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Analysis of unit-Weibull based on progressive type-II censored with optimal scheme

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Abstract Using progressive Type-II censoring data, this study deals with the estimation of parameters of the Unit-Weibull distribution using two classical methods and the Bayesian method. In the classical methods, maximum likelihood and the maximum product of spacing (MPS) methods are used to obtain the parameters of the model by utilizing the Newton–Raphson method. On the basis of observed Fisher information matrix, approximate confidence intervals for the unknown parameters are obtained. In addition, two bootstrap methods are used to obtain confidence intervals for the unknown parameters of the model. In the Bayesian estimation, we have considered both likelihood function as well as product of spacing function to estimate the model parameters. Bayes estimators are obtained under squared error loss function using independent gamma density priors for the unknown model parameters. Since closed-form of the Bayes estimators is not available, the Metropolis–Hastings algorithm is proposed to approximate the Bayes estimates. In addition, highest posterior density credible intervals are obtained. Further, using different optimally criteria, an optimal scheme has been proposed. A simulation study is conducted to assess the statistical performance of all the estimators. To demonstrate the proposed methodology a real data analysis is provided to illustrate all the statistical inferential procedures developed in the paper.

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1. Introduction

The two-parameter unit-Weibull (UW) distribution was introduced by Mazucheli et al. [1] by using exponential transformation. Suppose that the lifetime variable as random X of an individual testing item follows two-parameter $UW(\alpha,\beta)$, then the cumulative distribution function (CDF) $F(\cdot)$, the probability density function (PDF) $f(\cdot)$, reliability function (RF) $R(\cdot)$, hazard rate function (HRF)$h(\cdot)$and reversed HRF (RHRF) $rh(\cdot)$are respectively, given by

$$f(x;\alpha,\beta)=\alpha\beta x^{-1}(-\ln(x))^{\beta-1}e^{-\alpha(-\ln(x))^{\beta}},\ 0<x<1,\ \alpha,\beta>0, \tag{1}$$

$$F(x;\alpha,\beta)=e^{-\alpha(-\ln(x))^{\beta}},\ 0<x<1,\ \alpha,\beta>0, \tag{2}$$

$$R(x;\alpha,\beta)=1-e^{-\alpha(-\ln(x))^{\beta}},\ 0<x<1,\ \alpha,\beta>0, \tag{3}$$

$$h(x;\alpha,\beta)=\frac{\alpha\beta x^{-1}(-\ln(x))^{\beta-1}e^{-\alpha(-\ln(x))^{\beta}}}{1-e^{-\alpha(-\ln(x))^{\beta}}},\ 0<x<1,\ \alpha,\beta>0, \tag{4}$$

and

$$rh(x;\alpha,\beta)=\alpha\beta x^{-1}(-\ln(x))^{\beta-1},\quad 0<x<1,\ \alpha,\beta>0, \tag{5}$$

where α and β are the shape parameters. Mazucheli et al. [2] showed that the shape of the density function is quite similar to other well-known extensions of the exponential distribution including decreasing, increasing, unimodal, anti-unimodal. The flexibility of the shape of the pdf of UW distribution makes it a viable alternative for analyzing data on the unit interval. Furthermore, authors showed that the UW distribution outperforms several good known unit distributions including the Beta and Kumaraswamy distribution. Different shapes of the density and failure rate functions of the UW distribution are plotted in Fig. 1 using some specified values on the range of the parameters α and β. It is to be noted that when $\alpha=1$ and $\beta=1$ in (2) standard uniform distribution can be obtained while when $\beta=1$ power function distribution can be obtained as sub-models from UW distribution. Again, the unit-Rayleigh distribution can be obtained as a special case from the UW distribution when $\beta=2$.

In recent past, some scholars have studied UW distribution under different scenarios. For instance, Alotaibi et al. [3] discussed Bayesian and classical reliability estimation of multi-component stress-strength model of Unit-Weibull (UW) distribution. Mazucheli et al. [1] introduced unit-Weibull quantile regression model. Iliev et al. [4] studied one-sided Hausdorff approximation of the Heaviside step function using Unit-Logistic (UL), UW and Topp-Leone (TL) cumulative sigmoids.

Due to advancement in industrial design and technology, the lifespan of some items has increased significantly. Even with conventional Type-I and Type-II censored, the duration of the test takes a longer time and sometimes no or few failures occur during the test. For this reason, Balakrishnan and Aggarwala [5] proposed progressive Type-II censored schemes as an alternative to generalized Type-II censored sampling. This censoring strategy has become extremely common in survival analysis. For more information, see Balakrishnan and Cramer [6]. Briefly, the progressive Type-II censored scheme (PTIICS) can be characterized as follows: Suppose n independent and identical units, effective sample size $(m<n)$, and progressive censoring $(R_1,R_2,\ldots,R_m)$, are predetermined at the beginning of the experiment, then joint likelihood function of the PTIICS, where $\underline{\theta}$is the parameter vector, is defined as

$$L(\underline{\theta}|\underline{\mathbf{x}})=A_d\prod_{i=1}^{m}f(x_{i:m:n};\underline{\theta})\left[\overline{F}(x_{i:m:n};\underline{\theta})\right]^{R_i}, \tag{6}$$

where $A_d$is a constant, $\overline{F}(\cdot)=1-F(\cdot)$and $R^*_m=n-m-\sum_{i=1}^{d}R_i$. For extensive reviews of the literature on progressive censoring, readers may refer to the works of [7–12].

Besides the conventional likelihood function, the maximum product of spacing (PS) method is also considered as a method of competitive to the usual likelihood function (LF) under PTIICS presented in (6). Cheng and Amin [13] first inserted the maximum product of spacing (MPS) estimation style and later this was shown by Ranneby [14] as a method of competitive to the method of the likelihood. The MPS estimators (MPSEs) can be obtained by magnification the MPS function similar to the procedure followed for obtaining the maximum likelihood estimates (MLEs). Therefore, it is assumed that the MPSEs possesses most of the properties of the MLEs. Bbesides, a product of spacing is bounded, hence, even when likelihood breaks down due to unbounded LF, the MPS method can provide asymptotically optimal estimates. Further, Anatolyev and Kosenok (2005) stated that for skewed models or in small-sample cases for heavy-tailed models, MPSEs are more efficient than the MLEs. In regard to MPS method, readers may refer to the works of Ng et al. [15], Basu et al. [16], El-

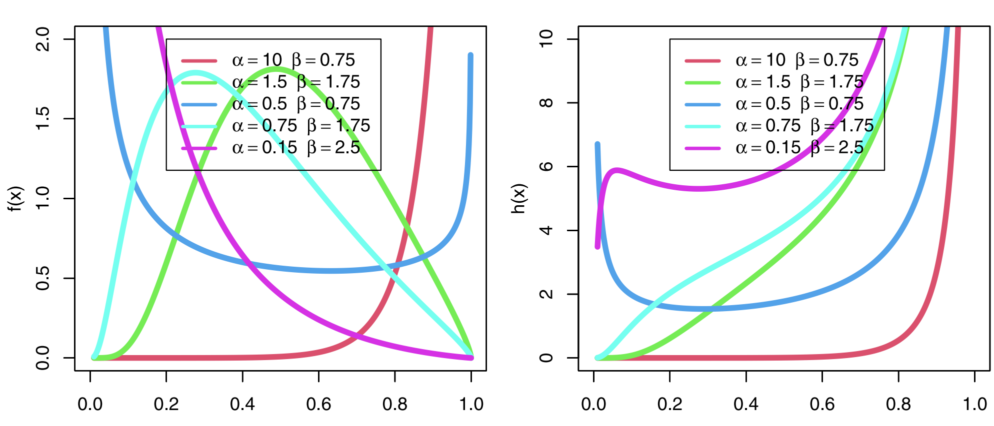

Fig. 1 Plot of the density and hazard rate functions of the UW distribution.

Sherpieny et al. [17], Almongy et al. [18] and Alshenawy et al. [19], Nassar et al. [20], Coolen and Newby [21], and many others.

Based on a PTIICS sample, we can write the PS function as follows

$$S(\underline{\theta}|\underline{\mathbf{x}}) = B_d \prod_{i=1}^{m+1} [F(x_{i:m:n}; \underline{\alpha}) - F(x_{i-1:m:n}; \underline{\alpha})] \left[\overline{F}(x_{i:m:n}; \underline{\alpha})\right]^{R_i}, \quad (7)$$

where $B_d$is a constant, $F(x_{0:m:n}; \underline{\theta}) \equiv 0$and $F(x_{m+1:m:n}; \underline{\theta}) \equiv 1$.

The best we know, no work has been done on the inference of model parameters, such as point and interval estimates and optimum censoring scheme for the UW distribution under PTIICS, which is more important in many practical situations as previously mentioned. In the premise of the above, in this study, we will only focus on developing point and interval estimates of unknown model parameters using both classical and Bayesian methods. Two traditional inferential procedures, LF and PS, as well as two Bayesian methods of estimation via these traditional approaches, are studied for this purpose. Two-sided approximate confidence intervals (ACIs) for the UW parameters are obtained using observed Fisher-information-matrix (FIM). In addition, two parametric bootstrap methods are also taken into account for obtaining confidence intervals. The Bayes estimates of the unknown parameters are obtained by LF and PS functions under squared-error loss function under independent gamma density priors. Since analytical solution to the Bayes estimators and associated credible intervals is intractable, Markov Chain Monte-Carlo (MCMC) techniques are used to generate samples from the appropriate posterior density functions. For classical estimates, we use 'maxLik' package, while for computing Bayesian estimates, R package with the Metropolis–Hastings (MH) algorithm is used. Further, optimal censoring plans based on three optimally criteria are presented. A comprehensive Monte-Carlo simulation study is undertaken to compare the performance of the considered methods. Furthermore, the behavior of point estimates is compared using mean squared errors (MSE) and relative absolute biases, whereas the behavior of interval estimates is examined using average lengths (ALS) and coverage probabilities (CPs). Two real-life data sets is analyzed to discuss how the applicability of the proposed methods in the real phenomena and to demonstrate the superiority and flexibility of the UW distribution. Finally, we draw some specific recommendations from numerical results.

The rest of the paper is arranged as follows: Section 2 provides the classical (point/interval) estimations of unknown parameters. Bootstrap confidence intervals are presented in Section 3. In Section 4, using each of the proposed frequentist functions, Bayesian (point/interval) estimations based on SEL function are developed. In Section 5, optimal progressive censoring plans are presented. The simulated results are presented in Section 6. Two real data analysis is provided for illustrative purpose as well as an optimum censoring plan is also presented in Section 7. Finally, we conclude the paper in Section 8.

2. Classical inference

In this section we discuss the point and interval estimators of the unknown parameters of the UW distribution based on the data gathered under the suggested censoring scheme using the LF and PS approach. Before progressing, suppose that $X_{1:m:n} < \cdots < X_{m:m:n}$is a PTIICS order statistics of size $(m < n)$ with censoring scheme $R_1, \ldots, R_m$ from $UW(\alpha, \beta)$.

