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Analytical solutions for time-fractional diffusion equation with heat absorption in spherical domains

Nehad Ali Shah [a], N. Ameer Ahammad [b], Dumitru Vieru [c], Se-Jin Yook [d,*], Hussam Alrabaiah [e,f]

[a] Department of Mechanical Engineering, Sejong University, Seoul 05006, Korea
[b] Department of Mathematics, Faculty of Science, University of Tabuk, P.O. Box741, Tabuk 71491, Saudi Arabia
[c] Department of Theoretical Mechanics, Technical University of Iasi, Romania
[d] School of Mechanical Engineering, Hanyang University, 222 Wangsimni-ro, Seongdong-gu, Seoul 04763, Republic of Korea
[e] College of Engineering, Al Ain University, Al Ain, United Arab Emirates
[f] Mathematics Department, Tafla Technical University, Tafla, Jordan

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ABSTRACT

Under the time-variable Dirichlet condition, the time-fractional diffusion equation with heat absorption in a sphere is taken into consideration. The time-fractional derivative with the power-law kernel is used in the generalized Cattaneo constitutive equation of the thermal flux. The Laplace transform and a suitable transformation of the independent variable and function are used to determine the analytical solution of the problem in the Laplace domain. To obtain the temperature distribution in the real domain, the inverse Laplace transforms of two functions of exponential type are obtained. These formulae are new in the literature. The particular cases of the classical Cattaneo law of heat conduction and of the classical Fourier's law are obtained from the solutions corresponding to the time-fractional generalized Cattaneo law.

1. Introduction

Even though the Fourier's law-based mathematical model of heat conduction assumes an impractical property—namely, that thermal disturbances can spread at an infinite rate—it is nonetheless effective for many engineering applications. Also, this model is unsuitable for thermal processes that take place at low temperatures or with ultra-fast heating. In the last years, many non-Fourier mathematical models have been proposed [1,2].

Due to the advanced technologies, very precise devices have been made for measuring experimental data. Significant discrepancies between the theoretical and experimental results were found by these measurements. The non-Fourier heat conduction phenomena in porous material heated by a microsecond laser pulse, for instance, were the subject of experiments by Jiang et al. [3], which have revealed numerous discrepancies between the experimental findings and the theoretical analyses provided by the Cattaneo-type and Jeffreys-type models.

Fractional calculus has recently found use in a variety of domains, including biology, economics, engineering, and physical sciences. Researchers proposed generalized models of the diffusion processes by using fractional calculus with different memory kernels.

A transient heat diffusion equation with a relaxation term stated by the Caputo-Fabrizio time-fractional derivative [5] has been developed by Hristov [4] beginning from Cattaneo's constitutive relation. This article explains the theoretical foundations of the Caputo-Fabrizio time-fractional derivative and illustrates how non-singular fading memories may be used to change the constitutive equations.

Compte and Metzler studied different generalizations of the Cattaneo equation to describe anomalous transport using the Rieman-Liouville time-fractional derivative [6]. The generalized heat transfer in living tissues was discussed by Hristov [7] presenting fractionalization by different constitutive approaches with different fractional differential operators. This study shows how

* Corresponding author.
E-mail addresses: nehadali199@sejong.ac.kr (N. Ali Shah), anaudalur@ut.edu.sa (N. Ameer Ahammad), dumitru_vieru@yahoo.com (D. Vieru), ysjnuri@hanyang.ac.kr (S.-J. Yook), hussam.alrabaiah@aau.ac.ae (H. Alrabaiah).

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crucial it is to formulate the bio-heat equation with memory correctly by selecting the damping (relaxation) function of the heat flux. Rukolaine has looked at a mathematical model of heat conduction that is based on the first/higher-order approximation to the dual-phase-lag constitutive relation [8,9]. He investigated a short-duration positive localized source initial value problem for the three-dimensional Jeffreys-type equations and reported illogical behavior for negative temperature values, leading one to believe that the dual-phase-lag model may not be the best one to describe heat conduction.

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In the context of the extended thermoelasticity theory with two time delays and kernel functions, El-Karamani and Ezzat [12] explored the thermoelastic diffusion in anisotropic/isotropic solid bodies. Using a convolutional variational approach, they demonstrated the reciprocity theorem and the uniqueness of the solution. Ezzat et al. [13] have created a novel fractional relaxation operator linear thermo-viscoelasticity model for isotropic media. Using the Laplace transform approach, certain specific issues, including the thermal shock problem and a problem for half-space exposed to ramp-type heating, have been resolved within the framework of the new theory. Based on fractional derivative heat transfer for perfectly conducting media in the presence of a constant magnetic field, a mathematically unified model of phase-lag Green-Naghdi magneto-thermoelastic theories has been established in [14]. Using a numerical approach based on Fourier expansion techniques, the analytical solutions were obtained in the Laplace domain and modified in the real domain. A new mathematical model for the two-temperature electro-thermo viscoelasticity theory with memory-dependent derivative has been studied in [15,16], as well as the two-temperature phase-lag Green-Naghdi theory of thermoelasticity with fractional derivatives for the half space subjected to a time-dependent boundary temperature. For an isotropic, perfect conducting thermoelastic body with temperature-dependent thermal conductivity, Ezzat and El-Barry [17] studied a fractional model of the generalized magneto-thermoelasticity. A issue involving an infinitely long hollow cylinder in the presence of an axial uniform magnetic field has been solved using the mathematical model. [18] has studied a fascinating fractional Fourier law with three-phase lag thermoelasticity model. Intriguing qualitative research for the flows of a fractional second-grade fluid defined by the fractional Caputo operator was conducted by Yavuz et al. [19]. The impact of the fractional operator on the fluid behavior is analyzed. Flows of the fractional Casson fluids described with the Caputo time-fractional derivative have been studied by Sene [20] using the Laplace and Fourier transforms. Other interesting and recent topics have been investigated in the papers [21–25].

In this article, the generalized time-fractional diffusion equation in spherical domains is studied. The situation of central symmetry under the time-variable Dirichlet condition is taken into account. The generalized Cattaneo's constitutive equation of the heat flow uses the time-fractional derivative with a power-law kernel. The analytical solution of the issue in the Laplace domain is found using the Laplace transform and appropriate transformations of the independent variable and function.

Two functions of the exponential type are given their inverse Laplace transforms in order to derive the temperature distribution in the real domain. These equations are brand-new to the literature. The solutions corresponding to the time-fractional generalized Cattaneo's law are used to determine the specific instances of the classical Cattaneo's law of heat conduction and the classical Fourier's law. Analyses of several specific scenarios are done, and the outcomes are visually displayed.

