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APPROXIMATION OF TWO GENERAL FUNCTIONAL EQUATIONS IN 2–BANACH SPACES

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Abstract. In this paper, we study the Ulam stability and hyperstability of two general functional equations in several variables in 2-Banach spaces. Multi-additive and multi-Jensen functions are particular cases of these functional equations. We also improve the main results of Theorem 3 and Theorem 4 of [Ciepliński, K. Ulam stability of functional equations in 2-Banach spaces via the fixed point method. *J. Fixed Point Theory Appl.* **23** (2021), no. 3, Paper No. 33, 14 pp.] and their consequences.

1. Introduction and preliminaries

Assume that X is a linear space over the field $\mathbb{F}$, and Y is a linear space over the field $\mathbb{K}$. Let $a_{11}, a_{12}, \ldots, a_{n1}, a_{n2} \in \mathbb{F}$, $a_{1,j_1,\cdots,j_n}, \ldots, a_{n,j_1,\cdots,j_n} \in \mathbb{F}$ for $j_1, \cdots, j_n \in \{-1,1\}$ and $A_{i_1,\ldots,i_n} \in \mathbb{K}$ for $i_1, \ldots, i_n \in \{1,2\}$ be given scalars. The following quite general functional equations were very recently introduced by Ciepliński [5, 6]:

$$\begin{aligned} &f(a_{11}x_{11} + a_{12}x_{12}, \ldots, a_{n1}x_{n1} + a_{n2}x_{n2}) \\ &= \textstyle\sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n} f(x_{1i_1}, \ldots, x_{ni_n}) \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\textstyle\sum_{j_1,\ldots,j_n \in \{-1,1\}} f(a_{1,j_1,\cdots,j_n}(x_{11} + j_1x_{12}) + \cdots + a_{n,j_1,\cdots,j_n}(x_{n1} + j_nx_{n2})) \\ &= \textstyle\sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n} f(x_{1i_1}, \ldots, x_{ni_n}) \end{aligned} \tag{2}$$

He studied the Ulam stability of the functional equations (1) and (2) in 2-Banach spaces [6]. The functional equation (1) generalizes among others the known functional equations

$$f(x_{11} + x_{12}, \ldots, x_{n1} + x_{n2}) = \sum_{i_1,\ldots,i_n \in \{1,2\}} f(x_{1i_1}, \ldots, x_{ni_n}),$$

$$f\left(\frac{x_{11} + x_{12}}{2}, \ldots, \frac{x_{n1} + x_{n2}}{2}\right) = \sum_{i_1,\ldots,i_n \in \{1,2\}} \frac{1}{2^n} f(x_{1i_1}, \ldots, x_{ni_n}).$$

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These functional equations and other special cases of the functional equation (1) have been investigated by some authors (see for example [1, 4, 13]). Let us also mention that for the case $n = 1$, we obtain the linear functional equation

$$f(\alpha x+\beta y) = Af(x)+Bf(y)$$

which includes, among others, the Cauchy equation and the Jensen functional equation.

The well-known Jordan-von Neumann equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y)$$

is a special case of (2) for $n = 1$. The following functional equation

$$\begin{aligned}\textstyle\sum_{j_1,\dots,j_n\in\{-1,1\}} f(x_{11}+j_1x_{12},\dots,x_{n1}+j_nx_{n2})\\ = \textstyle\sum_{i_1,\dots,i_n\in\{1,2\}} 2^n f(x_{1i_1},\dots,x_{ni_n})\end{aligned}$$

is a particular case of Eq. (2) which characterizes the so-called n-quadratic functions [9, 15]. Also, the functional equation

$$\sum_{j_1,j_2\in\{-1,1\}} f(x_{11}+j_1x_{12},x_{21}+j_2x_{22}) = \sum_{i,j\in\{1,2\}} A_{ij}f(x_{1i},x_{2j})$$

is another particular case of Eq. (2) which was very recently investigated in [8].

In this note, we prove the Ulam stability and hyperstability of functional equations (1) and (2) which improve Ciepliński's results [6, Theorems 3, 4] and their consequences.

2. Preliminaries

First, let us recall some basic definitions and facts concerning 2-normed spaces (see for instance [2, 11, 14]).