2.1. Maximum likelihood estimators

Substituting (2) and (1) into (6) by ignoring any additive constant, the likelihood function (6) becomes

$$L(\alpha, \beta|\underline{\mathbf{x}}) \propto \alpha^m \beta^m e^{-\alpha \sum_{i=1}^{m} [-\ln(x_i)]^\beta} \prod_{i=1}^{m} \frac{[-\ln(x_i)]^{\beta-1}}{x_i} \left\{1 - e^{-\alpha[-\ln(x_i)]^\beta}\right\}^{R_i}. \quad (8)$$

The corresponding log-likelihood function, $\ell(\cdot) = \log L(\cdot)$, of (8) can be written as

$$\begin{aligned} \ell(\alpha, \beta|d) = {} & m\ln(\alpha) + m\ln(\beta) - \alpha \sum_{i=1}^{m} [-\ln(x_i)]^\beta \\ & + (\beta - 1)\sum_{i=1}^{m} \ln[-\ln(x_i)] - \sum_{i=1}^{m} \ln(x_i) + \\ & \sum_{i=1}^{m} R_i \ln\left\{1 - e^{-\alpha[-\ln(x_i)]^\beta}\right\}. \end{aligned} \quad (9)$$

Differentiating (9) partially with respect to the parameters α and β, then equate them to zero, we have two normal equations are of the following form

$$\frac{\partial \ell}{\partial \alpha} = \frac{m}{\alpha} - \sum_{i=1}^{m} [-\ln(x_i)]^\beta + \sum_{i=1}^{m} R_i \frac{[-\ln(x_i)]^\beta e^{-\alpha[-\ln(x_i)]^\beta}}{1 - e^{-\alpha[-\ln(x_i)]^\beta}},$$

and

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} = {} & \frac{m}{\beta} - \alpha \sum_{i=1}^{m} [-\ln(x_i)]^\beta \ln[-\ln(x_i)] + \sum_{i=1}^{m} \ln[-\ln(x_i)] \\ & + \sum_{i=1}^{m} R_i \frac{\alpha[-\ln(x_i)]^\beta \ln[-\ln(x_i)] e^{-\alpha[-\ln(x_i)]^\beta}}{1 - e^{-\alpha[-\ln(x_i)]^\beta}}. \end{aligned} \quad (11)$$

As it seems, from (10) and (11), analytic solutions of MLEs of $\alpha$and $\beta$are not available, therefore, Newton–Raphson (NR) method may be used by implementing 'maxLik' package to obtain the MLEs $\hat{\alpha}$and $\hat{\beta}$for any given data set

2.2. Maximum PS estimators

Substituting (2) and (1) into (7), the PS function (7) becomes

$$S(\alpha, \beta|\underline{\mathbf{x}}) \propto \prod_{i=1}^{m+1} \left\{e^{-\alpha[-\ln(x_i)]^\beta} - e^{-\alpha[-\ln(x_{i-1})]^\beta}\right\} \left[1 - e^{-\alpha[-\ln(x_i)]^\beta}\right]^{R_i}. \quad (12)$$

From (12), the MPSEs $\widehat{\alpha}$ and $\widehat{\beta}$of α and β, respectively, can be obtained by maximizing the following log-PS function, $s(\cdot) \propto \log S(\cdot)$, as

$$\begin{aligned} s(\alpha, \beta|\underline{\mathbf{x}}) \propto {} & \sum_{i=1}^{m+1} \ln\left\{e^{-\alpha[-\ln(x_i)]^\beta} - e^{-\alpha[-\ln(x_{i-1})]^\beta}\right\} \\ & + R_i \sum_{i=1}^{m+1} \ln\left\{1 - e^{-\alpha[-\ln(x_i)]^\beta}\right\}. \end{aligned} \quad (13)$$

Differentiating (13) partially with respect to α and β, then equate them to zero, we have two non-linear equations that must be solved simultaneously to obtain respective $\widehat{\alpha}$ and $\widehat{\beta}$as

$$\frac{\partial s}{\partial \alpha}=\sum_{i=1}^{m+1}\frac{[-\ln(x_{i-1})]^{\beta}e^{-\alpha[-\ln(x_{i-1})]^{\beta}}-[-\ln(x_i)]^{\beta}e^{-\alpha[-\ln(x_i)]^{\beta}}}{e^{-\alpha[-\ln(x_i)]^{\beta}}-e^{-\alpha[-\ln(x_{i-1})]^{\beta}}}+\sum_{i=1}^{m}R_i\frac{[-\ln(x_i)]^{\beta}e^{-\alpha[-\ln(x_i)]^{\beta}}}{1-e^{-\alpha[-\ln(x_i)]^{\beta}}}, \quad (14)$$

and

$$\frac{\partial s}{\partial \beta}=\sum_{i=1}^{m+1}\frac{\alpha[-\ln(x_{i-1})]^{\beta}\ln[-\ln(x_{i-1})]e^{-\alpha[-\ln(x_{i-1})]^{\beta}}}{e^{-\alpha[-\ln(x_i)]^{\beta}}-e^{-\alpha[-\ln(x_{i-1})]^{\beta}}}-\sum_{i=1}^{m+1}\frac{\alpha[-\ln(x_i)]^{\beta}\ln[-\ln(x_i)]e^{-\alpha[-\ln(x_i)]^{\beta}}}{e^{-\alpha[-\ln(x_i)]^{\beta}}-e^{-\alpha[-\ln(x_{i-1})]^{\beta}}}+ +\sum_{i=1}^{m}R_i\frac{\alpha[-\ln(x_i)]^{\beta}\ln[-\ln(x_i)]e^{-\alpha[-\ln(x_i)]^{\beta}}}{1-e^{-\alpha[-\ln(x_i)]^{\beta}}}. \quad (15)$$

Since, there is no explicit form for the MPSEs, hence, for a given dataset, the NR method is used to find the MPSEs from (14) and (15) numerically. Regarding to the asymptotic properties of the MPSEs, Cheng and Amin [13] showed that the MPSEs are consistent and provide the same asymptotic properties of the MPSEs.

2.3. Asymptotic confidence intervals

To construct the $100(1-\gamma)\%$two-sided ACIs for the unknown parameters $\alpha$and β, say $\Omega=(\alpha,\beta)$, the FIM $\mathbf{I}_{ij}(\Omega)=E\left[-\frac{\partial^2\ell(\Omega|\underline{\mathbf{x}})}{\partial\Omega^2}\right]$, $i,j=1,2$. Since the exact solutions of the Fisher's expectation is tedious to obtain, hence the asymptotic-variance–covariance (V-C) matrix of the MLEs $\hat{\alpha}$and $\hat{\beta}$can be obtained by inverting $\mathbf{I}(\Omega)$and dropping $E$with replacing $\alpha$and $\beta$by their MLEs $\hat{\alpha}$and $\hat{\beta}$, respectively, see [22].

However, by second differentiating (9) partially with respect to $\alpha$and β, locally at their MLEs $\hat{\alpha}$and $\hat{\beta}$, the approximate V-C matrix, $\mathbf{I}^{-1}\left(\hat{\alpha},\hat{\beta}\right)$, is given by

$$\mathbf{I}^{-1}\left(\hat{\alpha},\hat{\beta}\right)=\begin{bmatrix}-\ell_{11} & -\ell_{12}\\ -\ell_{21} & -\ell_{22}\end{bmatrix}^{-1}_{(\hat{\alpha},\hat{\beta})}=\begin{bmatrix}\hat{\sigma}_{\hat{\alpha}\hat{\alpha}} & \hat{\sigma}_{\hat{\alpha}\hat{\beta}}\\ \hat{\sigma}_{\hat{\beta}\hat{\alpha}} & \hat{\sigma}_{\hat{\beta}\hat{\beta}}\end{bmatrix}. \quad (16)$$

Similarly, by differentiating (13) partially with respect to $\alpha$and β, locally at their MPSEs $\widehat{\alpha}$and $\widehat{\beta}$, the approximate V-C matrix, $\mathbf{I}^{-1}\left(\widehat{\alpha},\widehat{\beta}\right)$, is given by

$$\mathbf{I}^{-1}\left(\widehat{\alpha},\widehat{\beta}\right)=\begin{bmatrix}-s_{11} & -s_{12}\\ -s_{21} & -s_{22}\end{bmatrix}^{-1}_{(\widehat{\alpha},\widehat{\beta})}=\begin{bmatrix}\widehat{\sigma}_{\widehat{\alpha}\widehat{\alpha}} & \widehat{\sigma}_{\widehat{\alpha}\widehat{\beta}}\\ \widehat{\sigma}_{\widehat{\beta}\widehat{\alpha}} & \widehat{\sigma}_{\widehat{\beta}\widehat{\beta}}\end{bmatrix}. \quad (17)$$

3. Bootstrap confidence intervals

To obtain the ACIs of the unknown parameters, the normal approximation of the MLE will be adequate when the effective sample size is high. On the other hand, when the effective sample size is small, these intervals may not perform properly. In such a situation, re-sampling techniques such as the bootstrap method is more precise to obtain confidence intervals. Hence, in this section, two bootstrap resampling techniques namely: percentile bootstrap method (Boot-*p*) introduced by Efron [23] and bootstrap-t method (Boot-*t*) introduced by Efron [23], are considered to obtain the confidence intervals (CIs) for the unknown parameters α,β. Here, we briefly discuss how to obtain the bootstrap CIs of the unknown parameters using both Boot-*p*and Boot-*s*methods as

Step 1: Given n, m, and progressive censoring scheme $(R_1, R_2, \ldots, R_m)$, compute the MLEs of the unknown parameters α, and β based on PTIICS sample.

Step 2: Using the MLEs obtained in Step 1, generate a bootstrap sample $\left(X_1^*,\ldots,X_m^*\right)$ from $UW\left(\hat{\alpha},\hat{\beta}\right)$.

Step 3: Using outputs of Step 2, compute the bootstrap estimate of α, and β, say $\hat{\Omega}^b$, where $\hat{\Omega}^b=\left(\hat{\alpha}^b,\hat{\beta}^b\right)$is the bootstrap estimate of the unknown parameter Ω.

Step 4: Repeat Steps 2–3 $\mathscr{B}$times to get the bootstrap estimates $\hat{\Omega}_r^b$, $r=1,2,\ldots,\mathscr{B}$.

Step 5: Compute the bootstrap CIs of Ωas:

(a) **Boot-*p*method:**

1. Arrange the bootstrap estimates in ascending order to obtain $\left(\hat{\Omega}_{[1]}^b,\hat{\Omega}_{[2]}^b,\ldots,\hat{\Omega}_{[\mathscr{B}]}^b\right)$.
2. The $100(1-\gamma)\%$ Boot-*p*CI of $\Omega$is given by $\left\{\hat{\Omega}^b_{[\mathscr{B}(\gamma/2\,)]},\hat{\Omega}^b_{[\mathscr{B}(1-\gamma/2\,)]}\right\}$.

(b) **Boot-*t*method:**

1. Obtain the T-statistic for each unknown parameter as

$$T_r^{\Omega}=\frac{\hat{\Omega}_r^b-\hat{\Omega}}{\sqrt{\widehat{\mathrm{var}}\left(\hat{\Omega}^b\right)}},\ r=1,2,\ldots,\mathscr{B},$$

where $\hat{\Omega}$ is the MLE of $\Omega$and $\widehat{\mathrm{var}}\left(\hat{\Omega}^b\right)$ is the bootstrap variance of $\hat{\Omega}^b$.

2. Arrange the T-statistics to obtain $\left(T^{\Omega}_{[1]},T^{\Omega}_{[2]},\ldots,T^{\Omega}_{[\mathscr{B}]}\right)$.
3. The $100(1-\gamma)\%$ Boot-*t*CI of $\Omega$is given by

$$\left\{\hat{\Omega}+T^{\Omega}_{[\mathscr{B}\gamma/2]}\sqrt{\widehat{\mathrm{var}}\left(\hat{\Omega}^b\right)},\hat{\Omega}+T^{\Omega}_{[\mathscr{B}(1-\gamma/2)]}\sqrt{\widehat{\mathrm{var}}\left(\hat{\Omega}^b\right)}\right\}.$$

4. Bayesian Estimators

In this section, we take into account the Bayesian estimation of the UW distribution's unknown parameters when the data are PTIICS. More specifically, we concentrate on the MCMC and credible intervals for the unknown parameters associated with the Bayesian estimates using LF and PS functions. Other loss functions may be taken into account even though we have achieved Bayesian estimates using the squared-error loss function. In Bayesian inference, prior distribution selection is important. However, there is no set rule or instruction in the literature for selecting the appropriate priors for a model's unobserved parameters.

4.1. Prior information and loss function

We look into the piece-wise independent gamma priors for the model's parameters while keeping in mind the discussion from above. It has been noted that there is no conjugate found before the UW distribution. Additionally, the intricate forms of the FIM based on both LF and PS functions make it difficult to employ Jeffrey's priors. As a result, we consider the UW distribution to have independent gamma priors. Firstly, the gamma prior is flexible in nature and offers a variety of shapes dependent on parameter values, making it a suitable prior for model parameters. Independent gamma priors are quite simple and brief, therefore they may not give rise to many challenging inferential problems. See Dey et al. [24], and Nassar et al. [25] for more information. Let the UW parameters $\alpha$and $\beta$are assumed to have independent gamma PDFs as $Gamma(a_1, b_1)$and $Gamma(a_2, b_2)$, respectively. Then, the joint prior density of $\alpha$and βbecomes

$$\pi(\alpha,\beta) \propto \alpha^{a_1-1}\beta^{a_2-1}e^{-(b_1\alpha+b_2\beta)},\ \alpha,\beta>0, \tag{18}$$

where the hyperparameters $a_i, b_i,\ i=1,2,$are chosen to reflect prior information's about the unknown parameters $\alpha$and β, and they should be well-known and positive. Consequently, when setting $a_i=b_i=0,\ i=1,2,$in (18), the gamma improper function of $\alpha$and βbecomes $(\alpha\beta)^{-1}$.