2. Statement of the problem

In this paper, we consider the mathematical model of a thermal process described by [1,8]:

- the conservation law of energy

$$\rho c_p \frac{\partial \tilde{T}_1(\tilde{\boldsymbol{x}}_1, \tilde{t}_1)}{\partial \tilde{t}_1} = -div\tilde{\boldsymbol{q}}_1 - b\tilde{T}_1(\tilde{\boldsymbol{x}}_1, \tilde{t}_1) \tag{1}$$

- the constitutive Cattaneo thermal flux equation

$$\left(1 + \tilde{\tau}_q \frac{\partial}{\partial \tilde{t}_1}\right)\tilde{\boldsymbol{q}}_1(\tilde{\boldsymbol{x}}_1, \tilde{t}_1) = -k\nabla\tilde{T}_1(\tilde{\boldsymbol{x}}_1, \tilde{t}_1) \tag{2}$$

where,$(\tilde{\boldsymbol{x}}_1, \tilde{t}_1) \in \tilde{D} \times [0, \infty), \tilde{D} \subset \mathbb{R}^3$, ρ is the specific mass, c_p is the specific heat, $\tilde{T}_1(\tilde{\boldsymbol{x}}_1, \tilde{t}_1)$ is the temperature, $\tilde{\boldsymbol{q}}_1(\tilde{\boldsymbol{x}}_1, \tilde{t}_1)$ is the density vector of thermal flux, b is the heat absorption coefficient, k is the thermal conductivity, and $\tilde{\tau}_q$ is the thermal relaxation time and the divergence operator in spherical coordinate is given by$div(\overrightarrow{\mathrm{v}}) = \frac{1}{\mathrm{r}^2}\frac{\partial}{\partial \mathrm{r}}(\mathrm{r}^2\mathrm{v}_\mathrm{r}) + \frac{1}{\mathrm{r}\sin\theta}\frac{\partial}{\partial \mathrm{r}}(\mathrm{v}_\theta \sin\theta) + \frac{1}{\mathrm{r}\sin\theta}\frac{\partial \mathrm{v}_\varphi}{\partial \varphi}$.

In the following, we consider the domain $\tilde{D}$ is a spherical domain in$\mathbb{R}^3$, $\tilde{D} = \left\{\tilde{\boldsymbol{x}}_1(\tilde{r}_1, \tilde{\varphi}_1, \tilde{\theta}_1) \in \mathbb{R}^3 \middle| \quad 0 \leqslant \tilde{r}_1 \leqslant R,\ R > 0, \right.$ $\tilde{\varphi}_1 \in [0, 2\pi),\ \tilde{\theta}_1 \in [-\pi/2, \pi/2]\}$ reported to a spherical coordinate system$(\tilde{r}_1, \tilde{\varphi}_1, \tilde{\theta}_1)$, Fig. 1.

In this study, we consider the case of transient central symmetric diffusive process; therefore, all the functions involved in the description of the model are functions of $\tilde{r}_1$ and $\tilde{t}_1$ only, and the basic equations (1) and (2) are written in the simplified forms

$$\rho c_p \frac{\partial \tilde{T}_1(\tilde{r}_1, \tilde{t}_1)}{\partial \tilde{t}_1} = -\frac{1}{\tilde{r}_1^2}\frac{\partial}{\partial \tilde{r}_1}\left(\tilde{r}_1^2 \tilde{q}_{\tilde{r}_1}(\tilde{r}_1, \tilde{t}_1)\right) - b\tilde{T}_1(\tilde{r}_1, \tilde{t}_1) \tag{3}$$

$$\left(1 + \tilde{\tau}_q \frac{\partial}{\partial \tilde{t}_1}\right)\tilde{q}_{\tilde{r}_1}(\tilde{r}_1, \tilde{t}_1) = -k\frac{\partial \tilde{T}_1(\tilde{r}_1, \tilde{t}_1)}{\partial \tilde{r}_1} \tag{4}$$

Along with Eqs. (3) and (4), we consider the IB(initial and boundary) conditions

$$\tilde{T}_1(\tilde{r}_1, 0) = 0,\ \tilde{q}_{\tilde{r}_1}(\tilde{r}_1, 0) = 0,\ \tilde{r}_1 \in [0, R] \tag{5}$$

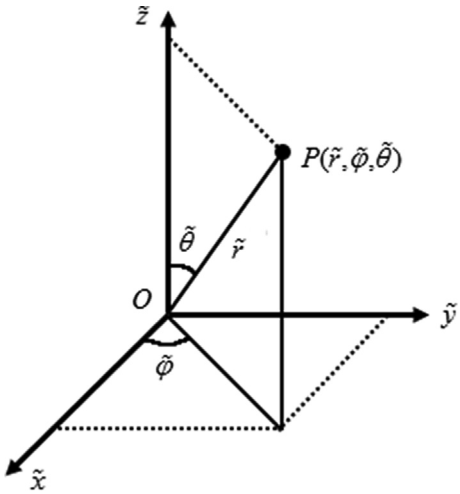

Fig. 1. The spherical coordinate system.

$$\widetilde{T}_1(R,\widetilde{t}_1)=T_0 f_1(\widetilde{t}_1),\ \widetilde{t}_1>0 \tag{6}$$

where, the function $f_1(\widetilde{t}_1)$ is a differentiable function of the exponential order to infinity.

Using the nondimensional functions and parameters

$$T=\frac{\widetilde{T}_1}{T_0},\ r=\frac{\widetilde{r}_1}{R},\ t=\frac{a_T\widetilde{t}_1}{R^2},\ q=\frac{R\widetilde{q}_{\widetilde{r}_1}}{kT_0},\ \tau=\frac{a_T\widetilde{\tau}_q}{R^2},\quad a_T=\frac{k}{\rho c_p},\ b=\frac{\widetilde{b}R^2}{k},\ f(t)=\widetilde{f}_1\left(\frac{R^2 t}{a_T}\right), \tag{7}$$

into Eqs. (4)-(6), we obtain the following dimensionless equations:

$$\frac{\partial T(r,t)}{\partial t}=-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 q(r,t)\right)-bT(r,t) \tag{8}$$

$$\left(1+\tau\frac{\partial}{\partial\tau}\right)q(r,t)=-\frac{\partial T(r,t)}{\partial r} \tag{9}$$

$$T(r,0)=0,\ q(r,0)=0,\ r\in[0,1] \tag{10}$$

$$T(1,t)=f(t) \tag{11}$$

2.1. The generalized fractional thermal process

To analyze a generalized heat diffusion process, we consider the thermal flux described by the following non-dimensional generalized Cattaneo's law:

$$\left(1+\tau_c D_t^{\omega}\right)q(r,t)=-\frac{\partial T(r,t)}{\partial r},\ 0<\omega\leqslant 1 \tag{12}$$

where, the operator ${}_eD_t^{\omega}-$ time-fractional Caputo derivative of orderω, defined as [26]

$$D_t^{\omega}q(r,t)=\begin{cases}\frac{1}{\Gamma(1-\omega)}\int_0^t(t-\lambda)^{-\omega}\dot{q}(r,\lambda)d\lambda,\ 0<\omega<1,\\ \dot{q}(r,t),\ \omega=1.\end{cases} \tag{13}$$

