{v-1}d\zeta\right] U^*(r)$$

$$||S_i(t) - S_i(t^*)|| \leqslant \left|\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\right|\left[L|t^*-t|t^* + \frac{|t-t^*|^v}{v}\right] U^*(r)$$

$$||S_i(t) - S_i(t^*)|| \leqslant \left|\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)}\right|\left[L|t^*-t|t^* + \frac{|t^*-t|^v}{v}\right] U^*(r)$$

$$||S_i(t) - S_i(t^*)|| \leqslant \left[\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)} U^{**}Lt^* + \frac{|t^*-t|^{v-1}}{AB(v)\Gamma(v)} U^*(r)\right]|t^*-t|$$

clearly as $|t^* - t| \to 0$ then $||S_i(t) - S_i(t^*)|| \to 0$

if $|t^* - t| \leqslant \delta ||S_i(t) - S_i(t^*)|| \leqslant D_1\delta = \epsilon$

where $D_1\delta = \epsilon$

$$\textit{then}||S_i(t) - S_i(t^*)|| < \epsilon \tag{3.13}$$

where

$$D_1 = \left[\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)} U^*(r)Lt^* + \frac{|t^*-t|^{v-1}}{\mathscr{A}\mathscr{B}(v)\Gamma(v)} U^*(r)\right]$$

in both cases (3.11, 3.13) we have the same result so, $S_i(t)$ is equicontinuous family then by Arzela-Ascoli Theorem [19] there exist a subsequence $S_{ij}(t)$of $S_i(t)$which is uniformly convergent. Hence $S(t)$is relatively compact. Similarly in the same line, one can show the equicontinuity for the remaining fixed point operator E, I, V.

as $|t^* - t| \to 0$then $||E_i(t) - E_i(t^*)|| \to 0$

so, if $|t^* - t| \leqslant \delta$where $D_2\delta = \epsilon$, then

$$||E_i(t) - E_i(t^*)|| \leqslant \epsilon \tag{3.14}$$

where

$$D_2 = \left[\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)} M^*(r)Lt^* + \frac{|t^*-t|^{v-1}}{\mathscr{A}\mathscr{B}(v)\Gamma(v)} M^*(r)\right]$$

clearly as $|t^* - t| \to 0$ then $||I_i(t) - I_i(t^*)|| \to 0$

if $|t^* - t| \leqslant \delta$

where $D_3\delta = \epsilon$, then

$$||I_i(t) - I_i(t^*)|| < \epsilon \tag{3.15}$$

where

$$D_3 = \left[\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)} N^*(r)Lt^* + \frac{|t^*-t|^{v-1}}{\mathscr{A}\mathscr{B}(v)\Gamma(v)} N^*(r)\right]$$

clearly as

$|t^* - t| \to 0\textit{then}||V_i(t) - V_i(t^*)|| \to 0$

so, if

$|t^* - t| \leqslant \delta$

where $D_4\delta = \epsilon$then

$$||V_i(t) - V_i(t^*)|| < \epsilon \tag{3.16}$$

where

$$D_4 = \left[\frac{v}{\mathscr{A}\mathscr{B}(v)\Gamma(v)} R^*(r)Lt^* + \frac{|t^*-t|^{v-1}}{\mathscr{A}\mathscr{B}(v)\Gamma(v)} R^*(r)\right]$$

Theorem 3.2. Hence in all four cases (((3.13)–(3.16)) Arzela-Ascoli theorem applicable. So, the following result is true for four operator (S, E, I, V).

Theorem 3.3. In view of above (3.1) and (3.2) the existence of fixed vectors (S, E, I, V) is guaranteed.

3.3. Positivity and Boundedness

Lemma 3.4. The solution associated to the model (2.5)–(2.8) is non-negative and bounded for all $(S(t), E(t), I(t), V(t)) \in R_+^4$, for $t > 0$.

Proof. To prove the non-negativity of the solution to the model (2.5)–(2.8), it is required that on each hyper-plane bounding the positive orthant, there is the vector field point R_+^4.

