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RANDOM STABILITY AND HYPERSTABILITY OF MULTI–QUADRATIC MAPPINGS

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Abstract. In this paper, we introduce a new quadratic functional equation. In light of this equation, we define the multi-quadratic mappings and reduce the system of n equations defining the multi-quadratic mappings to a single equation. We also obtain some stability and hyperstability results concerning multi-quadratic mappings in the setting of random normed spaces.

1. Introduction

The concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1940, Ulam [28] asked the question concerning the stability of group homomorphisms. The famous Ulam stability problem was partially solved by Hyers [13] for the linear functional equation of Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Th. M. Rassias [22] for linear mappings with considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruţa [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. Next, many mathematicians were attracted and motivated to investigate the stability problems of functional equations in various spaces; for more information and details, we refer to some papers and books such as [2], [3], [14], [15], [17], [19], [20], [21] and [23]. In particular, the stability problem for quadratic functional equation

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y) \tag{1.1}$$

has been studied in normed spaces. The generalized Hyers-Ulam stability theorem for (1.1) and miscellaneous versions of quadratic functional equations and their applications were proved by many authors which are available for instance in [6], [7], [16], [25] and [29] the references therein. More results on the stability of functional equations in random normed spaces can be found in [5] and [18].

For the set X, we denote $\overbrace{X\times X\times\cdots\times X}^{n\text{-times}}$ by X^n. Let V be a commutative group, W be a linear space, and $n\geqslant 2$ be an integer. Recall from [11] that a mapping

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$f: V^n \longrightarrow W$ is called *multi-quadratic* if it is quadratic (satisfying quadratic functional equation (1.1)) in each component. It is shown in [30] that the system of functional equations defining a multi-quadratic mappings can be unified as a single equation. Indeed, Zhao et al. proved that a mapping $f: V^n \longrightarrow W$ is multi-quadratic if and only if the relation

$$\sum_{s\in\{-1,1\}^n} f(x_1+sx_2) = 2^n \sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} f(x_{1j_1},x_{2j_2},\ldots,x_{nj_n}) \tag{1.2}$$

holds, where $x_j = (x_{1j},x_{2j},\ldots,x_{nj}) \in V^n$ with $j \in \{1,2\}$. Various versions of multi-quadratic mappings were introduced and studied in [4], [8], [9] and [24].

In this paper, we firstly show that the functional equation

$$Q(mx+y)+Q(mx-y) = Q(x+y)+Q(x-y)+2\left(m^2-1\right)Q(x) \tag{1.3}$$

is quadratic (m is a fixed integer with $m \neq 0,\pm 1$) and motivated by (1.3), we define the multi-quadratic mappings and present a characterization of such mappings. Then, we study some stability results concerning multi-quadratic mappings in the setting of random normed spaces. Furthermore, we show that every multi-quadratic mapping under some conditions can be hyperstable.

2. Preliminaries on random normed spaces

In this section, we state the usual terminology, notations and conventions of the theory of random normed spaces, as in [26] and [27]. The set of all probability distribution functions is denoted by

$$\Delta^+ := \{F : \mathbb{R}\cup\{-\infty,\infty\} \longrightarrow [0,1] \mid F \text{ is left-continuous and nondecreasing on } \mathbb{R}; \text{ where } F(0)=0 \text{ and } F(+\infty)=1\}.$$

Let us define $D^+ := \{F \in \Delta^+ \mid l^-F(+\infty)=1\}$, where $l^-F(x)$ denotes the left limit of the function f at the point x. The set Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leqslant G$ if and only if $F(t) \leqslant G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leqslant 0 \\ 1, & \text{if } t > 0. \end{cases}$$

DEFINITION 2.1. ([26]) A mapping $\tau : [0,1]\times[0,1] \longrightarrow [0,1]$ is said to be a continuous triangular norm (briefly, a continuous t-norm) if τ satisfies the following conditions:

(i) τ is commutative and associative;

(ii) τ is continuous;

(iii) $\tau(a,1) = a$ for all $a \in [0,1]$;

(iv) $\tau(a,b) \leqslant \tau(c,d)$ whenever $a \leqslant c$ and $b \leqslant d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $\tau_P(a,b) = ab$, $\tau_M(a,b) = \min\{a,b\}$ and $\tau_L(a,b) = \max\{a+b-1,0\}$.

DEFINITION 2.2. ([27]) A random normed space (RN-space, in short) is a triple $(\mathscr{X},\mu,\tau)$, where $\mathscr{X}$ is a vector space, τ is a continuous t-norm, and μ is a mapping from $\mathscr{X}$ into D^+ such that the following conditions hold:

(RN1) $\mu_x(t)=\varepsilon_0(t)$ for all $t>0$ if and only if $x=0$;

(RN2) $\mu_{\alpha x}(t)=\mu_x(t/|\alpha|)$ for all $x\in\mathscr{X}$, $\alpha\neq 0$ and all $t\geqslant 0$;

(RN3) $\mu_{x+y}(t+s)\geqslant\tau(\mu_x(t),\mu_y(s))$ for all $x,y\in\mathscr{X}$ and all $t,s\geqslant 0$.