In the above definition, $\Gamma(\zeta)=\int_0^{\infty}\exp(-\lambda)\lambda^{\zeta-1}d\lambda,\ Re(\zeta)>0$ is Gamma function.

Let ${}_ch(t,\omega)=\frac{t^{-\omega}}{\Gamma(1-\omega)},0<\omega<1$ be the kernel of time-fractional Caputo derivative.

Eq. (13) can be written as

$$D_t^{\omega}q(r,t)=\begin{cases}{}_ch(t,\omega)*\dot{q}(r,t)=\int_0^t ch(t-\lambda,\omega)\dot{q}(r,\lambda)d\lambda,0<\omega<1,\\ \dot{q}(r,t),\omega=1,\end{cases} \tag{14}$$

where, "$*$" the convolution product. Using Eq. (14), the initial condition (10), and the properties of the Laplace transform [27], we obtain that Laplace transform of the time-fractional Caputo derivative (13) is given by

$$L\{D_t^{\omega}q(r,t)\}(\hbar)=\hbar^{\omega}\hat{q}(r,\hbar),\ 0<\omega\leqslant 1 \tag{15}$$

where $L\{q(r,t)\}(\hbar)=\hat{q}(r,\hbar)=\int_0^{\infty}\exp(-\hbar t)q(r,t)dt$ is the Laplace transform of the function $q(r,t)$ and $\hbar$ is transform parameter.

Obviously, if the fractional parameter$\omega=1$, the generalized constitutive equation (12) becomes identically with the classical Cattaneo law (9).

3. Solution of the generalized thermal process

In this section, we will determine the analytical solution of the Eqs. (8) and (12) along with the IB conditions (10) and (11). To do this, we first use the Laplace transform to get the following problem in the transform domain:

$$\hbar\hat{T}(r,\hbar)=-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\hat{q}(r,\hbar)\right)-b\hat{T}(r,\hbar) \tag{16}$$

$$\left(1+\tau\hbar^{\omega}\right)\hat{q}(r,\hbar)=-\frac{\partial\hat{T}(r,\hbar)}{\partial r},\ 0<\omega\leqslant 1 \tag{17}$$

$$\hat{T}(1,\hbar)=\hat{f}(\hbar) \tag{18}$$

Eliminating function $\hat{q}(r,\hbar)$ between equations (16) and (17), we obtain that the function $\hat{T}(r,\hbar)$ is the solution of the differential equation

$$\frac{\partial^2\hat{T}(r,\hbar)}{\partial r^2}+\frac{2}{r}\frac{\partial\hat{T}(r,\hbar)}{\partial r}-\vartheta_1(\hbar)\hat{T}(r,\hbar)=0 \tag{19}$$

where,

$$\vartheta_1(\hbar)=\left(1+\tau\hbar^{\omega}\right)(\hbar+b) \tag{20}$$

Making the change of variable and function

$$z=r\sqrt{\vartheta_1(\hbar)},\ \hat{T}(r,\hbar)=\sqrt[4]{\vartheta_1(\hbar)}z^{-1/2}\hat{\Theta}(z,\hbar) \tag{21}$$

we obtain the following modified Bessel equation [28]:

$$z^2\frac{\partial^2\hat{\Theta}(z,\hbar)}{\partial z^2}+z\frac{\partial\hat{\Theta}(z,\hbar)}{\partial z}-\left(z^2+2^{-2}\right)\hat{\Theta}(z,\hbar)=0 \tag{22}$$

The general solution of Eq. (22) is

$$\hat{\Theta}(z,\hbar)=A(\hbar)I_{1/2}(z)+B(\hbar)K_{1/2}(z) \tag{23}$$

where $I_{1/2}(z),\ K_{1/2}(z)$ are the modified Bessel functions of order 1/2.

Given that$\lim\limits_{z\to 0}K_{1/2}(z)=\infty$, to have a finite temperature distribution inside the sphere, we must consider$B(s)=0$.

Now, using the property $I_{1/2}(z)=\sqrt{\frac{2}{\pi z}}\sinh(z)$ and transformations (21), we obtain that the temperature $\hat{T}(r,\hbar)$ is given by

$$\hat{T}(r,\hbar)=\frac{A(\hbar)\sqrt{2}}{r\sqrt{\pi\sqrt{\vartheta(\hbar)}}}\sinh(r\vartheta_1(\hbar)) \tag{24}$$

Imposing the boundary condition (18), we find the temperature in Laplace domain

$$\hat{T}(r,\hbar)=\hat{f}(\hbar)\frac{1}{r}\frac{\sinh(r\vartheta_1(\hbar))}{\sinh(\vartheta_1(\hbar))} \tag{25}$$

Let $\hat{\Psi}(r,\hbar)$ be the function

$$\hat{\Psi}(r,\hbar)=\frac{\sinh(r\vartheta_1(\hbar))}{\sinh(\vartheta_1(\hbar))} \tag{26}$$

which is written in the equivalent forms

$$\hat{\Psi}(r,\hbar)=\frac{\exp\left(r\sqrt{\vartheta_1(\hbar)}\right)-\exp\left(-r\sqrt{\vartheta_1(\hbar)}\right)}{\exp\left(\sqrt{\vartheta_1(\hbar)}\right)-\exp\left(-\sqrt{\vartheta_1(\hbar)}\right)}=\frac{\exp\left((r-1)\sqrt{\vartheta_1(\hbar)}\right)-\exp\left(-(r+1)\sqrt{\vartheta_1(\hbar)}\right)}{1-\exp\left(-2\sqrt{\vartheta_1(\hbar)}\right)}$$
$$=\sum_{k=0}^{\infty}\left[\exp\left(-(2k+1-r)\sqrt{\vartheta_1(\hbar)}\right)-\exp\left(-(2k+1-r)\sqrt{\vartheta_1(\hbar)}\right)\right] \tag{27}$$

