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ORIGINAL ARTICLE

Positivity preserving numerical method for epidemic model of hepatitis B disease dynamic with delay factor

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Abstract This work attempts to study the numerical solution of nonlinear delayed Immunized Susceptible Latent Infected and Recovered (MSLIR) epidemic model of HBV disease. Reproduction number, equilibria and stability are discussed. Three different numerical techniques, Euler, RK-4 and the non-standard finite difference (NSFD) techniques are used for the numerical solution of the model. The proposed technique is independent of the size of the time step, while forward Euler and RK-4 depend on the size of a time step and retains all essential characteristics of the continuous MSLIR epidemic model like positivity and stability of equilibrium, while well-known forward Euler and RK-4 cannot sustain these characteristics. Therefore, the proposed (NSFD)

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technique becomes a more efficient and reliable numerical technique than the forward Euler and RK-4 scheme. Numerical simulations are presented for the validation of the obtained results.

1. Introduction

The word "hepatitis" is known as "inflammation or burning" of the bile duct. So, "hepatitis B" mentioned the burning of the bile duct that leads to the hepatitis B virus. People suffering from hepatitis-B infection can be assumed to enjoy a hygienic life with appropriate follow-up medical care. HBV is propagated once blood or seminal fluid is infected from a person that is infected with virus when enters into the body of any individual that is not infected. Such infections can come off through sexual relations, splitting needles and narcotize injection, also transfer from mother to her baby during birth. Hepatitis B is a short-term ailment for immunized individuals. But for other individuals, it can be recommended as a chronic infection. Probably, 90 per cent of newborns that HBV infects become chronically infected, which is compared with (2–6) per cent of young ones. Vaccination is the best way the prevention this infectious disease. This active infection can lead to contamination of the bile duct. A person having HBV can transmit the virus to other individuals unaware of it. Some people are asymptomatic. Some individuals have an initial infection, which then recovers. For other individuals, the condition becomes chronic. In chronic cases, the virus continues to assault the liver over time without diagnosis [1,2]. The population infected with hepatitis B worldwide is in the millions. According to WHO, the hepatitis B patients were estimated about 296 million in 2019 with 1.5 million new infections each year. Moreover, it was causative of about 0.82 million deaths in 2019 [3].Fig. 1Table 1Table 2.

Delay differential equations allow previous activities to be added to numerical models, bringing the model closer to the actual occurrence [4]. For most transmittable sicknesses, there is a time interval between disease and the event of side effects (the incubation period); during this time, pathogens grow or develop. Even if they can transmit the disease, some affected people may never exhibit symptoms (in apparent infection). For example, Measles has a certain incubation period (10 to 14 days) and classified duration of infectivity for a taken patient (4 to 7 days). The time delay differential models are much more realistic because they account for time dynamics from infection to infectiousness. There are numerous models accessible in the literary study, which show the elements of this illness by arranging nonlinear differential equations without any delay, albeit the delay incorporation makes the model

Table 1 Parameters of the model.

Parameters	Representation
ϕ	Rate of vaccine efficacy
k	Rate of transformation from S towards L
μ	Rate of conversion from L towards I
ψ	Rate of conversion from I towards R
π	Rate of conversion from R towards S
q	Rate of change from L towards R
η	H.B. induced death
β	Natural death
cP	Immunized newborns
N	Total number of population

Table 2 Table of parameter values.

parameters	DF	EE
β	0.5	0.5
ϕ	0.01	0.01
μ	10.5	10.5
η	0.05	0.05
π	1	1
τ	5	3
P	0.1	0.1
ψ	0.5	0.5
q	1.5	1.5
C	0.01	0.01

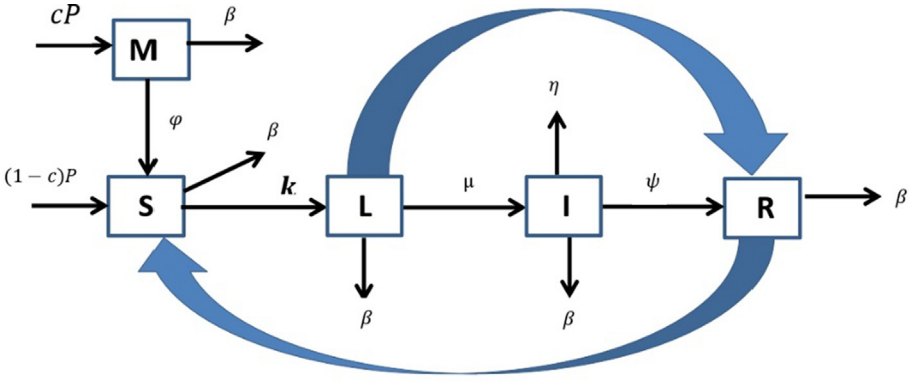

Fig. 1 Flow chart of the model.

more sensible. The demographic model's dynamical behavior delay is now becoming an important research area. In [5], a stochastic delayed model is developed to explain chronic hepatitis-B infection with HBV-DNA containing virions. In [6], Wang et al. explored a SEICRV scourge model with a time delay to investigate the transmission elements of Hepatitis B illness. Hu et al. considered cell and animal models to describe HBV [7]. Khan et al. studied an integer and fractional order hepatitis B model [8]. Existence, uniqueness and reproduction number are discussed and stability analysis are performed. Khan et al. analyzed a Hepatitis B fractional model under Caputo- Fabrizio derivative [9]. Existence and uniqueness are discussed with the help of fixed point theory. Adomian decomposition method coupled with the integral transform of Laplace is used for the semi analytical solutions. Khan et al. presented a model of hepatitis B virus by taking into account media coverage and the basic properties like reproductive number and stability of the model are discussed [9]. Zhong et al. presented integer-order and fractal-fractional model describing the Hepatitis B virus incorporating the well known Atangana-Baleanu derivative [10]. Moreover, Bashforth method is used for solving the model numerically. Hong studied an improved model of HBV and proposed an NSFD scheme for its solution [11]. The properties like stability, positivity and boundedness are discussed. Convergence analysis and error analysis are also included. Fatehi et al. proposed an age-structured HBV model [12]. Din and Li presented a delayed stochastic HBV model with noises and vaccination effect [13]. Din et al. presented a non integer order HBV model using Caputo derivative [14]. Laplace transformation and Adomian decompositions are being used for the semi analytical solution of the model. Khan et al. studied an HBV model using Caputo- Fabrizio derivative [15]. Zarin presented an HBV model using ABC (Atangana Baleanu Capotu) derivative [16]. Din and Li studied the transmission dynamics of HBV using AB derivative [17]. Fixed point theory was used in order to study the existence and uniqueness. Omame et al. presented a COVID-19 and HBV co-interaction model using AB derivative [18]. The Laypunov function is used for finding the stability of the model. Fractional and integer order derivatives are used for approximation of the solution.

Sometimes it is impossible or complicated to find the exact solution of many physical systems. Numerical techniques become essential tools to find the solutions to differential equations. Several dynamical systems depict some essential features like positivity, boundedness etc. These physical features should preserve by the numerical method. Various authors used different techniques to solve dynamical systems. Macías-Díaz and Szafrańska studied an NSFD based scheme preserving positivity, boundedness and spatio temporal monotonicity [19]. Ahmed et al. considered the Brusselator reactiondiffusion model and presented a finite difference method for its numerical solution [20]. The stability and consistency of the presented scheme also discussed using Neumann criteria and Taylor series respectively. Euler and Crank-Nicolson approaches are used for the comparison purposes. Asamoah et al. studied rabies transmission [21]. Shah et al. studied a fractional order HIV model with source term to give a better understanding of the disease [22].

In this article, a competitive mathematical approach is given to analyze a system of nonlinear coupled differential equations with the influence of time delay. The numerical scheme's unrestricted convergence and logical correctness with dynamical models are demonstrated through analysis. The novelty of the current work is the development, accomplishment and mathmatical analysis of the numerical technique in NSFD settings with delay factor. To our knowledge, the model under study has not been analyzed before in NSFD with delay in the literature and this is the first study of this model in this sense.