Let $(\mathscr{X},\|\cdot\|)$ be a normed space. Define the mapping $\mu:\mathscr{X}\longrightarrow D^+$ via $\mu_x(t)=\frac{t}{t+\|x\|}$ for all $x\in\mathscr{X}$ and all $t\geqslant 0$. Then $(\mathscr{X},\mu,\tau_M)$ is a random normed space.

DEFINITION 2.3. Let $(\mathscr{X},\mu,\tau)$ be an RN-space.

(1) A sequence $\{x_n\}$ in $\mathscr{X}$ is said to be *convergent* to a point $x\in\mathscr{X}$ if, for every $t>0$ and $\varepsilon>0$, there exists a positive integer N such that $\mu_{x_n-x}(t)>1-\varepsilon$ whenever $n\geqslant N$;

(2) A sequence $\{x_n\}$ in $\mathscr{X}$ is called a *Cauchy sequence* if, for every $t>0$ and $\varepsilon>0$, there exists a positive integer N such that $\mu_{x_n-x_m}(t)>1-\varepsilon$ whenever $n\geqslant m\geqslant N$;

(3) An RN-space $(\mathscr{X},\mu,\tau)$ is said to be *complete* if and only if every Cauchy sequence in $\mathscr{X}$ is convergent to a point in $\mathscr{X}$.

THEOREM 2.4. ([26]) *If $(\mathscr{X},\mu,\tau)$ is an RN-space and $\{x_n\}$ is a sequence such that $x_n\to x$, then* $\lim_{n\to\infty}\mu_{x_n}(t)=\mu_x(t)$.

For a t-norm τ and a given sequence $\{a_n\}$ in $[0,1]$, we define $\tau_{j=1}^{n}a_j$ recursively by $\tau_{j=1}^{1}a_j=a_1$ and $\tau_{j=1}^{n}a_j=\tau(\tau_{j=1}^{n-1}a_j,a_n)$ for all $n\geqslant 2$.

3. Characterization of multi-quadratic mappings

In this chapter, we introduce the multi-quadratic mappings and then characterize them. Here, we indicate an elementary result as follows.

PROPOSITION 3.1. *Let V and W be vector spaces over the rational numbers. Then, a mapping $Q:V\longrightarrow W$ satisfies functional equation (1.1) if and only if equation (1.3) is valid for Q, where m is a fixed integer with $m\neq 0,\pm 1$.*

Proof. (*Necessity*) Assume that Q satisfies (1.1). It is easy to check that $Q(0)=0$ and so $Q(2x)=4Q(x)$ for all $x\in V$. It is also routine to show that $Q(mx)=m^2Q(x)$ for all $x\in V$. Replacing x by mx in (1.1), we have

$$\begin{aligned}Q(mx+y)+Q(mx-y)&=2Q(mx)+2Q(y)\\&=2m^2Q(x)+2Q(y)\\&=2Q(x)+2Q(y)+2(m^2-1)Q(x)\\&=Q(x+y)+Q(x-y)+2\left(m^2-1\right)Q(x).\end{aligned}$$

Therefore, Q satisfies (1.3).

(*Sufficiency*) Putting $y = 0$ in (1.3), we find

$$Q(mx) = m^2 Q(x) \tag{3.1}$$

for all $x \in V$. On the other hand, $Q(-mx) = (-m)^2 Q(x) = m^2 Q(x) = Q(mx)$, and so $Q(-x) = Q(x)$. This means that Q is even. Interchanging y by my in (1.3) and using the eveness of Q, we get

$$\begin{aligned} Q(mx+my)+Q(mx-my) &= Q(x+my)+Q(x-my)+2\left(m^2-1\right)Q(x) \\ &= Q(x+my)+Q(my-x)+2\left(m^2-1\right)Q(x) \end{aligned} \tag{3.2}$$

for all $x, y \in V$. Substituting (x,y) by (y,x) in (3.2) and applying (3.1), we obtain

$$\begin{aligned} m^2[Q(x+y)+Q(x-y)] &= Q(mx+y)+Q(mx-y)+2\left(m^2-1\right)Q(y) \\ &= Q(x+y)+Q(x-y)+2\left(m^2-1\right)Q(x)+2\left(m^2-1\right)Q(y) \end{aligned}$$

for all $x, y \in V$. It now follows from the above relation that Q satisfies the functional equation (1.1). □

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$. For any $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $q = (q_1, \ldots, q_n) \in \{-1, 1\}^n$ and $x = (x_1, \ldots, x_n) \in V^n$ we write $lx := (lx_1, \ldots, lx_n)$ and $qx := (q_1x_1, \ldots, q_nx_n)$, where lx stands, as usual, for the lth power of an element x of the commutative group V.

