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Non-ideal sampling in shift-invariant spaces associated with quadratic-phase Fourier transforms

Waseem Z. Lone [a], Firdous A. Shah [a], Kottakkaran Sooppy Nisar [b,*], Wedad Albalawi [c], B. Alshahrani [d], Choonkil Park [e,*]

[a] *Department of Mathematics, University of Kashmir, South Campus, Anantnag 192101, Jammu and Kashmir, India*
[b] *Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, Wadi Aldawasir 11991, Saudi Arabia*
[c] *Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia*
[d] *Department of Physics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia*
[e] *Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea*

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Abstract Non-ideal sampling has nourished as one of the most attractive alternatives to classical sampling, which relies on shift-invariant spaces. The present study focuses on investigating the non-ideal sampling in shift-invariant spaces associated with the quadratic-phase Fourier transforms. The primary aim is to formulate novel convolution structures in quadratic-phase Fourier domains and invoke such structures to develop the generalized shift-invariant spaces. Moreover, we present the non-ideal sampling procedure via generalised shift-invariant spaces in the quadratic-phase Fourier domains by employing the proposed generalised convolutions.

1. Introduction

While working on the solution of the heat equation, Saitoh [1] obtained an extreme generalization of the classical Fourier transform coined as quadratic-phase Fourier transform (QPFT). Inspired by the work of Saitoh, Castro et al.[2] studied further possibilities for the QPFT by employing a general quadratic function in the exponent of the novel integral transform. The QPFT of $f \in L^2(\mathbb{R})$ with respect to real parameters $\Lambda = (A, B, C, D, E)$, $B \neq 0$, is given by

* Corresponding authors at: Department of Mathematics, College of Arts and Sciences, Prince Sattam bin Abdulaziz University, Wadi Aldawasir 11991, Saudi Arabia (K.S. Nisar). Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea (C. Park).
E-mail addresses: lone.waseem.393@gmail.com (W.Z. Lone), fashah@uok.edu.in (F.A. Shah), n.sooppy@psau.edu.sa (K.S. Nisar), wsalbalawi@pnu.edu.sa (W. Albalawi), bserhany@kku.edu.sa (B. Alshahrani), baak@hanyang.ac.kr (C. Park).
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$$\mathcal{Q}_\Lambda[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) \exp\left\{i\left(At^2 + Bt\omega + C\omega^2 + Dt + E\omega\right)\right\} dt. \quad (1.1)$$

It is worth noticing that (1.1) circumscribes several integral transforms ranging from the classical Fourier to the much recent special affine Fourier transforms [3]. Due to the exitence of free real parameters $\Lambda = (A, B, C, D, E)$, the QPFT is envisaged as a promising tool for investigating the signals whose energy is not well-concentrated within the Fourier domain, for instance, chirp-like signals, which are prevalent in nature. As a generalization of the celebrated Fourier transform, the QPFT gained its ground intermittently and profoundly influenced several disciplines of science and engineering, including harmonic analysis, quantum theory, differential equations, optics, pattern recognition, and so on [4,5].

Shift-invariant spaces (SISs) are regarded as one of the most attractive concepts in digital signal processing and they play a significant role in the framework of harmonic analysis [6]. Shift invariant spaces can be considered as the closed subspaces of $L^2(\mathbb{R})$ such that both the function f and the integer translates of f belong to the same space. Mathematically, the shift-invariant spaces are defined by:

$$\mathscr{V}(f) = \left\{ f(t) = \sum_{n\in\mathbb{Z}} x(n) f(t-n) : f \in L^2(\mathbb{R}), \{x(n)\} \in \ell^2(\mathbb{Z}) \right\}, \quad (1.2)$$

where $f \in L^2(\mathbb{R})$ is called the generator of the space. Sampling in shift-invariant spaces could be considered as representing the bandlimited signal using known generators or bases. The pioneering work on sampling in the framework of shift-invariant spaces is attributed to Bhandari et al.[7], whereas Zhao and his collaborators subsequently extended SISs to the generalized sampling in the fractional Fourier domains [8]. Xiao et al. [9] proposed a procedure for uniform and non-uniform sampling and the reconstruction of finite energy signals in function spaces. Aldroubi et al.[10] introduced the notion of dynamical sampling, which takes into account both the initial signal f and its multiple states at different times. Recently, we introduced an analogue of dynamical sampling in the realm of QPFT and demonstrated that the proposed method offers an effective signal reconstruction from dynamical sampling measures [11]. The aforementioned models follow the classical Shannon sampling theorem, in which an ideal band-limited signal is projected in the space. However, the majority of signals are not truly band-limited. Therefore, our goal is to utilize the generalized SISs to demonstrate the non-ideal sampling for the bandlimited signals whose energy is well concentrated in the QPFT domain.

Keeping recent trends of signal processing in the hindsight, it is both theoretically intriguing and practically beneficial to study the non-ideal sampling in the generalized SISs in the QPFT domain. The strategy adopted for the accomplishment of the objective includes the formulation of novel and elegant convolution structures which are both simple one-dimensional integral expressions and can be easily implemented for non-ideal sampling in the QPFT domain. The primary content of the present study are as under:

- To construct a practically reliable and efficient convolution structure for the QPFT.
- To establish a pair of discrete and semi-discrete convolution structures in the framework of QPFT domains.
- To present a novel generalized SISs in the realm of QPFT domain.
- To study the computationally efficient non-ideal sampling associated with the QPFT.

The main content of the paper is divided into two sections, viz; Section 2 and Section 3. Section 2 presents the formulation of novel convolution structures in the context of the QPFT domains. Section 3 is dedicated to present a generalized shift-invariant spaces for the band-limited signals followed by a thorough study of the non-ideal sampling in the QPFT domains. The article ends with an epilogue in Section 4.