Considering the following pair of functions

$$\varphi(a,t)=\frac{a}{2t\sqrt{\pi t}}\exp\left(\frac{-a^2}{4t}\right),$$
$$\hat{\Phi}(a,\hbar)=L\{\varphi(a,t)\}=\exp(-a\sqrt{\hbar}),\ Re(a^2)>0, \tag{28}$$

and using the property of the inverse Laplace transform of composite functions, we obtain [29]

$$L^{-1}\left\{\exp\left(-(2k+1-r)\sqrt{\vartheta_1(\hbar)}\right)\right\}=L^{-1}\left\{\hat{\Phi}((2k+1-r),\vartheta_1(\hbar))\right\}=\int_0^{\infty}\varphi((2k+1-r),u)\chi(t,u)du,$$
$$L^{-1}\left\{\exp\left(-(2k+1+r)\sqrt{\vartheta_1(\hbar)}\right)\right\}=L^{-1}\left\{\hat{\Phi}((2k+1+r),\vartheta_1(\hbar))\right\}=\int_0^{\infty}\varphi((2k+1+r),u)\chi(t,u)du, \tag{29}$$

where,

$$\chi(t,u)=L^{-1}\{\exp(-u\vartheta_1(\hbar))\} \tag{30}$$

Using (29) in Eq. (27) we obtain

$$\psi(r,t)=\sum_{k=0}^{\infty}\int_0^{\infty}[\varphi((2k+1-r),u)-\varphi((2k+1+r),u)]\chi(t,u)du \tag{31}$$

The temperature field is given by

$$T(r,t)=\frac{1}{r}f(t)*\psi(r,t) \tag{32}$$

where, "$*$" is convolution product.

To determine the function $\chi(t,u)$, we first prove the following.

Proposition. The inverse Laplace transform of function

$$\hat{W}(\hbar,u,a,b)=\exp\left[-u\left(a\hbar^{\omega+1}+b\hbar^{\omega}\right)\right],\ a,\ b>0,\ 0<\omega\leqslant 1 \tag{33}$$

is

$$W(t,u,a,b)=\sum_{k=1}^{\infty}\sum_{m=0}^{k}\frac{(-u)^{k+1}m!a^m b^{k-m}}{(k!)^2(k-m)!\Gamma(m+\omega k)}\int_0^{\infty}\sqrt{\frac{z}{t}}z^{m+\omega k-1}J_1\left(2\sqrt{tz}\right)dz \tag{34}$$

Proof. Let $\hat{U}(\hbar,u,a,b)$ be the function

$$\hat{U}(\hbar,u,a,b)=\exp\left[-u\left(\frac{a}{\hbar^{\omega+1}}+\frac{b}{\hbar^{\omega}}\right)\right] \tag{35}$$

Function (35) is written in the equivalent form

$$\hat{U}(\hbar,u,a,b)=1+\sum_{k=1}^{\infty}\frac{(-u)^k}{k!}\left(\frac{a}{\hbar^{\omega+1}}+\frac{b}{\hbar^{\omega}}\right)^k=1+\sum_{k=1}^{\infty}\sum_{m=0}^{k}\frac{(-u)^k m!a^m b^{k-m}}{(k!)^2(k-m)!}\frac{1}{\hbar^{m+\omega k}} \tag{36}$$

Using the following formula

$$L^{-1}\left\{\frac{1}{\hbar^{m+\omega k}}\right\}=\frac{t^{m+\omega k-1}}{\Gamma(m+\omega k)},\quad 0<\omega\leqslant 1,\ k\geqslant 1,\ m\in\{0,1,...,k\} \tag{37}$$

we obtain the inverse Laplace transform of the function $\hat{U}(\hbar,u,a,b)$ in Eq. (35) as

$$U(t,u,a,b)=\delta(t)+\sum_{k=1}^{\infty}\sum_{m=0}^{k}\frac{(-u)^k m!a^m b^{k-m}t^{m+\omega k-1}}{(k!)^2(k-m)!\Gamma(m+\omega k)} \tag{38}$$

where $\delta(t)$ is Dirac's distribution.

Now, we use the following property of Laplace transform [29]: if $\hat{G}(\hbar)=L\{F(t)\}$, then$L^{-1}\left\{\frac{1}{\hbar}\hat{G}\left(\frac{1}{\hbar}\right)\right\}=\int_0^{\infty}F(z)J_0(2\sqrt{tz})dz.$ (39).

We obtain

$$\frac{1}{\hbar}\hat{U}\left(\frac{1}{\hbar},u,a,b\right)=\frac{1}{\hbar}\exp\left(-u\left(a\hbar^{\omega+1}+b\hbar^{\omega}\right)\right)=\frac{1}{\hbar}\hat{W}(\hbar,u,a,b) \tag{40}$$

therefore,

$$L^{-1}\left\{\frac{1}{\hbar}\hat{U}\left(\frac{1}{\hbar},u,a,b\right)\right\}=L^{-1}\left\{\frac{1}{\hbar}\hat{W}(\hbar,u,a,b)\right\}=\int_0^{\infty}\delta(z)J_0(2\sqrt{tz})dz+\sum_{k=1}^{\infty}\sum_{m=0}^{k}\frac{(-u)^k m!a^m b^{k-m}t^{m+\omega k-1}}{(k!)^2(k-m)!\Gamma(m+\omega k)}\int_0^{\infty}z^{m+\omega k-1}J_0(2\sqrt{tz})dz. \tag{41}$$

Since$\int_0^{\infty}\delta(z)J_0(2\sqrt{tz})dz=J_0(2\sqrt{tz})|_{z=0}=J_0(0)=1$, we obtain

$$\int_0^t W(\sigma,u,a,b)d\sigma=1+\sum_{k=1}^{\infty}\sum_{m=0}^{k}\frac{(-u)^k m!a^m b^{k-m}t^{m+\omega k-1}}{(k!)^2(k-m)!\Gamma(m+\omega k)}\times\int_0^{\infty}z^{m+\omega k-1}J_0\left(2\sqrt{tz}\right)dz \tag{42}$$

Deriving relation (41) concerning the time t and using the following relation$\frac{dJ_0(h(t))}{dt}=-J_1(h(t))h'(t)$, we obtain (33), therefore, the proposition is demonstrated.