2. Mathematical model

This section gives rise to the model of HBV using the MSLIR model in [23] with a time delay factor. In MSLIR Model, the total population is categorized into five sections, Immunized$\widehat{M}(t)$, Susceptible$\widehat{S}(t)$, Latent$\widehat{L}(t)$, Infected $\widehat{I}(t)$ and Recovered$\widehat{R}(t)$. We accept the whole human population as a constant population that we have$N=\widehat{M}+\widehat{S}+\widehat{L}+\widehat{I}+\widehat{R}$. The flow chart of the MSLIR model and parameter table [23] are:

2.1. System of differential equation with delay factors

To draw up the HBV transference, MSLIR model is given below:

$$\frac{d\widehat{M}(t)}{dt}=cP-\phi\widehat{M}(t)-\beta\widehat{M}(t), \tag{1}$$

$$\frac{d\widehat{S}(t)}{dt}=(1-c)P+\phi\widehat{M}(t)+\pi\widehat{R}(t)-\left(k\widehat{I}(t)+\beta\right)\widehat{S}(t), \tag{2}$$

$$\frac{d\widehat{L}(t)}{dt}=k\widehat{I}(t-\tau)\widehat{S}(t-\tau)e^{-\beta\tau}-q\widehat{L}(t)-\mu\widehat{L}(t)-\beta\widehat{L}(t), \tag{3}$$

$$\frac{d\widehat{I}(t)}{dt}=\mu\widehat{L}(t)-\psi\widehat{I}(t)-\eta\widehat{I}(t)-\beta\widehat{I}(t), \tag{4}$$

$$\frac{d\widehat{R}(t)}{dt}=q\widehat{L}(t)+\psi\widehat{I}(t)-\pi\widehat{R}(t)-\beta\widehat{R}(t). \tag{5}$$

2.2. Equilibrium analysis

At equilibrium states we have,

$$\frac{d\widehat{M}(t)}{dt}=\frac{d\widehat{S}(t)}{dt}=\frac{d\widehat{L}(t)}{dt}=\frac{d\widehat{I}(t)}{dt}=\frac{d\widehat{R}(t)}{dt}=0$$

Therefore:

$$cP-(\phi+\beta)\widehat{M}(t)=0, \tag{6}$$

$$(1-c)P+\phi\widehat{M}(t)+\pi\widehat{R}(t)-\left(k\widehat{I}(t)+\beta\right)\widehat{S}(t)=0, \tag{7}$$

$$k\widehat{I}(t-\tau)\widehat{S}(t-\tau)e^{-\beta\tau}-(q+\mu+\beta)\widehat{L}(t)=0, \tag{8}$$

$$\mu\widehat{L}(t)-(\psi+\eta+\beta)\widehat{I}(t)=0, \tag{9}$$

$$q\,\widehat{L}(t)+\psi\,\widehat{I}(t)-(\pi+\beta)\,\widehat{R}(t)=0. \tag{10}$$

2.2.1. Existence of a trivial equilibrium state (T.E.S.)

Assume that $E_0\left(\widehat{M}_0,\widehat{S}_0,\widehat{L}_0,\widehat{I}_0,\widehat{R}_o\right)$ be T.E.S. of (6) to (10) of the model, there exist no T.E.S. after all the individuals cannot be disappeared, then in a long time, newborn enter into a total number of individuals (i.e.$cP\neq0$ and $(1-c)P\neq0$).

That is,

$$E_0\left(\widehat{M}_0,\widehat{S}_0,\widehat{L}_0,\widehat{I}_0,\widehat{R}_o\right)\neq0 \tag{11}$$

2.2.2. Disease-Free equilibrium (DFE) point

DFE condition is the total elimination of the disease. Let $E^0\left(\widehat{M}^0,\widehat{S}^0,\widehat{L}^0,\widehat{I}^0,\widehat{R}^0\right)$ become a D.F.E. state. Consider $\widehat{I}^0$,$\widehat{L}^0$, and both are equal to zero so, for DFE state:$\widehat{L}^0=\widehat{I}^0=0$. On putting these terms in Eqs. (6) to (10) and on solving simultaneously, we get the DFE state of the model:

$$E^0\left(\widehat{M}^0,\widehat{S}^0,\widehat{L}^0,\widehat{I}^0,\widehat{R}^0\right)=\left(\frac{cP}{\phi+\beta},\frac{(\phi+\beta-c\beta)P}{(\phi+\beta)\beta},0,0,0\right) \tag{12}$$

2.2.3. Endemic equilibrium (EE) point

This is the stage when the disease will persist into the population, i.e.$L^0=I^0\neq0$, Let $E^*\left(\widehat{M}^*,\widehat{S}^*,\widehat{L}^*,\widehat{I}^*,\widehat{R}^*\right)$ be the EE state.

$$E^*\left(\widehat{M}^*,\widehat{S}^*,\widehat{L}^*,\widehat{I}^*,\widehat{R}^*\right)=\left(\begin{array}{c}\frac{cP}{\phi+\beta},\frac{(q+\mu+\beta)(\psi+\eta+\beta)e^{\beta\tau}}{\mu k},\\ \frac{\left[\beta\left((q+\mu+\beta)(\psi+\eta+\beta)e^{\beta\tau}\right)+\mu kP(-\phi-\beta-c\beta)\right](\pi+\beta)(\psi+\eta+\beta)}{\mu k(\phi+\beta)\left[\pi(q(\psi+\eta+\beta)+\mu\psi)-(\pi+\beta)(q+\mu+\beta)(\psi+\eta+\beta)e^{-\beta\tau}\right]},\\ \frac{\left[\beta\left((q+\mu+\beta)(\psi+\eta+\beta)e^{\beta\tau}\right)+\mu kP(-\phi-\beta-c\beta)\right](\pi+\beta)}{k(\phi+\beta)\left[\pi(q(\psi+\eta+\beta)+\mu\psi)-(\pi+\beta)(q+\mu+\beta)(\psi+\eta+\beta)e^{-\beta\tau}\right]},\\ \frac{\left[\beta\left((q+\mu+\beta)(\psi+\eta+\beta)e^{\beta\tau}\right)+\mu kP(-\phi-\beta-c\beta)\right][q(\psi+\eta+\beta)+\mu\psi]}{\mu^2k(\phi+\beta)\left[\pi(q(\psi+\eta+\beta)+\mu\psi)-(\pi+\beta)(q+\mu+\beta)(\psi+\eta+\beta)e^{-\beta\tau}\right]}\end{array}\right).$$

2.3. Basic reproduction number

The disease will exist in the population if an infected individual appears in it. So to analyze the transmission of dynamics of illness, we find out the basic reproductive number using next-generation matrix approach [24]. For this, we considered the two disease Eqs. (3) and (4) as follows:

$$k\,\widehat{I}(t-\tau)\,\widehat{S}(t-\tau)e^{-\beta\tau}-(q+\mu+\beta)\,\widehat{L}(t)=0, \tag{13}$$

$$\mu\,\widehat{L}(t)-(\psi+\eta+\beta)\,\widehat{I}(t)=0. \tag{14}$$

We obtained the transmission matrix F, as well as the transition matrix V as given below,

$$F=\begin{bmatrix}0 & K\frac{(\phi+\beta-c\beta)P}{(\phi+\beta)\beta}e^{-\beta\tau}\\ 0 & 0\end{bmatrix}$$

$$\text{and}\,V=\begin{bmatrix}q+\mu+\beta & 0\\ -\mu & \psi+\eta+\beta\end{bmatrix}.$$

We get therefore the basic reproductive number as:

$$R_0=\frac{\mu kP(\phi+\beta-c\beta)e^{-\beta\tau}}{\beta(q+\mu+\beta)(\psi+\eta+\beta)(\phi+\beta)} \tag{15}$$

If $R_0<1$, then disease dies as well as infection dies and if $R_0>1$, then the spread of the disease will continue in the population.