2. Novel Convolution Structures in the Quadratic-Phase Fourier Domain

The sole aim of this section is to gain deeper insights into the notion of convolution structures in the QPFT domain. Primarily, we provide a stimulus to the notion of QPFT, and then formulate the novel convolution structures in the context of QPFT. With minor modifications to (1.1), the QPFT of f is defined as follows:

Definition 2.1. The quadratic-phase Fourier transform $\mathcal{Q}_\Lambda[f](\omega)$ of any square integrable function f with respect to a collection $\Lambda = (A, B, C, D, E)$, $B > 0$, is defined by

$$\mathcal{Q}_\Lambda[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)\, \mathscr{K}_\Lambda(t,\omega)\, dt, \quad (2.1)$$

where $\mathscr{K}(t,\omega)$ represents the QPFT kernel and is given as

$$\mathscr{K}_\Lambda(t,\omega) = \exp\left\{-i\left(At^2 + Bt\omega + C\omega^2 + Dt + E\omega\right)\right\}. \quad (2.2)$$

The inversion formula corresponding to the QPFT defined in (2.1) is given by

$$f(t) = \frac{B}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{Q}_\Lambda[f](\omega)\, \overline{\mathscr{K}_\Lambda(t,\omega)}\, \mathrm{d}\omega. \quad (2.3)$$

Moreover, the Plancheral theorem corresponding to QPFT reads:

$$\langle f_1, f_2 \rangle_2 = B \langle \mathcal{Q}_\Lambda[f_1], \mathcal{Q}_\Lambda[f_2] \rangle_2, \quad \forall f_1, f_2 \in L^2(\mathbb{R}). \quad (2.4)$$

Remark 2.2. By appropriately choosing the parameters $\Lambda = (A, B, C, D, E)$, Definition 2.1 can be transformed to Fourier, fractional Fourier, Fresnal transform and linear canonical transforms.

The notion of convolution is one of the most extensively used concepts in mathematics with applications across diverse fields of signal and image processing, including quantum physics, operator theory, optics, denoising and filter designing [12]. Here, we shall introduce the notion of chirp-free convolution associated with the QPFT, which uphold the classical convolution and product theorems in the ordinary Fourier domain in the sense that, the quadratic-phase Fourier convolution of two functions is equal to the product of their respective quadratic-phase Fourier transforms. However, as we shall demonstrate in the sequel, such a convolution does not satisfy the commutativity and associative properties. Nevertheless, the distributive property holds good and hence, this convolution plays a significant role in the realm of generalized sampling and reconstruction procedures.

Definition 2.3. For any $f_1, f_2 \in L^2(\mathbb{R})$, the convolution operator $\star_\Lambda$ pertaining to QPFT is defined by

$$(f_1 \star_\Lambda f_2)(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_1(t) f_2(z-t)\, e^{iA\left(z^2-t^2\right)} dt. \tag{2.5}$$

Some important characteristics of the quadratic-phase convolution operation (2.5) are presented in the following theorem.

Theorem 2.4. For any trio $f_1, f_2, f_3 \in L^2(\mathbb{R})$ and the scalars $k \in \mathbb{R}$, $\lambda \in \mathbb{R} \setminus \{0\}$, the quadratic-phase convolution operation $\star_\Lambda$ has the following properties:(i). Non-commutative: $(f_1 \star_\Lambda f_2)(z) \neq (f_2 \star_\Lambda f_1)(z)$, (ii). Non-Associative: $((f_1 \star_\Lambda f_2) \star_\Lambda f_3)(z) \neq (f_1 \star_\Lambda (f_2 \star_\Lambda f_3))(z)$, (iii). Distributive: $(f_1 \star_\Lambda (f_2 + f_3))(z) = (f_1 \star_\Lambda f_3)(z) + (f_1 \star_\Lambda f_3)(z)$ (iv). Translation: $(f_1 \star_\Lambda f_2)(z-k) = (f_1(t-k) \star_\Lambda G(t))(z), \quad G(t) = e^{-2iAkt} f_2(t)$, (v). Scaling: $(f_1 \star_\Lambda f_2)(\lambda z) = |\lambda| (f_1(\lambda t) \star_{\Lambda'} f_2(\lambda t))\ (z), \quad \Lambda' = \left(\lambda^2 A, B, C, D, E\right)$, (vi). Parity: $(f_1 \star_\Lambda f_2)(-z) = (f_1(-t) \star_\Lambda f_2(-t))(z)$.

Proof. (i) In order to demonstrate that the quadratic-phase convolution operation $\star_\Lambda$ is non-commutative, we proceed as

$$\begin{aligned}(f_2 \star_\Lambda f_1)(z) &= \textstyle\int_{\mathbb{R}} f_2(t) f_1(z-t)\, e^{iA\left(z^2-t^2\right)} dt \\ &= \textstyle\int_{\mathbb{R}} f_2(z-x) f_1(x)\, e^{iA\left(z^2-(z-x)^2\right)} dx \\ &= \textstyle\int_{\mathbb{R}} f_2(z-x) f_1(x)\, e^{iA\left(x^2-2xz\right)} dx \\ &\neq (f_1 \star_\Lambda f_2)(z).\end{aligned}$$

(ii) By a straightforward computation, we can show that the quadrtic-phase convolution operation $\star_\Lambda$ is also non-associative.(iii) To examine the distributive property of the quadrtic-phase convolution operation $\star_\Lambda$, we consider a trio of functions $f, g, h \in L^2(\mathbb{R})$ and proceed as

$$\begin{aligned}((f_1+f_2) \star_\Lambda f_3)(z) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (f_1+f_2)(t) f_3(z-t)\, e^{iA\left(z^2-t^2\right)} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_1(t) f_3(z-t)\, e^{iA\left(z^2-t^2\right)} dt \\ &\quad + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_2(t) f_3(z-t)\, e^{iA\left(z^2-t^2\right)} dt \\ &= (f_1 \star_\Lambda f_3)(z) + (f_2 \star_\Lambda f_3)(z).\end{aligned}$$

Similarly, we can show that

$$(f_1 \star_\Lambda (f_2 + f_3))(z) = (f_1 \star_\Lambda f_2)(z) + (f_1 \star_\Lambda f_3)(z).$$