According to the model (2.5)–(2.8),

$${}^{AB}D_t^v S(t)|_{S=0} = A \geqslant 0$$
$${}^{AB}D_t^v E(t)|_{E=0} = \beta SI \geqslant 0$$
$${}^{AB}D_t^v I(t)|_{I=0} = (bE + v_1)E \geqslant 0$$
$${}^{AB}D_t^v V(t)|_{V=0} = vS \geqslant 0.$$

The solution is:

$$N(t) \leqslant N(0)E_{v,1}(vt^v) + At^v E_{v,v+1}(-vt^v).$$

Since, for all $t > 0$, Mittag–Leffler function is bounded. so,

$$\lim_{t\to\infty} \leqslant \frac{A}{v}.$$

So one can conclude that the solution of the system (2.5)–(2.8) will remain in R_+^4 and this biologically invariant region is established as:

$$\Phi = \left\{(S, E, I, V) \in R_+^4 : N \leqslant \frac{A}{v}\right\}.$$

Since, all the terms are non-negative, hence solution of system (2.5)–(2.8) is bounded. □

4. Equilibrium Points

Following are two equilibrium states of the Polio virus system:

4.1. Disease-Free Equilibrium point

It is represented by $\mathscr{E}_0(S_0, E_0, I_0, V_0)$and in this case disease-free equilibrium will be

$$E_0(S_0, E_0, I_0, V_0) = E_0\left(\frac{A}{\mu+v}, 0, 0, \frac{v(A)}{\mu(\mu+v)}\right)$$

where

$$S_0 = \frac{A}{\mu+v}, E_0 = 0, I_0 = 0, V_0 = \frac{vA}{\mu(\mu+v)}$$

We can easily find disease-free equilibrium $\mathscr{E}_0$ by taking all the fractal fractional derivatives of the system zero. The benefit of disease-free equilibrium gives us the solution converges to that equilibrium point. If there is no infected and exposed population then the susceptible population will be $\frac{A}{\mu+v}$ and the immunized population will be $V_0 = \frac{v(A)}{\mu(\mu+v)}$.

4.2. Endemic Equilibrium point:

It is represented by $\mathscr{E}^\tau(S^\tau, E^\tau, I^\tau, V^\tau)$and in this case endemic equilibrium will be

$$\mathscr{E}^\tau(S^\tau, E^\tau, I^\tau, V^\tau) = E^\tau\left(\frac{A}{(\mu+v)}\frac{1}{R}, \frac{A}{(b+\mu+v_1)}\left(1-\frac{1}{R}\right), \frac{(b+v_1)}{(\mu+\alpha)}E^\tau, \frac{v}{\mu}S^\tau\right) \tag{4.1}$$

with

$$R = \frac{A\beta(b+v_1+r(\mu+\alpha))}{(\mu+v)(b+\mu+v_1)(\mu+\alpha)}. \tag{4.2}$$

The value of equilibrium of various variables may be written as $S^\tau, E^\tau, I^\tau, V^\tau$. Eq. (4.1) clearly expresses these equilibrium values.

Remark: It is observed that E^τ and I^τ exist if $R > 1$, where

$$R = \frac{A\beta(b+v_1+r(\mu+\alpha))}{(\mu+v)(b+\mu+v_1)(\mu+\alpha)}.$$

is called the basic reproduction ratio.

4.3. Stability Analysis of Equilibrium Points

The different matrices estimated at a given equilibrium are set on by using the symbols of the real part of the eigenvalues. The differentiating of the r.h.s of the system w.r.t S, E, I, Vgives the entries of the general variational matrix i.e.,

$$V(S, E, I, V) = \begin{bmatrix} -\beta I - r\beta E - (\mu+v) & -r\beta S & -\beta S & 0 \\ \beta I + r\beta E & r\beta S - (b+\mu+v_1) & \beta S & 0 \\ 0 & b+v_1 & -(\mu+\alpha) & 0 \\ v & 0 & 0 & -\mu \end{bmatrix}$$

We represent the variational matrix corresponding to $E_0$by mean of $V(E_0)$and similar to $E^\tau$by $V(E^\tau)$.

4.4. Global Stability

Theorem 4.1. The disease free equilibrium for the model (2.5)–(2.8) is globally asymptotically stable (GAS), if reproductive number $R < 1$.