2.3.1. Stability analysis of DFE state

We now calculate the Jacobian matrix corresponding to the linear system of Eqs. (1)–(5) as.

$$J=\begin{bmatrix}-(\phi+\beta) & 0 & 0 & 0 & 0\\ \phi & -\left(k\widehat{I}^0+\beta\right) & 0 & -k\widehat{S}^0 & \pi\\ 0 & k\widehat{I}^0 & -(q+\mu+\beta) & k\widehat{S}^0e^{-\beta\tau} & 0\\ 0 & 0 & \mu & -(\psi+\eta+\beta) & 0\\ 0 & 0 & q & \psi & -(\pi+\beta)\end{bmatrix}$$

At DFE point $E^o\left(\widehat{M}^o,\widehat{S}^o,\widehat{L}^o,\widehat{I}^o,\widehat{R}^o\right)$ the Jacobian matrix becomes:

$$J=\begin{bmatrix}w_1 & 0 & 0 & 0 & 0\\ \phi & -\beta & 0 & w_2 & \pi\\ 0 & 0 & w_3 & w_4 & 0\\ 0 & 0 & \mu & w_5 & 0\\ 0 & 0 & q & \psi & w_6\end{bmatrix} \tag{16}$$

where $w_1=-(\phi+\beta)$,$w_2=-k\frac{(\phi+\beta-c\beta)Pe^{-\beta\tau}}{(\phi+\beta)\beta}$,$w_3=-(q+\mu+\beta)$, $w_4=k\frac{(\phi+\beta-c\beta)Pe^{-\beta\tau}}{(\phi+\beta)\beta}$,$w_5=-(\psi+\eta+\beta)$ and$w_6=-(\pi+\beta)$.

The characteristic equation of the matrix J is.

$$(\phi+\beta+\lambda)(\beta+\lambda)(-(\pi+\beta)-\lambda)[A]=0 \tag{17}$$

$$(\phi+\beta+\lambda)(\beta+\lambda)(-(\pi+\beta)-\lambda)=0$$

Or.

$$[A]=\begin{bmatrix}-(q+\mu+\beta)-\lambda & -k\frac{(\phi+\beta-c\beta)Pe^{-\beta\tau}}{(\phi+\beta)\beta}\\ \mu & -(\psi+\eta+\beta)-\lambda\end{bmatrix}=0$$

We get the eigen values$\lambda_1=-(\phi+\beta)$,$\lambda_2=-\beta$ and$\lambda_3=-(\pi+\beta)$. The DFE is asymptotically stable, if trace of the matrix A is less than 0 and determinant of the matrix A is greater than 0.

Here $trace(A)=-(q+\mu+\beta+\lambda)-(\psi+\beta+\eta+\lambda)$ and.

We can see that$Trace(A)<0$. To show that$|A|>0$, we must have,

$$(q+\mu+\beta+\lambda)(\psi+\beta+\eta+\lambda)-\mu k\frac{(\phi+\beta-c\beta)Pe^{-\beta\tau}}{(\phi+\beta)\beta}>0,$$

$$(q+\mu+\beta+\lambda)(\psi+\beta+\eta+\lambda)>\mu k\frac{(\phi+\beta-c\beta)Pe^{-\beta\tau}}{(\phi+\beta)\beta}.$$

$$(q+\mu+\beta+\lambda)>\mu k\frac{(\phi+\beta-c\beta)Pe^{-\beta\tau}}{\beta(\phi+\beta)(\psi+\beta+\eta+\lambda)}. \tag{18}$$

2.3.2. Local stability of EE point

To analyze the local stability at EEP, $E^*\left(\widehat{M}^*, \widehat{S}^*, \widehat{L}^*, \widehat{I}^*, \widehat{R}^*\right)$. we considered.

$$J=\begin{bmatrix} -(\phi+\beta) & 0 & 0 & 0 & 0 \\ \phi & -\left(kI^0+\beta\right) & 0 & -kS^0 & \pi \\ 0 & kI^0 & -(q+\mu+\beta) & kS^0e^{-\beta\tau} & 0 \\ 0 & 0 & \mu & -(\psi+\eta+\beta) & 0 \\ 0 & 0 & q & \psi & -(\pi+\beta) \end{bmatrix}$$

$$|J_0-\lambda I|=\begin{bmatrix} -(\phi+\beta)-\lambda & 0 & 0 & 0 & 0 \\ \phi & -\left(kI^0+\beta\right)-\lambda & 0 & -kS^0 & \pi \\ 0 & kI^0 & -(q+\mu+\beta)-\lambda & kS^0e^{-\beta\tau} & 0 \\ 0 & 0 & \mu & -(\psi+\eta+\beta)-\lambda & 0 \\ 0 & 0 & q & \psi & -(\pi+\beta)-\lambda \end{bmatrix}$$

$$|J_0-\lambda I|=\begin{bmatrix} -a-\lambda & 0 & 0 & 0 & 0 \\ \phi & -b-\lambda & 0 & c & \pi \\ 0 & d & -e-\lambda & f & 0 \\ 0 & 0 & \mu & -g-\lambda & 0 \\ 0 & 0 & q & \psi & -h-\lambda \end{bmatrix}$$

$$\lambda=-a<0$$

$$|J_0-\lambda I|=\begin{bmatrix} -b-\lambda & 0 & c & \pi \\ d & -e-\lambda & f & 0 \\ 0 & \mu & -g-\lambda & o \\ 0 & q & \psi & -h-\lambda \end{bmatrix},$$

$$|J_0-\lambda I|=-(b+\lambda)\begin{vmatrix} -e-\lambda & f & 0 \\ \mu & -g-\lambda & 0 \\ q & \psi & -h-\lambda \end{vmatrix} - d\begin{vmatrix} 0 & c & \pi \\ \mu & -g-\lambda & 0 \\ q & \psi & -h-\lambda \end{vmatrix},$$

$$|J_0-\lambda I|=(b+\lambda)(h+\lambda)(e+\lambda)(g+\lambda)-\mu f(b+\lambda)(h+\lambda) - d\mu\{ch+c\lambda+\psi\pi\}-dq\pi g-dq\pi\lambda,$$

$$|J_0-\lambda I|=\lambda^4+\lambda^3(b+h+e+g) +\lambda^2(bh+be+bg+he+hg+eg-\mu f) +\lambda(bhe+bhg+beg+heg-\mu fb-\mu fh-d\mu c-dq\pi) +(bheg-\mu fbh-d\mu ch-d\mu\pi\psi-dq\pi g).$$

$$|J_0-\lambda I|=0,$$

$$a_4\lambda^4+a_3\lambda^3+a_2\lambda^2+a_1\lambda+a_0=0,$$

$$a_4=1,\ \ a_3=(b+h+e+g),$$
$$a_2=(bh+be+bg+he+hg+eg-\mu f),$$
$$a_1=(bhe+bhg+beg+heg-\mu fb-\mu fh-d\mu c-dq\pi),$$
$$a_0=(bheg-\mu fbh-d\mu ch-d\mu\pi\psi-dq\pi g).$$

2.3.3. Global stability at DFE point

We considered the Eqs. (1) to (5) of the MSLIR model for global stability at the DFE point. Let Lyapunov function of a given system at DFE $E^0\left(\widehat{M}^0, \widehat{S}^0, \widehat{L}^0, \widehat{I}^0, \widehat{R}^0\right)=\left(\frac{cP}{\phi+\beta}, \frac{(\phi+\beta-c\beta)P}{(\phi+\beta)\beta}, 0, 0, 0\right)$ be:

$$\widehat{N}=\left(\widehat{M}-\widehat{M}^0-\widehat{M}^0\log\frac{\widehat{M}}{\widehat{M}^0}\right)+\left(\widehat{S}-\widehat{S}^0-\widehat{S}^0\log\frac{\widehat{S}}{\widehat{S}^0}\right) +\widehat{L}+\widehat{I}+\widehat{R},$$

$$\frac{d\widehat{N}}{dt}=\left(\frac{\widehat{M}-\widehat{M}^0}{\widehat{M}}\right)\widehat{M}'+\left(\frac{\widehat{S}-\widehat{S}^0}{\widehat{S}}\right)\widehat{S}'+\widehat{L}'+\widehat{I}'+\widehat{R}',$$

$$\frac{d\widehat{N}}{dt}=\left(\widehat{M}-\widehat{M}^0\right)\left(\frac{cP}{\widehat{M}}-\phi-\beta\right)+\left(\widehat{S}-\widehat{S}^0\right) \left(\frac{(1-c)P}{\widehat{S}}+\phi\frac{\widehat{M}}{\widehat{S}}+\pi\frac{\widehat{R}}{\widehat{S}}-\left(k\widehat{I}+\beta\right)\right)+\left(k\widehat{I}\,\widehat{S}e^{-\beta\tau}-q\widehat{L}-\mu\widehat{L}-\beta\widehat{L}\right) +\left(\mu\widehat{L}-\psi\widehat{I}-\eta\widehat{I}-\beta\widehat{I}\right)+\left(q\widehat{L}+\psi\widehat{I}-\pi\widehat{R}-\beta\widehat{R}\right),$$

$$\frac{d\widehat{N}}{dt}=\frac{-cP\left(\widehat{M}-\widehat{M}^0\right)^2}{\widehat{M}\,\widehat{M}^0}-\frac{\left(\widehat{S}-\widehat{S}^0\right)^2}{\widehat{S}\,\widehat{S}^0}\left((1-c)P+\phi\widehat{M}+\pi\widehat{R}\right) +0+0+0.$$