DEFINITION 1. Let $\mathscr{Y}$ be an at least 2-dimensional real linear space. A function $\|.,.\| : \mathscr{Y}^2 \to \mathbb{R}$ is called a 2*-norm* on $\mathscr{Y}^2$ if it fulfils the following four conditions:

(*i*) $\|x,y\| = 0$ if and only if x,y are linearly dependent;

(*ii*) $\|x,y\| = \|y,x\|$;

(*iii*) $\|\alpha x,y\| = |\alpha|\|x,y\|$;

(*iv*) $\|x+y,z\| \leqslant \|x,z\|+\|y,z\|$,

for any $\alpha\in\mathbb{R}$ and $x,y,z\in\mathscr{Y}$. The pair $(\mathscr{Y},\|.,.\|)$ is called a 2*-normed space*.

It follows from $(i),(iii)$ and (iv) that the function $\|.,.\|$ is non-negative.

We say that a sequence $\{x_n\}_n$ of elements of a 2-normed space $(\mathscr{Y},\|.,.\|)$ is *Cauchy sequence* provided

$$\lim_{n,k\to\infty}\|x_n - x_k, y\| = 0, \qquad y \in \mathscr{Y}.$$

The sequence $\{x_n\}_n$ is called *convergent* if there is a $y\in\mathscr{Y}$ such that

$$\lim_{n\to\infty}\|x_n - y, z\| = 0, \qquad z \in \mathscr{Y}.$$

In this case we say that y is the limit of $\{x_n\}_n$ and it is denoted by

$$\lim_{n\to\infty} x_n = y.$$

By a 2-*Banach space* we mean a 2-normed space such that each its Cauchy sequence is convergent.

In 2011, W. G. Park [12] introduces a basic property of linear 2-normed spaces as follows:

LEMMA 1. *Let $(\mathscr{Y},\|.,.\|)$ be a 2-normed space.*

(a) *If $x\in\mathscr{Y}$ and $\|x,y\|=0$ for all $y\in\mathscr{Y}$, then $x=0$.*

(b) *For a convergent sequence $\{x_n\}$ in $\mathscr{Y}$,*

$$\lim_{n\to\infty}\|x_n, y\| = \left\|\lim_{n\to\infty} x_n, y\right\|, \quad y\in\mathscr{Y}.$$

By Lemma 1 (a) and (iv), it is obvious that each convergent sequence has exactly one limit and the standard properties of the limit of a sum and a scalar product hold true.

LEMMA 2. *Let $(\mathscr{Y},\|.,.\|)$ be a 2-normed space and $x_1,\cdots,x_n\in\mathscr{Y}\setminus\{0\}$. Suppose that $\varphi:\mathscr{Y}^n\to\mathscr{Y}$ is a function such that $\|\varphi(x_1,\cdots,x_n),y\|=0$ for all $y\in\mathscr{Y}$ with $\|x_i,y\|\neq 0$ for all $1\leqslant i\leqslant n$. Then $\varphi(x_1,\cdots,x_n)=0$.*

Proof. We can choose linearly independent elements $y,z\in\mathscr{Y}$ such that

$$\|x_i,y\|\neq 0 \quad \text{and} \quad \|x_i,z\|\neq 0, \quad 1\leqslant i\leqslant n.$$

Since $\|\varphi(x_1,\cdots,x_n),y\|=0$ and $\|\varphi(x_1,\cdots,x_n),z\|=0$, there exist scalars λ,μ such that $\varphi(x_1,\cdots,x_n)=\lambda y$ and $\varphi(x_1,\cdots,x_n)=\mu z$. Then $\lambda y-\mu z=0$, and we conclude that $\lambda=\mu=0$. Hence $\varphi(x_1,\cdots,x_n)=0$. $\square$

Finally, it should be noted that more information on 2-normed spaces as well as on some problems investigated in them can be found for example in [2, 3, 10, 11, 14].

3. Main results

We recall that a pair (G,d) is said to be a generalized metric space provided G is a nonempty set and $d : G\times G \to [0,+\infty]$ is a function satisfying the standard metric axioms.

We will use the following key theorem to prove our results.