That is, both addition as well as convolution operations are distribute over each other.(iv) For any $k \in \mathbb{R}$, we have

$$\begin{aligned}(f_1 \star_\Lambda f_2)(z-k) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_1(t) f_2(z-k-t)\, e^{iA\left((z-k)^2-t^2\right)} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_1(t) f_2(z-(t+k))\, e^{iA\left(z^2+k^2-2zk-t^2\right)} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_1(x-k) f_2(z-x)\, e^{iA\left(z^2+k^2-2zk-x^2-k^2+2xk\right)} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_1(x-k) f_2(z-x)\, e^{iA\left(z^2-x^2\right)} e^{2iAk(x-z)} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_1(x-k) \left(e^{-2iAk(z-x)} f_2(z-x)\right) e^{iA\left(z^2-x^2\right)} dx \\ &= (f_1(x-k) \star_\Lambda G)(z), \quad G(x) = e^{-2iAkx} g(x).\end{aligned}$$

(v) For $\lambda \in \mathbb{R} \setminus \{0\}$, we observe that

$$\begin{aligned}(f_1 \star_\Lambda f_2)(\lambda z) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_1(t) f_2(\lambda z - t)\, e^{iA\left(\lambda^2 z^2 - t^2\right)} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_1(t) f_2\left(\lambda\left(z - \frac{t}{\lambda}\right)\right) e^{iA\left(\lambda^2 z^2 - t^2\right)} dt \\ &= \frac{|\lambda|}{\sqrt{2\pi}} \int_{\mathbb{R}} f_1(\lambda x) f_2(\lambda(z-x)) e^{iA\lambda^2\left(z^2-x^2\right)} dx \\ &= |\lambda| (f_1(\lambda x) \star_{\Lambda'} f_2(\lambda x))(z), \quad \Lambda' = \left(\lambda^2 A, B, C, D, E\right).\end{aligned}$$

(vi) For $\lambda = -1$, the scaling property (v) yields

$(f_1 \star_\Lambda f_2)(-z) = (f_1(-x) \star_\Lambda f_2(-x))(z)$.

That is, the axis reversal of the convolution corresponds directly to an axis-reversal of the individual functions.

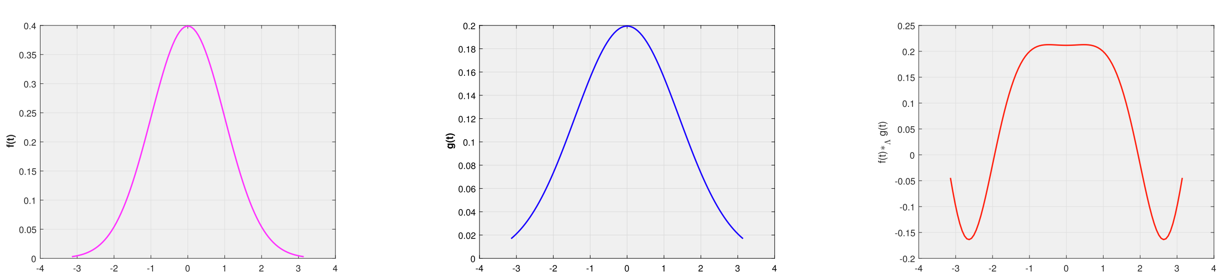
We now illustrate the notion of convolution structure (2.5) via a lucid example.

Example 2.5. Consider the following pair of Gaussian functions:

$$f_1(t) = \frac{1}{\sigma_1 \sqrt{2\pi}}\, e^{-t^2/2\sigma_1^2} \qquad \text{and}$$

$$f_2(t) = \frac{1}{\sigma_2 \sqrt{2\pi}}\, e^{-t^2/2\sigma_2^2}, \qquad \sigma_1, \sigma_2 > 0. \tag{2.6}$$

Using Definition 2.3 and invoking the standard Gaussian integral, we can compute the convolution of the pair of functions defined in (2.6) as



(a) (b) (c)

Fig. 1 (a) Gaussian function $f_1(t)$, (b) Gaussian function $f_2(t)$, (c) Convolution of $f_1(t)$ and $f_2(t)$.

$$(f_1 \star_\Lambda f_2)(z) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{2\pi}} \int_{\mathbb{R}} e^{-t^2/2\sigma_1^2} e^{-(z-t)^2/2\sigma_2^2} e^{iA\left(z^2-t^2\right)} dt$$
$$= \frac{\exp\left\{\left(-\frac{1}{2\sigma_2^2}+iA\right)z^2\right\}}{2\pi\sigma_1\sigma_2\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left\{-\left(\frac{1}{2\sigma_1^2}+\frac{1}{2\sigma_2^2}+iA\right)t^2\right.$$
$$\left.+\left(\frac{z}{\sigma_2^2}\right)t\right\} dt = \frac{\exp\left\{\left(-\frac{1}{2\sigma_2^2}+iA\right)z^2\right\}}{2\pi\sigma_1\sigma_2\sqrt{2\pi}}$$
$$\times\left(\frac{\sqrt{\pi}}{\sqrt{\frac{1}{2\sigma_1^2}+\frac{1}{2\sigma_2^2}+iA}} \exp\left\{\frac{\left(\frac{z}{\sigma_2^2}\right)^2}{4\left(\frac{1}{2\sigma_1^2}+\frac{1}{2\sigma_2^2}+iA\right)}\right\}\right)$$
$$= \frac{1}{2\sqrt{\sigma_1^2+\sigma_2^2+2iA\sigma_1\sigma_2}}$$
$$\exp\left\{-\frac{z^2}{2}\left(\frac{1}{\sigma_2^2}-\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2+2iA\sigma_1\sigma_2}-2iA\right)\right\}, \tag{2.7}$$

which is again a Gaussian function. For $\sigma_1 = 1, \sigma_2 = 2$ and $\Lambda = (1/2, 4, 2/3, 0, 5/7), f_1(t), f_2(t)$ and $f_1(t) \star_\Lambda f_2(t)$ are plotted in Fig. 1.

Next, we examine the nature of convolution theorem associated with the convolution operation $\star_\Lambda$ as defined in (2.5).