The most often employed symmetric loss function is the squared-error loss function, $\mathscr{L}(\cdot)$, which is defined as:

$$\mathscr{L}(\eta,\tilde{\eta}) = (\tilde{\eta}-\eta)^2, \tag{19}$$

where $\tilde{\eta}$being an estimate of η. Under (19), objective estimate $\tilde{\eta}$is given by the posterior mean of η.

4.2. Posterior analysis by LF

Observing the PTIICS sample data from the LF $L(\cdot)$and the prior information given by $\pi(\cdot)$, the joint posterior density function, $\pi_L(\cdot)$of $\alpha$and $\beta$is given by

$$\pi_L(\alpha,\beta|\underline{\mathbf{x}}) = K_1^{-1}\pi(\alpha,\beta)L(\alpha,\beta|\underline{\mathbf{x}}), \tag{20}$$

where $K_1=\int_0^\infty\int_0^\infty \pi_L(\alpha,\beta|\underline{\mathbf{x}})d\alpha d\beta$is the normalizing constant.

Substituting (8) and (18) into (20), the joint posterior PDF of $\alpha$and βbecomes

$$\pi_L(\alpha,\beta|\underline{\mathbf{x}}) = \frac{1}{K_1}\alpha^{m+a_1-1}\beta^{m+a2-1}e^{-\beta b_2}e^{-\alpha\left\{b_1+\sum_{i=1}^{m}[-\ln(x_i)]^\beta\right\}}\prod_{i=1}^{m}\frac{[-\ln(x_i)]^{\beta-1}}{x_i}\left\{1-e^{-\alpha[-\ln(x_i)]^\beta}\right\}^{R_i}. \tag{21}$$

Under SEL function (19), the Bayes estimator for any function of $\alpha$and β, say ω, is the posterior expectation of ω. So, to develop these estimates, the marginal posterior distributions of $\alpha$and βmust be obtained. But due to implicit mathematical expressions of (21), it is clear that it is not possible to get explicit forms for the marginal PDFs for each unknown parameter. Thus, we apply some simulation algorithms, such as the MCMC methods, to compute the Bayesian estimates and corresponding credible intervals.

First, from (21), we obtain the conditional posterior distributions of $\alpha$and β, respectively, as

$$\pi_L(\alpha|\beta,\underline{\mathbf{x}}) \propto \alpha^{m+a_1-1}e^{-\alpha\left\{b_1+\sum_{i=1}^{m}[-\ln(x_i)]^\beta\right\}}\prod_{i=1}^{m}\frac{[-\ln(x_i)]^{\beta-1}}{x_i}\left\{1-e^{-\alpha[-\ln(x_i)]^\beta}\right\}^{R_i}, \tag{22}$$

and

$$\pi_L(\beta|\alpha,\underline{\mathbf{x}}) \propto \beta^{m+a2-1}e^{-\beta b_2}e^{-\alpha\sum_{i=1}^{m}[-\ln(x_i)]^\beta}\prod_{i=1}^{m}\frac{[-\ln(x_i)]^{\beta-1}}{x_i}\left\{1-e^{-\alpha[-\ln(x_i)]^\beta}\right\}^{R_i}. \tag{23}$$

From (22) and (23), it is clear that the conditional posterior distributions of $\alpha$and βcannot be reduced analytically to any well-known distribution and therefore cannot be sampled directly via standard methods. Therefore, the MH algorithm with normal proposal distributions is proposed to simulate samples from (22) and (23). The plot of the conditional PDFs of $\alpha$and βshows that these distributions behave similarly to the normal distribution.

4.3. Posterior analysis by PS

Analogous to the case of Bayesian inference using the LF discussed in the previous section, here we obtain the conditional posterior distributions of $\alpha$and βusing the PS function. Therefore, based on PS function, the joint posterior density $\pi_S(\cdot)$of $\alpha$and β, given the Type-II progressively censored sample data and the prior knowledge given by $\pi(\cdot)$, is given by

$$\pi_S(\alpha,\beta|\underline{\mathbf{x}}) = K_2^{-1}\pi(\alpha,\beta)S(\alpha,\beta|\underline{\mathbf{x}}), \tag{24}$$

where $K_2=\int_0^\infty\int_0^\infty \pi_S(\alpha,\beta|\underline{\mathbf{x}})d\alpha d\beta$is the normalizing constant.

Substituting (12) and (18) into (24), the joint posterior PDF of $\alpha$and βcan be written as

$$\pi_S(\alpha,\beta|\underline{\mathbf{x}}) = K_2^{-1}\alpha^{a_1-1}\beta^{a_2-1}e^{-(b_1\alpha+b_2\beta)}\prod_{i=1}^{m+1}\left\{e^{-\alpha[-\ln(x_i)]^\beta}-e^{-\alpha[-\ln(x_{i-1})]^\beta}\right\}\left[1-e^{-\alpha[-\ln(x_i)]^\beta}\right]^{R_i}. \tag{25}$$

From (25), to obtain the Bayes estimates as well as the corresponding HPD credible intervals of any parametric function $\alpha$and β, we need to derive the conditional distributions of $\alpha$and β, respectively, as

$$\pi_S(\alpha|\beta,\underline{\mathbf{x}}) \propto \alpha^{a_1-1}e^{-b_1\alpha}\prod_{i=1}^{m+1}\left\{e^{-\alpha[-\ln(x_i)]^\beta}-e^{-\alpha[-\ln(x_{i-1})]^\beta}\right\}\left[1-e^{-\alpha[-\ln(x_i)]^\beta}\right]^{R_i}, \tag{26}$$

and

$$\pi_S(\beta|\alpha,\underline{\mathbf{x}}) \propto \beta^{a_2-1}e^{-b_2\beta}\prod_{i=1}^{m+1}\left\{e^{-\alpha[-\ln(x_i)]^\beta}-e^{-\alpha[-\ln(x_{i-1})]^\beta}\right\}\left[1-e^{-\alpha[-\ln(x_i)]^\beta}\right]^{R_i}, \tag{27}$$

As expected, similar to Bayesian inference in this instance using LF approach, the conditional distributions (26) and (27) of $\alpha$and β, respectively, cannot be reduced analytically to any standard distribution. Therefore, we consider to use MCMC simulation technique to approximate the Bayes (point/interval) estimates of α,β.

4.4. M-H algorithm sampler

One of the most useful MCMC techniques is the MH algorithm, in order to approximate the Bayes estimates and create the matching HPD credible intervals, which is utilised to create random samples using the posterior density distribution and an independent proposal distribution. For more details related to this algorithm, we refer to [26,27]. To generate MCMC samples from (21) of α and β, or any function of them, using the MH algorithm, conduct the following process:

Table 1 Some optimality criteria of progressive censoring plan

Criterion	Objective
OA	Minimize $\det\left(\mathbf{I}^{-1}\left(\hat{\alpha},\hat{\beta}\right)\right)$
OB	Minimize $\text{trace}\left(\mathbf{I}^{-1}\left(\hat{\alpha},\hat{\beta}\right)\right)$
OC	Maximize $\text{trace}\left(\mathbf{I}\left(\hat{\alpha},\hat{\beta}\right)\right)$

Step 1: Start with initial guess $\left(\alpha^{(0)},\beta^{(0)}\right)=\left(\hat{\alpha},\hat{\beta}\right)$.

Step 2: Set $j=1$.

Step 3: Generate α^* and β^* from (22) and (23) with normal distributions $N(\hat{\alpha},\sigma_{\hat{\alpha}\hat{\alpha}})$ and $N\left(\hat{\beta},\sigma_{\hat{\beta}\hat{\beta}}\right)$, respectively, as

(a) Calculate $\vartheta_1=\frac{\pi_L\left(\alpha^*|\beta^{(j-1)},\underline{\mathbf{x}}\right)}{\pi_L\left(\alpha^{(j-1)}\left|\beta^{(j-1)},\underline{\mathbf{x}}\right.\right)}$ and $\vartheta_2=\frac{\pi_L\left(\beta^*|\alpha^{(j)},\underline{\mathbf{x}}\right)}{\pi_L\left(\beta^{(j-1)}\left|\alpha^{(j)},\underline{\mathbf{x}}\right.\right)}$.

(b) Obtain $Q_\alpha=\min\{1,\vartheta_1\}$ and $Q_\beta=\min\{1,\vartheta_2\}$.

(c) Generate sample variates u_1 and u_2 from the uniform $U(0,1)$ distribution.

(d) If $u_1\leqslant Q_\alpha$, set $\alpha^{(j)}=\alpha^*$, else set $\alpha^{(j)}=\alpha^{(j-1)}$. If $u_2\leqslant Q_\beta$, set $\beta^{(j)}=\beta^*$, else set $\beta^{(j)}=\beta^{(j-1)}$.

Step 4: Set $j=j+1$.

Step 5: Redo steps 2–5 for $\mathcal{N}$ times to collect $\mathcal{N}$ draws of α, and β, as

Table 2 Bayesian and non-bayesian estimators bsed on PTII censored sample when $\alpha=0.5,\beta=0.5$.

$\alpha=0.5,\beta=0.5$				Non-Bayeisna				Bayesian			
				MLE		MPS		MLE		MPS	
P	n	m		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.15	30	20	α	−0.0135	0.0202	0.0284	0.0194	0.0208	0.0057	0.0240	0.0050
			β	0.0348	0.0103	−0.0283	0.0080	0.0088	0.0019	−0.0213	0.0018
		25	α	0.0031	0.0175	0.0358	0.0173	0.0238	0.0052	0.0218	0.0046
			β	0.0251	0.0083	−0.0314	0.0071	0.0080	0.0016	−0.0193	0.0017
	70	50	α	−0.0054	0.0081	0.0179	0.0081	0.0086	0.0019	0.0115	0.0019
			β	0.0162	0.0037	−0.0173	0.0033	0.0060	0.0008	−0.0105	0.0008
		60	α	0.0043	0.0078	0.0221	0.0079	0.0121	0.0021	0.0130	0.0019
			β	0.0114	0.0031	−0.0183	0.0030	0.0044	0.0007	−0.0108	0.0007
	150	110	α	0.0033	0.0041	0.0157	0.0043	0.0070	0.0010	0.0075	0.0009
			β	0.0050	0.0013	−0.0133	0.0014	0.0020	0.0003	−0.0070	0.0003
		130	α	−0.0017	0.0037	0.0083	0.0037	0.0031	0.0008	0.0033	0.0007
			β	0.0053	0.0015	−0.0110	0.0015	0.0017	0.0003	−0.0062	0.0003
0.5	30	20	α	−0.0115	0.0225	0.0326	0.0218	0.0215	0.0060	0.0229	0.0054
			β	0.0342	0.0107	−0.0307	0.0084	0.0099	0.0020	−0.0212	0.0019
		25	α	−0.0089	0.0169	0.0241	0.0158	0.0166	0.0045	0.0173	0.0039
			β	0.0296	0.0089	−0.0277	0.0072	0.0101	0.0019	−0.0183	0.0017
	70	50	α	0.0037	0.0077	0.0254	0.0080	0.0116	0.0019	0.0139	0.0020
			β	0.0095	0.0031	−0.0235	0.0032	0.0036	0.0007	−0.0134	0.0008
		60	α	0.0041	0.0069	0.0212	0.0070	0.0104	0.0019	0.0104	0.0018
			β	0.0050	0.0023	−0.0241	0.0026	0.0010	0.0005	−0.0135	0.0006
	150	110	α	−0.0030	0.0038	0.0085	0.0037	0.0045	0.0008	0.0050	0.0008
			β	0.0072	0.0014	−0.0110	0.0014	0.0025	0.0003	−0.0065	0.0003
		130	α	−0.0024	0.0039	0.0074	0.0039	0.0036	0.0009	0.0037	0.0008
			β	0.0064	0.0013	−0.0101	0.0013	0.0020	0.0002	−0.0061	0.0003
0.8	30	20	α	−0.0071	0.0204	0.0372	0.0203	0.0222	0.0055	0.0243	0.0047
			β	0.0323	0.0104	−0.0324	0.0083	0.0107	0.0019	−0.0202	0.0018
		25	α	0.0002	0.0174	0.0323	0.0168	0.0198	0.0052	0.0216	0.0047
			β	0.0277	0.0085	−0.0287	0.0070	0.0107	0.0020	−0.0174	0.0018
	70	50	α	0.0012	0.0088	0.0225	0.0089	0.0118	0.0023	0.0129	0.0022
			β	0.0104	0.0029	−0.0222	0.0029	0.0032	0.0005	−0.0132	0.0006
		60	α	−0.0005	0.0080	0.0164	0.0078	0.0090	0.0020	0.0099	0.0019
			β	0.0107	0.0029	−0.0186	0.0028	0.0036	0.0006	−0.0115	0.0007
	150	110	α	−0.0004	0.0041	0.0111	0.0042	0.0051	0.0009	0.0047	0.0009
			β	0.0056	0.0016	−0.0127	0.0016	0.0019	0.0003	−0.0068	0.0004
		130	α	−0.0025	0.0038	0.0083	0.0037	0.0045	0.0008	0.0045	0.0008
			β	0.0073	0.0014	−0.0103	0.0014	0.0027	0.0003	−0.0060	0.0003

$$\omega^{(j)} = \left(\alpha^{(j)}, \beta^{(j)}\right),\ j = 1, 2, \ldots, \mathcal{N}.$$