THEOREM 1. [7] *Let (G,d) be a complete generalized metric space and let $J : G \to G$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. If there exists a nonnegative integer k such that $d(J^k x, J^{k+1}x) < \infty$ for some $x \in X$, then the following are true.*

(i) *the sequence $\{J^n x\}$ converges to a fixed point x^* of J;*

(ii) *x^* is the unique fixed point of J in*

$$G^* = \{y \in G : d(J^k x, y) < \infty\};$$

(iii) *$d(y,x^*) \leqslant \frac{1}{1-L} d(y,Jy)$ for all $y \in G^*$.*

For convenience, we set

$$Df(x_{11},x_{12},\cdots,x_{n1},x_{n2}) := f(a_{11}x_{11}+a_{12}x_{12},\ldots,a_{n1}x_{n1}+a_{n2}x_{n2})$$

$$-\textstyle\sum_{i_1,\ldots,i_n\in\{1,2\}} A_{i_1,\ldots,i_n} f(x_{1i_1},\ldots,x_{ni_n}).$$

The following theorem presents a more general result than Theorem 3 of [6].

THEOREM 2. *Assume that $\mathscr{Y}$ is a 2-normed space. Let $\varphi : X^{2n} \to [0,+\infty)$ and $f : X^n \to \mathscr{Y}$ be functions such that*

$$\|Df(x_{11},x_{12},\cdots,x_{n1},x_{n2}),z\| \leqslant \varphi(x_{11},x_{12},\cdots,x_{n1},x_{n2}) \tag{3}$$

for $x_{11},x_{12},\ldots,x_{n1},x_{n2} \in X$ and $z \in \mathscr{Y}$. Then f fulfills equation (1).

Proof. Replacing z by kz in (3) and dividing the resultant inequality by k, we obtain

$$\|Df(x_{11},x_{12},\cdots,x_{n1},x_{n2}),z\| \leqslant \tfrac{1}{k}\varphi(x_{11},x_{12},\cdots,x_{n1},x_{n2}) \tag{4}$$

for $x_{11},x_{12},\ldots,x_{n1},x_{n2} \in X$, $z \in \mathscr{Y}$ and $k \in \mathbb{N}$. Allowing k tending to infinity, we get

$$\|Df(x_{11},x_{12},\cdots,x_{n1},x_{n2}),z\| = 0$$

for $x_{11},x_{12},\ldots,x_{n1},x_{n2} \in X$ and $z \in \mathscr{Y}$. Hence by Lemma 1, f satisfies (1). $\square$

COROLLARY 1. *Assume that $\varepsilon > 0$ and $\mathscr{Y}$ is a 2-normed space. If $f : X^n \to \mathscr{Y}$ is a function satisfying*

$$\|Df(x_{11},x_{12},\cdots,x_{n1},x_{n2}),z\| \leqslant \varepsilon \tag{5}$$

for $x_{11},x_{12},\ldots,x_{n1},x_{n2} \in X$ and $z \in \mathscr{Y}$, then f fulfills equation (1) for $x_1,\ldots,x_n \in X$.

Proof. The result follows from Theorem 2 by letting

$$\varphi(x_{11},x_{12},\cdots,x_{n1},x_{n2}) = \varepsilon. \quad \square$$

THEOREM 3. *Assume that $\mathscr{Y}$ is a 2-Banach space, $g : X \to \mathscr{Y}$ is a surjective function and*

$$\left| \sum_{i_1,\dots,i_n \in\{1,2\}} A_{i_1,\dots,i_n} \right| > 1. \tag{6}$$

Let $\varphi : X \to [0,+\infty)$ and $f : X^n \to \mathscr{Y}$ be a function satisfying

$$\|Df(x_{11},x_{12},\cdots,x_{n1},x_{n2}), g(z)\| \leqslant \varphi(z) \tag{7}$$

for $x_{11},x_{12},\dots,x_{n1},x_{n2}$, $z\in X$. Then there is a unique function $F : X^n \to \mathscr{Y}$ fulfilling equation (1) and

$$\|f(x_1,\dots,x_n) - F(x_1,\dots,x_n), g(z)\| \leqslant \frac{\varphi(z)}{|\sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n}| - 1} \tag{8}$$

for $x_1,\dots,x_n$, $z\in X$.

