

A quasilinear operator retaining magnetic drift effects in tokamak geometry

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The interaction of radio frequency waves with charged particles in a magnetized plasma is usually described by the quasilinear operator that was originally formulated by Kennel & Engelmann (*Phys. Fluids*, vol. 9, 1966, pp. 2377–2388). In their formulation the plasma is assumed to be homogenous and embedded in a uniform magnetic field. In tokamak plasmas the Kennel–Engelmann operator does not capture the magnetic drifts of the particles that are inherent to the non-uniform magnetic field. To overcome this deficiency a combined drift and gyrokinetic derivation is employed to derive the quasilinear operator for radio frequency heating and current drive in a tokamak with magnetic drifts retained. The derivation requires retaining the magnetic moment to higher order in both the unperturbed and perturbed kinetic equations. The formal prescription for determining the perturbed distribution function then follows a novel procedure in which two non-resonant terms must be evaluated explicitly. The systematic analysis leads to a diffusion equation that is compact and completely expressed in terms of the drift kinetic variables. The equation is not transit averaged, and satisfies the entropy principle, while retaining the full poloidal angle variation without resorting to Fourier decomposition. As the diffusion equation is in physical variables, it can be implemented in any computational code. In the Kennel–Engelmann formalism, the wave–particle resonant delta function is either for the Landau resonance or the Doppler shifted cyclotron resonance. In the combined gyro and drift kinetic approach, a term related to the magnetic drift modifies the resonance condition.

Key words: fusion plasma, plasma heating, plasma nonlinear phenomena

1. Introduction

The quasilinear operator derived by Kennel & Engelmann (1966) is widely used to treat radio frequency (rf) heating and current drive in tokamaks even though it was derived for plane electromagnetic waves in a constant magnetic field and homogeneous plasma. It successfully captures the gyromotion departure of charged particles from flux surfaces through its Bessel function dependence. However, it does not retain the magnetic drift effects associated with tokamak confinement. These magnetic drift effects enter by altering the resonance, which is then expected to modify the quasilinear diffusivity. The absence of this geometrical effect is a key

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limitation of the Kennel–Engelmann operator, that retains the wave–particle Landau and Doppler shifted cyclotron resonances, but not drift resonances. The main purpose of the work herein is to show how this deficiency can be corrected by a simple modification of the standard quasilinear diffusivity while preserving negative definite entropy production. Even though this alteration is simple, the detailed proof is rather involved since it requires using higher-order drift kinetic and high frequency gyrokinetic variables (Lee, Myra & Catto 1983). In particular, the magnetic moment must be employed to higher order and the non-resonant portions of the linearized distribution must be evaluated explicitly.

The derivation of the quasilinear operator presented here takes advantage of high frequency gyrokinetics to obtain the perturbed distribution function f_1 with magnetic drift effects retained in full tokamak geometry. We retain drift effects by extending the high frequency electrostatic gyrokinetic treatment by Lee *et al.* (1983) to electromagnetic waves. Their treatment employs gyrokinetic variables to higher order than the standard lowest-order gyrokinetic variables introduced by Catto (1978). We find that to retain drift modifications to the quasilinear operator we need to employ their higher-order definition of the magnetic moment for the gyrokinetic and drift kinetic changes of variables. Higher-order gyrokinetic variables are sometimes required for low frequency gyrokinetic treatments as well (Catto, Tang & Baldwin 1981; Kagan & Catto 2008; Parra & Catto 2008; Parra & Calvo 2011; Calvo & Parra 2012).

Our high frequency gyrokinetic treatment also has the virtue that it avoids Fourier transforming the perturbed distribution function f_1 in poloidal angle. We are thereby able to obtain a compact modified Fourier representation for f_1 with poloidally varying coefficients so geometric coupling of modes is treated in a natural manner in poloidally varying, axisymmetric tokamak magnetic fields. Our procedure provides a more general treatment of geometric effects than has been possible in the past (Faulconer 1987; Smithe *et al.* 1988; Catto, Lashmore-Davies & Martin 1993).

Once f_1 is obtained in gyrokinetic variables, with the magnetic moment retained to higher order, it must be transformed back to drift kinetic variables to form the quasilinear operator appearing in the drift kinetic equation for the unperturbed distribution function f_0 . Consequently, a key difference between our gyrokinetic and drift kinetic treatments is that our gyrokinetic variables retain the distinction between the guiding centre and charge location, while our drift kinetic ones do not.