It is clear from above that $\frac{d\widehat{N}}{dt}\leqslant 0$ for $R_0<1$, and $\frac{d\widehat{N}}{dt}=0$, only if $\widehat{M}=\widehat{M}^0$, $\widehat{S}=\widehat{S}^0$, $\widehat{L}=0$, $\widehat{I}=0$ and $\widehat{R}=0$.

2.3.4. Global stability at EE point

We again considered the Eqs. (1) to (5) of the MSLIR model. Let Lyapunov function of the given system at EE point $E^*\left(\widehat{M}^*, \widehat{S}^*, \widehat{L}^*, \widehat{I}^*, \widehat{R}^*\right)$ be:

$$\widehat{N}=\left(\widehat{M}-\widehat{M}^*-\widehat{M}^*\log\frac{\widehat{M}}{\widehat{M}^*}\right)+\left(\widehat{S}-\widehat{S}^*-\widehat{S}^*\log\frac{\widehat{S}}{\widehat{S}^*}\right) +\left(\widehat{L}-\widehat{L}^*-\widehat{L}^*\log\frac{\widehat{L}}{\widehat{L}^*}\right)+\left(\widehat{I}-\widehat{I}^*-\widehat{I}^*\log\frac{\widehat{I}}{\widehat{I}^*}\right) +\left(\widehat{R}-\widehat{R}^*-\widehat{R}^*\log\frac{\widehat{R}}{\widehat{R}^*}\right),$$

$$\frac{d\widehat{N}}{dt}=\left(\frac{\widehat{M}-\widehat{M}^*}{\widehat{M}}\right)\widehat{M}'+\left(\frac{\widehat{S}-\widehat{S}^*}{\widehat{S}}\right)\widehat{S}'+\left(\frac{\widehat{L}-\widehat{L}^*}{\widehat{L}}\right)\widehat{L}' +\left(\frac{\widehat{I}-\widehat{I}^*}{\widehat{I}}\right)\widehat{I}'+\left(\frac{\widehat{R}-\widehat{R}^*}{\widehat{R}}\right)\widehat{R}',$$

$$\frac{d\widehat{N}}{dt}=\left(\widehat{M}-\widehat{M}^*\right)\left(\frac{cP}{\widehat{M}}-\phi-\beta\right)+\left(\widehat{S}-\widehat{S}^*\right)\left(\frac{(1-c)P}{\widehat{S}}+\phi\frac{\widehat{M}}{\widehat{S}}+\pi\frac{\widehat{R}}{\widehat{S}}-\left(k\widehat{I}+\beta\right)\right)+\left(\widehat{L}-\widehat{L}^*\right)\left(\frac{k\widehat{I}\,\widehat{S}e^{-\beta\tau}}{\widehat{L}}-q-\mu-\beta\right)+\left(\widehat{I}-\widehat{I}^*\right) \left(\frac{\mu\widehat{L}}{\widehat{I}}-\psi-\eta-\beta\right)+\left(\widehat{R}-\widehat{R}^*\right)\left(\frac{q\widehat{L}}{\widehat{R}}+\frac{\psi\widehat{I}}{\widehat{R}}-\pi-\beta\right),\ \frac{d\widehat{N}}{dt}=\frac{-cP\left(\widehat{M}-\widehat{M}^*\right)^2}{\widehat{M}\,\widehat{M}^*}-\frac{\left(\widehat{S}-\widehat{S}^*\right)^2}{\widehat{S}\,\widehat{S}^*}\left((1-c)P+\phi\widehat{M}+\pi\widehat{R}\right)+\frac{\left(\widehat{L}-\widehat{L}^*\right)^2}{\widehat{L}} \widehat{L}^*\left(\widehat{L}\ \widehat{I}\ \widehat{S}e^{-\beta\tau}\right)+\frac{\left(\widehat{I}-\widehat{I}^*\right)\mu\widehat{L}}{\widehat{I}\ \widehat{I}^*}+\frac{\left(\widehat{R}-\widehat{R}^*\right)^2}{\widehat{R}\,\widehat{R}^*}\left(q\widehat{L}-\psi\widehat{I}\right),\frac{d\widehat{N}}{dt}\leqslant 0,$$

Obviously, for $R_0<1$, and $\frac{d\widehat{N}}{dt}=0$, only if $\widehat{M}=\widehat{M}^*$, $\widehat{S}=\widehat{S}^*$, $\widehat{L}=\widehat{L}^*$, $\widehat{I}=\widehat{I}^*$ and $\widehat{R}=\widehat{R}^*$. It is concluded that the assumed system is globally asymptotically stable via Lasalle's invariance principle.

3. Numerical modelling of delayed MSLIR on HBV

For the last few decades, great interest has been observed in treating the delay differential equation numerically. This interest is the flexibility in the procedure of mathematical modelling and its application in various fields [25–27]. Some numerical methods are Picard's method, the Predictor-corrector method, Taylor series, forward Euler method, Runge Kutta method, NSFD, etc. In this research, we will only compare our results obtained by using NSFD scheme with forward Euler and RK-4 schemes respectively.

The discretization of the variable $t \geq 0$ at point $t_n = nh(n = 0, 1, 2, 3, \cdots,)$ and $h > 0$ with constant step size. $M(t_n)$, $S(t_n)$, $L(t_n)$, $I(t_n)$ and $R(t_n)$ are the solutions of the Eqs. (1)–(5) at $t = t_n$. Solutions at t_n of the numerical methods are denoted by M^n, S^n, L^n, I^n and R^n respectively. The construction of the following methods are based on first order approximations.

$\frac{dM(t)}{dt} = \frac{M(t+h)-M(t)}{h} + O(h)$ as.$h \to 0$
$\frac{dS(t)}{dt} = \frac{S(t+h)-S(t)}{h} + O(h)$ as.$h \to 0$
$\frac{dL(t)}{dt} = \frac{L(t+h)-L(t)}{h} + O(h)$ as.$h \to 0$
$\frac{dI(t)}{dt} = \frac{I(t+h)-I(t)}{h} + O(h)$ as.$h \to 0$
$\frac{dR(t)}{dt} = \frac{R(t+h)-R(t)}{h} + O(h)$ as $h \to 0$.