Proof. Put

$$A := \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n}, \qquad a_i := a_{i1}+a_{i2}, \quad i\in\{1,\dots,n\}.$$

Let us first note that (7) with $x_{i1} = x_{i2} = z_i$ for $i \in \{1,\dots,n\}$ gives

$$\|f(a_1z_1,\dots,a_nz_n) - Af(z_1,\dots,z_n), g(z)\| \leqslant \varphi(z), \qquad (z_1,\dots,z_n,z)\in X^{n+1}. \tag{9}$$

Set $\mathscr{G} := \{T : X^n \to \mathscr{Y}\}$ and define $d : \mathscr{G}\times\mathscr{G} \to [0,+\infty]$ by

$$d(T,S) := \inf\{C\in[0,+\infty] : \ \|(T-S)(x_1,\cdots,x_n), g(z)\| \leqslant C\varphi(z), \ x_1,\cdots,x_n, \ z\in X\}.$$

It can be shown that $(\mathscr{G},d)$ is a complete generalized metric space. Let us define

$$Q : \mathscr{G}\to\mathscr{G}, \quad QT(x_1,\cdots,x_n) = \frac{1}{A}T(a_1x_1,\cdots,a_nx_n).$$

We show that $Q : \mathscr{G}\to\mathscr{G}$ is a strictly contractive operator with the Lipschitz constant $\frac{1}{|A|}$. Let $T,S\in\mathscr{G}$ with $d(T,S)<\infty$ and $\varepsilon>0$. Then

$$\|(T-S)(x_1,\cdots,x_n), g(z)\| \leqslant (d(T,S)+\varepsilon)\varphi(z), \quad x_1,\cdots,x_n, \ z\in X.$$

Consequently

$$\begin{aligned}\|QT(x_1,\cdots,x_n) - QS(x_1,\cdots,x_n), g(z)\| &= \frac{1}{|A|}\|(T-S)(a_1x_1,\cdots,a_nx_n), g(z)\| \\ &\leqslant \frac{1}{|A|}(d(T,S)+\varepsilon)\varphi(z)\end{aligned}$$

for all $x_1,\cdots,x_n,\ z\in X$. Therefore $d(QT,QS)\leqslant \frac{1}{|A|}(d(T,S)+\varepsilon)$. Since $\varepsilon>0$ is arbitrary, we get $d(QT,QS)\leqslant \frac{1}{|A|}d(T,S)$, as claimed. On the other hand, (9) yields

$$\begin{aligned}\|Qf(x_1,\cdots,x_n)-f(x_1,\cdots,x_n),g(z)\| &= \left\|\frac{1}{A}f(a_1x_1,\cdots,a_nx_n)-f(x_1,\cdots,x_n),g(z)\right\|\\ &\leqslant \frac{1}{|A|}\varphi(z),\quad x_1,\cdots,x_n,\ z\in X.\end{aligned}$$

Thus $d(Qf,f)\leqslant\frac{1}{|A|}$. Hence by Theorem 1 (i), we deduce that the sequence $\{Q^mf\}_m$ is convergent in $(\mathscr{G},d)$ and $F=\lim_{m\to\infty}Q^mf$ is a fixed point of Q. Thus

$$\begin{aligned}F(x_1,\cdots,x_n)=\lim_{m\to\infty}Q^mf(x_1,\cdots,x_n)&=\lim_{m\to\infty}\frac{f(a_1^mx_1,\cdots,a_n^mx_n)}{A^m},\\ \frac{1}{A}F(a_1x_1,\cdots,a_nx_n)&=F(x_1,\cdots,x_n),\quad x_1,\cdots,x_n\in X.\end{aligned}$$

Since $f\in\mathscr{G}^*$, Theorem 1 (iii) implies

$$d(f,F)\leqslant\frac{1}{1-\frac{1}{|A|}}d(Qf,f)\leqslant\frac{1}{|A|-1}$$

which proves (8). Now, we show that the function $F:X^n\to\mathscr{Y}$ fulfilling equation (1). Indeed, from (7), we get

$$\left\|\frac{Df(a_1^mx_{11},a_1^mx_{12},\cdots,a_n^mx_{n1},a_n^mx_{n2})}{A^m},g(z)\right\|\leqslant\frac{1}{A^m}\varphi(z)$$

for $x_{11},x_{12},\ldots,x_{n1},x_{n2},\ z\in X$. Letting $m\to\infty$, and applying the definition of F we infer that

$$\|DF(x_{11},x_{12},\cdots,x_{n1},x_{n2}),g(z)\|=0,\quad x_{11},x_{12},\ldots,x_{n1},x_{n2},\ z\in X.$$

Since g is surjective, we deduce that F fulfils equation (1) by Lemma 1 (a).