Theorem 2.6. For any pair of functions $f_1, f_2 \in L^2(\mathbb{R})$, we have

$$\mathcal{Q}_\Lambda[(f_1 \star_\Lambda f_2)(z)](\omega) = \mathcal{Q}_\Lambda[f_1](\omega)\mathcal{F}[G](B\omega), \tag{2.8}$$

where $\mathcal{F}[G]$ represents the Fourier transform of $G(t) = e^{-iDt} f_2(t)$.

Proof. Using Definition 2.1, the QPFT corresponding to (2.5) is computed as follows:

$$\mathcal{Q}_\Lambda[(f_1 \star_\Lambda f_2)(z)](\omega)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (f_1 \star_\Lambda f_2)(z) \mathcal{K}_\Lambda(z,\omega)\, dz$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} f_1(t) f_2(z-t) e^{iA\left(z^2-t^2\right)} dt \right\} \mathcal{K}_\Lambda(z,\omega)\, dz$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} f_1(t) \left\{ \int_{\mathbb{R}} f_2(z-t) e^{iA\left(z^2-t^2\right)}\right.$$
$$\left.\times e^{-i\left(Az^2+Bz\omega+C\omega^2+Dz+E\omega\right)} dz \right\} dt$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} f_1(t) \left\{ \int_{\mathbb{R}} f_2(x) e^{iAx(x+2t)} e^{-i\left(A(t+x)^2+B(t+x)\omega+C\omega^2+D(t+x)+E\omega\right)} dx \right\} dt$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} f_1(t) e^{-i\left(At^2+Bt\omega+c\omega^2+Dt+E\omega\right)} \left\{ \int_{\mathbb{R}} e^{-iDx} f_2(x) e^{-iBx\omega} dx \right\} dt$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} f_1(t) \left\{ \int_{\mathbb{R}} G(x) e^{-iB\omega x} dx \right\} e^{-i\left(At^2+Bt\omega+C\omega^2+Dt+E\omega\right)} dt$$
$$= \frac{1}{\sqrt{2\pi}} \mathcal{F}[G](\omega) \int_{\mathbb{R}} f_1(t) \mathcal{K}_\Lambda(t,\omega)\, dt$$
$$= \mathcal{Q}_\Lambda[f_1](\omega)\mathcal{F}[G](B\omega), \quad G(t) = e^{-iDt} f_2(t),$$

which is the desired result.

Remark 2.7. For the case $\Lambda = (0, 1, 0, 0, 0)$, Definition 2.3 yields the traditional convolution operator $*$ as

$$(f_1 * f_2)(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_1(t) f_2(z-t)\, dt \tag{2.9}$$

and the corresponding convolution theorem is obtained from (2.8).

Abreast to the notion of quadratic-phase convolution operation (2.5), we introduce the notion of discrete and semi-discrete quadratic-phase convolution structures and then present the corresponding convolution theorems, respectively. Prior to that, we shall first revisit the formal definition of the discrete-time QPFT.

Definition 2.8. [11] Given a sequence $x(n) \in \ell^2(\mathbb{Z})$, the discrete-time QPFT of $x(n)$ corresponding to a parametric set $\Lambda = (A, B, C, D, E)$, $B > 0$ is defined as

$$\mathbf{D}_\Lambda[x(n)](\omega) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} x(n) \exp\left\{-i\left(An^2 + Bn\omega + C\omega^2 + Dn + E\omega\right)\right\}. \tag{2.10}$$

Moreover, any signal can be retracted from the corresponding discrete-time QPFT (2.10) via the following relation:

$$u(n) = \int_0^{2\pi/B} \mathbf{D}_\Lambda[u(n)](\omega) \mathcal{K}_\Lambda(n,\omega)\, d\omega, \tag{2.11}$$

where $\mathcal{K}_\Lambda(n,\omega)$ is given by (2.2).

We are now in a position to introduce the convolution structure of two sequences $u(n), v(n) \in \ell^2(\mathbb{Z})$ associated with the QPFT.

Definition 2.9. Given a parametric set $\Lambda = (A, B, C, D, E)$, $B > 0$ and a pair of sequences $x(n), y(n) \in \ell^2(\mathbb{Z})$, the chirp free convolution operation $\star_{\mathbf{D}}$ associated with the discrete-time QPFT is defined by

$$x(n) \star_{\mathbf{D}} y(n) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} x(n) y(k-n) e^{iA\left(k^2-n^2\right)}. \tag{2.12}$$

The following theorem assembles the fundamnetal properties pertaining to the convolution operation $\star_{\mathbf{D}}$ as defined by (2.12).

Theorem 2.10. Let the scalars $m \in \mathbb{R}$, $\lambda \in \mathbb{R} \setminus \{0\}$ and the sequences $w(n), x(n), y(n) \in \ell^2(\mathbb{Z})$. Then, the convolution operation $\star_{\mathbf{D}}$ as defined in (2.12) satisfies:(i). Non-commutative: $x(n) \star_{\mathbf{D}} y(n) \neq y(n) \star_{\mathbf{D}} x(n)$,(ii). Non-associative: $(w(n) \star_{\mathbf{D}} x(n)) \star_{\mathbf{D}} y(n) \neq w(n) \star_{\mathbf{D}} (x(n) \star_{\mathbf{D}} y(n))$, (iii). Distributive: $w(n) \star_{\mathbf{D}} (x(n) + y(n)) = w(n) \star_{\mathbf{D}} x(n) + w(n) \star_{\mathbf{D}} y(n)$,(iv). Translation: $(x(n) \star_{\mathbf{D}} y(n))(k-m) = (x(n-m) \star_{\mathbf{D}} y_\Lambda(n))(k)$, $y_\Lambda(n) = e^{-2iAkn} y(n)$,(v). Scaling: $(x(n) \star_{\mathbf{D}} y(n))(\lambda k) = |\lambda|\left(x(\lambda n) \star_{\mathbf{D}}^{\Lambda'} y(\lambda n)\right)(k)$, $\Lambda' = (\lambda^2 A, B, C, D, E)$,(vi). Parity: $(x(n) \star_{\mathbf{D}} y(n))(-k) = (x(-n) \star_{\mathbf{D}} y(-n))(k)$.