3.1. Forward Euler's scheme

To develop an explicit Euler technique, it is enough to consider the Eqs. (2) and (3) of the MSLIR model.

$$\frac{\widehat{S}^{n+1} - \widehat{S}^n}{h} = (1-c)\ P + \phi\widehat{M}^n(t) + \pi\widehat{R}^n(t) - \left(k\,\widehat{I}(t) + \beta\right)\widehat{S}^n(t), \tag{19}$$

$$\frac{\widehat{L}^{n+1} - \widehat{L}^n}{h} = k\widehat{I}^n(t-\tau)\widehat{S}^n(t-\tau)e^{-\beta\tau} - (q+\mu+\beta)\widehat{L}^n(t), \tag{20}$$

After some simplification, we obtain.

$$\widehat{S}^{n+1} = \widehat{S}^n + h\left[(1-c)\,P + \phi\widehat{M}^n(t) + \pi\widehat{R}^n(t) - \left(k\,\widehat{I}(t) + \beta\right)\widehat{S}^n(t)\right], \tag{21}$$

$$\widehat{L}^{n+1} = \widehat{L}^n + h\left[k\widehat{I}^n(t-\tau)\widehat{S}^n(t-\tau)e^{-\beta\tau} - (q+\mu+\beta)\widehat{L}^n(t)\right], \tag{22}$$

3.2. Runga Kutta (RK-4) scheme

To develop an explicit RK-4 technique we need to consider the all equations of MSLIR model.

$$\widehat{M}^{n+1} - \widehat{M}^n = h\left(cP - (\phi+\beta)\widehat{M}^n(t)\right), \tag{23}$$

$$\widehat{S}^{n+1} - \widehat{S}^n = h\left((1-c)\,P + \phi\widehat{M}^n(t) + \pi\widehat{R}^n(t) - \left(k\,\widehat{I}(t) + \beta\right)\widehat{S}^n(t)\right), \tag{24}$$

$$\widehat{L}^{n+1} - \widehat{L}^n = h\left(k\widehat{I}^n(t-\tau)\widehat{S}^n(t-\tau)e^{-\beta\tau} - (q+\mu+\beta)\widehat{L}^n(t)\right), \tag{25}$$

$$\widehat{I}^{n+1} - \widehat{I}^n = h\left(\mu\widehat{L}^n(t) - (\psi+\eta+\beta)\widehat{I}^n(t)\right), \tag{26}$$

$$\widehat{R}^{n+1} - \widehat{R}^n = h\left(q\widehat{L}^n(t) + \psi\widehat{I}^n(t) - (\pi+\beta)\widehat{R}^n(t)\right). \tag{27}$$

We can write it as,

$$k_1 = h\left(cP - (\phi+\beta)\widehat{M}^n(t)\right), \tag{28}$$

$$m_1 = h\left((1-c)\,P + \phi\widehat{M}^n(t) + \pi\widehat{R}^n(t) - \left(k\,\widehat{I}(t) + \beta\right)\widehat{S}^n(t)\right), \tag{29}$$

$$n_1 = h\left(k\widehat{I}^n(t-\tau)\widehat{S}^n(t-\tau)e^{-\beta\tau} - (q+\mu+\beta)\widehat{L}^n(t)\right), \tag{30}$$

$$p_1 = h\left(\mu\widehat{L}^n(t) - (\psi+\eta+\beta)\widehat{I}^n(t)\right), \tag{31}$$

$$q_1 = h\left(q\widehat{L}^n(t) + \psi\widehat{I}^n(t) - (\pi+\beta)\widehat{R}^n(t)\right). \tag{32}$$

$$k_2 = h\left(cP - (\phi+\beta)\widehat{M}^n + \frac{k_1}{2}\right), \tag{33}$$

$$m_2 = h\left((1-c)P + \phi\left(\widehat{M}^n + \frac{k_1}{2}\right) + \pi\left(\widehat{R}^n + \frac{q_1}{2}\right) - \left(k\left(\widehat{I} + \frac{p_1}{2}\right) + \beta\right)\left(\widehat{S}^n + \frac{m_1}{2}\right)\right), \tag{34}$$

$$n_2 = h\left(k\left(\widehat{I}^n + \frac{p_1}{2}\right)\left(\widehat{S}^n + \frac{m_1}{2}\right)e^{-\beta\tau} - (q+\mu+\beta)\left(\widehat{L}^n + \frac{n_1}{2}\right)\right), \tag{35}$$

$$p_2 = h\left(\mu\left(\widehat{L}^n + \frac{n_1}{2}\right) - (\psi+\eta+\beta)\left(\widehat{I}^n + \frac{p_1}{2}\right)\right), \tag{36}$$

$$q_2 = h\left(q\left(\widehat{L}^n + \frac{n_1}{2}\right) + \psi\left(\widehat{I}^n + \frac{p_1}{2}\right) - (\pi+\beta)\left(\widehat{R}^n + \frac{q_1}{2}\right)\right). \tag{37}$$

$$k_3 = h\left(cP - (\phi+\beta)\widehat{M}^n + \frac{k_2}{2}\right), \tag{38}$$

$$m_3 = h\left((1-c)P + \phi\left(\widehat{M}^n + \frac{k_2}{2}\right) + \pi\left(\widehat{R}^n + \frac{q_2}{2}\right) - \left(k\left(\widehat{I} + \frac{p_2}{2}\right) + \beta\right)\left(\widehat{S}^n + \frac{m_2}{2}\right)\right), \tag{39}$$

$$n_3 = h\left(k\left(\widehat{I}^n + \frac{p_2}{2}\right)\left(\widehat{S}^n + \frac{m_2}{2}\right)e^{-\beta\tau} - (q+\mu+\beta)\left(\widehat{L}^n + \frac{n_2}{2}\right)\right), \tag{40}$$

$$p_3 = h\left(\mu\left(\widehat{L}^n + \frac{n_2}{2}\right) - (\psi+\eta+\beta)\left(\widehat{I}^n + \frac{p_2}{2}\right)\right), \tag{41}$$

$$q_3 = h\left(q\left(\widehat{L}^n + \frac{n_2}{2}\right) + \psi\left(\widehat{I}^n + \frac{p_2}{2}\right) - (\pi+\beta)\left(\widehat{R}^n + \frac{q_2}{2}\right)\right). \tag{42}$$

$$k_4 = h\left(cP - (\phi+\beta)\widehat{M}^n + k_3\right), \tag{43}$$

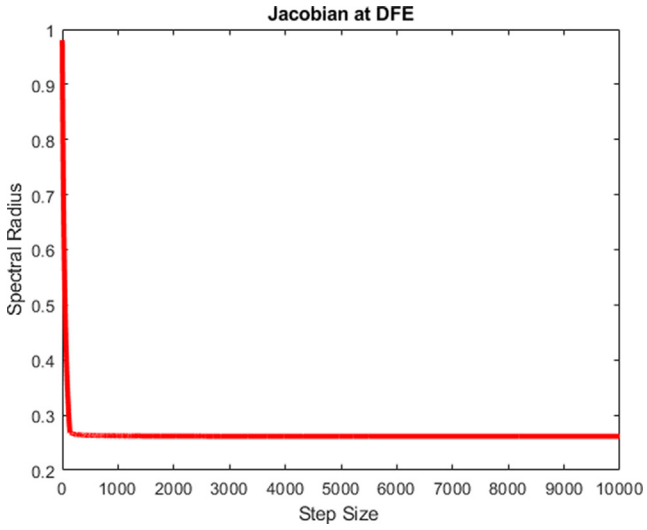

Fig. 2 Spectral radius of the matrix A.

$$m_4 = h\Big((1-c)P + \phi\Big(\widehat{M}^n + k_3\Big) + \pi(\widehat{R}^n + q_3) - \Big(k\Big(\widehat{I} + p_3\Big) + \beta\Big)\ \Big(\widehat{S}^n + m_2\Big)\ \Big), \tag{44}$$

$$n_4 = h\Big(k\Big(\widehat{I}^n + p_3\Big)\Big(\widehat{S}^n + m_3\Big)e^{-\beta\tau} - (q+\mu+\beta)\Big(\widehat{L}^n + n_3\Big)\Big), \tag{45}$$

$$p_4 = h\Big(\mu\Big(\widehat{L}^n + n_3\Big) - (\psi+\eta+\beta)\Big(\widehat{I}^n + p_3\Big)\Big), \tag{46}$$

$$q_4 = h\Big(q\Big(\widehat{L}^n + n_3\Big) + \psi\Big(\widehat{I}^n + p_3\Big) - (\pi+\beta)\Big(\widehat{R}^n + q_3\Big)\Big). \tag{47}$$

$$\widehat{M}^{n+1} = \widehat{M}^n + \frac{1}{6}\ [k_1 + 2k_2 + 2k_3 + k_4], \tag{48}$$

$$\widehat{S}^{n+1} = \widehat{S}^n + \frac{1}{6}\ [m_1 + 2m_2 + 2m_3 + m_4], \tag{49}$$

$$\widehat{L}^{n+1} = \widehat{L}^n + \frac{1}{6}\ [n_1 + 2n_2 + 2n_3 + n_4], \tag{50}$$