In the sequel, let V and W be vector spaces over the rational numbers, $n \in \mathbb{N}$ and $x_i^n = (x_{i1}, x_{i2}, \ldots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. We shall denote x_i^n by x_i when no confusion can arise. Let $x_1, x_2 \in V^n$ and $k \in \mathbb{N}_0$ with $0 \leqslant k \leqslant n$. Put $\mathscr{M} = \left\{\mathfrak{N}_n = (N_1, \ldots, N_n) | \, N_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\right\}$, where $j \in \{1, \ldots, n\}$. Consider

$$\mathscr{M}_k^n := \left\{\mathfrak{N}_n \in \mathscr{M} | \, \mathrm{Card}\{N_j : N_j = x_{1j}\} = k\right\}.$$

DEFINITION 3.2. A mapping $f : V^n \longrightarrow W$ is said to be n*-quadratic* or briefly *multi-quadratic* if f satisfies (1.3) in each variable.

For such mappings, we use the following notations:

$$f\left(\mathscr{M}_k^n\right) := \sum_{\mathfrak{N}_n \in \mathscr{M}_k^n} f(\mathfrak{N}_n), \tag{3.3}$$

$$f\left(\mathscr{M}_k^n, z\right) := \sum_{\mathfrak{N}_n \in \mathscr{M}_k^n} f(\mathfrak{N}_n, z) \qquad (z \in V).$$

We are going to show that if a mapping $f : V^n \longrightarrow W$ satisfies the equation

$$\sum_{q \in \{-1,1\}^n} f(mx_1 + qx_2) = \sum_{k=0}^{n} \left(2m^2 - 2\right)^k f\left(\mathscr{M}_k^n\right), \tag{3.4}$$

where $f\left(\mathscr{M}_k^n\right)$ is defined in (3.3) and m is a fixed integer with $m \neq 0, \pm 1$, then it is multi-quadratic and vice versa.

Let m be as in (1.3). We say a mapping $f : V^n \longrightarrow W$ satisfies *the r-power condition* in the jth component if

$$f(z_1, \ldots, z_{j-1}, mz_j, z_{j+1}, \ldots, z_n) = m^r f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n),$$

for all $(z_1, \ldots, z_n) \in V^n$. The 2-power condition is sometimes called the *quadratic condition*.

We remember that the binomial coefficient for all $n, r \in \mathbb{N}_0$ with $n \geqslant r$ is defined and denoted by $\binom{n}{r} := \frac{n!}{r!(n-r)!}$.

THEOREM 3.3. *For a mapping $f: V^n \longrightarrow W$, the following assertions are equivalent:*

(i) f is multi-quadratic;

(ii) f satisfies equation (3.4) and the quadratic condition in each variable.

Proof. (i)⇒(ii) It is easily verified that f satisfies the quadratic condition in all variables. We now prove that f satisfies equation (3.4) by induction on n. For $n=1$, it is trivial that f satisfies equation (1.3). If (3.4) is valid for some positive integer $n>1$, then,

$$\begin{aligned}\sum_{q\in\{-1,1\}^{n+1}} f\left(mx_1^{n+1}+qx_2^{n+1}\right) &= \sum_{q\in\{-1,1\}^{n}} f\left(mx_1^{n}+qx_2^{n},x_{1,n+1}+x_{2,n+1}\right)\\ &\quad+\sum_{q\in\{-1,1\}^{n}} f\left(mx_1^{n}+qx_2^{n},x_{1,n+1}-x_{2,n+1}\right)\\ &\quad+2\left(m^2-1\right)\sum_{q\in\{-1,1\}^{n}} f\left(mx_1^{n}+qx_2^{n},x_{1,n+1}\right)\\ &=\sum_{k=0}^{n}\sum_{q\in\{-1,1\}}\left(2m^2-2\right)^k f\left(\mathscr{M}_k^n,x_{1,n+1}+qx_{2,n+1}\right)\\ &\quad+2\left(m^2-1\right)\sum_{k=0}^{n}\left(2m^2-2\right)^k f\left(\mathscr{M}_k^n,x_{1,n+1}\right)\\ &=\sum_{k=0}^{n+1}\left(2m^2-2\right)^k f\left(\mathscr{M}_k^{n+1}\right).\end{aligned}$$

This means that (3.4) holds for $n+1$.