Proof. Consider the following Lyapunov function $\mathscr{H}_1 : \mathscr{B} \to \mathscr{R}$ defined as:

$$\mathscr{H}_1 = \left(S - S_0 - S_0 ln\frac{S}{S_0}\right) + E + I + \left(V - V_0 - V_0 ln\frac{V}{V_0}\right),$$
$$\text{forall}(S, E, I, V) \in \mathscr{B}^4,$$

We have

$$
\begin{aligned}
{}^{AB}D_t^v\mathscr{H}_1 &\leqslant \left(1-\tfrac{S_0}{S}\right)\ {}^{AB}D_t^vS+\ {}^{AB}D_t^vE+\ {}^{AB}D_t^vI+\left(1-\tfrac{V_0}{V}\right)\ {}^{AB}D_t^vV,\\
{}^{AB}D_t^v\mathscr{H}_1 &\leqslant \left(\tfrac{S-S_0}{S}\right)\ {}^{AB}D_t^vS+\ {}^{AB}D_t^vE+\ {}^{AB}D_t^vI+\left(\tfrac{V-V_0}{V}\right)\ {}^{AB}D_t^vV,
\end{aligned}
$$

$$
\begin{aligned}
{}^{AB}D_t^v\mathscr{H}_1 \leqslant{}& \left(\tfrac{S-S_0}{S}\right)(A-\beta SI-r\beta SE-(\mu+v)S)\\
&+(\beta SI+r\beta SE-(b+\mu+v_1)E)+((b+v_1)E-(\mu+\alpha_1)I)+\\
&\left(\tfrac{V-V_0}{V}\right)(vS-\mu V),
\end{aligned}
$$

$$
\begin{aligned}
{}^{AB}D_t^v\mathscr{H}_1 \leqslant{}& (S-S_0)\left(\tfrac{A}{S}-\beta SI-r\beta SE-(\mu+v)S\right)+(\beta SI+r\beta SE)\\
&-\mu E-\mu I-\alpha_1 I+(V-V_0)\left(\tfrac{vS}{V}-\mu\right),
\end{aligned}
$$

$$
\begin{aligned}
{}^{AB}D_t^v\mathscr{H}_1 \leqslant{}& (S-S_0)\left(\tfrac{1}{S}-\tfrac{1}{S_0}\right)A-\mu E\left(1-\tfrac{r\beta S}{\mu}\right)-\mu I\left(1-\tfrac{\beta S}{\mu}\right)-\alpha_1 I+\\
&(V-V_0)\left(\tfrac{1}{V}-\tfrac{1}{V_0}\right)vS,
\end{aligned}
$$

$$
\begin{aligned}
{}^{AB}D_t^v\mathscr{H}_1 \leqslant{}& -\tfrac{(S-S_0)^2}{SS_0}A-\mu E\left(1-\tfrac{r\beta S}{\mu}\right)-\mu I\left(1-\tfrac{\beta S}{\mu}\right)-\\
&\alpha_1 I-\left(\tfrac{(V-V_0)^2}{VV_0}\right)vS.
\end{aligned}
$$

From the above calculations, it can be observed that ${}^{AB}D_t^v\mathscr{H}_1<0$, if $R<1$.

Hence, $\mathscr{E}_0$ is globally asymptotically stable. □

Theorem 4.2. The endemic equilibrium $\mathscr{E}^\tau$ for the system (2.5)–(2.8) is globally asymptotically stable (GAS), if $R>1$.

Proof. Considering the Volterra Lyapunov function $\mathscr{H}_2:\mathscr{B}\to\mathscr{R}$, where

$$
\begin{aligned}
\mathscr{H}_2 = {}& \left(S-S^*-S^*ln\tfrac{S}{S^*}\right)+\left(E-E^*-E^*ln\tfrac{E}{E^*}\right)+\left(I-I^*-I^*ln\tfrac{I}{I^*}\right)+\\
&\left(V-V^*-V^*ln\tfrac{V}{V^*}\right),\text{forall}(S,E,I,V)\in\mathscr{B}^4.
\end{aligned}
$$

We have

$$
\begin{aligned}
{}^{AB}D_t^v\mathscr{H}_2 \leqslant{}& \frac{S-S^*}{S}\ {}^{AB}D_t^vS+\frac{E-E^*}{E}\ {}^{AB}D_t^vE\\
&+\frac{I-I^*}{I}\ {}^{AB}D_t^vI+\frac{V-V^*}{V}\ {}^{AB}D_t^vV,
\end{aligned}
$$

$$
\begin{aligned}
{}^{AB}D_t^v\mathscr{H}_2 \leqslant{}& \tfrac{S-S^*}{S}(A-\beta SI-r\beta SE-(\mu+v)S)+\\
&\tfrac{E-E^*}{E}(\beta SI-r\beta SE-(v+\mu+v_1)E)+\\
&\tfrac{I-I^*}{I}((b+v_1)E-(\mu+\alpha_1)I)+\tfrac{V-V^*}{V}(vS-\mu V),
\end{aligned}
$$