To avoid the effect of selection of an initial value, the first $\mathcal{N}_0$generated samples are discarded as burn-in period and then the remaining $\mathcal{N} - \mathcal{N}_0$samples will be utilized to carry out the Bayesian inference. Thus, the selected samples of $\phi^{(j)}$, where $j = \mathcal{N}_0 + 1, \mathcal{N}_0 + 2, \ldots, \mathcal{N}$for sufficiently large $\mathcal{N}$, will be used. However, based on the SEL function, the approximate Bayes estimate of α, and β, as in ω, is given by

$$\tilde{\omega} = \sum_{j=\mathcal{N}_0+1}^{\mathcal{N}} \frac{\omega^{(j)}}{\mathcal{N} - \mathcal{N}_0}.$$

Now, to construct the HPD credible intervals of $\alpha$and β, the associated simulated MCMC variates $\omega^{(j)} = \left(\alpha^{(j)}, \beta^{(j)}\right)$for $j = \mathcal{N}_0 + 1, \ldots, \mathcal{N}$must be ordered as $\omega_{(\mathcal{N}_0+1)}, \omega_{(\mathcal{N}_0+2)}, \ldots, \omega_{(\mathcal{N})}$. Hence, following [28], the $100(1-\gamma)\%$two-sided HPD credible interval for $\alpha$and βor any function of them $R(t)$or $h(t)$can be constructed as

$$\left[\omega_{(j^*)}, \omega_{(j^*+(1-\gamma)(\mathcal{N}-\mathcal{N}_0))}\right],$$

where $j^*$is chosen such that

$$\begin{aligned}&\omega_{(j^*+\{(1-\gamma)(\mathcal{N}-\mathcal{N}_0)\})} - \omega_{(j^*)} \\ &\quad = \min_{1\leqslant j\leqslant \gamma(\mathcal{N}-\mathcal{N}_0)} \left(\omega_{(j+\{(1-\gamma)(\mathcal{N}-\mathcal{N}_0)\})} - \omega_{(j)}\right), j^* \\ &\quad = \mathcal{N}_0 + 1, \mathcal{N}_0 + 2, \ldots, \mathcal{N}.\end{aligned}$$

Here $[x]$denotes the largest integer less than or equal to x. In this case, the HPD credible interval of ωhas the shortest length.

Similarly, to carry out the Bayesian estimates and associated HPD credible intervals of $\alpha, \beta, R(t)$and $h(t)$using the MPS approach, one can easily perform the same above steps of the M-H algorithm.

Table 3 Bayesian and non-bayesian estimators bsed on PTII censored sample when $\alpha = 3, \beta = 0.5$.

$\alpha = 3, \beta = 0.5$				Non-Bayeisna				Bayesian			
				MLE		MPS		MLE		MPS	
P	n	m		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.15	30	20	α	0.2054	0.6046	−0.1882	0.3943	0.2198	0.2533	0.0610	0.2272
			β	0.0246	0.0089	−0.0361	0.0079	0.0224	0.0029	−0.0057	0.0024
		25	α	0.2362	0.5203	−0.1364	0.3590	0.2315	0.2581	0.0634	0.1697
			β	0.0255	0.0063	−0.0238	0.0053	0.0198	0.0021	−0.0028	0.0015
	70	50	α	0.1048	0.2139	−0.1108	0.1674	0.1007	0.0706	0.0068	0.0498
			β	0.0103	0.0031	−0.0226	0.0032	0.0089	0.0009	−0.0064	0.0008
		60	α	0.0931	0.2101	−0.1098	0.1704	0.0951	0.0701	0.0065	0.0496
			β	0.0082	0.0028	−0.0208	0.0028	0.0076	0.0008	−0.0056	0.0007
	150	110	α	0.0243	0.0883	−0.0925	0.0861	0.0375	0.0255	−0.0112	0.0222
			β	0.0020	0.0015	−0.0160	0.0016	0.0028	0.0004	−0.0049	0.0004
		130	α	0.0762	0.0864	−0.0376	0.0740	0.0636	0.0291	0.0177	0.0238
			β	0.0046	0.0012	−0.0116	0.0013	0.0035	0.0003	−0.0037	0.0003
0.5	30	20	α	0.2126	0.5748	−0.1892	0.3775	0.2509	0.3421	0.0648	0.1724
			β	0.0220	0.0083	−0.0402	0.0077	0.0204	0.0029	−0.0074	0.0021
		25	α	0.2170	0.4709	−0.1568	0.3407	0.2314	0.2528	0.0715	0.1686
			β	0.0198	0.0070	−0.0338	0.0066	0.0184	0.0025	−0.0056	0.0018
	70	50	α	0.0963	0.2482	−0.1225	0.2008	0.0910	0.0790	0.0013	0.0600
			β	0.0146	0.0034	−0.0182	0.0032	0.0110	0.0010	−0.0036	0.0008
		60	α	0.1201	0.2291	−0.0930	0.1717	0.1047	0.0746	0.0243	0.0585
			β	0.0095	0.0026	−0.0198	0.0026	0.0074	0.0007	−0.0051	0.0006
	150	110	α	0.0387	0.0931	−0.0820	0.0875	0.0470	0.0284	−0.0036	0.0239
			β	0.0025	0.0013	−0.0153	0.0015	0.0033	0.0003	−0.0048	0.0003
		130	α	0.0343	0.0794	−0.0770	0.0790	0.0396	0.0248	−0.0046	0.0210
			β	0.0038	0.0011	−0.0123	0.0012	0.0033	0.0003	−0.0035	0.0003
0.8	30	20	α	0.2331	0.6912	−0.1761	0.4148	0.2526	0.3483	0.0669	0.1832
			β	0.0347	0.0117	−0.0290	0.0090	0.0283	0.0042	−0.0009	0.0028
		25	α	0.2070	0.4692	−0.1743	0.3280	0.2368	0.2648	0.0419	0.1493
			β	0.0237	0.0069	−0.0310	0.0061	0.0204	0.0023	−0.0054	0.0016
	70	50	α	0.1268	0.2525	−0.0958	0.1898	0.1127	0.0828	0.0201	0.0600
			β	0.0121	0.0037	−0.0201	0.0036	0.0096	0.0011	−0.0046	0.0009
		60	α	0.0947	0.1973	−0.1155	0.1567	0.0942	0.0677	0.0086	0.0522
			β	0.0090	0.0026	−0.0202	0.0026	0.0080	0.0007	−0.0051	0.0006
	150	110	α	0.0386	0.0856	−0.0818	0.0821	0.0463	0.0268	−0.0058	0.0211
			β	0.0026	0.0014	−0.0151	0.0015	0.0030	0.0004	−0.0049	0.0004
		130	α	0.0245	0.0789	−0.0886	0.0780	0.0340	0.0242	−0.0123	0.0216
			β	0.0054	0.0012	−0.0109	0.0012	0.0046	0.0003	−0.0026	0.0003

5. Optimal progressive censored plans

The MLE, MPS, and Bayes estimations of the $UW(\alpha, \beta)$ distribution parameters when samples are gathered under PTIICS have all been examined in earlier sections. In the context of dependability, the experimenter may wish to choose an 'optimal' censoring scheme from a set of all conceivable schemes in order to obtain the most information about the unknown parameters of interest. Balakrishnan and Aggarwala [5] initially highlighted the topic of determining the appropriate censoring plan in many situations. However, numerous optimality criteria as well as numerous conclusions on optimal censoring schemes have been discussed in literature. The challenge of comparing two (or more) alternative censoring schemes has received great attention in statistical literature, for example, see Ng et al. [29], Pradhan et al. [30], Lee and Cho [31], and Ashour et al. [32]. However, some commonly-used criteria are used and are given in Table 1 in order to develop the optimum censoring scheme.

In regard to criteria $OA$and OB, our objective is to minimize the determinant and the trace of approximate variance–covariance (AVC) matrices $\mathbf{I}^{-1}\left(\hat{\Omega}\right)$and $\mathbf{I}^{-1}\left(\widehat{\Omega}\right)$with respect to MLEs $\hat{\Omega}$and MPSEs $\widehat{\Omega}$, respectively. Similarly, in regard to criterion OC, maximize the main diagonal elements of the FIM $\mathbf{I}\left(\hat{\Omega}\right)$and $\mathbf{I}\left(\widehat{\Omega}\right)$with respect to MLEs $\hat{\Omega}$and MPSEs $\widehat{\Omega}$, respectively. Clearly, the optimum censoring plan provides more information corresponding to the smallest value of $OA$and OBoptimality criteria and the highest value of OC-optimality criteria.

6. Monte-Carlo Simulation

To compare the performance of the different estimators proposed in the previous sections, we conducted the Monte-Carlo simulation to obtain the MLE and MPSEs when each parameter has non-prior distribution (Non-Bayesian) and has the prior distribution (Bayesian) of unknown parameters

Table 4 Bayesian and non-bayesian estimators bsed on PTII censored sample when $\alpha = 3, \beta = 3$.