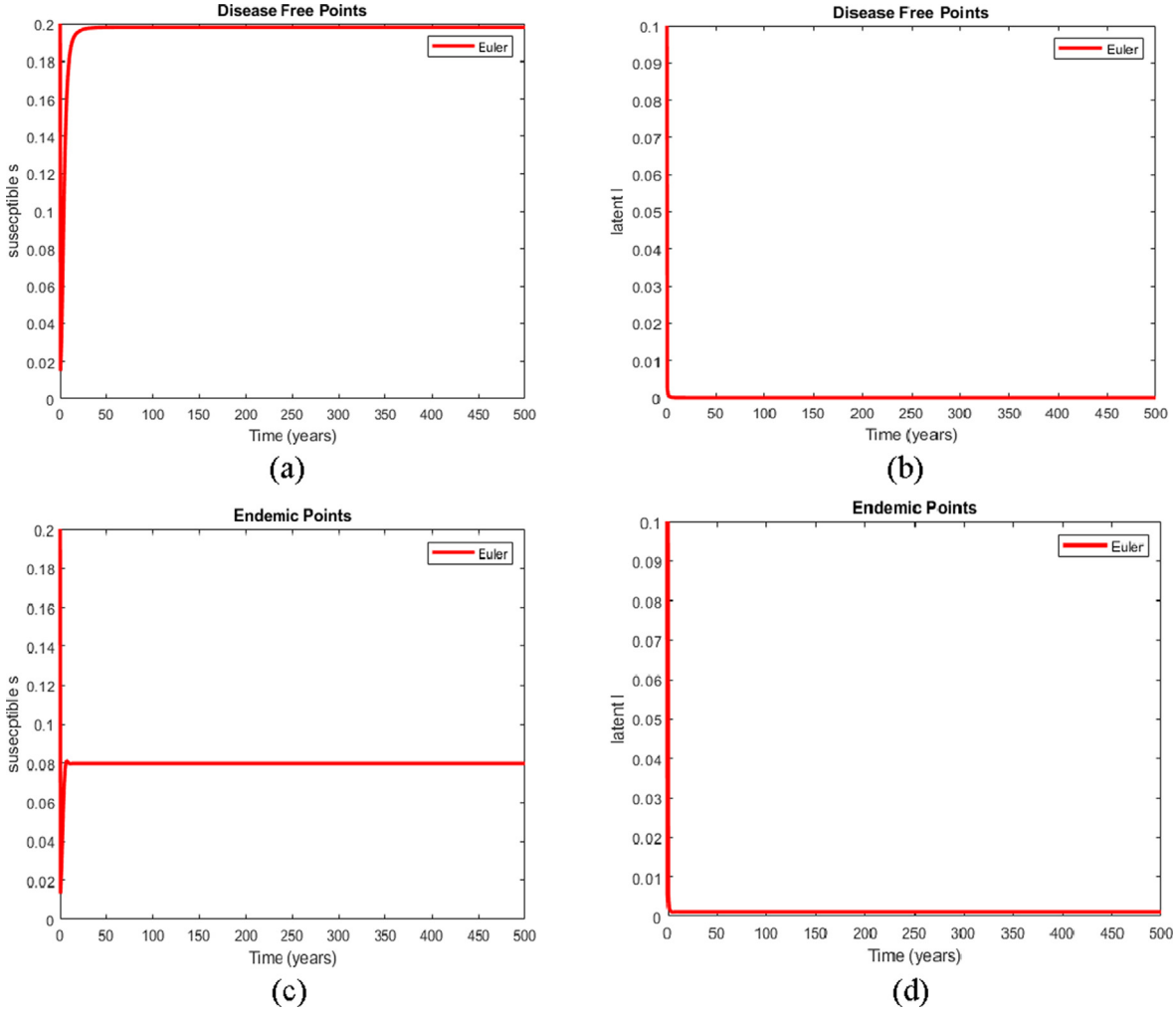

Fig. 3 Subpopulations using Euler at h = 0.001.

$$\widehat{I}^{n+1} = \widehat{I}^{n} + \frac{1}{6}\,[p_1 + 2p_2 + 2p_3 + p_4], \tag{51}$$

$$\widehat{R}^{n+1} = \widehat{R}^{n} + \frac{1}{6}\,[q_1 + 2q_2 + 2q_3 + q_4], \tag{52}$$

3.3. NSFD scheme

In numerical analysis, NSFD scheme is used to produce a general set of methods to find the numerical solution of ODE's by constructing a brief model [28]. To develop an explicit NSFD method we just need to consider the Eqs. (2) and (3) of MSLIR model.

$$\widehat{M}^{n+1} = \widehat{M}^{n} + h\Big(cP - (\phi+\beta)\widehat{M}^{n+1}\Big), \quad \widehat{M}^{n+1} = \frac{\widehat{M}^{n}+hcP}{1+h(\phi+\beta)}, \tag{53}$$

$$\widehat{S}^{n+1} = \widehat{S}^{n} + h\Big[(1-c)P + \phi\widehat{M}^{n}(t) + \pi\widehat{R}^{n}(t) - \Big(k\,\widehat{I}(t) + \beta\Big)\widehat{S}^{n+1}(t)\Big], \ \widehat{S}^{n+1} = \frac{\widehat{S}^{n}+h\Big[(1-c)P+\phi\widehat{M}^{n}+\pi\widehat{R}^{n}\Big]}{1+h\Big(k\,\widehat{I}^{n}+\beta\Big)}, \tag{54}$$

$$\widehat{L}^{n+1} = \widehat{L}^{n} + h\Big[k\,\widehat{I}^{n}\widehat{S}^{n}e^{-\beta\tau} - (q+\mu+\beta)\widehat{L}^{n+1}\Big], \quad \widehat{L}^{n+1} = \frac{\widehat{L}^{n}+hk\,\widehat{I}^{n}\widehat{S}^{n}e^{-\beta\tau}}{1+h(q+\mu+\beta)}. \tag{55}$$

$$\widehat{I}^{n+1} = \widehat{I}^{n} + h\Big(\mu\widehat{L}^{n}(t) - (\psi+\eta+\beta)\widehat{I}^{n+1}\Big), \quad \widehat{I}^{n+1} = \frac{\widehat{I}^{n}+h\mu\widehat{L}^{n}}{1+h(\psi+\eta+\beta)}, \tag{56}$$

$$\widehat{R}^{n+1} = \widehat{R}^{n} + h\Big(q\widehat{L}^{n}(t) + \psi\,\widehat{I}^{n}(t) - (\pi+\beta)\widehat{R}^{n+1}\Big). \quad \widehat{R}^{n+1} = \frac{\widehat{R}^{n}+h\Big(q\widehat{L}^{n}(t)+\psi\,\widehat{I}^{n}(t)\Big)}{1+h(\pi+\beta)}. \tag{57}$$

3.4. Convergence analysis of NSFD

Convergence analysis of proposed NSFD scheme of delayed MSLIR model is performed at a DFE point $E^0\Big(\widehat{M}^0, \widehat{S}^0, \widehat{L}^0, \widehat{I}^0, \widehat{R}^0\Big) = \Big(\frac{cP}{\phi+\beta}, \frac{(\phi+\beta-c\beta)P}{(\phi+\beta)\beta}, 0, 0, 0\Big)$.