(ii)⇒(i) Fix $j\in\{1,\dots,n\}$. Putting $x_{2k}=0$ for all $k\in\{1,\dots,n\}\backslash\{j\}$ in the left side of (3.4) and using the assumption, we get

$$\begin{aligned}&2^{n-1}\times m^{2(n-1)}[f\left(x_{11},\dots,x_{1,j-1},mx_{1j}+x_{2j},x_{1,j+1},\dots,x_{1n}\right)\\ &\quad+f\left(x_{11},\dots,x_{1,j-1},mx_{1j}-x_{2j},x_{1,j+1},\dots,x_{1n}\right)]\\ &=2^{n-1}[f\left(mx_{11},\dots,mx_{1,j-1},mx_{1j}+x_{2j},mx_{1,j+1},\dots,mx_{1n}\right)\\ &\quad+f\left(mx_{11},\dots,mx_{1,j-1},mx_{1j}-x_{2j},mx_{1,j+1},\dots,mx_{1n}\right)].\end{aligned}\tag{3.5}$$

Set

$$\begin{aligned}f^*(x_{1j},x_{2j}):=f&\left(x_{11},\dots,x_{1,j-1},x_{1j}+x_{2j},x_{1,j+1},\dots,x_{1n}\right)\\ &+f\left(x_{11},\dots,x_{1,j-1},x_{1j}-x_{2j},x_{1,j+1},\dots,x_{1n}\right).\end{aligned}$$

By the above replacements in (3.4), it follows from (3.5) that

$$\begin{aligned}&2^{n-1}\times m^{2(n-1)}[f\left(x_{11},\dots,x_{1,j-1},mx_{1j}+x_{2j},x_{1,j+1},\dots,x_{1n}\right)\\ &\quad+f\left(x_{11},\dots,x_{1,j-1},mx_{1j}-x_{2j},x_{1,j+1},\dots,x_{1n}\right)]\\ &=\sum_{k=0}^{n-1}\left[\binom{n-1}{k}2^{n-k-1}(2m^2-2)^k\right]f^*(x_{1j},x_{2j})\\ &\quad+\sum_{k=1}^{n-1}\left[\binom{n-1}{k-1}2^{n-k}(2m^2-2)^k\right]f\left(x_{11},\dots,x_{1n}\right)+(2m^2-2)^n f\left(x_{11},\dots,x_{1n}\right)\end{aligned}$$

$$\begin{aligned}
&= (2m^2-2+2)^{n-1} f^*(x_{1j},x_{2j}) \\
&\quad + (2m^2-2)\left[(2m^2-2)^{n-1} + \sum_{k=0}^{n-2} \binom{n-1}{k} 2^{n-k-1} \times (2m^2-2)^k\right] f(x_{11},\ldots,x_{1n}) \\
&= (2m^2)^{n-1} f^*(x_{1j},x_{2j}) + (2m^2-2)(2m^2-2+2)^{n-1} f(x_{11},\ldots,x_{1n}) \\
&= 2^{n-1} m^{2(n-1)} [f^*(x_{1j},x_{2j}) + (2m^2-2) f(x_{11},\ldots,x_{1n})]. \qquad (3.6)
\end{aligned}$$

Now, relation (3.6) implies that

$$\begin{aligned}
& f\left(x_{11},\ldots,x_{1,j-1}, mx_{1j}+x_{2j}, x_{1,j+1},\ldots,x_{1n}\right) \\
&\quad + f\left(x_{11},\ldots,x_{1,j-1}, mx_{1j}-x_{2j}, x_{1,j+1},\ldots,x_{1n}\right) \\
&= f^*(x_{1j},x_{2j}) + (2m^2-2) f(x_{11},\ldots,x_{1n}).
\end{aligned}$$

This means that f is quadratic in the jth variable. Since j is arbitrary, we obtain the desired result. □

4. Random stability of multi-quadratic mappings

In this chapter, we prove the Hyers-Ulam stability of multi-quadratic mappings in the setting of random normed spaces.

From now on, for a mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathfrak{D}f : V^n \times V^n \longrightarrow W$ by

$$\mathfrak{D}f(x_1,x_2) := \sum_{q\in\{-1,1\}^n} f(mx_1+qx_2) - \sum_{k=0}^{n} \left(2m^2-2\right)^k f\left(\mathscr{M}_k^n\right),$$

where $f\left(\mathscr{M}_k^n\right)$ is defined in (3.3) and m is a fixed integer with $m \neq 0, \pm 1$. With this notation, we have the next stability result for functional equation (3.4).

THEOREM 4.1. *Let V be a linear space, $(\mathscr{Z},\Lambda,\tau_M)$ be an RN-space and (W,μ,τ_M) be a complete RN-space. Suppose that $\psi : V^n \times V^n \longrightarrow \mathscr{Z}$ is a mapping such that for some $0<\alpha<m^{2n}$,*

$$\Lambda_{\psi(mx,0)}(t) \geqslant \Lambda_{\alpha\psi(x,0)}(t) \qquad (x\in V^n,\ t>0) \tag{4.1}$$

and

$$\lim_{p\to\infty} \Lambda_{\psi(m^p x_1, m^p x_2)}(m^{2np} t) = 1 \qquad (x_1,x_2\in V^n,\ t>0). \tag{4.2}$$

If $f : V^n \longrightarrow W$ is a mapping satisfying

$$\mu_{\mathfrak{D}f(x_1,x_2)}(t) \geqslant \Lambda_{\psi(x_1,x_2)}(t), \tag{4.3}$$

for all $x_1,x_2\in V^n$ and all $t>0$, then there exists a unique solution $\mathscr{Q} : V^n \longrightarrow W$ of (3.4) such that

$$\mu_{f(x)-\mathscr{Q}(x)}(t) \geqslant \Lambda_{\psi(x,0)}\left(2^n(m^{2n}-\alpha)t\right), \tag{4.4}$$

for all $x\in V^n$ and all $t>0$. Moreover, if $\mathscr{Q}$ has the quadratic condition in each variable, then it is a multi-quadratic mapping.