$\alpha = 3, \beta = 3$				Non-Bayeisna				Bayesian			
				MLE		MPS		MLE		MPS	
P	n	m		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.15	30	20	α	0.3722	1.4519	−0.0895	0.7188	0.2833	0.4482	0.1209	0.3951
			β	0.2170	0.4509	−0.1652	0.3377	0.1632	0.1488	−0.0052	0.1040
		25	α	0.2334	0.7305	−0.1805	0.4344	0.2208	0.4170	0.0467	0.1887
			β	0.1481	0.2811	−0.1901	0.2431	0.1170	0.0908	−0.0273	0.0671
	70	50	α	0.1331	0.2814	−0.0871	0.2081	0.1152	0.0893	0.0213	0.0631
			β	0.0801	0.1344	−0.1173	0.1274	0.0600	0.0393	−0.0294	0.0328
		60	α	0.1383	0.2595	−0.0770	0.1905	0.1125	0.0814	0.0241	0.0616
			β	0.0841	0.1142	−0.0946	0.1044	0.0591	0.0330	−0.0173	0.0271
	150	110	α	0.0553	0.1031	−0.0663	0.0901	0.0482	0.0305	−0.0024	0.0247
			β	0.0440	0.0488	−0.0654	0.0479	0.0326	0.0133	−0.0174	0.0122
		130	α	0.0340	0.0847	−0.0822	0.0789	0.0368	0.0228	−0.0136	0.0189
			β	0.0291	0.0477	−0.0700	0.0486	0.0241	0.0122	−0.0198	0.0113
0.5	30	20	α	0.2512	0.9048	−0.1797	0.4957	0.2380	0.3645	0.0544	0.2616
			β	0.1838	0.3536	−0.2038	0.2862	0.1496	0.1238	−0.0243	0.0822
		25	α	0.3278	1.0493	−0.1073	0.5730	0.2636	0.3410	0.0832	0.2067
			β	0.1932	0.3567	−0.1495	0.2779	0.1454	0.1217	−0.0104	0.0823
	70	50	α	0.1342	0.2591	−0.0913	0.1911	0.1103	0.0833	0.0186	0.0628
			β	0.1060	0.1398	−0.0929	0.1214	0.0740	0.0396	−0.0135	0.0312
		60	α	0.1222	0.2470	−0.0948	0.1884	0.1024	0.0803	0.0134	0.0618
			β	0.0823	0.1171	−0.0966	0.1071	0.0617	0.0335	−0.0208	0.0271
	150	110	α	0.0413	0.1074	−0.0834	0.0979	0.0446	0.0290	−0.0037	0.0266
			β	0.0231	0.0519	−0.0854	0.0554	0.0226	0.0135	−0.0258	0.0127
		130	α	0.0647	0.0962	−0.0550	0.0823	0.0519	0.0273	0.0024	0.0222
			β	0.0329	0.0417	−0.0657	0.0425	0.0239	0.0109	−0.0196	0.0101
0.8	30	20	α	0.3223	1.2426	−0.1303	0.6170	0.2926	0.5414	0.0961	0.2785
			β	0.2143	0.4378	−0.1748	0.3274	0.1695	0.1519	−0.0038	0.1127
		25	α	0.3381	1.0678	−0.1051	0.5579	0.2955	0.5212	0.0897	0.2171
			β	0.1577	0.3223	−0.1844	0.2687	0.1243	0.1081	−0.0341	0.0684
	70	50	α	0.0890	0.2450	−0.1317	0.1975	0.0960	0.0823	−0.0017	0.0621
			β	0.0563	0.1070	−0.1379	0.1112	0.0501	0.0299	−0.0358	0.0256
		60	α	0.1253	0.2480	−0.0917	0.1868	0.1032	0.0788	0.0144	0.0570
			β	0.0710	0.0997	−0.1062	0.0950	0.0540	0.0280	−0.0264	0.0234
	150	110	α	0.0592	0.1078	−0.0676	0.0940	0.0518	0.0307	0.0026	0.0252
			β	0.0258	0.0486	−0.0827	0.0516	0.0225	0.0122	−0.0259	0.0115
		130	α	−0.0047	0.1968	−0.0910	0.1527	0.0666	0.0623	0.0070	0.0498
			β	0.2051	0.4204	−0.1900	0.3262	0.1531	0.1504	−0.0595	0.0937

for UW distribution based on Type-II progressive samples obtained under different schemes.

6.1. Simulation Design

In the simulation study, the censored sample has been generated for the UW model in the following manner:

1. The sample size (n) of the complete sample was manipulated, resulting in three sizes n = 30, 70, and 150.
2. The size of the progressive censored sample (m) where $m < n$ and $\sum_{i=1}^{m} R_i + m = n$ was manipulated, resulting in two levels m = 20, and 25 when n = 30; m = 50, and 60 when n = 70 and m = 110, and 130 when n = 150.
3. We have arbitrarily chosen true values of (β, α) as (0.5, 0.5), (3, 0.5), and (3, 3).
4. Parameter of binomial removal (P) was manipulated as 0.15, 0.5 and 0.8.
5. Elective hyper parameters are used to assign values to hyper parameters for these three sets of parameter values. For more information see Dey et al. [33], El-Sherpieny et al. [34].

6.2. Simulation Analysis

The different methods outlined in Sections 2–4 are used to estimate model parameters for the simulated datasets. The iterative NR method may be used numerically by implement 'maxLik' package to obtain the desired MLEs and MPSEs. The asymptotic confidence intervals has been obtained by using approximate normal distribution. Bootstrap confidence intervals have been obtained by using Boot-p method and Boot-t method. Bayesian estimators based on gamma priors have been obtained by using MCMC techniques and the MH algorithm.

Table 5 Length of interval estimation for different estimation methods: $\alpha = 0.5, \beta = 0.5$.

$\alpha = 0.5, \beta = 0.5$				Non-Bayesian						Bayesian	
				MLE			MPS			MLE	MPS
P	n	m		LACI	LCIBP	LCIBT	LACI	LCIBP	LCIBT	LCCI	LCCI
0.15	30	20	α	0.5554	0.0261	0.0262	0.4931	0.0238	0.0240	0.2468	0.2318
			β	0.3749	0.0167	0.0167	0.3972	0.0157	0.0158	0.1448	0.1349
		25	α	0.5183	0.0236	0.0234	0.4644	0.0222	0.0225	0.2449	0.2172
			β	0.3428	0.0150	0.0149	0.3643	0.0139	0.0141	0.1460	0.1290
	70	50	α	0.3516	0.0157	0.0158	0.3222	0.0154	0.0152	0.1571	0.1528
			β	0.2291	0.0101	0.0101	0.2485	0.0093	0.0093	0.1009	0.0924
		60	α	0.3457	0.0155	0.0156	0.3199	0.0148	0.0149	0.1667	0.1556
			β	0.2145	0.0102	0.0103	0.2320	0.0093	0.0094	0.0916	0.0893
	150	110	α	0.2518	0.0116	0.0117	0.2361	0.0117	0.0119	0.1186	0.1094
			β	0.1409	0.0064	0.0064	0.1543	0.0059	0.0059	0.0613	0.0592
		130	α	0.2378	0.0110	0.0111	0.2252	0.0107	0.0110	0.1082	0.1011
			β	0.1480	0.0066	0.0067	0.1601	0.0063	0.0062	0.0646	0.0631
0.5	30	20	α	0.5865	0.0264	0.0263	0.5207	0.0268	0.0270	0.2838	0.2337
			β	0.3839	0.0174	0.0176	0.4032	0.0148	0.0150	0.1575	0.1376
		25	α	0.5090	0.0227	0.0228	0.4516	0.0215	0.0215	0.2274	0.2094
			β	0.3511	0.0162	0.0161	0.3710	0.0135	0.0136	0.1505	0.1376
	70	50	α	0.3449	0.0251	0.0250	0.3159	0.0243	0.0242	0.1571	0.1555
			β	0.2174	0.0153	0.0153	0.2370	0.0149	0.0149	0.0921	0.0914
		60	α	0.3248	0.0141	0.0142	0.2997	0.0151	0.0151	0.1645	0.1528
			β	0.1853	0.0082	0.0082	0.2046	0.0076	0.0077	0.0876	0.0845
	150	110	α	0.2406	0.0111	0.0111	0.2257	0.0111	0.0112	0.1104	0.1117
			β	0.1465	0.0065	0.0064	0.1595	0.0059	0.0059	0.0656	0.0656
		130	α	0.2449	0.0108	0.0109	0.2321	0.0110	0.0108	0.1050	0.1043
			β	0.1391	0.0060	0.0060	0.1516	0.0060	0.0060	0.0618	0.0604
0.8	30	20	α	0.5592	0.0252	0.0252	0.4954	0.0235	0.0233	0.2586	0.2270
			β	0.3802	0.0173	0.0174	0.3982	0.0145	0.0147	0.1527	0.1413
		25	α	0.5173	0.0242	0.0250	0.4600	0.0217	0.0215	0.2551	0.2294
			β	0.3443	0.0151	0.0155	0.3645	0.0143	0.0142	0.1588	0.1298
	70	50	α	0.3680	0.0165	0.0162	0.3383	0.0168	0.0168	0.1569	0.1586
			β	0.2062	0.0099	0.0099	0.2262	0.0084	0.0088	0.0878	0.0840
		60	α	0.3504	0.0157	0.0156	0.3244	0.0156	0.0156	0.1631	0.1561
			β	0.2060	0.0094	0.0094	0.2243	0.0089	0.0089	0.0985	0.0899
	150	110	α	0.2527	0.0114	0.0114	0.2378	0.0117	0.0115	0.1100	0.1055
			β	0.1564	0.0070	0.0069	0.1691	0.0068	0.0068	0.0709	0.0672
		130	α	0.2405	0.0114	0.0114	0.2262	0.0103	0.0103	0.1042	0.0991
			β	0.1461	0.0067	0.0067	0.1586	0.0061	0.0061	0.0663	0.0634

6.3. Simulation Outcome

The bias, mean squared error (MSE) and length of confidence intervals (LCI) are computed for each estimated model parameter. Tables 1, and 3, show the results of different scheme for point estimation of the parameters. Tables 2, and 4, show the results of different scheme for intervals estimation of the parameters. The results are shown in Tables 1–4, which include some intriguing data. The estimates get more accurate as the sample size increases, suggesting that they are asymptotically unbiased. Furthermore, in all cases, the MSE decreases as the sample size increases, suggesting that the various estimates are consistent. When comparing the various estimates, we can observe that in most of the cases, the Bayes estimates based on MPS method have the lowest MSE. Also, in terms of minimum LCI, it is noted that the LCIs of α, superior to those produced from the LF approach when using the PS strategy, while the LCIs of β using the LF approach perform better than those obtained from the PS approach. Additionally, the BCIs created using the PS approach typically outperform those created using the LF approach. (see Tables 5–10).

7. Real-life applications

In this section, we give two examples utilizing real-world datasets to demonstrate the flexibility and adaptation of the suggested approaches to real phenomena.

7.1. Flood data

This data set was reported in Dumonceaux and Antle [35], contains 20 observations of the maximum flood level (in millions of cubic feet per second) for the Susquehanna River at Harrisburg, Pennsylvania. The data are: 0.26, 0.27, 0.30, 0.32, 0.32, 0.34, 0.38, 0.38, 0.39, 0.40, 0.41, 0.42, 0.42, 0.42, 0.45, 0.48, 0.49, 0.61, 0.65, 0.74. We first fit the UW distribu-

Table 6 Length of interval estimation for different estimation methods: $\alpha = 3, \beta = 3$

$\alpha = 3, \beta = 3$				Non-Bayesian						Bayesian	
				MLE			MPS			MLE	MPS
P	n	m		LACI	LCIBP	LCIBT	LACI	LCIBP	LCIBT	LCCI	LCCI
0.15	30	20	α	4.4969	0.2002	0.1980	3.7699	0.1549	0.1526	1.7499	1.5988
			β	2.4936	0.1119	0.1119	2.5686	0.0977	0.0950	1.2089	1.1536
		25	α	3.2263	0.1448	0.1460	2.9011	0.1179	0.1161	1.5481	1.3371
			β	1.9978	0.0909	0.0893	2.1233	0.0766	0.0761	0.9920	0.9345
	70	50	α	2.0150	0.0899	0.0891	1.9773	0.0745	0.0770	0.9576	0.8968
			β	1.4039	0.0659	0.0646	1.5204	0.0590	0.0579	0.6883	0.6272
		60	α	1.9236	0.0871	0.0870	1.9013	0.0781	0.0781	0.9808	0.8817
			β	1.2845	0.0591	0.0592	1.3912	0.0529	0.0531	0.5982	0.5757
	150	110	α	1.2412	0.0551	0.0550	1.2702	0.0520	0.0517	0.6400	0.5870
			β	0.8494	0.0398	0.0391	0.9293	0.0361	0.0362	0.4371	0.4156
		130	α	1.1340	0.0504	0.0500	1.1698	0.0451	0.0461	0.5586	0.5205
			β	0.8490	0.0366	0.0368	0.9197	0.0347	0.0355	0.4200	0.4034
0.5	30	20	α	3.5999	0.1601	0.1564	3.1021	0.1200	0.1210	1.6857	1.3777
			β	2.2192	0.0952	0.0953	2.3284	0.0867	0.0852	1.1372	0.9291
		25	α	3.8082	0.1699	0.1652	3.3755	0.1367	0.1359	1.7793	1.4656
			β	2.2175	0.0984	0.0983	2.3262	0.0881	0.0879	1.1562	0.9702
	70	50	α	1.9268	0.0828	0.0824	1.9031	0.0763	0.0761	0.9435	0.8386
			β	1.4069	0.0591	0.0598	1.5169	0.0560	0.0560	0.7148	0.6691
		60	α	1.8904	0.0879	0.0869	1.8792	0.0723	0.0735	0.9660	0.8465
			β	1.3031	0.0601	0.0597	1.4061	0.0533	0.0543	0.6461	0.6120
	150	110	α	1.2759	0.0533	0.0537	1.3077	0.0526	0.0528	0.6026	0.5700
			β	0.8890	0.0383	0.0379	0.9692	0.0372	0.0366	0.4465	0.4297
		130	α	1.1905	0.0537	0.0526	1.2245	0.0492	0.0495	0.5955	0.5766
			β	0.7908	0.0349	0.0353	0.8656	0.0333	0.0339	0.3849	0.3756
0.8	30	20	α	4.1872	0.1878	0.1854	3.4920	0.1358	0.1368	1.8872	1.6826
			β	2.4564	0.1092	0.1105	2.5269	0.0941	0.0932	1.1986	1.0548
		25	α	3.8315	0.1744	0.1652	3.3450	0.1305	0.1275	1.7652	1.5038
			β	2.1401	0.0935	0.0939	2.2432	0.0885	0.0886	1.0270	0.9219
	70	50	α	1.9105	0.0860	0.0859	1.8863	0.0762	0.0761	0.9641	0.8621
			β	1.2643	0.0564	0.0562	1.3852	0.0557	0.0560	0.6094	0.5611
		60	α	1.8913	0.0873	0.0871	1.8745	0.0740	0.0744	0.8720	0.8215
			β	1.2069	0.0550	0.0549	1.3128	0.0525	0.0528	0.6062	0.5841
	150	110	α	1.2670	0.0581	0.0575	1.3000	0.0531	0.0533	0.6339	0.6007
			β	0.8592	0.0386	0.0378	0.9385	0.0378	0.0375	0.4111	0.3976
		130	α	1.7406	0.0790	0.0781	1.5774	0.0679	0.0653	0.8740	0.8510
			β	2.4137	0.1103	0.1084	2.5084	0.0979	0.0983	1.2654	1.0397