Now, using the result given by the above proposition, we obtain the following expression of the function$\chi(t,u)$:

$$\chi(t,u)=L^{-1}\{\exp(-u\vartheta_1(\hbar))\}=e^{-ub}L^{-1}\left\{e^{-u\hbar}e^{-u(\tau\hbar^{\omega+1}+b\tau\hbar^{\omega})}\right\}$$
$$=e^{-ub}L^{-1}\{e^{-u\hbar}\}L^{-1}\left\{\hat{W}(\hbar,u,\tau,b\tau)\right\}$$
$$=e^{-ub}\delta(t-u)*W(t,u,\tau,b\tau)=e^{-ub}W(t-u,u,\tau,b\tau)$$
$$=\sum_{k=1}^{\infty}\sum_{m=0}^{k}\frac{m!b^{k-m}\tau^k(-u)^{k+1}e^{-bu}}{(k!)^2(k-m)!\Gamma(m+\omega k)}\int_0^{\infty}\sqrt{\frac{z}{t-u}}z^{m+\omega k-1}J_1(2\sqrt{(t-u)z})dz. \tag{43}$$

where, we used the property of the Dirac distribution $\delta(t-t_0)*f(t)=f(t-t_0)$.

3.1. Particular case $\omega=1$

It is easy to see that all the results presented in the previous section remain true if the fractional parameter ω is equal to 1. Therefore, the solution of the problem of a thermal process described by the classical Cattaneo law of thermal flux is given by Eq. (32) with the fractional parameter ω replaced by 1. However, in this section, we determine another form of the analytical solution corresponding to the classical Cattaneo thermal process.

To do this, let's observe that for$\omega=1$, function $\vartheta(s)$ is written in the equivalent forms

$$\vartheta_1(\hbar)=\tau\hbar^2+(1+b\tau)\hbar+b=\tau\left[(\hbar+a_0)^2-b_0^2\right] \tag{44}$$

where $a_0=\frac{1+b\tau}{2\tau}$, $b_0=\frac{1-b\tau}{2\tau}$.

The inverse Laplace transform of function $\hat{H}(\hbar, p) = e^{-p\sqrt{\vartheta_1(\hbar)}}$ is given by

$$\begin{aligned} H(t,p) &= L^{-1}\left\{\hat{H}(\hbar,p)\right\} = L^{-1}\left\{e^{-p\sqrt{\vartheta_1(\hbar)}}\right\} = L^{-1}\left\{e^{-p\sqrt{\tau}\sqrt{(\hbar+a_0)^2-b_0^2}}\right\} \\ &= L^{-1}\left\{\left[e^{p\sqrt{\tau}\left[(\hbar+a_0)-\sqrt{(\hbar+a_0)^2-b_0^2}\right]}-1\right]e^{-p\sqrt{\tau}(\hbar+a_0)}+e^{-p\sqrt{\tau}(\hbar+a_0)}\right\} \\ &= e^{-a_0p\sqrt{\tau}}L^{-1}\left\{e^{-p\sqrt{\tau}\hbar}\left[e^{p\sqrt{\tau}\left[(\hbar+a_0)-\sqrt{(\hbar+a_0)^2-b_0^2}\right]}-1\right]\right\}+L^{-1}\left\{e^{-p\sqrt{\tau}(\hbar+a_0)}\right\} \\ &= e^{-a_0p\sqrt{\tau}}\delta(t-p\sqrt{\tau}) * \frac{b_0p\sqrt{\tau}e^{-a_0t}}{\sqrt{t^2+2p\sqrt{\tau}t}}I_1\left(b_0\sqrt{t^2+2p\sqrt{\tau}t}\right)+e^{-a_0p\sqrt{\tau}}\delta(t-p\sqrt{\tau}) \\ &= \frac{b_0p\sqrt{\tau}e^{-a_0t}}{\sqrt{t^2-p^2\tau}}I_1\left(b_0\sqrt{t^2-p^2\tau}\right)+e^{-a_0p\sqrt{\tau}}\delta(t-p\sqrt{\tau}). \end{aligned} \tag{45}$$

Using (25)-(27) and (45), we obtain

$$\begin{aligned} T(r,t) &= \frac{1}{r}\sum_{k=0}^{\infty}[f(t)*H(t,2k+1-r)-f(t)*H(t,2k+1+r)] \\ &= \frac{1}{r}\sum_{k=0}^{\infty}\left[e^{-a_0(2k+1+r)\sqrt{\tau}}f(t-(2k+1+r))-e^{-a_0(2k+1-r)\sqrt{\tau}}f(t-(2k+1-r))\right] \\ &\quad+\frac{1}{r}\sum_{k=0}^{\infty}\int_0^t f(t-\sigma)[\varphi_1(\sigma,2k+1-r)-\varphi_1(\sigma,2k+1+r)]d\sigma, \end{aligned} \tag{46}$$

where

$$\varphi_1(t,p) = \frac{b_0p\sqrt{\tau}e^{-a_0t}}{\sqrt{t^2-p^2\tau}}I_1\left(b_0\sqrt{t^2-p^2\tau}\right) \tag{47}$$

3.2. Particular case $\tau = 0$

This case corresponds to a thermal process described by the classical Fourier's law. The solution is given by

$$\begin{aligned} &L^{-1}\left\{e^{-p\sqrt{\vartheta_1(\hbar)}}\right\} = L^{-1}\left\{e^{-p\sqrt{\hbar+b}}\right\} = \frac{pe^{-bt}}{2t\sqrt{\pi t}}e^{-\frac{p^2}{4t}}, \\ &T(r,t) = \frac{1}{r}f(t)*\sum_{k=0}^{\infty}\left[\frac{(2k+1-r)}{2t\sqrt{\pi t}}e^{-bt-\frac{(2k+1-r)^2}{4t}}-\frac{(2k+1+r)}{2t\sqrt{\pi t}}e^{-bt-\frac{(2k+1+r)^2}{4t}}\right]. \end{aligned} \tag{48}$$

4. Numerical examples and discussions

A generalized thermal transport in spherical domains with heated boundaries has been studied in the case of central symmetry. Using the generalized fractional Cattaneo's law, the thermal transport is influenced by the heat transfer history. The boundary temperature is described by a time-variable differentiable function$f(t)$. In this section, two cases are discussed namely, the case of constant temperature on the sphere,$f(t) = 4,\ t > 0$, respectively