Our treatment is not transit or bounce averaged, but does not retain radial spatial derivatives in the quasilinear operator. Therefore, it does not treat large departures of charges from flux surfaces by retaining the radial derivative terms of Kaufman (1972) and Eriksson & Helander (1994). Their elegant treatments are in canonical variables so the transit averaged collision operator must be transformed to the same canonical variables. Then once the equation for the unperturbed distribution function is solved the solution must be changed back to drift kinetic variables for use in full wave codes or for the evaluation of energy input or current driven. In addition to the drift corrections evaluated here, these radial derivative terms can become important for energetic ions produced by minority heating (Eriksson *et al.* 1998; Mantsinen *et al.* 2002) when they have very large departures from a flux surface. Moreover, minority ion cyclotron current drive plays a role in sawtooth control (Chapman *et al.* 2015) where spatial diffusion is known to result in additional current drive (Hellsten, Carlsson & Eriksson 1995). Our small poloidal gyroradius treatment neglects spatial diffusion, which may also be important for minority current drive in the core, as well as for heating and current drive in the pedestal.

The retention of the magnetic drift effects in the non-transit averaged description considered here is expected to be relevant – and perhaps even important – for ions whenever perpendicular wavelengths become small or for charges in the vicinity of a turning point. For example, the ion cyclotron radio frequency mode conversion process to an ion Bernstein wave and an ion cyclotron wave (Perkins 1977) involves perpendicular wavelengths comparable to or smaller than the gyroradius (Jaeger *et al.* 2003; Lin *et al.* 2003; Wright *et al.* 2004). Consequently, magnetic drift effects might be expected to play a role in this case. These drift effects matter because they alter the resonance. In particular, both the velocity dependent magnetic drift and the usual parallel streaming velocity term broaden the cyclotron resonance especially for energetic minority ions. However, we will find that the drift terms typically become important as the perpendicular wavelengths become comparable to or smaller than the ion gyroradius ρ_i . In this limit the argument of the usual Bessel functions is large so the quasilinear diffusivity is reduced. Indeed, this is likely the main reason that the quasilinear form of Kennel & Engelmann (1966) works so well. Even though the drift correction derived herein may not substantially affect heating and current drive by most rf waves, it may shift the resonant interaction to a significantly different poloidal location for minority current drive when the energetic ions have large drift velocities. Most existing quasilinear codes are transit averaged, but a non-transit averaged implementation is necessary to evaluate some effects associated with toroidal geometry. In particular, a non-transit averaged quasilinear code (Kapper *et al.* 2016) has recently verified electron cyclotron current drive enhancement due to symmetric spectrum effects associated with toroidal geometry as described by Helander & Catto (2001). In addition to velocity space diffusion, it would ultimately be desirable to generalize our quasilinear treatment to retain radial derivative modifications due to finite orbit effects for minority ions.

In the following we present a background discussion in § 2 and formulate our combined drift and gyrokinetic descriptions in § 3 with the higher-order magnetic moment retained. In § 4 we solve the gyrokinetic equation for the perturbed distribution function after explicitly extracting two non-resonant contributions to it. Section 5 derives the non-transit averaged quasilinear operator with drifts retained in a form that manifestly satisfies the entropy production principle. We close with a brief summary in § 6, and give some mathematical details in the appendix A.

2. Background

The quasilinear operator must be valid for a wide range of wave frequencies ω . For waves harmonic in time and space, our orderings must allow us to consider low frequencies of order

$$\omega \sim k_{\parallel} v_{\parallel} \sim \mathbf{k}_{\perp} \cdot \mathbf{v}_d, \tag{2.1}$$

to recover the unmagnetized and low frequency limit of Kennel & Engelmann (1966). Here and hereafter, k_{\parallel} and \mathbf{k}_{\perp} are the parallel and perpendicular components of the wave vector, and v_{\parallel} and \mathbf{v}_d are the parallel and magnetic drift velocities. For higher frequencies we must allow

$$\omega - k_{\parallel} v_{\parallel} - \mathbf{k}_{\perp} \cdot \mathbf{v}_d \sim \Omega \gg \mathbf{k}_{\perp} \cdot \mathbf{v}_d \tag{2.2}$$

to recover magnetized Kennel & Engelmann (1966), where Ω is the gyrofrequency. The large range of frequencies allows our quasilinear result to treat frequencies from drift waves to ion and then electron cyclotron frequencies and above.