Table 7 Length of interval estimation for different estimation methods: $\alpha = 3, \beta = 0.5$

$\alpha = 3, \beta = 0.5$				Non-Bayesian						Bayesian	
				MLE			MPS			MLE	MPS
P	n	m		LACI	LCIBP	LCIBT	LACI	LCIBP	LCIBT	LCCI	LCCI
0.15	30	20	α	2.9427	0.1237	0.1255	2.7444	0.1047	0.1064	1.6002	1.3478
			β	0.3577	0.0163	0.0162	0.3784	0.0143	0.0144	0.1763	0.1608
		25	α	2.6744	0.1209	0.1206	2.6618	0.1002	0.1007	1.5589	1.3117
			β	0.2939	0.0136	0.0137	0.3206	0.0120	0.0121	0.1612	0.1422
	70	50	α	1.7676	0.0830	0.0810	1.7613	0.0687	0.0687	0.8764	0.7957
			β	0.2149	0.0095	0.0097	0.2346	0.0087	0.0087	0.1022	0.0968
		60	α	1.7612	0.0804	0.0812	1.7643	0.0706	0.0717	0.8293	0.8033
			β	0.2038	0.0092	0.0092	0.2206	0.0088	0.0087	0.1026	0.0939
	150	110	α	1.1620	0.0545	0.0557	1.2096	0.0481	0.0477	0.5862	0.5646
			β	0.1495	0.0063	0.0064	0.1632	0.0065	0.0065	0.0708	0.0697
		130	α	1.1141	0.0508	0.0509	1.1708	0.0462	0.0462	0.6077	0.5690
			β	0.1352	0.0063	0.0063	0.1474	0.0058	0.0056	0.0675	0.0638
0.5	30	20	α	2.8555	0.1294	0.1235	2.6954	0.1049	0.1028	1.6766	1.4109
			β	0.3465	0.0164	0.0164	0.3680	0.0134	0.0135	0.1811	0.1667
		25	α	2.5545	0.1190	0.1160	2.5800	0.1003	0.1019	1.5981	1.3853
			β	0.3179	0.0145	0.0146	0.3444	0.0124	0.0124	0.1688	0.1553
	70	50	α	1.9181	0.0826	0.0849	1.9102	0.0744	0.0722	0.9330	0.8429
			β	0.2227	0.0103	0.0102	0.2417	0.0091	0.0091	0.1116	0.1067
		60	α	1.8180	0.0784	0.0780	1.7976	0.0677	0.0672	0.9176	0.8440
			β	0.1970	0.0083	0.0083	0.2141	0.0082	0.0083	0.0985	0.0940
	150	110	α	1.1874	0.0526	0.0527	1.2360	0.0500	0.0499	0.6266	0.5840
			β	0.1427	0.0059	0.0060	0.1554	0.0059	0.0059	0.0658	0.0626
		130	α	1.0978	0.0493	0.0493	1.1720	0.0472	0.0466	0.5703	0.5492
			β	0.1303	0.0056	0.0056	0.1432	0.0058	0.0057	0.0644	0.0613
0.8	30	20	α	3.1316	0.1379	0.1391	2.8399	0.1078	0.1063	1.7540	1.4054
			β	0.4017	0.0176	0.0180	0.4179	0.0163	0.0159	0.1953	0.1820
		25	α	2.5622	0.1193	0.1190	2.5219	0.0955	0.0953	1.6192	1.2846
			β	0.3116	0.0138	0.0141	0.3363	0.0132	0.0122	0.1607	0.1489
	70	50	α	1.9079	0.0891	0.0870	1.8904	0.0760	0.0757	0.9277	0.8935
			β	0.2352	0.0107	0.0109	0.2540	0.0099	0.0100	0.1199	0.1088
		60	α	1.7031	0.0808	0.0813	1.6961	0.0655	0.0658	0.8623	0.8459
			β	0.1965	0.0086	0.0087	0.2132	0.0082	0.0081	0.1002	0.0947
	150	110	α	1.1378	0.0507	0.0506	1.1978	0.0475	0.0488	0.6001	0.5466
			β	0.1460	0.0065	0.0064	0.1592	0.0061	0.0060	0.0733	0.0689
		130	α	1.0980	0.0505	0.0494	1.1523	0.0453	0.0458	0.5677	0.5462
			β	0.1315	0.0055	0.0056	0.1440	0.0056	0.0057	0.0669	0.0645

tion to the entire data set along with four unit lifetime distributions as competitors, namely Kumaraswamy (K), Beta, Kumaraswamy Kumaraswamy (KK) distribution proposed by El-Sherpieny et al. [36], and unit-Gompertz (UG) distribution proposed by Mazucheli et al. [37].

The Kolmogorov–Smirnov distance (DKS) goodness-of-fit test statistic with p-values (PVKS) is used to verify the validity of the given model. The values of MLEs of the model parameters, corresponding standard error (SE), and the statistic values of DKS along with PVKS are reported in Table 11. Moreover, we use a graphical method for goodness-of-fit of UW distribution. We draw estimated pdf with histogram, cdf with empirical and P-P plots of UW distribution in Fig. 2. Fig. 3 shows the log-likelihood function of parameters for the UW distribution as an example, demonstrating the existence and uniqueness of MLEs.

Using the real data set, we generate some PTIICS samples with $m = 15$ using three different sampling schemes when P = 0.15, 0.5 and 0.8. The generated data and the corresponding censored schemes are reported as follows:

When m = 15 and P = 0.15, the data is 0.26 0.27 0.32 0.32 0.34 0.38 0.38 0.41 0.42 0.42 0.45 0.48 0.49 0.61 0.74. R is 1 2 0 1 0 0 1 0 0 0 0 0 0 0 0.

When m = 15 and P = 0.5, the data is 0.26 0.27 0.32 0.38 0.38 0.39 0.40 0.41 0.42 0.42 0.45 0.48 0.49 0.61 0.74. R is 1 3 0 1 0 0 0 0 0 0 0 0 0 0 0.

When m = 15 and P = 0.8, the data is 0.26 0.27 0.32 0.32 0.38 0.39 0.40 0.41 0.42 0.42 0.45 0.48 0.61 0.65 0.74. R is 1 2 2 0 0 0 0 0 0 0 0 0 0 0 0.

By using these data, we have obtained MLEs and Bayes estimates for α and β for UW distribution in Table 12. We produce 10,000 MCMC samples using the MCMC algorithm as described earlier. The MLEs $\hat{\alpha}$ and $\hat{\beta}$ are used as the starting values of the unknown parameters in α and β for conducting the MCMC sampler algorithm. Table 12 also includes two-sided 95 percent ACI/HPD credible intervals. Some optimally

Table 8 Optimality measures baesd on PTII censored sample: $\alpha = 0.5, \beta = 0.5$

$\alpha = 0.5, \beta = 0.5$			Non-Bayheain						Bayesian	
			MLE			MPS			MLE	MPS
P	n	m	OA	OB	OC	OA	OB	OC	PC	PC
0.15	30	20	0.02850	0.0000695	461.24682	0.02871	0.0000699	445.55332	0.13743	0.14203
			0.02426	0.0000584	458.92626	0.02416	0.0000571	459.09608	0.13841	0.14240
	70	50	0.01164	0.0000123	986.66143	0.01169	0.0000122	993.88556	0.14011	0.14258
			0.01001	0.0000101	1033.98240	0.00999	0.0000099	1045.37686	0.14066	0.14291
	150	110	0.00534	0.0000027	2037.51870	0.00535	0.0000026	2064.26971	0.14238	0.14371
			0.00454	0.0000021	2209.77400	0.00453	0.0000021	2233.20329	0.14159	0.14274
0.5	30	20	0.02983	0.0000763	448.46834	0.02997	0.0000751	439.79503	0.13744	0.14197
			0.02418	0.0000574	462.85506	0.02408	0.0000561	461.50294	0.13667	0.14083
	70	50	0.01191	0.0000131	942.88970	0.01193	0.0000128	962.87376	0.14143	0.14397
			0.00997	0.0000101	1027.44388	0.00996	0.0000099	1044.15630	0.14211	0.14414
	150	110	0.00533	0.0000028	1969.95688	0.00533	0.0000027	2005.11916	0.14149	0.14279
			0.00455	0.0000021	2159.41840	0.00455	0.0000021	2186.70742	0.14123	0.14241
0.8	30	20	0.03011	0.0000775	433.34810	0.03025	0.0000762	429.71528	0.13730	0.14193
			0.02464	0.0000604	454.21395	0.02448	0.0000585	455.42892	0.13744	0.14158
	70	50	0.01188	0.0000131	943.54773	0.01187	0.0000128	963.79630	0.13984	0.14223
			0.00997	0.0000101	1027.92436	0.00995	0.0000099	1043.85282	0.14044	0.14264
	150	110	0.00535	0.0000028	1953.30552	0.00535	0.0000027	1991.65593	0.14186	0.14310
			0.00505	0.0000026	2021.89534	0.00505	0.0000025	2056.10811	0.14138	0.14264

Table 9 Optimality measures baesd on PTII censored sample: $\alpha = 3, \beta = 0.5$.

$\alpha = 3, \beta = 0.5$			Non-Bayheain						Bayesian	
			MLE			MPS			MLE	MPS
P	n	m	OA	OB	OC	OA	OB	OC	PC	PC
0.15	30	20	0.5882	0.0036807	177.6308	0.4321	0.0023757	202.2323	0.0448	0.0491
			0.5329	0.0022213	254.3353	0.4112	0.0015504	283.2846	0.0431	0.0472
	70	50	0.2107	0.0005103	429.6049	0.1777	0.0003986	464.9271	0.0429	0.0453
			0.1995	0.0004018	515.8358	0.1708	0.0003212	552.9168	0.0435	0.0458
	150	110	0.0914	0.0000985	948.2673	0.0832	0.0000857	993.0855	0.0434	0.0448
			0.0899	0.0000818	1116.3997	0.0824	0.0000721	1162.2078	0.0413	0.0425
0.5	30	20	0.5941	0.0037450	174.9001	0.4336	0.0023656	201.8834	0.0442	0.0484
			0.5495	0.0027302	217.4690	0.4205	0.0018626	245.8926	0.0435	0.0473
	70	50	0.2212	0.0005476	422.3044	0.1850	0.0004244	456.7848	0.0438	0.0462
			0.2091	0.0004185	515.9126	0.1769	0.0003311	552.7153	0.0423	0.0445
	150	110	0.0976	0.0001045	951.0981	0.0882	0.0000904	995.0904	0.0429	0.0443
			0.0889	0.0000812	1111.4804	0.0815	0.0000716	1156.9771	0.0430	0.0442
0.8	30	20	0.6260	0.0043338	167.9800	0.4463	0.0026596	194.2324	0.0443	0.0486
			0.5458	0.0027376	213.7845	0.4114	0.0018368	242.0178	0.0436	0.0481
	70	50	0.2285	0.0005597	430.3215	0.1902	0.0004320	464.9486	0.0426	0.0450
			0.2043	0.0004104	514.5521	0.1732	0.0003249	551.3688	0.0429	0.0452
	150	110	0.0984	0.0001056	951.3981	0.0889	0.0000912	995.3363	0.0429	0.0443
			0.0885	0.0000815	1102.8441	0.0809	0.0000715	1148.5550	0.0432	0.0445

criteria of different schemes have been obtained in Table 13. Fig. 4 shows histogram for Bayesian MCMC results with HDI intervals based on complete sample. To check convergence, we draw the trace plot of MCMC results in Fig. 5. One of the primary difficulties with likelihood iterations is that it is usually impossible to validate the existence and uniqueness of the MLEs $\hat{\alpha}$ and $\hat{\beta}$ using analytical processes. Figs. 6 and 10 shows a contour map of the log-likelihood function for α and β acting on the entire data set to solve this problem. It indicates that the most suitable starting values of α and β are close to $\hat{\alpha}$ and $\hat{\beta}$, respectively. In addition, it indicates that the MLEs $\hat{\alpha}$ and $\hat{\beta}$ are exist and unique.