By using Eqs. (52) to (57), the Jacobian matrix is:

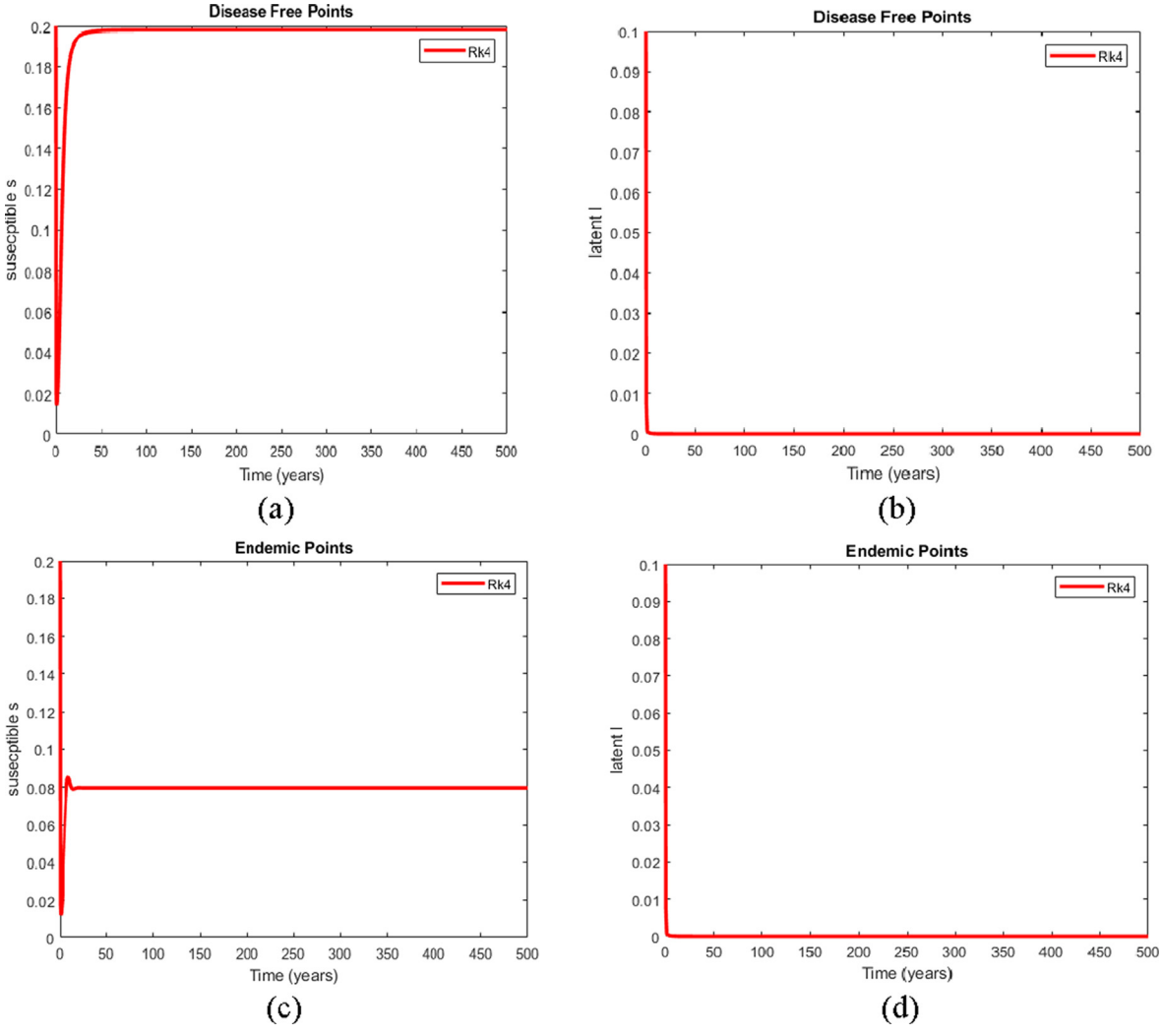

Fig. 4 Subpopulations using RK-4 at h = 0.001.

$$j=\begin{pmatrix} \frac{\partial E}{\partial \widehat{M}} & \frac{\partial E}{\partial \widehat{S}} & \frac{\partial E}{\partial \widehat{L}} & \frac{\partial E}{\partial \widehat{I}} & \frac{\partial E}{\partial \widehat{R}} \\ \frac{\partial F}{\partial \widehat{M}} & \frac{\partial F}{\partial \widehat{S}} & \frac{\partial F}{\partial \widehat{L}} & \frac{\partial F}{\partial \widehat{I}} & \frac{\partial F}{\partial \widehat{R}} \\ \frac{\partial G}{\partial \widehat{M}} & \frac{\partial G}{\partial \widehat{S}} & \frac{\partial G}{\partial \widehat{L}} & \frac{\partial G}{\partial \widehat{I}} & \frac{\partial G}{\partial \widehat{R}} \\ \frac{\partial H}{\partial \widehat{M}} & \frac{\partial H}{\partial \widehat{S}} & \frac{\partial H}{\partial \widehat{L}} & \frac{\partial H}{\partial \widehat{I}} & \frac{\partial H}{\partial \widehat{R}} \\ \frac{\partial J}{\partial \widehat{M}} & \frac{\partial J}{\partial \widehat{S}} & \frac{\partial J}{\partial \widehat{L}} & \frac{\partial J}{\partial \widehat{I}} & \frac{\partial J}{\partial \widehat{R}} \end{pmatrix}$$

At DFE points, it becomes,

$$j=\begin{pmatrix} \frac{1}{1+h(\phi+\beta)} & 0 & 0 & 0 & 0 \\ \frac{h\phi}{1+h\beta} & \frac{1}{1+h\beta} & 0 & \frac{P(1+h)(\phi+\beta-c\beta)}{(\phi+\beta)(1+h\beta)^2} & \frac{h\pi}{1+h\beta} \\ 0 & 0 & \frac{1}{1+h(q+\mu+\beta)} & \frac{hkP(\phi+\beta-c\beta)e^{-\beta\tau}}{\beta(\phi+\beta)(1+h(q+\mu+\beta))} & 0 \\ 0 & 0 & \frac{1}{1+h(\psi+\eta+\beta)} & \frac{1}{1+h(\psi+\eta+\beta)} & 0 \\ 0 & 0 & \frac{hq}{1+h(\pi+\beta)} & \frac{-h\psi}{1+h(\pi+\beta)} & \frac{1}{1+h(\pi+\beta)} \end{pmatrix}$$

$$=\left(\frac{1}{1+h(\phi+\beta)}-\lambda\right)\left(\frac{1}{1+h\beta}-\lambda\right)\left(\frac{1}{1+h(\pi+\beta)}-\lambda\right)$$

$$\begin{vmatrix} \frac{1}{1+h(q+\mu+\beta)}-\lambda & \frac{hkP(\phi+\beta-c\beta)e^{-\beta\tau}}{\beta(\phi+\beta)(1+h(q+\mu+\beta))} \\ \frac{1}{1+h(\psi+\eta+\beta)} & \frac{1}{1+h(\psi+\eta+\beta)}-\lambda \end{vmatrix}$$

Let $A=\begin{vmatrix} \frac{1}{1+h(q+\mu+\beta)}-\lambda & \frac{hkP(\phi+\beta-c\beta)e^{-\beta\tau}}{\beta(\phi+\beta)(1+h(q+\mu+\beta))} \\ \frac{1}{1+h(\psi+\eta+\beta)} & \frac{1}{1+h(\psi+\eta+\beta)}-\lambda \end{vmatrix}$.

The proposed NSFD method will be unconditionally convergent if all eigenvalues of the Jacobian matrix A are not greater or equal to 1. We demonstrate this graphically by plotting the spectral radius with the help of MATLAB in Fig. 2.

4. Results and discussions

Numerical investigations are achieved by using the values given in the table below.