To prove the uniqueness of F, let $H:X^n\to\mathscr{Y}$ be a solution of (1) satisfying (8). Since H satisfies (1), we get

$$H(a_1x_1,a_2x_2,\cdots,a_nx_n)=AH(x_1,x_2,\cdots,x_n),\quad x_1,x_2,\cdots,x_n\in X.$$

Hence H is a fixed point of Q. On the other hand, (8) yields $d(f,H)\leqslant\frac{1}{|A|-1}$. Hence $H\in\mathscr{G}^*$, and consequently $H=F$ by Theorem 1 (ii). $\Box$

In the following results, $\mathscr{X}$ is a normed linear space.

COROLLARY 2. *Assume that* $\varepsilon,\theta\geqslant 0$ *and* $\mathscr{Y}$ *is an* 2*-Banach space. Let* $g:\mathscr{X}\to\mathscr{Y}$ *be a surjective function and*

$$\left|\sum_{i_1,\ldots,i_n\in\{1,2\}}A_{i_1,\ldots,i_n}\right|>1. \tag{10}$$

If $f:\mathscr{X}^n\to\mathscr{Y}$ *is a function satisfying*

$$\|Df(x_{11},x_{12},\cdots,x_{n1},x_{n2}),g(z)\|\leqslant\varepsilon+\theta\|z\| \tag{11}$$

for $x_{11},x_{12},\ldots,x_{n1},x_{n2},\ z\in\mathscr{X}$, *then there is a unique function* $F:X^n\to\mathscr{Y}$ *fulfilling* (1) *and*

$$\|f(x_1,\ldots,x_n)-F(x_1,\ldots,x_n),g(z)\|\leqslant\frac{\varepsilon+\theta\|z\|}{|\sum_{i_1,\ldots,i_n\in\{1,2\}}A_{i_1,\ldots,i_n}|-1} \tag{12}$$

for $x_1,\ldots,x_n,\ z\in X$.

THEOREM 4. *Assume that* $\varepsilon\geqslant 0$ *and* $\mathscr{Y}$ *is an* 2*-normed space. Let* $\{\alpha_i\}_{i=1}^n$, $\{\beta_i\}_{i=1}^n$ *and* $\{r_i\}_{i=1}^n$ *be nonnegative real numbers with* $\max_{1\leqslant i\leqslant n}r_i<1$, *and let* $f:\mathscr{Y}^n\to\mathscr{Y}$ *be a function such that*

$$\|Df(x_{11},x_{12},\cdots,x_{n1},x_{n2}),z\|\leqslant\varepsilon+\textstyle\sum_{i=1}^n[\alpha_i\|x_{i1},z\|^{r_i}+\beta_i\|x_{i2},z\|^{r_i}] \tag{13}$$

for $x_{11},x_{12},\ldots,x_{n1},x_{n2},\ z\in\mathscr{Y}$. *Then* f *satisfies* (1).

Proof. Replacing z by kz in (13) and dividing the resultant inequality by k, we obtain

$$\|Df(x_{11},x_{12},\cdots,x_{n1},x_{n2}),z\|\leqslant\frac{\varepsilon}{k}+\sum_{i=1}^n k^{r_i-1}[\alpha_i\|x_{i1},z\|^{r_i}+\beta_i\|x_{i2},z\|^{r_i}].$$

Letting now $k\to\infty$, we get

$$\|Df(x_{11},x_{12},\cdots,x_{n1},x_{n2}),z\|=0,\quad x_{11},\ldots,x_{n2},\ z\in\mathscr{Y}.$$

Hence by Lemma 1, f satisfies (1). □

THEOREM 5. *Assume that* $\varepsilon\geqslant 0$ *and* $\mathscr{Y}$ *is an* 2*-normed space. Let* $\{\alpha_i\}_{i=1}^n$, $\{\beta_i\}_{i=1}^n$ *be nonnegative real numbers and* $\{r_i\}_{i=1}^n$ *be real numbers with* $\max_{1\leqslant i\leqslant n}r_i<1$. *Suppose* $f:\mathscr{Y}^n\to\mathscr{Y}$ *satisfies* (13) *for* $x_{11},x_{12},\ldots,x_{n1},x_{n2},\ z\in\mathscr{Y}\setminus\{0\}$ *with* $\|x_{ij},z\|\neq 0$ *for* $1\leqslant i\leqslant n$ *and* $j=1,2$. *Then* f *satisfies* (1) *for* $x_{11},x_{12},\ldots,x_{n1},x_{n2}\in\mathscr{Y}\setminus\{0\}$.