$$
\begin{aligned}
{}^{AB}D_t^v\mathscr{H}_2 \leqslant{}& (S-S^*)\left(\tfrac{A}{S}-\beta I-r\beta E-(\mu+v)S\right)+\\
&(E-E^*)\left(\tfrac{\beta SI}{E}+r\beta S-(b+\mu+v_1)\right)+\\
&(I-I^*)\left(\left(\tfrac{bE}{I}+\tfrac{v_1E}{I}\right)-(\mu+\alpha_1)\right)+(V-V^*)\left(\left(\tfrac{vS}{V}-\mu\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
{}^{AB}D_t^v\mathscr{H}_2 \leqslant{}& (S-S^*)\left(\tfrac{A}{S}-\tfrac{A}{S^*}-\beta I+\beta I^*-r\beta E+r\beta E^*-v+v\right)+\\
&(E-E^*)\left(\tfrac{\beta SI}{E}-\tfrac{\beta SI}{E^*}+r\beta S-r\beta S-b+b-v_1+v_1\right)+\\
&(I-I^*)\left(\tfrac{bE}{I}-\tfrac{bE}{I^*}+\tfrac{v_1E}{I}-\tfrac{v_1E}{I^*}-\alpha_1+\alpha_1\right)+(V-V^*)\left(\tfrac{V^*-V}{VV^*}\right)vS,
\end{aligned}
$$

$$
\begin{aligned}
{}^{AB}D_t^v\mathscr{H}_2 \leqslant{}& -\tfrac{(S-S^*)^2}{SS^*}A-\tfrac{(E-E^*)^2}{EE^*}\beta SI-\tfrac{(I-I^*)^2}{II^*}bE-\\
&\tfrac{(I-I^*)^2}{II^*}\beta-\tfrac{(V-V^*)^2}{VV^*}rS.
\end{aligned}
$$

It can be concluded from the above equation that ${}^{AB}D_t^v\mathscr{H}_2\leqslant 0$ for $R>1$.

Hence $\mathscr{E}^\tau$ is globally asymptotically stable (GAS). □

5. Analysis of the Model with Mittag–Leffler Kernel

We consider the following problem with the Atangana-Baleanu derivative as:

$$
\begin{aligned}
{}^{\mathscr{AB}}_{a}D_t^vS(t) &= A-\beta SI-r\beta SE-(\mu+v)S,\\
{}^{\mathscr{AB}}_{a}D_t^vE(t) &= \beta SI+r\beta SE-(b+\mu+v_1)E,\quad \forall t\geqslant 0,\\
{}^{\mathscr{AB}}_{a}D_t^vI(t) &= (b+v_1)E-(\mu+\alpha)I,\\
{}^{\mathscr{AB}}_{a}D_t^vV(t) &= vS-\mu V,\quad \forall t\geqslant 0.
\end{aligned}
$$

For simplicity, we define

$$
\begin{aligned}
K_1(t,S,E,I,V) &= A-\beta SI-r\beta SE-(\mu+v)S,\quad \forall t\geqslant 0,\\
K_2(t,S,E,I,V) &= \beta SI+r\beta SE-(b+\mu+v_1)E,\quad \forall t\geqslant 0,\\
K_3(t,S,E,I,V) &= (b+v_1)E-(\mu+\alpha)I,\quad \forall t\geqslant 0,\\
K_4(t,S,E,I,V) &= vS-\mu V,\quad \forall t\geqslant 0.
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
\tfrac{\mathscr{AB}(v)}{1-v}\tfrac{d}{dt}\int_0^t S(\zeta)E_\alpha\left(\tfrac{-v}{1-v}(t-\zeta)^v\right)d\zeta &= K_1(t,S,E,I,V)\\
\tfrac{\mathscr{AB}(v)}{1-v}\tfrac{d}{dt}\int_0^t E(\zeta)E_\alpha\left(\tfrac{-v}{1-v}(t-\zeta)^v\right)d\zeta &= K_2(t,S,E,I,V)\\
\tfrac{\mathscr{AB}(v)}{1-v}\tfrac{d}{dt}\int_0^t I(\zeta)E_\alpha\left(\tfrac{-v}{1-v}(t-\zeta)^v\right)d\zeta &= K_3(t,S,E,I,V)\\
\tfrac{\mathscr{AB}(v)}{1-v}\tfrac{d}{dt}\int_0^t V(\zeta)E_\alpha\left(\tfrac{-v}{1-v}(t-\zeta)^v\right)d\zeta &= K_4(t,S,E,I,V)
\end{aligned}
$$