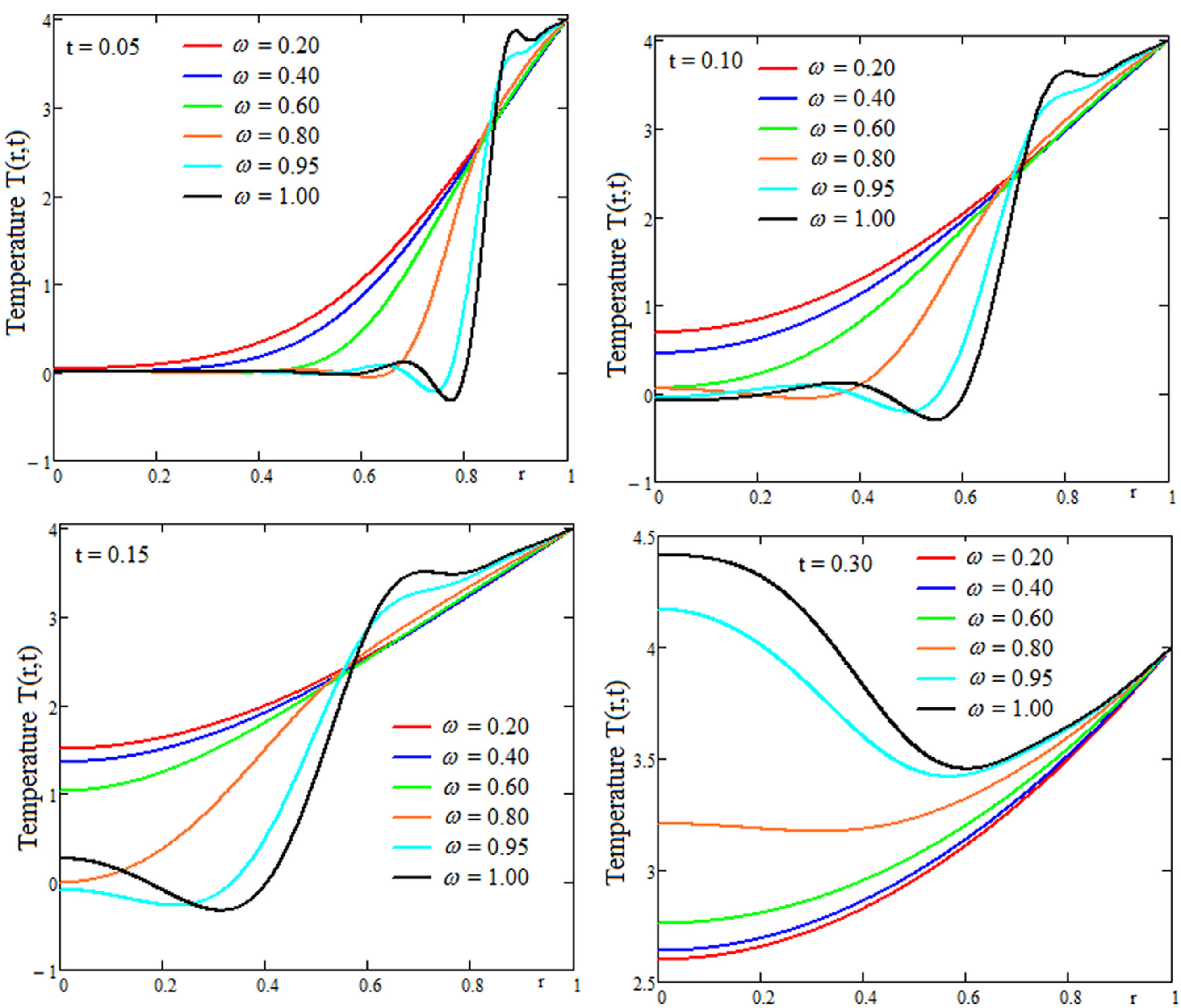

Fig. 2. The profiles of $T(r,t)$ versus r for different values ofω.

an exponential time-variation of the temperature on the sphere's surface, $f(t) = \frac{1}{2}(1 + e^{-t}),\ t > 0$.

The numerical results corresponding to the first case are plotted in Figs. 2 and 3 for the non-dimensional thermal relaxation time $\tau = 0.1$ and the parameter of heat absorption $b = 2$.

The profiles of the $T(r,t)$, versus the radial coordinate r are sketched in Fig. 2 for starting of the time t and for different values of the fractional parameter ω. It can be seen in Fig. 1 that the temperature values decrease with the memory parameter ω. It should be noted that, in the classic Cattaneo case of heat transfer ($\omega = 1$), the temperature profile is significantly different from the fractional case. The difference is due to the presence of the Caputo kernel in the fractional derivative. Caputo kernel plays the role of a weight function for the heat flux. The history of the thermal process influences the heat transfer at the instant t.

Fig. 3 was plotted to show the evolution of temperature over time in different positions inside the sphere and for different values of the fractional parameter ω. As in the previous figure, note that the influence of the fractional parameter is significant only for small values of time t. For time values t greater than 0.6, the differences between the temperature profiles are insignificant. Note that for high values of time t, the temperature attains a constant value over time.

This property is theoretically proved because we have:

$$\lim_{t\to\infty} T(r,t) = \lim_{\hbar\to 0} \hbar \hat{T}(r,\hbar) = \lim_{\hbar\to 0} \hbar \frac{4}{\hbar}\frac{1}{r}\frac{\sinh\left(r\sqrt{\vartheta_1(\hbar)}\right)}{\sinh\left(\sqrt{\vartheta_1(\hbar)}\right)} = \frac{4}{r}\frac{\sinh\left(r\sqrt{b}\right)}{\sinh\left(\sqrt{b}\right)}$$

The influence of the non-dimensional relaxation time τ on the temperature profiles is shown in Fig. 4. It is observed that the temperature is decreased with the parameter τ.

A comparison between the mathematical models of heat transfer described by the fractional Cattaneo law, classical Cattaneo law, and the classical Fourier law is given in Figs. 5 and 6. It is observed in Figs. 4 and 5 that the maximum values of the temperature are given by the Fourier model, while the fractional Cattaneo model generates temperatures smaller than corresponding to the Fourier model. The temperature of the classical Cattaneo model has