Proof. Putting $x_2 = 0$ in (4.3), we have

$$\mu_{\left(2^n f(mx) - \left(\sum_{k=0}^{n} \binom{n}{k} 2^{n-k}(2m^2-2)^k\right) f(x)\right)}(t) \geqslant \Lambda_{\psi(x,0)}(t), \tag{4.5}$$

for all $x := x_1 \in V^n$ and $t > 0$. An easy computation shows that

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k}(2m^2-2)^k = (2m^2)^n. \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$\mu_{\left(2^n f(mx) - (2m^2)^n f(x)\right)}(t) \geqslant \Lambda_{\psi(x,0)}(t), \tag{4.7}$$

for all $x \in V^n$ and $t > 0$. Hence, relation (4.5) implies that

$$\mu_{\left(\frac{1}{m^{2n}} f(mx) - f(x)\right)}(t) \geqslant \Lambda_{\psi(x,0)}\left(2^n m^{2n} t\right), \tag{4.8}$$

for all $x \in V^n$. Substituting x by $m^p x$ in (4.8) and applying (4.1), we get

$$\begin{aligned} \mu_{\left(\frac{f(m^{p+1}x)}{m^{(p+1)2n}} - \frac{f(m^p x)}{m^{2np}}\right)}(t) &\geqslant \Lambda_{\psi(m^p x,0)}\left(2^n m^{2n(p+1)} t\right) \\ &\geqslant \Lambda_{\alpha^p \psi(x,0)}\left(2^n m^{2n(p+1)} t\right) \\ &\geqslant \Lambda_{\psi(x,0)}\left(2^n m^{2n}\left(\frac{m^{2n}}{\alpha}\right)^p t\right), \end{aligned} \tag{4.9}$$

for all $x \in V^n$ and all non-negative integers p. Using inequalities (4.8) and (4.9), we obtain

$$\begin{aligned} &\mu_{\left(\frac{f(m^p x)}{m^{2np}} - f(x)\right)}\left(\frac{1}{2^n m^{2n}} \sum_{j=0}^{p-1} \left(\frac{\alpha}{m^{2n}}\right)^j t\right) \\ &= \mu_{\left(\sum_{j=0}^{p-1}\left(\frac{f(m^{j+1}x)}{m^{(j+1)2n}} - \frac{f(m^j x)}{m^{2nj}}\right)\right)}\left(\frac{1}{2^n m^{2n}} \sum_{j=0}^{p-1} \left(\frac{\alpha}{m^{2n}}\right)^j t\right) \\ &\geqslant (\tau_M)_{j=0}^{p-1}\left(\mu_{\left(\frac{f(m^{j+1}x)}{m^{(j+1)2n}} - \frac{f(m^j x)}{m^{2nj}}\right)}\left(\frac{1}{2^n m^{2n}}\left(\frac{\alpha}{m^{2n}}\right)^j t\right)\right) \\ &= \mu_{\left(\frac{1}{m^{2n}} f(mx) - f(x)\right)}\left(\frac{1}{2^n m^{2n}} t\right) \\ &\geqslant \Lambda_{\psi(x,0)}(t), \end{aligned}$$

for all $x \in V^n$ and all non-negative integers p. In other words,

$$\mu_{\left(\frac{f(m^p x)}{m^{2np}} - f(x)\right)}(t) \geqslant \Lambda_{\psi(x,0)}\left(\frac{t}{\frac{1}{2^n m^{2n}} \sum_{j=0}^{p-1}\left(\frac{\alpha}{m^{2n}}\right)^j}\right). \tag{4.10}$$