Applying the AB integral gives,

$$
\begin{aligned}
S(t)-S(0) &= \tfrac{1-v}{\mathscr{AB}(v)}K_1(t,S,E,I,V)+\tfrac{v}{\mathscr{AB}(v)\Gamma(v)}\int_0^t(t-\zeta)^{v-1}K_1(\zeta,S,E,I,V)d\zeta\\
E(t)-E(0) &= \tfrac{1-v}{\mathscr{AB}(v)}K_2(t,S,E,I,V)+\tfrac{v}{\mathscr{AB}(v)\Gamma(v)}\int_0^t(t-\zeta)^{v-1}K_2(\zeta,S,E,I,V)d\zeta\\
I(t)-I(0) &= \tfrac{1-v}{\mathscr{AB}(v)}K_3(t,S,E,I,V)+\tfrac{v}{\mathscr{AB}(v)\Gamma(v)}\int_0^t(t-\zeta)^{v-1}K_3(\zeta,S,E,I,V)d\zeta\\
V(t)-V(0) &= \tfrac{1-v}{\mathscr{AB}(v)}K_4(t,S,E,I,V)+\tfrac{v}{\mathscr{AB}(v)\Gamma(v)}\int_0^t(t-\zeta)^{v-1}K_4(\zeta,S,E,I,V)d\zeta
\end{aligned}
$$

We discretize these equations at t_{n+1} as:

$$
\begin{aligned}
S^{n+1} = {}& S^0+\tfrac{1-v}{\mathscr{AB}(v)}K_1(t_{n+1},S^n,E^n,I^n,V^n)\\
&+\tfrac{v}{\mathscr{AB}(v)\Gamma(v)}\int_0^{t_{n+1}}(t_{n+1}-\zeta)^{v-1}K_1(\zeta,S,E,I,V)d\zeta\\
E^{n+1} = {}& E^0+\tfrac{1-v}{\mathscr{AB}(v)}K_2(t_{n+1},S^n,E^n,I^n,V^n)\\
&+\tfrac{v}{\mathscr{AB}(v)\Gamma(v)}\int_0^{t_{n+1}}(t_{n+1}-\zeta)^{v-1}K_2(\zeta,S,E,I,V)d\zeta\\
I^{n+1} = {}& I^0+\tfrac{1-v}{\mathscr{AB}(v)}K_3(t_{n+1},S^n,E^n,I^n,V^n)\\
&+\tfrac{v}{\mathscr{AB}(v)\Gamma(v)}\int_0^{t_{n+1}}(t_{n+1}-\zeta)^{v-1}K_3(\zeta,S,E,I,V)d\zeta\\
V^{n+1} = {}& V^0+\tfrac{1-v}{\mathscr{AB}(v)}K_4(t_{n+1},S^n,E^n,I^n,V^n)\\
&+\tfrac{v}{\mathscr{AB}(v)\Gamma(v)}\int_0^{t_{n+1}}(t-\zeta)^{v-1}K_4(\zeta,S,E,I,V)d\zeta
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
S^{n+1} = {}& S^0+\tfrac{1-v}{\mathscr{AB}(v)}K_1(t_{n+1},S^n,E^n,I^n,V^n)\\
&+\tfrac{v}{\mathscr{AB}(v)}\sum_{j=0}^{n}\left[\tfrac{h^vK_1(t_j,S^n,E^n,I^n,V^n)}{\Gamma(v+2)}((n+1-j)^v(n-j+2+v)\right.\\
&\left.-(n-j)^v(n-j+2+2v))\right]\\
&-\tfrac{v}{\mathscr{AB}(v)}\sum_{j=0}^{n}\left[\tfrac{h^vK_1(t_{j-1},S^{n-1},E^{n-1},I^{n-1},V^{n-1})}{\Gamma(v+2)}\left((n+1-j)^{v+1}\right.\right.\\
&\left.\left.-(n-j)^v(n-j+1+v)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
E^{n+1} = {} & E^0 + \tfrac{1-v}{\mathscr{AB}(v)} K_2(t_{n+1}, S^n, E^n, I^n, V^n) \\
& + \tfrac{v}{\mathscr{AB}(v)} \sum_{j=0}^{n} \left[\tfrac{h^v K_2(t_j, S^n, E^n, I^n, V^n)}{\Gamma(v+2)} ((n+1-j)^v (n-j+2+v) \right. \\
& \left. -(n-j)^v (n-j+2+2v)) \right] \\
& - \tfrac{v}{\mathscr{AB}(v)} \sum_{j=0}^{n} \left[\tfrac{h^v K_2(t_{j-1}, S^{n-1}, E^{n-1}, I^{n-1}, V^{n-1})}{\Gamma(v+2)} \left((n+1-j)^{v+1} \right. \right. \\
& \left. \left. -(n-j)^v (n-j+1+v)) \right] \right.
\end{aligned}
$$