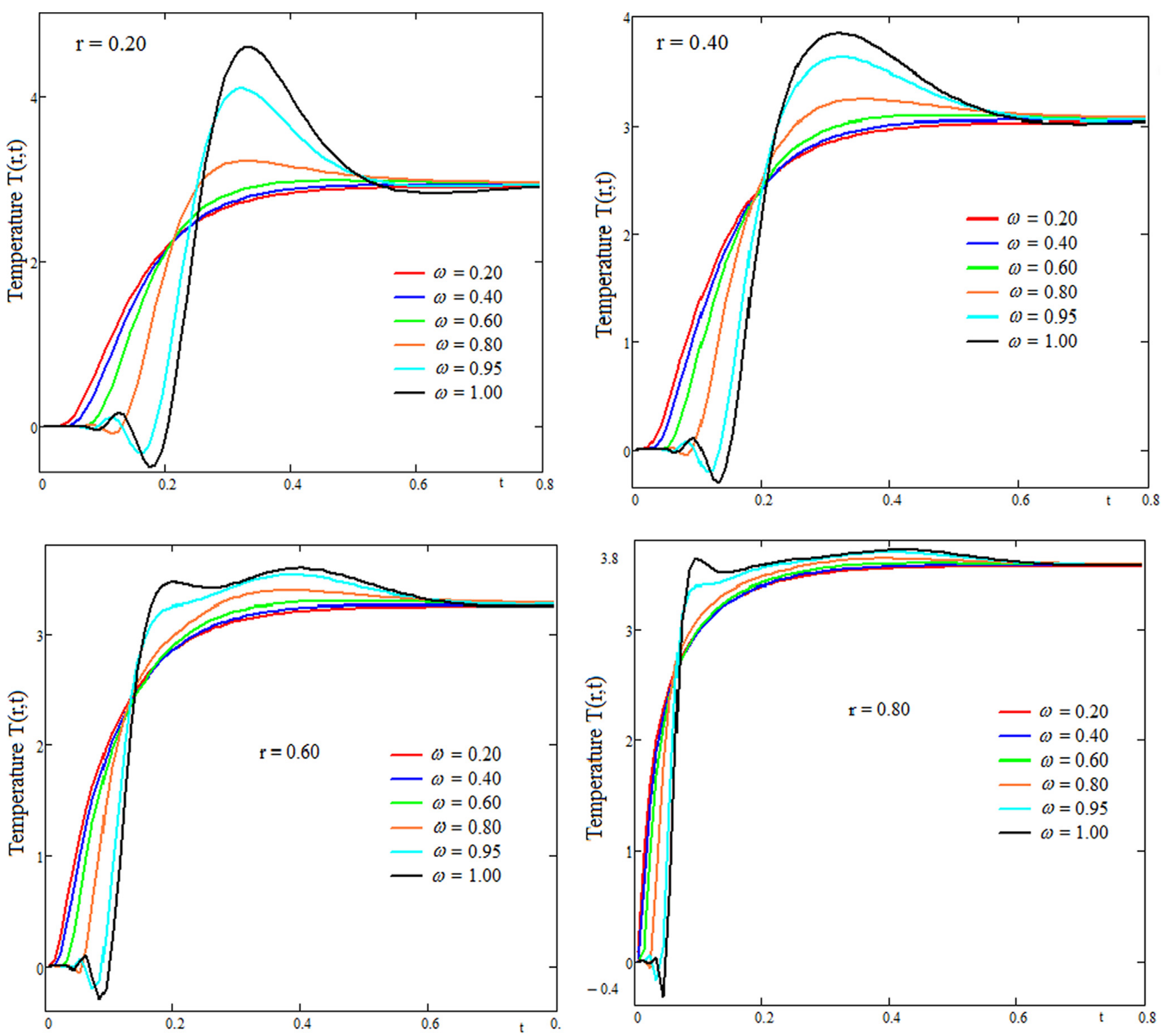

Fig. 3. The profiles of $T(r,t)$ versus t for different values of ω.

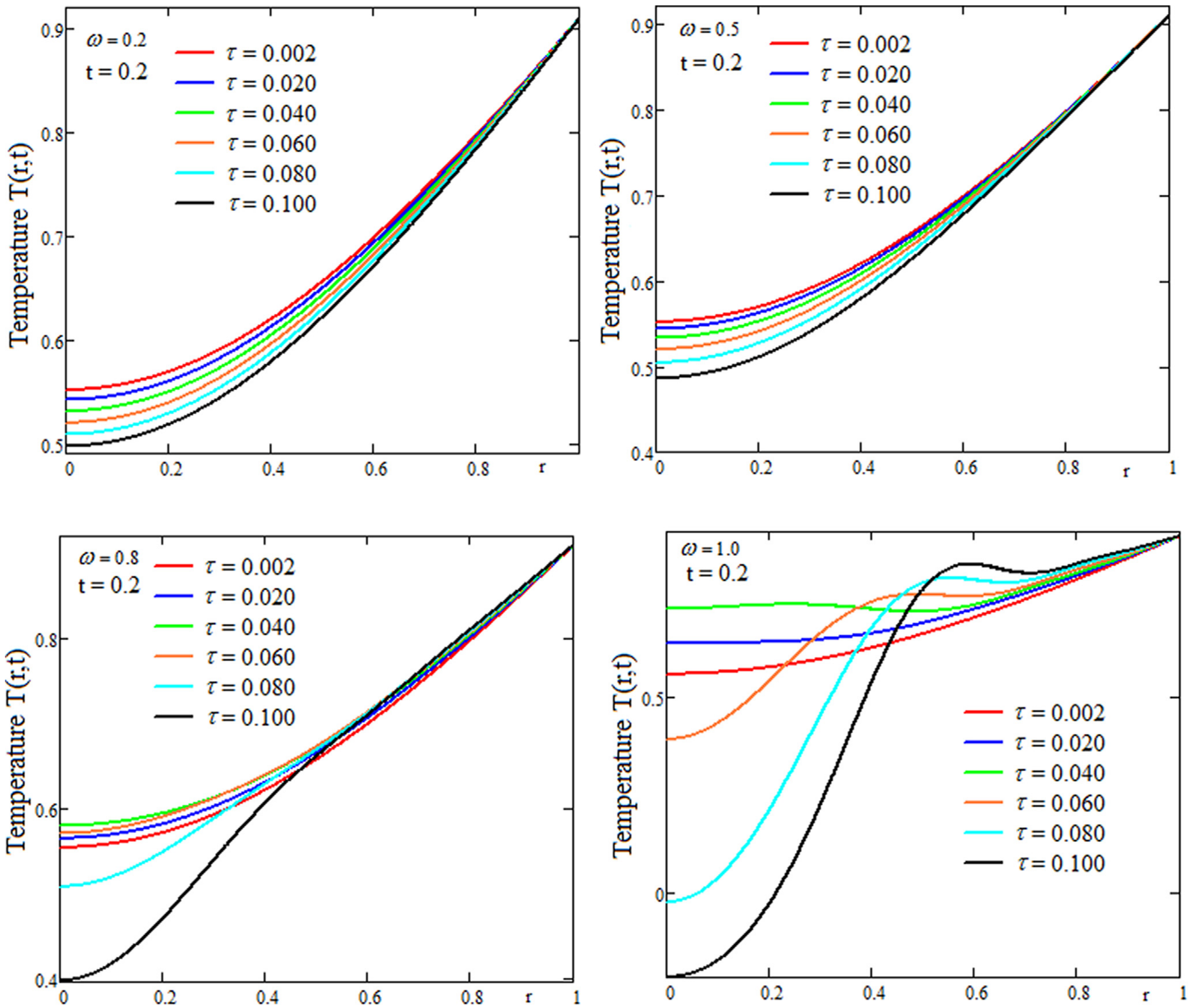

Fig. 4. The profiles of $T(r,t)$ versus r for variation of the thermal relaxation timeτ.