It is observed from Table 12 that the Bayes estimates performs better than MLE where SE is the smallest and the length of CI is shorter. Table 13 shows that when P = 0.8, *OA* provides the optimum progressive Type-II censored scheme while

Table 10 Optimality measures baesd on PTII censored sample: $\alpha = 3, \beta = 3$.

$\alpha = 3, \beta = 3$			Non-Bayheain						Bayesian	
			MLE			MPS			MLE	MPS
P	n	m	OA	OB	OC	OA	OB	OC	PC	PC
0.15	30	20	1.1065	0.1842	7.7581	0.7952	0.1092	9.2764	0.0148	0.0161
			0.8256	0.1080	9.0645	0.6278	0.0689	10.7061	0.0151	0.0162
	70	50	0.3276	0.0193	18.3796	0.2830	0.0150	20.3719	0.0149	0.0156
			0.3027	0.0153	21.0303	0.2634	0.0122	23.0684	0.0149	0.0155
	150	110	0.1418	0.0037	39.5188	0.1304	0.0032	41.9505	0.0149	0.0153
			0.1279	0.0029	45.7303	0.1183	0.0025	48.2365	0.0151	0.0154
0.5	30	20	0.9734	0.1599	7.4367	0.7146	0.0953	9.0195	0.0152	0.0166
			0.9335	0.1288	8.7990	0.6964	0.0809	10.4168	0.0147	0.0159
	70	50	0.3381	0.0205	17.7713	0.2899	0.0158	19.7533	0.0148	0.0154
			0.3030	0.0154	20.9695	0.2628	0.0122	23.0215	0.0148	0.0155
	150	110	0.1451	0.0038	39.3697	0.1327	0.0033	41.8313	0.0152	0.0156
			0.1324	0.0030	45.2478	0.1221	0.0026	47.7226	0.0150	0.0153
0.8	30	20	1.0657	0.1842	7.4297	0.7629	0.1063	8.9926	0.0147	0.0160
			0.9301	0.1269	8.8761	0.6864	0.0783	10.5211	0.0150	0.0162
	70	50	0.3274	0.0193	18.1365	0.2806	0.0149	20.1596	0.0151	0.0158
			0.3032	0.0154	20.9445	0.2629	0.0121	22.9859	0.0149	0.0156
	150	110	0.1474	0.0039	39.1611	0.1346	0.0033	41.6080	0.0151	0.0155
			0.5518	0.0665	9.2596	0.4637	0.0471	10.8213	0.0149	0.0164

Table 11 MLE with SE and KS statistics for different distributions.

distributions		α	β	θ	λ	DKS	PVKS
UW	estimates	1.0249	3.9037			0.1448	0.7956
	SE	0.2387	0.7077				
K	estimates	3.3777	12.0060			0.2175	0.3004
	SE	0.6043	5.4739				
Beta	estimates	6.8315	9.2373			0.2063	0.3625
	SE	2.1181	2.8915				
UG	estimates	0.0151	4.1148			0.1522	0.7428
	SE	0.0130	0.7375				
KK	estimates	12.1697	2.3118	47.8991	0.1626	0.1865	0.4897
	SE	6.1200	0.0149	0.0138	0.0381		

for P = 0.15, *OB* and *OC* provides the optimum progressive Type-II censored schemes.

7.2. COVID-19 data

The Second data set represents mortality rate due to COVID-19 from 3 November 2021 to 11 November 2021 in France. The data are: 0.4823, 0.6861, 0.1458, 0.3640, 0.1163, 0.7850, 0.6095, 0.4491, 0.2458, 0.6360, 0.2308, 0.2448.

The DKS goodness-of-fit test statistic with PVKS is used to verify the validity of the given model. The values of MLEs of the model parameters and the corresponding SE are reported in Table 14. Moreover, we use a graphical method for goodness-of-fit of UW distribution. We draw estimated pdf with histogram, cdf with empirical and P-P plots of UW distribution in Fig. 7.

By using COVID-19 data set, we have obtained MLEs, MPSEs and Bayes estimates for the unknown parameters α and β for UW distribution and presented in Table 15. The MLEs $\hat{\alpha}$ and $\hat{\beta}$ are used as the starting values of the unknown parameters in α and β for conducting the MCMC sampler algorithm for COVID-19 data. Table 15 includes two-sided 95 percent ACI/HPD credible intervals for COVID-19 data.

Some optimally criteria of different schemes have been obtained for MLE and MPSE in Table 16. Fig. 8 shows histogram for Bayesian MCMC results with HDI intervals based on complete sample. To check convergence, we draw the trace plot of MCMC results in Fig. 9. Fig. 11 shows the log-likelihood function of parameters for the UW distribution as an example, demonstrating the existence and uniqueness of MLEs.

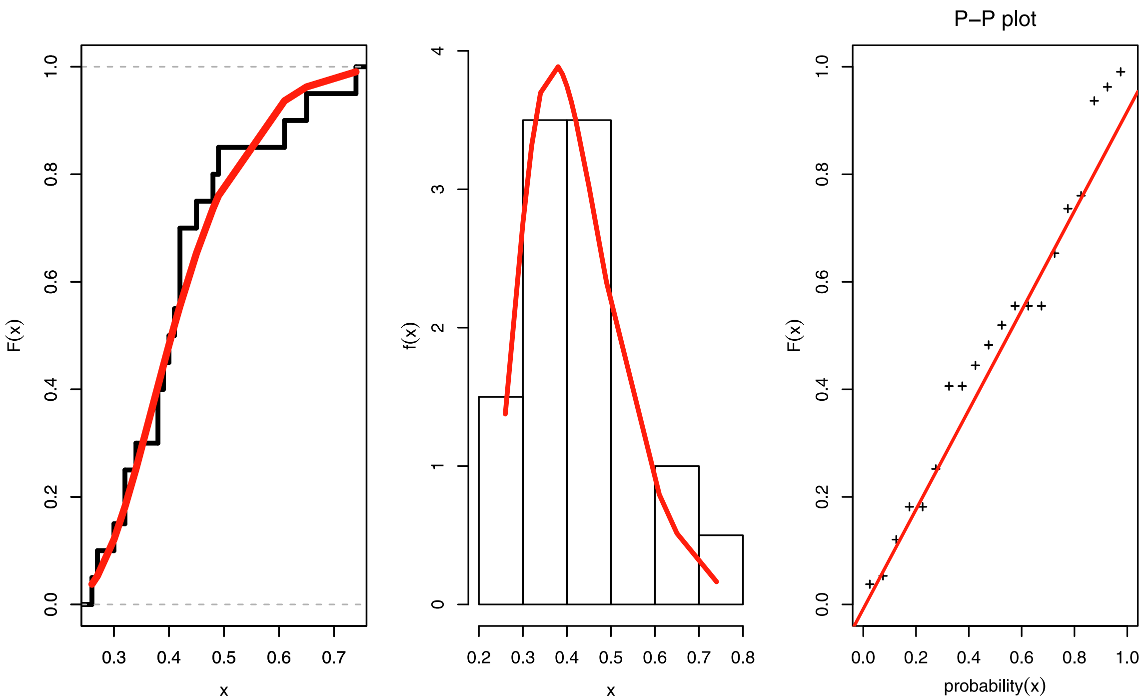

Fig. 2 Estimated pdf, cdf and p-p plot for UW distribution.

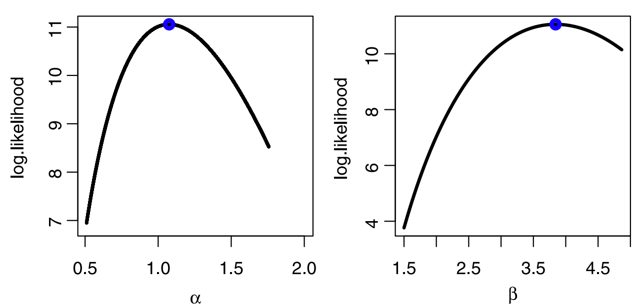

Fig. 3 global maximums of log-likelihood values by parameters in complete sample.

8. Conclusion

This study uses a progressively Type-II censoring technique to statistically deduce the unknown parameters of the UW lifetime model. The MLEs, MPSEs as well as associated ACIs

Table 13 Some optimally criteria of different schemes.

P	OA	OB	OC
0.15	0.6841	0.0391	17.5122
0.5	0.6539	0.0442	14.7937
0.8	0.6030	0.0393	15.3396

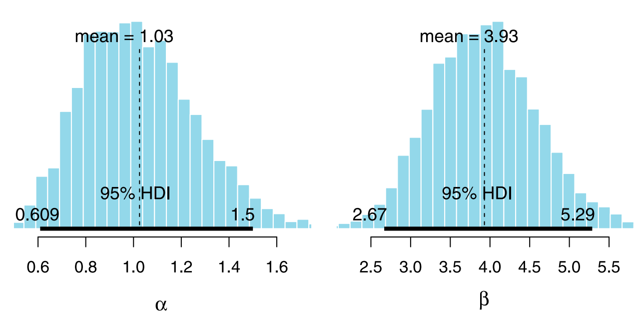

Fig. 4 histogram for Bayesian MCMC results with HDI intervals based on complete sample.

Table 12 MLE and Bayesian with confidence intervals with different schemes.