$$
\begin{aligned}
I^{n+1} = {} & I^0 + \tfrac{1-v}{\mathscr{AB}(v)} K_3(t_{n+1}, S^n, E^n, I^n, V^n) \\
& + \tfrac{v}{\mathscr{AB}(v)} \sum_{j=0}^{n} \left[\tfrac{h^v K_3(t_j, S^n, E^n, I^n, V^n)}{\Gamma(v+2)} ((n+1-j)^v (n-j+2+v) \right. \\
& \left. -(n-j)^v (n-j+2+2v)) \right] \\
& - \tfrac{v}{\mathscr{AB}(v)} \sum_{j=0}^{n} \left[\tfrac{h^v K_3(t_{j-1}, S^{n-1}, E^{n-1}, I^{n-1}, V^{n-1})}{\Gamma(v+2)} \left((n+1-j)^{v+1} \right. \right. \\
& \left. \left. -(n-j)^v (n-j+1+v)) \right] \right.
\end{aligned}
$$

$$
\begin{aligned}
R^{n+1} = {} & R^0 + \tfrac{1-v}{\mathscr{AB}(v)} K_4(t_{n+1}, S^n, E^n, I^n, V^n) \\
& + \tfrac{v}{\mathscr{AB}(v)} \sum_{j=0}^{n} \left[\tfrac{h^v K_4(t_j, S^n, E^n, I^n, V^n)}{\Gamma(v+2)} ((n+1-j)^v (n-j+2+v) \right. \\
& \left. -(n-j)^v (n-j+2+2v)) \right] \\
& - \tfrac{v}{\mathscr{AB}(v)} \sum_{j=0}^{n} \left[\tfrac{h^v K_4(t_{j-1}, S^{n-1}, E^{n-1}, I^{n-1}, V^{n-1})}{\Gamma(v+2)} \left((n+1-j)^{v+1} \right. \right. \\
& \left. \left. -(n-j)^v (n-j+1+v)) \right] \right.
\end{aligned}
$$

by the method using in [16].

6. Numerical simulations

The values of parameters involved in this model are, $A = 0.5, \mu = 0.5, v = 0.6, \alpha = 0.0001, v1 = 0.001, b = 0.9, r = 0.5$. For Disease Free, $\beta = 1.002$, For EE, $\beta = 2.002$. The initial conditions are, $S(0) = 0.5, E(0) = 0.2, I(0) = 0.1, V(0) = 0.1$.

All the curve patterns in Fig. 1 represent the convergence behavior of the susceptible populace at a disease-free equilibrium state (DFES). The behavior of the graphs are investigated for different values of v, which is the order of the fractional derivative. Each graph adopts a non-linear path to reach the DFES. It is noticeable that each path follows a slightly different route to reach the same fixed point i.e. the equilibrium point for the disease-free state. Moreover, each graph shows the dynamics against the specific value of v (fractional order of the derivative). The role of fractional order v is evident from each of the graphs. The rate of convergence is higher for the larger value of v. Fig. 2 is the simulation of the exposed population for the poliovirus. All the curved lines follow different trajectories to reach the DFES. But it is important that each curve goes to the same point, which is the DFE point. Also, this point coincides with the point which is already calculated analytically in Section 6. Moreover, the curve with a greater value of v attains the fixed point fast. The graphical patterns in Fig. 3 express the behavior and dynamics of the phenomenon to gain the disease-free stable state. All the curve lines move towards the true steady- state against the different values of v, but the pathes and rate of convergence varies for the different values of the fractional-order v. This shows that the fractional-order of the derivative plays an important role in describing the path that the event adopts to attain the stable state. Fig. 4 expresses the graphical situation of the vaccinated

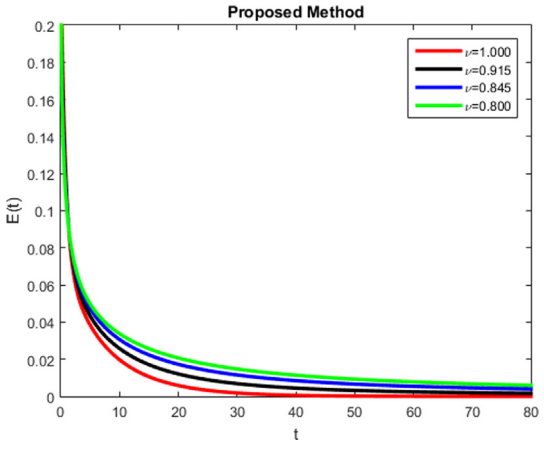

Fig. 2 Graphical behavior of exposed population at disease free equilibrium of fractional order v for different values.