Proof. Let $x_{11},x_{12},\ldots,x_{n1},x_{n2}\in\mathscr{Y}\setminus\{0\}$. By the same argument as above, we get

$$\|Df(x_{11},x_{12},\cdots,x_{n1},x_{n2}),z\|=0$$

for all $z\in\mathscr{Y}$ with $\|x_{ij},z\|\neq 0$ for $1\leqslant i\leqslant n,\ j=1,2$. Thus the result follows from Lemma 2. □

THEOREM 6. *Assume that* $\mathscr{Y}$ *is an* 2*-normed space. Let* $\{\alpha_i\}_{i=1}^n$, $\{\beta_i\}_{i=1}^n$ *and* $\{r_i\}_{i=1}^n$ *be nonnegative real numbers with* $\min_{1\leqslant i\leqslant n}r_i>1$, *and* $f:\mathscr{Y}^n\to\mathscr{Y}$ *be a function such that*

$$\|Df(x_{11},x_{12},\cdots,x_{n1},x_{n2}),z\|\leqslant\textstyle\sum_{i=1}^n[\alpha_i\|x_{i1},z\|^{r_i}+\beta_i\|x_{i2},z\|^{r_i}] \tag{14}$$

for $x_{11},x_{12},\ldots,x_{n1},x_{n2},\ z\in\mathscr{Y}$. *Then* f *satisfies* (1).

Proof. By replacing z by $\frac{z}{k}$ in (14) and applying a similar argument as in the proof of Theorem 4, the result is achieved. □

4. Stability and hyperstability of the functional equation (2)

For convenience, we set

$$\begin{aligned}
&\Delta f(x_{11},x_{12},\cdots,x_{n1},x_{n2})\\
&= \sum_{j_1,\dots,j_n\in\{-1,1\}} f(a_{1,j_1,\cdots,j_n}(x_{11}+j_1x_{12})+\cdots+a_{n,j_1,\cdots,j_n}(x_{n1}+j_nx_{n2}))\\
&\quad - \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} f(x_{1i_1},\dots,x_{ni_n}).
\end{aligned}$$

By an argument similar to the proof of Theorem 2, it can be shown that the following result improves Theorem 4 of [6].

THEOREM 7. *Assume that $\mathscr{Y}$ is a 2-normed space. Let $\varphi : X^{2n} \to [0,+\infty)$ and $f : X^n \to \mathscr{Y}$ be functions such that*

$$\|\Delta f(x_{11},x_{12},\cdots,x_{n1},x_{n2}),z\| \leqslant \varphi(x_{11},x_{12},\cdots,x_{n1},x_{n2})$$

for $x_{11},x_{12},\dots,x_{n1},x_{n2} \in X$ and $z \in \mathscr{Y}$. Then f fulfills equation (2).

The proof of the following theorem is similar to the proof of Theorem 3. Hence, we omit the proof.

THEOREM 8. *Assume that $\mathscr{Y}$ is a 2-Banach space, $g : X \to \mathscr{Y}$ is a surjective function and*

$$\left|\sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n}\right| > 1.$$

Let $\varphi : X \to [0,+\infty)$ and $f : X^n \to \mathscr{Y}$ be a function such that $f(x_1,\cdots,x_n) = 0$ for any $(x_1,\cdots,x_n) \in X^n$ with at least one component which is equal to zero, and satisfying

$$\|\Delta f(x_{11},x_{12},\cdots,x_{n1},x_{n2}),g(z)\| \leqslant \varphi(z)$$

for $x_{11},x_{12},\dots,x_{n1},x_{n2}$, $z \in X$. Then there is a unique function $F : X^n \to \mathscr{Y}$ fulfilling equation (2) *and*

$$\|f(x_1,\dots,x_n) - F(x_1,\dots,x_n),g(z)\| \leqslant \frac{\varphi(z)}{|\sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n}| - 1}$$

for $x_1,\dots,x_n$, $z \in X$.

In the following results, $\mathscr{X}$ is a normed linear space.