Proof. The proof can be obtained similarly as Theorem 2.4 and is thus omitted.

In the sequel, we demonstrate that indeed the convolution theorem pertaining to the discret convolution operation $\star_{\mathbf{D}}$ defined in (2.12) is chirp-free.

Theorem 2.11. For any pair of sequences $u(n), v(n) \in \ell^2(\mathbb{Z})$, we have

$$\mathbf{D}_\Lambda[x(n) \star_{\mathbf{D}} y(n)](\omega) = \mathbf{D}_\Lambda[x(n)](\omega)\mathcal{F}[y_\Lambda(n)](B\omega), \tag{2.13}$$

where $\mathscr{F}[y(n)]$ represents the discrete-time Fourier transform of $y_\Lambda(n) = e^{-iDn}y(n)$.

Proof. Invoking the definition of discrete-time QPFT (2.10), we have

$$\begin{aligned}
&\mathbf{D}_\Lambda[x(n)\star_{\mathbf{D}}y(n)](\omega)\\
&=\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}(x(n)\star_{\mathbf{D}}y(n))(k)\,e^{-i\left(Ak^2+Bk\omega+C\omega^2+Dk+E\omega\right)}\\
&=\frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}x(n)\,y(k-n)\,e^{iA\left(k^2-n^2\right)}\,e^{-i\left(Ak^2+Bk\omega+C\omega^2+Dk+E\omega\right)}\\
&=\frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}x(n)y(m)\,e^{iA\left((n+m)^2-n^2\right)}\,e^{-i\left(A(n+m)^2+B(n+m)\omega+C\omega^2+D(n+m)+E\omega\right)}\\
&=\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}x(n)\,e^{-i\left(An^2+Bn\omega+C\omega^2+Dn+E\omega\right)}\sum_{m\in\mathbb{Z}}e^{-iDm}y(m)\,e^{-iBm\omega}\\
&=\mathbf{D}_\Lambda[x(n)](\omega)\,\mathscr{F}[y_\Lambda(n)](B\omega),
\end{aligned}$$

where $\mathscr{F}[y(n)]$ is the discrete-time Fourier transform of $y_\Lambda(n) = e^{-iDn}y(n)$.

Towards the culmination, we formulate the semi-discrete quadratic-phase convolution structure for a sequence $x(n)\in\ell^2(\mathbb{Z})$ and a function $f\in L^2(\mathbb{R})$, and then obtain the corresponding convolution theorem.

Definition 2.12. The semi-discrete convolution $\star_{\mathbf{S}}$ of a sequence $x(n)\in\ell^2(\mathbb{Z})$ and a function $\phi\in L^2(\mathbb{R})$ with respect to a parametric set $\Lambda=(A,B,C,D,E)$, $B>0$ is defined by

$$x(n)\star_{\mathbf{S}}\phi(t)=\frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}x(n)\,\phi(t-n)\,e^{iA\left(t^2-n^2\right)}. \tag{2.14}$$

The convolution theorem corresponding to the semi-discrete convolution defined by (2.14) is given below:

Theorem 2.13. For a pair $x(n)\in\ell^2(\mathbb{Z})$ and $\phi(t)\in L^2(\mathbb{R})$, we have

$$\mathcal{Q}_\Lambda[x(n)\star_{\mathbf{S}}\phi](\omega)=\mathbf{D}_\Lambda[x(n)](\omega)\,\mathscr{F}[\Phi](B\omega), \tag{2.15}$$

where $\mathscr{F}[\Phi]$ represents the Fourier transform of $\Phi(t)=e^{-iDt}\phi(t)$.

Proof. The proof is analogous to Theorem 2.4 and is therefore omitted.

3. Non-ideal Sampling in Shift-invariant Spaces Associated with Quadratic-phase Fourier Transforms

This section is dedicated to demonstrate the application of the novel convolution structures defined in Section 2 to the generalized SISs and non-ideal sampling in the framework of QPFT.

3.1. Generalized shift-invariant spaces associated with quadratic-phase Fourier transform

A shift-invariant space is a closed subspace $\mathscr{V}$ of the space of square intergable functions $L^2(\mathbb{R})$, which is invariant under all integer translates of the constituent functions. A function $\phi\in L^2(\mathbb{R})$ is said to be a generator of a given SIS if the integer tranlates of ϕ span the entire closed subspace $\mathscr{V}$. With the aim to introduce the notion of generalized shift-invariant spaces, we formulate the shift-invariant system $\mathscr{V}_\Lambda(\phi_1,\ldots,\phi_M)$ associated the QPFT. Nevertheless, our main concern is to find a necessary and sufficient condition for a square integrable function $\phi_\ell, 1\leqslant\ell\leqslant M$ to act as a generator for the generalized shift-invariant space.

Definition 3.1. For any sequence $u(n)\in\ell^2(\mathbb{Z})$ and a collection of functions $\phi_\ell(t)\in L^2(\mathbb{R})$, $1\leqslant\ell\leqslant M$, the generalized SISs of $L^2(\mathbb{R})$ with respect to a parametric set $\Lambda=(A,B,C,D,E)$, $B>0$ is denoted as $\mathscr{V}_\Lambda(\phi_1,\ldots,\phi_M)$ and is defined by

$$\mathscr{V}_\Lambda(\phi_1,\ldots,\phi_M)=\text{closure}\left\{f\in L^2(\mathbb{R}):f(t)=\sum_{\ell=1}^{M}\sum_{n\in\mathbb{Z}}u_\ell(n)\star_{\mathbf{S}}\phi_\ell(t)\right\}, \tag{3.1}$$

where $\star_{\mathbf{S}}$ denoted the semi-discrete convolution as defined by (2.14).