			MLE				Bayesian			
m	p		estimates	SE	Lower	Upper	estimates	SE	Lower	Upper
20	0	α	1.0249	0.2387	0.5571	1.4928	1.0252	0.2359	0.6090	1.4999
		β	3.9037	0.7077	2.5166	5.2908	3.9298	0.6714	2.6678	5.2880
15	0.15	α	1.0754	0.2697	0.9328	1.2180	1.0628	0.2445	0.6047	1.5359
		β	3.8431	0.7819	2.6449	5.0413	3.9162	0.7699	2.4792	5.4547
	0.5	α	1.1688	0.2909	1.0030	1.3346	1.1863	0.2975	0.6333	1.7518
		β	3.8237	0.7545	2.7080	4.9395	3.8380	0.7545	2.3702	5.2632
	0.8	α	1.1644	0.2879	1.0019	1.3268	1.1794	0.2776	0.6525	1.7018
		β	3.5172	0.7212	2.4979	4.5365	3.5521	0.7156	2.2326	5.0038

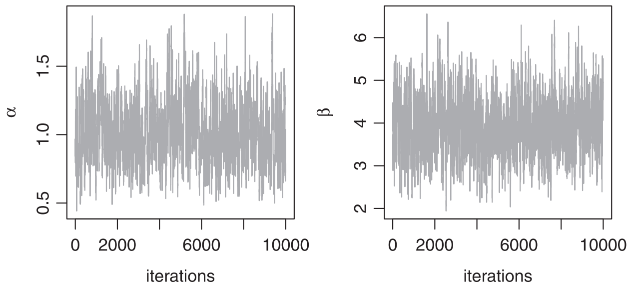

Fig. 5 Trace for Bayesian MCMC results based on complete sample.

are obtained. Gamma informative priors with regard to the squared-error loss function are used to produce the Bayes estimates based on LF and the product of spacing functions. The point Bayesian estimates and their corresponding credible intervals were obtained using the MH method. The results of Monte Carlo simulations demonstrated that the Bayes approach-based point and interval estimates outperformed the frequentist approach in terms of the fewest MSEs and LCIs. An ideal gradually censoring scheme has been presented using several optimality criteria measurements. To show how suggested estimations are applied in practice, two actual data sets pertaining to flood level and COVID-19 are investigated. When such a life test is required, we expect that the approaches

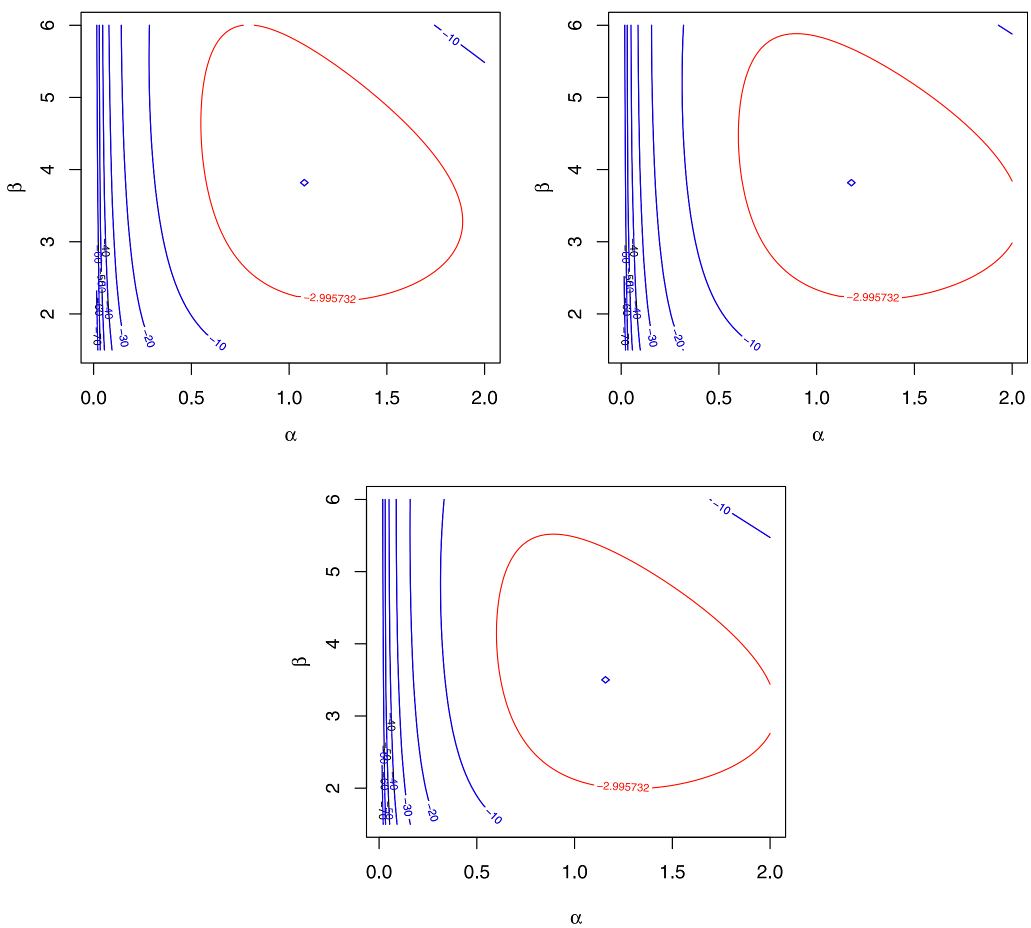

Fig. 6 Contour plot for log-likelihood with different values of parameters when m = 15.

Table 14 MLE with SE and KS statistics for different distributions for COVID-19 data.

distributions		α	β	θ	λ	DKS	PVKS
UW	estimates	0.7502	1.8102			0.1663	0.8416
	SE	0.2490	0.4187				
K	estimates	1.7464	2.9521			0.1828	0.7536
	SE	0.5196	1.3761				
Beta	estimates	1.9863	2.7769			0.1823	0.7565
	SE	0.7632	1.1047				
UG	estimates	0.3385	1.1030			0.1309	0.9697
	SE	0.3405	0.4954				
KK	estimates	27.1400	0.9495	22.6990	0.0782	0.1675	0.8358
	SE	2.1265	0.0556	3.1565	0.0233		

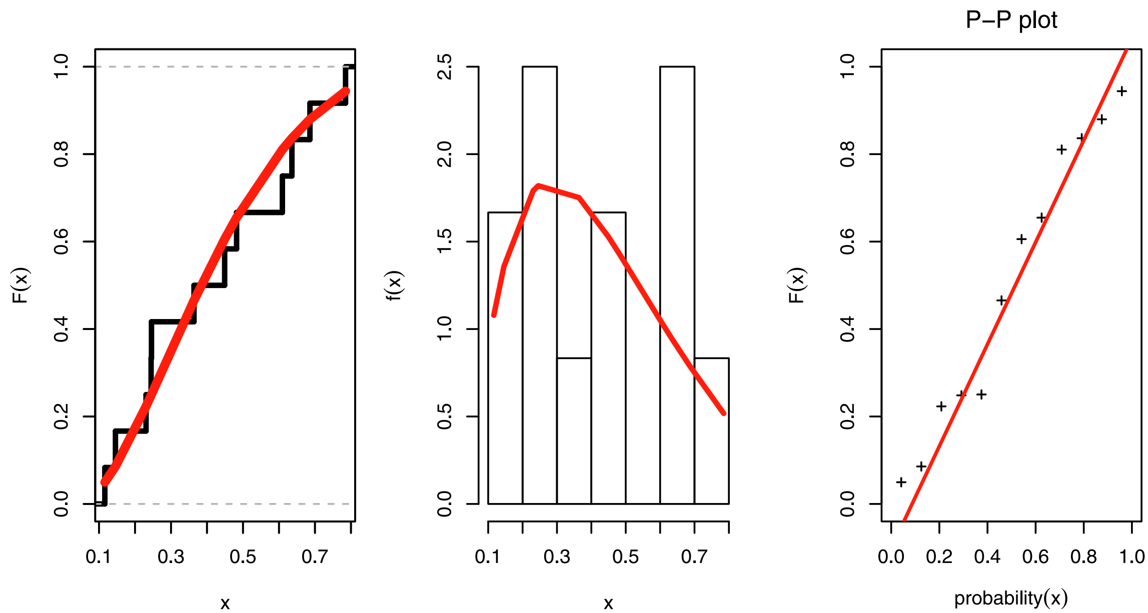

Fig. 7 Estimated pdf, cdf and p-p plot for UW distribution for COVID-19 data.

Table 15 MLE, MPSE and Bayes estimates along with confidence intervals under different schemes for COVID-19 data.

			ML				MPS				Bayesian			
m	p		estimates	SE	Lower	Upper	estimates	SE	Lower	Upper	estimates	SE	Lower	Upper
12	0	α	0.7502	0.2490	0.2621	1.2383	0.9276	0.2625	0.7925	1.0626	0.7375	0.2161	0.3299	1.1633
		β	1.8102	0.4187	0.9895	2.6309	1.3047	0.3193	1.1048	1.5046	1.8504	0.4118	1.1120	2.6728
10	0.2	α	1.0140	0.2817	0.8585	1.1695	0.8577	0.2630	0.7221	0.9932	1.0447	0.2758	0.5403	1.5644
		β	1.4325	0.3255	1.2248	1.6402	1.3303	0.3629	1.0722	1.5885	1.4283	0.3135	0.8593	2.0594
	0.5	α	1.0140	0.2817	0.8585	1.1695	0.8577	0.2630	0.7221	0.9932	1.0447	0.2758	0.5403	1.5644
		β	1.4325	0.3255	1.2248	1.6402	1.3303	0.3629	1.0722	1.5885	1.4283	0.3135	0.8593	2.0594
	0.8	α	0.9299	0.2708	0.7862	1.0737	0.7640	0.2499	0.6416	0.8864	0.9375	0.2544	0.4666	1.4290
		β	1.5610	0.3573	1.3109	1.8112	1.4962	0.4097	1.1673	1.8252	1.5447	0.3403	0.9118	2.2185

Table 16 Some optimally criteria of MLE and MPSE for COVID-19 data.

p	0.2		0.5		0.8	
	MLE	MPS	MLE	MPS	MLE	MPS
OA	0.1853	0.2008	0.1853	0.2008	0.2010	0.2303
OB	0.0072	0.0073	0.0072	0.0073	0.0077	0.0078
OC	25.6298	27.5074	25.6298	27.5074	26.2272	29.4059

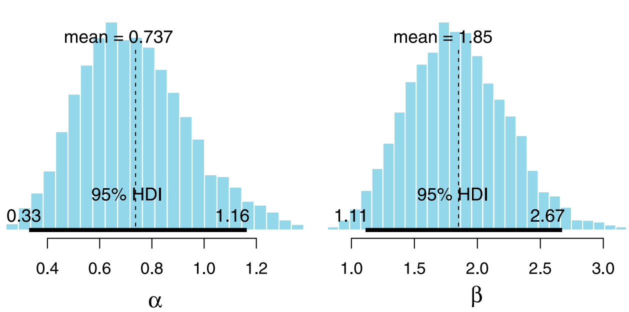

Fig. 8 histogram for Bayesian MCMC results with HDI intervals based on complete sample for COVID-19 data.

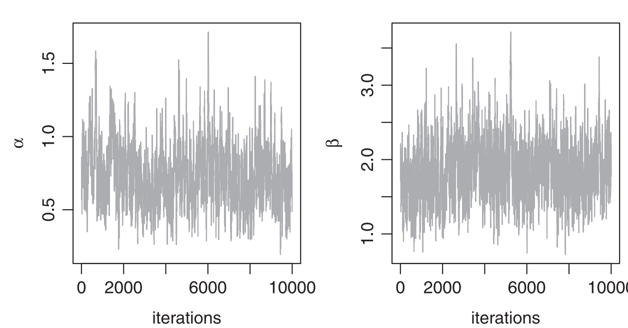

Fig. 9 Trace for Bayesian MCMC results based on complete sample for COVID-19 data.

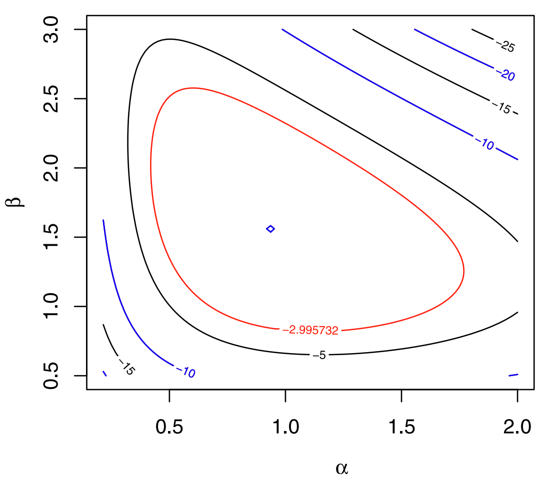

Fig. 10 Contour plot for log-likelihood with different values of parameters for COVID-19 data when m = 12 and p = 0.8.

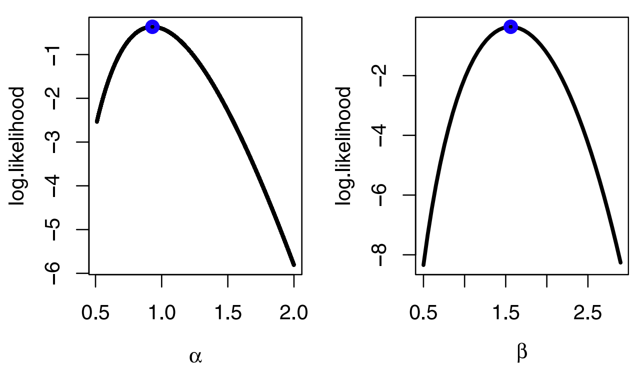

Fig. 11 global maximums of log-likelihood values by parameters for COVID-19 data when m = 12 and p = 0.8.

provided in this study will be helpful to statisticians and academics.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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