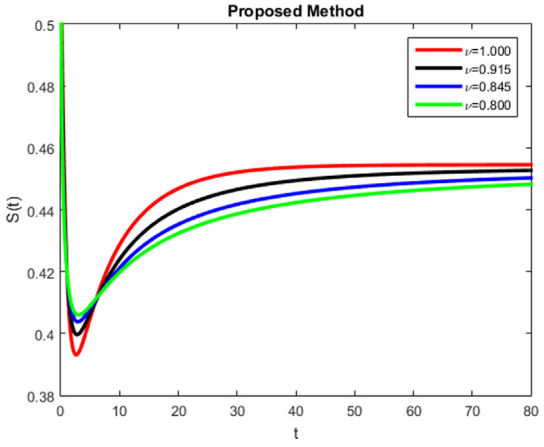

Fig. 1 Graphical behavior of susceptible population at disease free equilibrium of fractional order v for different values.

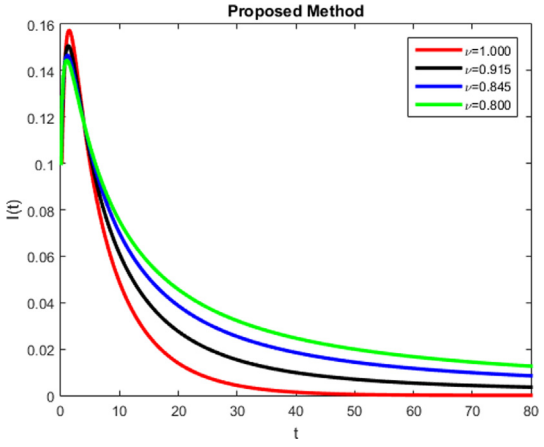

Fig. 3 Graphical behavior of infected population at disease free equilibrium of fractional order v for different values.

populace at DFES. The different curves in the figure also point out towards the fractional-order v, of the derivative. It is clear from the curved graphs that each graph follows some specific path depending upon the value of v. All the simulations in the figure reveal that every graph attains the steady-state of the system by following the different routes of convergence. Figs. 5–8 show the behavior of the state variables involved in the model. Fig. 5 shows, how the susceptible portion of the population converges towards the endemic equilibrium, for the various values of v. It is evident that each graph converges towards the exact equilibrium point that is the endemic equilibria point. The graphical sketches in Fig. 6 depict the behavior of the state variable E. Every graph is sketched for a specific value of v, but each graph attains the endemic equilibrium point irrespective of the value of v. Also, the converging value of each of the curved graphs is the same as computed analytically. Fig. 7 is the face of the graphical behavior of the infected population for different values of v. Every graph in the figure adopts a particular route and reaches the steady state of the model. Similarly, the last figure provides the same result as obtained analytically. Furthermore, each sketch describes the role of the fractional-order v. Each graph follows

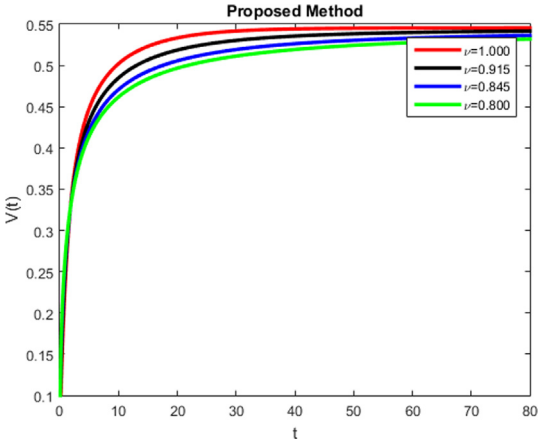

Fig. 4 Graphical behavior of vaccinated population at disease free equilibrium of fractional order v for different values.