Interchanging x into $m^l x$ in (4.10), we have

$$\mu_{\left(\frac{f(m^{p+l}x)}{m^{2n(p+l)}} - \frac{f(m^l x)}{m^{2nl}}\right)}(t) \geqslant \Lambda_{\psi(x,0)}\left(\frac{t}{\frac{1}{2^n m^{2n}} \sum_{j=l}^{l+p}\left(\frac{\alpha}{m^{2n}}\right)^j}\right), \tag{4.11}$$

for all $x \in V^n$ and all integers $p \geqslant l \geqslant 0$. Due to the convergence of $\sum_{j=l}^{\infty}\left(\frac{\alpha}{m^{2n}}\right)^j$, we see that $\Lambda_{\psi(x,0)}\left(\frac{t}{\frac{1}{2^n m^{2n}}\sum_{j=l}^{l+p}\left(\frac{\alpha}{m^{2n}}\right)^j}\right)$ goes to 1 as l and n tend to infinity, and so $\left\{\frac{f(m^p x)}{m^{2np}}\right\}$ is a Cauchy sequence in (W,μ,τ_M). The completeness of (W,μ,τ_M) as a RN-space implies that the mentioned sequence converges to some point $\mathscr{Q}(x) \in W$. It follows from (4.10) that for each $\varepsilon > 0$

$$\begin{aligned}\mu_{(\mathscr{Q}(x)-f(x))}(t+\varepsilon) &\geqslant \tau_M\left(\mu_{\left(\mathscr{Q}(x)-\frac{f(m^p x)}{m^{2np}}\right)}(\varepsilon), \mu_{\left(\frac{f(m^p x)}{m^{2np}}-f(x)\right)}(t)\right)\\ &\geqslant \tau_M\left(\mu_{\left(\mathscr{Q}(x)-\frac{f(m^p x)}{m^{2np}}\right)}(\varepsilon), \Lambda_{\psi(0,x)}\left(\frac{t}{\frac{1}{2^n m^{2n}}\sum_{j=0}^{p-1}\left(\frac{\alpha}{m^{2n}}\right)^j}\right)\right),\end{aligned}$$

for all $x \in V^n$. Letting p to infinity in the above inequality, we deduce that

$$\mu_{(\mathscr{Q}(x)-f(x))}(t+\varepsilon) \geqslant \Lambda_{\psi(x,0)}\left(2^n(m^{2n}-\alpha)t\right). \tag{4.12}$$

Taking $\varepsilon \to 0$ in (4.12), we get (4.4). Moreover, inequality (4.3) implies that

$$\mu_{\frac{1}{m^{2np}}\mathfrak{D}f(m^p x_1, m^p x_2)}(t) \geqslant \Lambda_{\psi(m^p x_1, m^p x_2)}(m^{2np}t), \tag{4.13}$$

for all $x_1, x_2 \in V$ and all $t > 0$. Once more, Letting p to infinity in (4.13), by (4.2), we observe that the mapping $\mathscr{Q}$ satisfies (3.4) . If $\mathfrak{Q}: V^n \longrightarrow W$ is another mapping satisfies (3.4) and (4.4), then

$$\begin{aligned}\mu_{\left(\frac{\mathfrak{Q}(m^p x)}{m^{2np}}-\frac{\mathscr{Q}(m^p x)}{m^{2np}}\right)}(t) &\geqslant \min\left\{\mu_{\left(\frac{f(m^p x)}{m^{2np}}-\frac{\mathscr{Q}(m^p x)}{m^{2np}}\right)}\left(\frac{t}{2}\right), \mu_{\left(\frac{\mathfrak{Q}(m^p x)}{m^{2np}}-\frac{f(m^p x)}{m^{2np}}\right)}\left(\frac{t}{2}\right)\right\}\\ &\geqslant \Lambda_{(\psi(m^p x,0))}\left(2^{n-1}m^{2np}(m^{2n}-\alpha)t\right)\\ &\geqslant \Lambda_{(\psi(x,0))}\left(\left(\frac{m^{2n}}{\alpha}\right)^p 2^{n-1}(m^{2n}-\alpha)t\right),\end{aligned}$$

for all $x \in V^n$. Therefore

$$\begin{aligned}\mu_{\mathfrak{Q}(x)-\mathscr{Q}(x)}(t) &= \lim_{p\to\infty}\mu_{\left(\frac{\mathfrak{Q}(m^p x)}{m^{2np}}-\frac{\mathscr{Q}(m^p x)}{m^{2np}}\right)}(t)\\ &\geqslant \lim_{p\to\infty}\Lambda_{(\psi(x,0))}\left(\left(\frac{m^{2n}}{\alpha}\right)^p 2^{n-1}(m^{2n}-\alpha)t\right) = 1.\end{aligned}$$

The relation above shows that $\mathscr{Q}(x) = \mathfrak{Q}(x)$ for all $x \in V^n$. This completes the proof. □

The following corollary is a direct consequences of Theorem 4.1 concerning the stability of (3.4).