COROLLARY 3. *Assume that* $\varepsilon, \theta \geqslant 0$ *and* $\mathscr{Y}$ *is an* 2*-Banach space. Let* $g : \mathscr{X} \to \mathscr{Y}$ *be a surjective function and*

$$\left| \sum_{i_1,\dots,i_n \in \{1,2\}} A_{i_1,\dots,i_n} \right| > 1.$$

If $f : \mathscr{X}^n \to \mathscr{Y}$ *is a function such that* $f(x_1,\cdots,x_n) = 0$ *for any* $x_1,\cdots,x_n$ *in* X *with at least one component which is equal to zero, and satisfying*

$$\|\Delta f(x_{11},x_{12},\cdots,x_{n1},x_{n2}), g(z)\| \leqslant \varepsilon + \theta\|z\| \tag{15}$$

for $x_{11},x_{12},\dots,x_{n1},x_{n2},\ z \in \mathscr{X}$*, then there is a unique function* $F : X^n \to \mathscr{Y}$ *fulfilling* (2) *and*

$$\|f(x_1,\dots,x_n) - F(x_1,\dots,x_n), g(z)\| \leqslant \frac{\varepsilon + \theta\|z\|}{|\sum_{i_1,\dots,i_n \in \{1,2\}} A_{i_1,\dots,i_n}| - 1}$$

for $x_1,\dots,x_n,\ z \in X$.

The proof of the following theorems are similar to the proof of Theorems 4, 5 and 6. Hence, we omit the proofs.

THEOREM 9. *Assume that* $\varepsilon \geqslant 0$ *and* $\mathscr{Y}$ *is an* 2*-normed space. Let* $\{\alpha_i\}_{i=1}^n$, $\{\beta_i\}_{i=1}^n$ *and* $\{r_i\}_{i=1}^n$ *be nonnegative real numbers with* $\max_{1\leqslant i\leqslant n} r_i < 1$*, and let* $f : \mathscr{Y}^n \to \mathscr{Y}$ *be a function such that*

$$\|\Delta f(x_{11},x_{12},\cdots,x_{n1},x_{n2}), z\| \leqslant \varepsilon + \textstyle\sum_{i=1}^n [\alpha_i\|x_{i1},z\|^{r_i} + \beta_i\|x_{i2},z\|^{r_i}] \tag{16}$$

for $x_{11},x_{12},\dots,x_{n1},x_{n2},\ z \in \mathscr{Y}$*. Then* f *satisfies* (2).

THEOREM 10. *Assume that* $\varepsilon \geqslant 0$ *and* $\mathscr{Y}$ *is an* 2*-normed space. Let* $\{\alpha_i\}_{i=1}^n$, $\{\beta_i\}_{i=1}^n$ *be nonnegative real numbers and* $\{r_i\}_{i=1}^n$ *be real numbers with* $\max_{1\leqslant i\leqslant n} r_i < 1$*. Suppose* $f : \mathscr{Y}^n \to \mathscr{Y}$ *satisfies* (16) *for* $x_{11},x_{12},\dots,x_{n1},x_{n2},\ z \in \mathscr{Y}\setminus\{0\}$ *with* $\|x_{ij},z\| \neq 0$ *for* $1 \leqslant i \leqslant n$ *and* $j = 1,2$*. Then* f *satisfies* (2) *for* $x_{11},x_{12},\dots,x_{n1},x_{n2} \in \mathscr{Y}\setminus\{0\}$.

THEOREM 11. *Assume that* $\mathscr{Y}$ *is an* 2*-normed space. Let* $\{\alpha_i\}_{i=1}^n, \{\beta_i\}_{i=1}^n$ *and* $\{r_i\}_{i=1}^n$ *be nonnegative real numbers with* $\min_{1\leqslant i\leqslant n} r_i > 1$*, and* $f : \mathscr{Y}^n \to \mathscr{Y}$ *be a function such that*

$$\|\Delta f(x_{11},x_{12},\cdots,x_{n1},x_{n2}), z\| \leqslant \textstyle\sum_{i=1}^n [\alpha_i\|x_{i1},z\|^{r_i} + \beta_i\|x_{i2},z\|^{r_i}]$$

for $x_{11},x_{12},\dots,x_{n1},x_{n2},\ z \in \mathscr{Y}$*. Then* f *satisfies* (2).

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