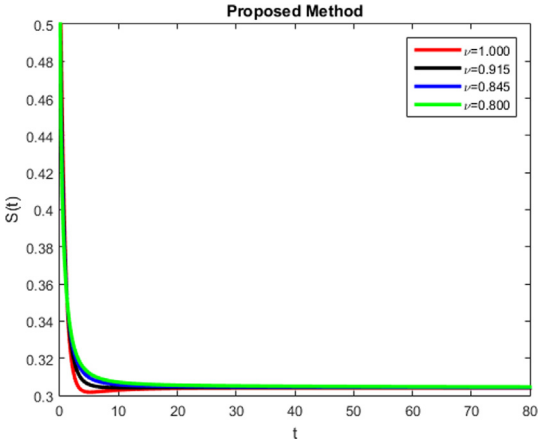

Fig. 5 Graphical behavior of susceptible population at endemic equilibrium of fractional order v for different values.

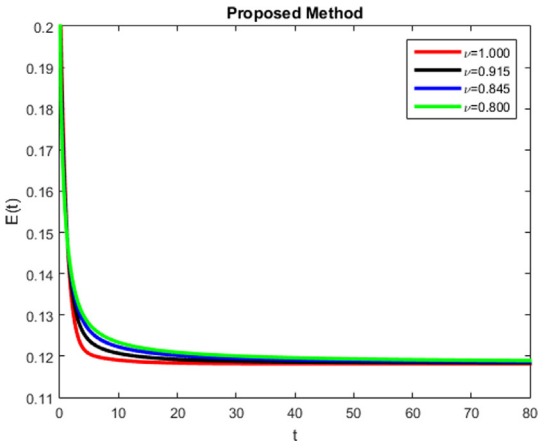

Fig. 6 Graphical behavior of exposed population at endemic equilibrium of fractional order v for different values.

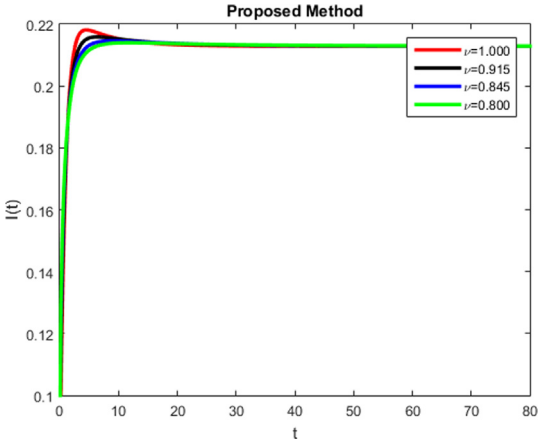

Fig. 7 Graphical behavior of infected population at endemic equilibrium of fractional order v for different values.

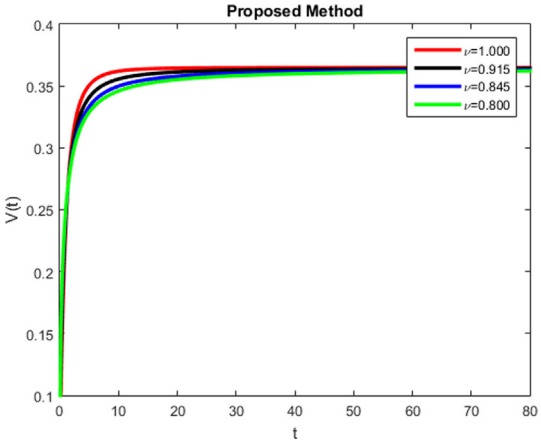

Fig. 8 Graphical behavior of vaccinated population at endemic equilibrium of fractional order v for different values.

a slightly different path, but ultimately, all the graph converges at the true steady state.

7. Conclusion and Future Perspectives

In this study, a fractal fractional-order mathematical model of Polio-virus is studied. The existence of the solution for the underlying model is guaranteed by applying the fixed point theory. Two results for the existence of the solution are presented. It is observed that this model possesses the two steady states namely, the disease-free and endemic steady states. The global stability of the model is studied and a benchmark result is established to ensure the global stability. The model is analyzed with the help of Mittag Leffler Kernel. The basic reproduction number for the model of Polio-virus is presented and its key role is elaborated. The disease transmission is predicted on the basis of the value of R_0 (the basic reproduction number). Moreover, it is R_0 used for describing the stability of the model. The numerical graphs are drawn and it is ascertained that the graphs are in line with the exact results. That is, the graphs of each state variable converge towards the true steady state. Numerical simulations are elaborated to reach a fruitful conclusion. For a future perspective, the current work may be applied to solve the stochastic fractal fractional models and delay the fractal fractional infection disease model.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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