COROLLARY 4.2. *Let V be a linear space, $(\mathscr{Z},\Lambda,\tau_M)$ be an RN-space and (W,μ,τ_M) be a complete RN-space. Let also s be a real number such that $s \in [0,2n)$ and $z_0 \in \mathscr{Z}$. If $f: V^n \longrightarrow W$ is a mapping such that*

$$\mu_{\mathfrak{D}f(x_1,x_2)}(t) \geqslant \Lambda_{\left(\sum_{i=1}^2\sum_{j=1}^n \|x_{ij}\|^s\right)z_0}(t), \tag{4.14}$$

for all $x_1,x_2 \in V^n$ *and all* $t>0$*, then there exists a unique solution* $\mathscr{Q}: V^n \longrightarrow W$ *of* (3.4) *satisfying*

$$\mu_{f(x)-\mathscr{Q}(x)}(t) \geqslant \Lambda_{\sum_{j=1}^n \|x_{1j}\|^s z_0}\left(2^n(m^{2n}-m^s)t\right),$$

for all $x\in V^n$ *and all* $t>0$*. In particular, if* $\mathscr{Q}$ *has the quadratic condition in each variable, then it is a multi-quadratic mapping.*

Proof. Putting $\psi(x_1,x_2) := \left(\sum_{i=1}^2 \sum_{j=1}^n \|x_{ij}\|^s\right) z_0$ and applying Theorem 4.1 when $\alpha = m^s$, we get the desired result. □

We have the next stability theorem which is analogous to Theorem 4.1 with somewhat different method in the proof.

THEOREM 4.3. *Let* V *be a linear space,* $(\mathscr{Z},\Lambda,\tau_M)$ *be an RN-space and* (W,μ,τ_M) *be a complete RN-space. Suppose that* $\psi: V^n\times V^n \longrightarrow \mathscr{Z}$ *is a mapping such that for some* $\alpha > m^{2n}$,

$$\Lambda_{\psi(m^{-1}x,0)}(t) \geqslant \Lambda_{\psi(x,0)}(\alpha t) \qquad (x\in V^n,\ t>0) \tag{4.15}$$

and

$$\lim_{n\to\infty} \Lambda_{m^{2np}\psi(m^p x_1, m^p x_2)}(t) = 1 \qquad (x_1,x_2\in V^n,\ t>0). \tag{4.16}$$

If $f: V^n \longrightarrow W$ *is a mapping satisfying*

$$\mu_{\mathfrak{D}f(x_1,x_2)}(t) \geqslant \Lambda_{\psi(x_1,x_2)}(t), \tag{4.17}$$

for all $x_1,x_2 \in V^n$ *and all* $t>0$*, then there exists a unique solution* $\mathscr{Q}: V^n \longrightarrow W$ *of* (3.4) *such that*

$$\mu_{f(x)-\mathscr{Q}(x)}(t) \geqslant \Lambda_{\psi(x,0)}\left(\frac{\alpha - 2^n(m^{2n})}{\alpha} t\right), \tag{4.18}$$

for all $x\in V^n$ *and* $t>0$.

Proof. Putting $x_2=0$ in (4.3), we arrive at relation (4.7) and so

$$\mu_{\left(f(x)-m^{2n}f\left(\frac{x}{m}\right)\right)}(t) \geqslant \Lambda_{\psi\left(\frac{x}{m},0\right)}(2^n t), \tag{4.19}$$

for all $x:=x_1\in V^n$ and $t>0$. It follows from (4.15) and (4.19) that

$$\mu_{\left(f(x)-m^{2n}f\left(\frac{x}{m}\right)\right)}(t) \geqslant \Lambda_{\psi(x,0)}(2^n\alpha t). \tag{4.20}$$

Replacing x by $\frac{x}{m^p}$ in (4.20), we get

$$\mu_{\left(m^{2np}f\left(\frac{x}{m^p}\right)-m^{2n(p+1)}f\left(\frac{x}{m^{p+1}}\right)\right)}(t) \geqslant \Lambda_{\psi(x,0)}\left(2^n\left(\frac{\alpha}{m^{2n}}\right)^p t\right),$$

for all $x\in V^n$ and $t>0$ and all non-negative integers p. Since

$$f(x) - m^{2np}f\left(\frac{x}{m^p}\right) = \sum_{j=0}^{p-1} m^{2nj}f\left(\frac{x}{m^j}\right) - m^{2n(j+1)}f\left(\frac{x}{m^{j+1}}\right),$$

we have

$$\mu_{\left(f(x)-m^{2np}f\left(\frac{x}{m^p}\right)\right)}\left(\frac{1}{2^n}\sum_{j=0}^{p-1}\left(\frac{m^{2n}}{\alpha}\right)^j t\right) \geqslant \Lambda_{\psi(x,0)}(t), \tag{4.21}$$

for all $x \in V^n$ and $t > 0$. In other words,

$$\mu_{\left(f(x)-m^{2np}f\left(\frac{x}{m^p}\right)\right)}(t) \geqslant \Lambda_{\psi(x,0)}\left(\frac{t}{\frac{1}{2^n}\sum_{j=0}^{p-1}\left(\frac{m^{2n}}{\alpha}\right)^j}\right), \tag{4.22}$$

for all $x \in V^n$ and $t > 0$. Substituting x by $\frac{x}{m^l}$ in (4.22), we have