Theorem 3.2. Let $\mathscr{V}_\Lambda(\phi_1,\ldots,\phi_M)\subset L^2(\mathbb{R})$ be the generalized SISs associated with the QPFT. Then, the family $\left\{e^{An\left(t^2-n^2\right)}\,\phi_\ell(t-n):1\leqslant\ell\leqslant M\right\}$ forms a Riesz basis for $\mathscr{V}_\Lambda(\phi_1,\ldots,\phi_M)$ if and only if there exists a pair of constants $\Gamma_1,\Gamma_2>0$ such that

$$\Gamma_1\leqslant\sum_{\ell=1}^{M}\sum_{n\in\mathbb{Z}}\left|\mathscr{F}[\Phi_\ell](B\omega+n)\right|^2\leqslant\Gamma_2,\quad 1\leqslant\ell\leqslant M,\forall\omega\in[0,2\pi/B], \tag{3.2}$$

where $\mathscr{F}[\Phi_\ell]$ is the Fourier transform of $\Phi_\ell(t)=e^{-iDt}\phi_\ell(t)$.

Proof. For any function $f(t)\in\mathscr{V}_\Lambda(\phi_1,\ldots,\phi_M)$, we have

$$f(t)=\sum_{\ell=1}^{M}u_\ell(n)\star_{\mathbf{S}}\phi_\ell(t).$$

Then by implementing QPFT 2.1, we obtain

$$\begin{aligned}
&\mathcal{Q}_\Lambda[f](\omega)\\
&=\frac{1}{2\pi}\int_{\mathbb{R}}\sum_{\ell=1}^{M}\sum_{n\in\mathbb{Z}}u_\ell(n)\phi_\ell(t-n)\,e^{iA\left(t^2-n^2\right)}\,e^{-i\left(At^2+Bt\omega+C\omega^2+Dt+E\omega\right)}dt\\
&=\frac{1}{2\pi}\sum_{\ell=1}^{M}\sum_{n\in\mathbb{Z}}u_\ell(n)\int_{\mathbb{R}}\phi_\ell(t-n)\,e^{iA\left(t^2-n^2\right)}\,e^{-i\left(At^2+Bt\omega+C\omega^2+Dt+E\omega\right)}dt\\
&=\frac{1}{2\pi}\sum_{\ell=1}^{M}\sum_{n\in\mathbb{Z}}u_\ell(n)\int_{\mathbb{R}}\phi_\ell(z)\,e^{iA\left((z+n)^2-n^2\right)}\,e^{-i\left(A(z+n)^2+B(z+n)\omega+C\omega^2+D(z+n)+E\omega\right)}dz\\
&=\frac{1}{2\pi}\sum_{\ell=1}^{M}\sum_{n\in\mathbb{Z}}u_\ell(n)\,e^{-i\left(An^2+Bn\omega+C\omega+Dn+E\omega\right)}\int_{\mathbb{R}}e^{-iDz}\phi(z)\,e^{-iBz\omega}dz\\
&=\sum_{\ell=1}^{M}\mathbf{D}_\Lambda[u_\ell(n)](\omega)\,\mathscr{F}[\Phi_\ell](B\omega),
\end{aligned}$$

where $\mathscr{F}[\Phi_\ell]$ denotes the Fourier transform of $\Phi_\ell(t)=e^{-iDt}\phi_\ell(t)$. Since $e^{-2n\pi i}=1$, we have

$$\begin{aligned}
\mathbf{D}_\Lambda[u(n)](\omega+2\pi/B)&=\tfrac{1}{\sqrt{2\pi}}um_{n\in\mathbb{Z}}u(n)\,e^{-i\left(An^2+Bn(\omega+2\pi/B)+C(\omega+2\pi/B)^2+Dn+E(\omega+2\pi/B)\right)}\\
&=\tfrac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}u(n)\,e^{-i\left(An^2+Bn\omega+C\omega+Dn+E\omega\right)}\,e^{-i\left(4\pi^2C/B^2+4\pi C\omega/B+2\pi E/B\right)}\\
&=e^{-i\left(4\pi^2C/B^2+4\pi C\omega/B+2\pi E/B\right)}\,\tfrac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}u(n)\,e^{-i\left(An^2+Bn\omega+C\omega+Dn+E\omega\right)}\\
&=e^{-i\left(4\pi^2C/B^2+4\pi C\omega/B+2\pi E/B\right)}\,\mathbf{D}_\Lambda[u(n)](\omega),
\end{aligned}$$

which yields

$$|\mathbf{D}_\Lambda[u(n)](\omega + 2\pi/B)| = |\mathbf{D}_\Lambda[u(n)](\omega)|.$$

Invoking Parseval's formula for the QPFT as given by (2.5), we have

$$\begin{aligned}\|f(t)\|^2 &= |B|\,\|\mathcal{Q}_\Lambda[f](B\omega)\|^2\\ &= |B|\sum_{\ell=1}^{M}\int_{\mathbb{R}}|\mathbf{D}_\Lambda[u_\ell(n)](\omega)|^2|\mathcal{F}[\Phi_\ell](B\omega)|^2d\omega\\ &= |B|\sum_{\ell=1}^{M}\sum_{n\in\mathbb{Z}}\int_{2n\pi/B}^{2\pi(n+1)/B}|\mathbf{D}_\Lambda[u_\ell(n)](\omega)|^2|\mathcal{F}[\Phi](B\omega)|^2d\omega\\ &= |B|\sum_{\ell=1}^{M}\sum_{n\in\mathbb{Z}}\int_0^{2\pi/B}|\mathbf{D}_\Lambda[u_\ell(n)](\omega+2n\pi/B)|^2|\mathcal{F}[\Phi](B\omega+2n\pi)|^2d\omega\\ &= |B|\sum_{\ell=1}^{M}\sum_{n\in\mathbb{Z}}\int_0^{2\pi/B}|\mathbf{D}_\Lambda[u_\ell(n)](\omega)|^2|\mathcal{F}[\Phi_\ell](\omega+2n\pi)|^2d\omega\\ &= |B|\sum_{\ell=1}^{M}\int_0^{2\pi/B}|\mathbf{D}_\Lambda[u_\ell(n)](\omega)|^2\sum_{n\in\mathbb{Z}}|\mathcal{F}[\Phi](\omega+2n\pi)|^2d\omega.\end{aligned}\tag{3.3}$$