Figs. 3, 4, and 5 are plotted for susceptible and latently infected populations by forward Euler, RK-4 and the proposed NSFD schemes at both DFE and endemic EE points. At $h = 0.001$, it can be clearly seen that all of above numerical methods are converging towards steady states. It can also be seen from Fig. 6 that the forward Euler and RK-4 are failed to give accurate solution even at very small step sizes, i.e. $h = 0.2$. On the other hand, the proposed NSFD method gives the convergent solution even at a very large step size i.e. $h = 500$ which is shown in Fig. 7. Conventional standard difference methods in the literature can cause chaos and misleading

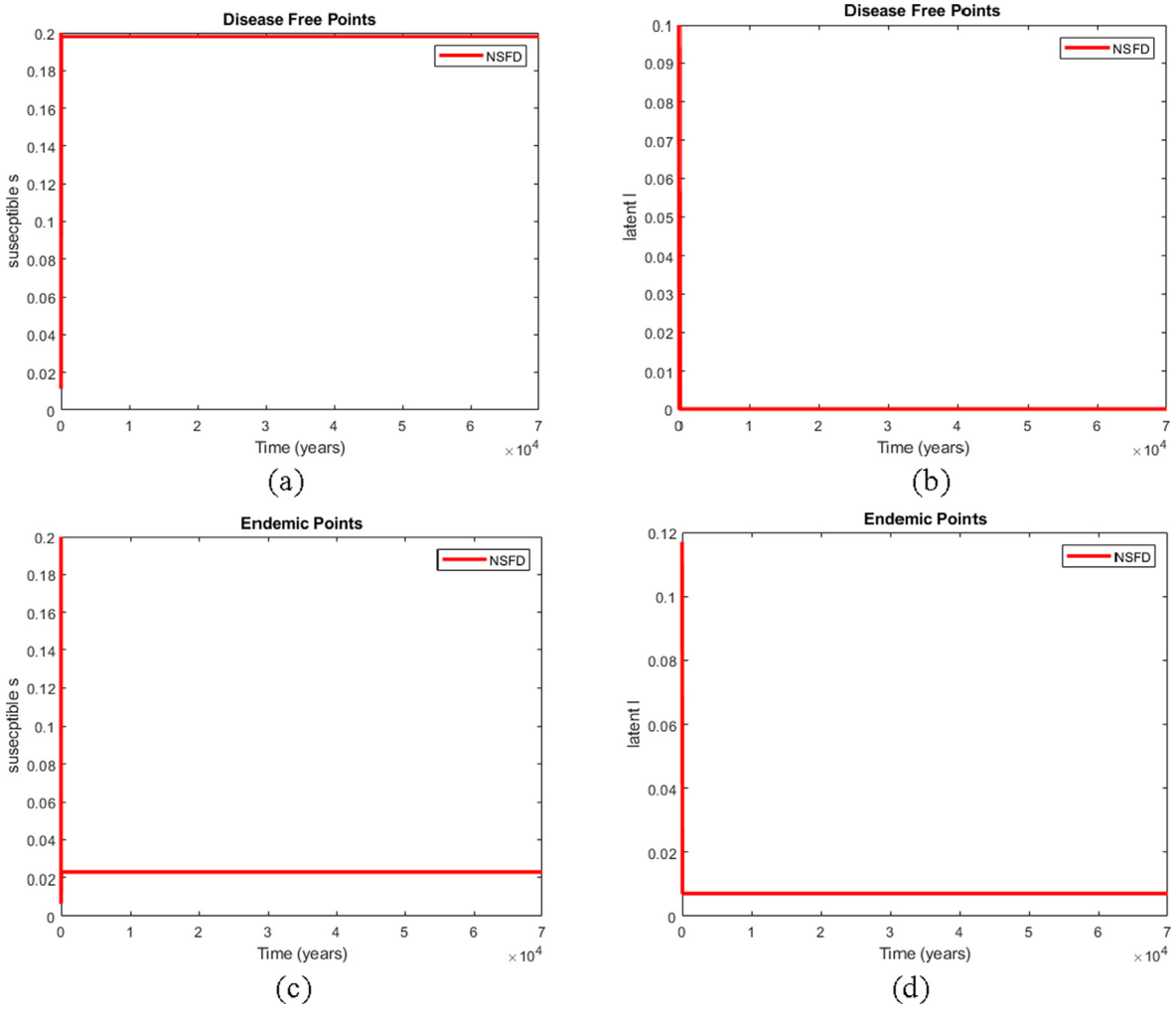

Fig. 5 Subpopulations using NSFD at h = 0.001.

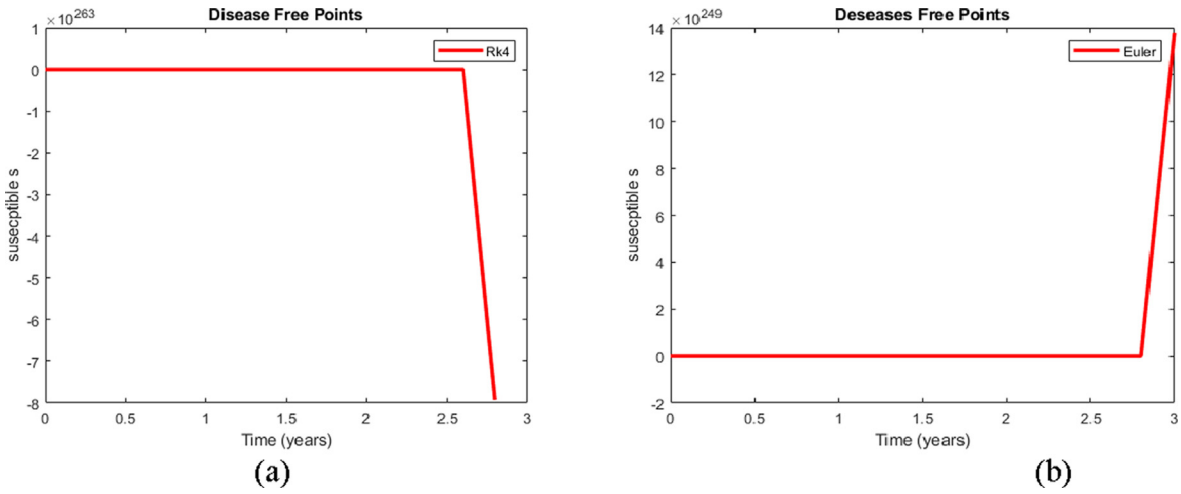

Fig. 6 Susceptible compartment at $h = 0.2$ using RK-4 and Euler schemes.

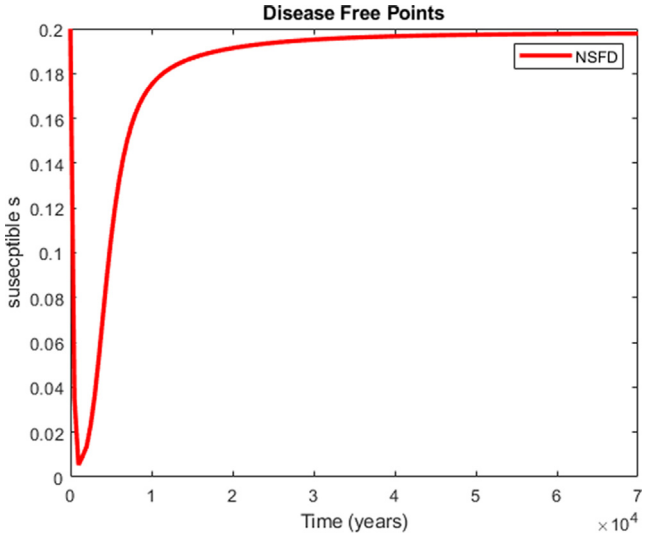

Fig. 7 Susceptible compartment at $h = 0.2$ using NSFD scheme $h = 500$.

ing hesitations for some passions of discretization constraints. Many classical methods such as Euler, Stochastic Euler, RK-4 and Stochastic RK-4 etc. do not keep it at comparatively large time step values [29,30]. The NSFD method remains convergent and it is independent to the increase in the value of the step size.

5. Conclusions

In this research work, a reliable numerical solution of an HBV model is presented with the help of NSFD method. It is found that the delay factor has a significant impact on the transmission of hepatitis B virus. Equilibrium analysis are performed for the studied model.

Local and global stability of the DFE and EE points are also discussed. Euler, RK-4 and NSFD schemes are constructed for the numerical solution of the model. The convergence of the NSFD at DFE point is also examined. The NSFD method retains all important characteristics possessed by the HBV epidemic model, which shows its efficacy. The relation of the proposed method is made with the forward Euler method and RK-4 method. From stability analysis, it is concluded that if $R_0 < 1$ then disease dies as well as infection dies and if $R_0 > 1$ then the spread of the disease will continue in the population. From the simulations, it is concluded that the two well-known classical techniques do not provide an accurate solution even for small steps, while our proposed method gives reliable solutions at all step sizes and it is independent of the step size. Hence the proposed scheme becomes one of the best choices of all other classical finite difference schemes. Delayed, stochastic, fractional and fuzzy extension of the current work are some of the future directions.

Author Contributions

All authors contributed equally to this manuscript.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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