$$\mu_{\left(m^{2nl}f\left(\frac{x}{m^l}\right)-m^{2n(p+l)}f\left(\frac{x}{m^{p+l}}\right)\right)}(t) \geqslant \Lambda_{\psi(x,0)}\left(\frac{t}{\frac{1}{2^n}\sum_{j=l}^{l+p}\left(\frac{m^{2n}}{\alpha}\right)^j}\right), \tag{4.23}$$

for all $x \in V^n$ and all integers $p \geqslant l \geqslant 0$. Since the series $\sum_{j=l}^{\infty}\left(\frac{m^{2n}}{\alpha}\right)^j$ is convergent, we see that $\Lambda_{\psi(x,0)}\left(\frac{t}{\frac{1}{2^n}\sum_{j=l}^{l+p}\left(\frac{m^{2n}}{\alpha}\right)^j}\right)$ goes to 1 as l and n tend to infinity, and so $\left\{m^{2np}f\left(\frac{x}{m^p}\right)\right\}$ is a Cauchy sequence in (W,μ,τ_M). The completeness of (W,μ,τ_M) as a RN-space necessitates that the last sequence converges to some point $\mathscr{Q}(x) \in W$. The remaining assertion goes through in a similar method to the corresponding part of Theorem 4.1. This finishes the proof. □

COROLLARY 4.4. *Let V be a linear space, $(\mathscr{Z},\Lambda,\tau_M)$ be an RN-space and (W,μ,τ_M) be a complete RN-space. Suppose that s is a real number such that $s \in [2n,\infty)$ and $z_0 \in \mathscr{Z}$. If $f : V^n \longrightarrow W$ is a mapping such that*

$$\mu_{\mathfrak{D}f(x_1,x_2)}(t) \geqslant \Lambda_{\left(\sum_{i=1}^{2}\sum_{j=1}^{n}\|x_{ij}\|^s\right)z_0}(t) \tag{4.24}$$

for all $x_1,x_2 \in V^n$ and all $t > 0$, then there exists a unique solution $\mathscr{Q} : V^n \longrightarrow W$ of (3.4) satisfying

$$\mu_{f(x)-\mathscr{Q}(x)}(t) \geqslant \Lambda_{\sum_{j=1}^{n}\|x_{1j}\|^{s}{}_{1j}z_0}\left(\frac{2^n(m^s-m^{2n})}{m^s}t\right)$$

for all $x \in V^n$ and all $t > 0$.

Proof. Putting $\psi(x_1,x_2) := \left(\sum_{i=1}^{2}\sum_{j=1}^{n}\|x_{ij}\|^s\right)z_0$ and applying Theorem 4.1 when $\alpha = m^s$, we get the desired result. □

For two sets X and Y, the set of all mappings from X to Y is denoted by Y^X. Let A be a nonempty set, (X,d) be a metric space, $\psi \in \mathbb{R}_+^{A^n}$, and $\mathscr{F}_1,\mathscr{F}_2$ be operators mapping a nonempty set $D \subset X^A$ into X^{A^n}. We say that operator equation

$$\mathscr{F}_1\varphi(a_1,\ldots,a_n) = \mathscr{F}_2\varphi(a_1,\ldots,a_n) \tag{4.25}$$

is ψ-hyperstable provided every $\varphi_0 \in D$ satisfying inequality

$$d(\mathscr{F}_1\varphi_0(a_1,\ldots,a_n),\mathscr{F}_2\varphi_0(a_1,\ldots,a_n)) \leqslant \psi(a_1,\ldots,a_n), \qquad a_1,\ldots,a_n \in A,$$

fulfils (4.25); this definition is introduced in [10]. In other words, a functional equation $\mathscr{F}$ is *hyperstable* if any mapping f satisfying the equation $\mathscr{F}$ approximately is a true solution of $\mathscr{F}$. Under some mild conditions, the equation (3.4) can be hyperstable as follows.

COROLLARY 4.5. *Let V be a linear space, $(\mathscr{X},\Lambda,\tau_M)$ be an RN-space and (W,μ,τ_M) be a complete RN-space. Let s_{ij} be non-negative real numbers such that $\sum_{i=1}^{2}\sum_{j=1}^{n} s_{ij} \neq 2n$ and $z_0 \in \mathscr{X}$. If $f: V^n \longrightarrow W$ is a mapping such that*

$$\mu_{\mathscr{D}f(x_1,x_2)}(t) \geqslant \Lambda_{\prod_{i=1}^{2}\prod_{j=1}^{n}\|x_{ij}\|^{s_{ij}} z_0}(t)$$

for all $x_1,x_2 \in V^n$ and all $t>0$, then f satisfies (3.4). Furthermore, if $\mathscr{Q}$ has the quadratic condition in each variable, then it is a multi-quadratic mapping.

Proof. Putting $\psi(x_1,x_2) := \prod_{i=1}^{2}\prod_{j=1}^{n}\|x_{ij}\|^{s_{ij}} z_0$ in Theorem 4.1 and Theorem 4.3, we obtain the result. $\square$

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