Moreover, we have

$$\begin{aligned}&\int_0^{2\pi/B}|\mathbf{D}_\Lambda[u_\ell(n)](\omega)|^2d\omega\\ &= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}u_\ell(m)\,\overline{u_\ell(n)}\,e^{-i\left(A\left(m^2-n^2\right)+D(m-n)\right)}\int_0^{2\pi/B}e^{-iB(m-n)\omega}d\omega\\ &= \frac{1}{\sqrt{2\pi|B|}}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}u_\ell(m)\,\overline{u_\ell(n)}\,e^{-i\left(A\left(m^2-n^2\right)+D(m-n)\right)}\delta_{m,n}\\ &= \frac{1}{\sqrt{2\pi|B|}}\sum_{n\in\mathbb{Z}}|u_\ell(n)|^2\\ &= \frac{1}{|B|}\,\|u_\ell(n)\|^2.\end{aligned}$$

Using the fact $0 < \Gamma_1 \leqslant \sum_{\ell=1}^{M}\sum_{n\in\mathbb{Z}}|\mathcal{F}[\Phi_\ell](B\omega+n)|^2 \leqslant \Gamma_2 < \infty$, relation (3.3) gives

$$\begin{aligned}\Gamma_1\sum_{\ell=1}^{M}\|\mathbf{D}_\Lambda[u_\ell(n)](\omega)\|^2 &= \frac{\Gamma_1}{|B|}\,\|u_\ell(n)\|^2\\ &\leqslant \|\mathcal{F}[\Phi](B\omega+n)\|^2\\ &\leqslant \frac{\Gamma_2}{|B|}\,\|u_\ell(n)\|^2\\ &= \Gamma_2\sum_{\ell=1}^{M}\|\mathbf{D}_\Lambda[u_\ell(n)](\omega)\|^2,\end{aligned}$$

which yields the desired result.

3.2. Non-ideal sampling associated with quadratic-phase Fourier transform

Sampling theory pertaining to QPFT is particularly interesting as many chirp-like signals arising in different phenomena are not bandlimited in usual Fourier domain but turn to be bandlimited in the generalized Fourier domains, in particular in the QPFT domain. Among various sampling procedures, non-ideal sampling method is the most exclusively applied concepts in signal processing, which wash out the unwanted components and then returns the desired output signal [8]. In this subsection, we shall demonstrate that the proposed convolution structures associated with the QPFT can be employed to design a computationally efficient non-ideal sampling method in the QPFT domain.

Non-ideal sampling in the chirp modulated SISs $\mathcal{V}_\Lambda(\phi_1,\ldots,\phi_M) \subset L^2(\mathbb{R})$ associated with the QPFT is demonstrated in Fig. 2. For any $f(t) \in \mathcal{V}_\Lambda(\phi_1,\ldots,\phi_M)$, we have

$$f(t) = \sum_{\ell=1}^{M}u_\ell(n)\star_{\mathbf{S}}\phi_\ell(t).$$

Application of QPFT yields

$$\mathcal{Q}_\Lambda[f](\omega) = \sum_{\ell=1}^{M}\mathbf{D}_\Lambda[u_\ell(n)](\omega)\,\mathcal{F}[\Phi_\ell](B\omega),\tag{3.4}$$

where $\mathcal{F}[\Phi_\ell]$ represents the Fourier transform of $\Phi_\ell(t) = e^{-iDt}\phi_\ell(t)$, $1 \leqslant \ell \leqslant M$.

By virtue of Fig. 2, $d_\ell(n)$ is demonstrated by

$$d_\ell(z) = \left(f(t)\star_\Lambda\bar{\check{h}}_\ell(t)\right)(z),\quad \check{h}(t) = h(-t).\tag{3.5}$$

Implementing QPFT $\mathcal{Q}_\Lambda[f]$ as given by (2.1) on both sides of (3.5), we obtain

$$\mathcal{Q}_\Lambda[d_\ell(z)](\omega) = \mathcal{Q}_\Lambda\left[\left(f(t)\star_\Lambda\bar{\check{h}}_\ell(t)\right)(z)\right](\omega),$$

which can be recast as

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}d_\ell(z)\exp\left\{-i\left(Az^2+Bz\omega+C\omega^2+Dz+E\omega\right)\right\}dz\\ &\quad= \mathcal{Q}_\Lambda\left[\left(f(t)\star_\Lambda\bar{\check{h}}_\ell(t)\right)(z)\right](\omega).\end{aligned}$$

Or equivalently,

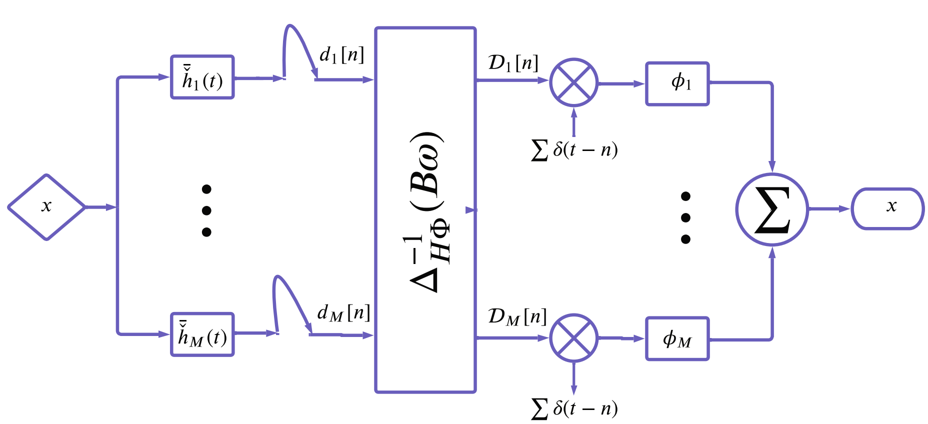


Fig. 2 Non-ideal sampling in the SISs associated with the QPFT.

$$\frac{e^{-i\left(D\omega^2+E\omega\right)}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\left(Az^2+Dz\right)}\, d_\ell(z)\, e^{-iB\omega z} dz$$
$$= \mathcal{Q}_\Lambda\left[\left(f(t)\star_\Lambda \bar{\tilde{h}}_\ell(t)\right)(z)\right](\omega).$$