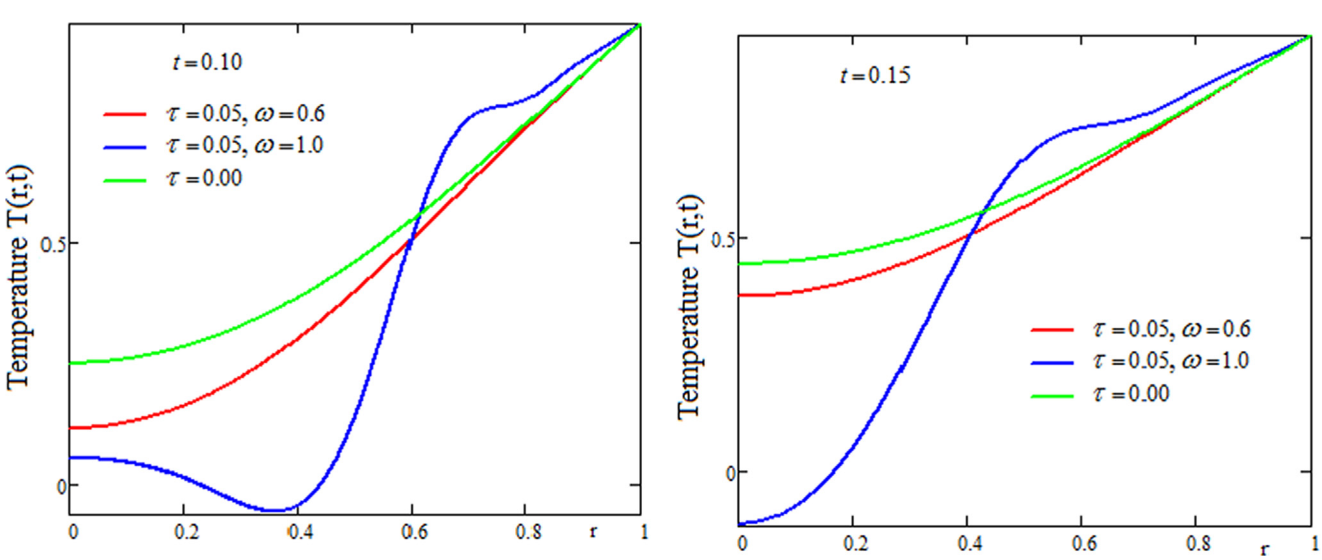

Fig. 5. The comparison between temperatures corresponding to fractional Cattaneo, classical Cattaneo, and classical Fourier model of heat transfer. Variation with the radial coordinate.

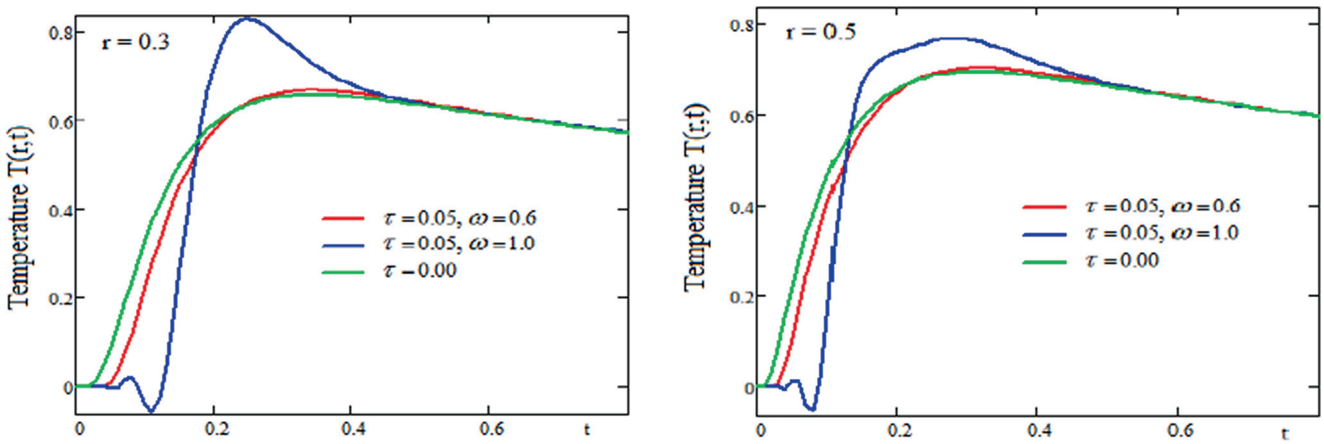

Fig. 6. The comparison between temperatures corresponding to fractional Cattaneo, classical Cattaneo, and classical Fourier model of heat transfer. Variation with the time *t*.

different behavior. In the central area of the spherical domain, the temperature has the smallest values, and in the exterior area the highest values. Also, in Fig. 5 is observed that, for large values of the time t, the differences between temperature values are negligible; therefore, the fractional model is significantly different from the classical model for small values of the time t.

5. Conclusions

In a spherical domain with the time-dependent Dirichlet boundary condition, the time-fractional diffusion equation with the Caputo fractional derivative of the order $0 < \omega \leqslant 1$ with heat absorption has been explored.

The Laplace transform and an appropriate transformation of the independent variable and unknown function were used to arrive at the analytical solution to the problem.

For the first time in the literature, the novel inverse Laplace transforms of two exponential-order functions have been demonstrated.

The effect of the key issue factors, including the value of the order of fractional derivative on the spatial–temporal development of the temperature, is shown graphically by the generated analytical solutions.

Results provided by fractional and traditional mathematical models are compared.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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NEHAD ALI SHAH is currently working as an in Department of Mechanical Engineering, Sejong University, Seoul, South Korea. He has published more than 200 research articles in different high-reputed journals. He is the Editor and reviewer for many national and international journals.