Then, employing Theorem 2.6 yields

$$e^{-i\left(D\omega^2+E\omega\right)}\mathscr{F}[D_\ell(t)](B\omega) = \mathcal{Q}_\Lambda[f](\omega)\,\mathscr{F}\left[\bar{\tilde{H}}_\ell(t)\right](B\omega).$$

Equivalently,

$$\mathscr{F}[D_\ell(t)](B\omega) = \Psi(\omega)\,\mathcal{Q}_\Lambda[f](\omega)\,\mathscr{F}\left[\bar{\tilde{H}}_\ell(t)\right](B\omega),$$
$$\Psi(\omega) = e^{i\left(D\omega^2+E\omega\right)}, \tag{3.6}$$

where $\mathscr{F}[D_\ell](\omega)$ and $\mathscr{F}\left[\bar{\tilde{H}}_\ell\right](\omega)$ are the Fourier transforms of $D_\ell(t) = e^{-i\left(At^2+Dt\right)}\, d_\ell(t)$ and $\bar{\tilde{H}}_\ell(t) = e^{-iDt}\,\bar{\tilde{h}}_\ell(t)$, respectively.

Plugging (3.4) in (3.6), we obtain

$$\mathscr{F}[D_\ell(t)](B\omega) = \sum_{k\in\mathbb{Z}}\Psi(\omega-\omega_k)\,\mathcal{Q}_\Lambda[f](\omega-\omega_k)\,\mathscr{F}\left[\bar{\tilde{H}}_\ell(t)\right](B(\omega-\omega_k)$$
$$= \sum_{\ell=1}^{M}\mathbf{D}_\Lambda[u_\ell(n)](\omega)\sum_{k\in\mathbb{Z}}\Psi(\omega-\omega_k)\,\mathscr{F}\left[\bar{\tilde{H}}_\ell(t)\right]$$
$$(B(\omega-\omega_k)\,\mathscr{F}[\Phi_\ell](B(\omega-\omega_k)). \tag{3.7}$$

Moreover, setting

$$\varDelta_{H\Phi}(B\omega) = \begin{pmatrix} \xi_{H_1\Phi_1} & \xi_{H_1\Phi_2} & \cdots & \xi_{H_1\Phi_M} \\ \xi_{H_2\Phi_1} & \xi_{H_2\Phi_2} & \cdots & \xi_{H_2\Phi_M} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{H_M\Phi_1} & \xi_{H_M\Phi_2} & \cdots & \xi_{H_M\Phi_M} \end{pmatrix}, \tag{3.8}$$

where $\xi_{H_\ell\Phi_i} = \mathscr{F}\left[\bar{\tilde{H}}_\ell(t)\right](B\omega)\,\mathscr{F}[\Phi_i](B\omega)$.

Thus (3.7) can be written more explicitly as follows:

$$\mathscr{F}[\mathbf{D}](B\omega) = \Psi(\omega)\,\Delta_{H\Phi}(B\omega)\,\mathbf{D}_\Lambda[\mathbf{u}](\omega), \tag{3.9}$$

where $\mathscr{F}[\mathbf{D}](B\omega) = [\mathscr{F}[D_1](B\omega),\ldots,\mathscr{F}[D_M](B\omega)]^T$ and $\mathbf{D}_\Lambda[\mathbf{u}](\omega) = [\mathbf{D}_\Lambda[u_1](\omega),\ldots,\mathbf{D}_\Lambda[u_M](\omega)]^T$.

Therefore from (3.9), we infer that

$$\mathbf{D}_\Lambda[\mathbf{u}](\omega) = \Psi^{-1}(\omega)\,\Delta^{-1}_{H\Phi}(B\omega)\,\mathscr{F}[\mathbf{D}](B\omega). \tag{3.10}$$

By virtue of (3.6) and (3.10), $x(t)$ and $\mathscr{F}\left[\bar{\tilde{H}}_\ell(t)\right](B\omega)$ shares the follwing relationship:

$$\mathbf{D}_\Lambda[\mathbf{u}](\omega) = \mathscr{F}\left[\bar{\tilde{\mathbf{H}}}\right](B\omega)\,\Delta^{-1}_{H\Phi}(B\omega)\,\mathcal{Q}_\Lambda[f](\omega)$$
$$= \mathcal{Q}_\Lambda[f](\omega)\,\mathbf{V}(B\omega), \tag{3.11}$$

where $\mathbf{V}(B\omega) = \mathscr{F}\left[\bar{\tilde{\mathbf{H}}}\right](B\omega)\,\Delta^{-1}_{H\Phi}(B\omega)$. Here $\mathbf{V}(B\omega)$ and $\mathscr{F}\left[\bar{\tilde{\mathbf{H}}}\right](B\omega)$ are vectors with ℓ elements given by $V_\ell(B\omega)$ and $\mathscr{F}\left[\bar{\tilde{H}}_\ell(t)\right](B\omega)$, respectively.

Moreover, relation (3.11) can be represented in time domain as:

$$\mathbf{D}_\Lambda[\mathbf{u}](\omega) = f(t)\star_\Lambda\bar{\tilde{v}}_\ell(t), \quad 1 \leqslant \ell \leqslant M, \tag{3.12}$$

where $\star_\Lambda$ represents the quadratic-phase convolution given by (2.5).

Theorem 3.3. The pair of collections $\{v_\ell(t-n) : 1 \leqslant \ell \leqslant M, n\in\mathbb{Z}\}$ and $\{\phi_\ell : 1 \leqslant \ell \leqslant M\}$ are orthogonal to each other.

Proof. The proof is quite straight forward and is therefore omitted,

4. Conclusion

In the present study, we accomplished two objectives: first, we formulated a novel convolution structures associated with the QPFT; second, we introduced the generalized SISs in the QPFT domains. We implemented both the proposed convolution structures and generalized shift-invariant spaces to present a practically reliable and computationally efficient non-ideal sampling procedure in the realm of QPFT domain.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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