For magnetic drifts to modify the usual quasilinear (QL) resonance condition $\omega - p\Omega - k_{\parallel}v_{\parallel} = 0$ requires $\mathbf{k}_{\perp} \cdot \mathbf{v}_d \sim k_{\parallel}v_{\parallel}$, where p is an integer or zero. Letting ρ and R denote the gyroradius and major radius of the tokamak, we find that

$$k_{\perp}\rho_p \sim k_{\parallel}R, \quad (2.3)$$

with $\rho_p = \rho B/B_p$ the poloidal gyroradius and B_p the poloidal magnetic field. Typically,

$$k_{\parallel}R \gg 1, \quad (2.4)$$

so we expect magnetic drift effects will be of most interest when

$$k_{\perp}\rho_p \gg 1. \quad (2.5)$$

For large values of $k_{\perp}\rho_i$ we expect the QL diffusivity to be small since it depends on Bessel functions that for large argument behave as $J_p(w) \sim w^{-1/2}$, with $w \sim k_{\perp}\rho_i$. As a result, the magnetic drift corrections we evaluate may be important when $k_{\perp}\rho_i \sim 1$ for $B_p \ll B$. Indeed, this seems to explain why the Kennel & Engelmann (1966) limit of the QL operator works so well in a tokamak.

The quasilinear diffusivity in the constant magnetic field case is proportional to a delta function whose argument is $\omega - p\Omega - k_{\parallel}v_{\parallel}$. We desire to generalize this QL diffusivity to include magnetic drifts. We expect to find that generalized diffusivity contains a drift modified resonance condition so the argument of the delta function depends on $\omega - p\Omega - k_{\parallel}v_{\parallel} - \mathbf{k}_{\perp} \cdot \mathbf{v}_d$. This new resonance condition is now what must be used to gather terms in the proper way to obtain the required form of the quasilinear diffusivity with magnetic drift effects retained.

3. Formulation of the quasilinear and linear equations

Consider the full kinetic equation

$$\partial f / \partial t + \mathbf{v} \cdot \nabla f + (Ze/M)[\mathbf{e} + c^{-1}\mathbf{v} \times (\mathbf{B} + \mathbf{b})] \cdot \nabla_{\mathbf{v}} f = C\{f\}, \quad (3.1)$$

with \mathbf{e} and \mathbf{b} the applied radio frequency (rf) wave fields and $\mathbf{B} = B\mathbf{n}$ the unperturbed magnetic field. We then take

$$f = f_0 + f_1 + \dots \quad (3.2)$$

with

$$f_1 \ll f_0, \quad (3.3)$$

and f_0 gyrophase independent so that

$$\partial f_0 / \partial \varphi = 0, \quad (3.4)$$

with φ the gyrophase. We are interested in situations for which collisions and quasilinear diffusion are equally important in the evolution of f_0 . We also require that f_0 does not have any fast time or space dependence so take it to be the coarse grain average of f :

$$f_0 = \langle f \rangle_{cg} = \frac{1}{2T} \int_{t-T}^{t+T} d\tau \frac{1}{2\Delta} \int_{\psi-\Delta}^{\psi+\Delta} d\psi' \oint \frac{d\zeta}{2\pi} \langle f \rangle_{\vartheta}, \quad (3.5)$$

where the integral over toroidal angle ζ insures axisymmetry, and the average $\langle \dots \rangle_\vartheta$ over poloidal angle ϑ removes the high poloidal mode number variation. The integral over τ removes fast time variation by taking $\omega^{-1} \ll T \ll \nu^{-1}$, and the integral over the poloidal flux function ψ removes rapid radial variation by assuming $\lambda_{rf}^{\text{radial}} \ll \Delta \ll a$, with ν and ω being the collision frequency and typical rf wave frequency, respectively, and a and $\lambda_{rf}^{\text{radial}}$ being the minor radius and the typical radial wavelength of the rf wave, respectively. More details on how $\langle \dots \rangle_\vartheta$ deals with rapid variation will be given in § 5, but for now it is sufficient to know that f_0 is allowed to be a slow function of ϑ , as well as ψ and t . Consequently, we are assuming

$$\Omega f_0 \gg \partial f_0 / \partial t \sim \mathbf{v} \cdot \nabla f_0 \sim \nu f_0 \sim Q, \tag{3.6}$$

with $\Omega = ZeB/Mc$ and Q the quasilinear operator to be derived. We allow f_1 to rapidly vary in time and space by permitting

$$\Omega \partial f_1 / \partial \varphi \sim \partial f_1 / \partial t \sim \mathbf{v}_\perp \cdot \nabla f_1 \sim v_\parallel \mathbf{n} \cdot \nabla f_1. \tag{3.7}$$

These assumptions mean that f_0 satisfies the quasilinear (QL) equation

$$\partial f_0 / \partial t + \mathbf{v} \cdot \nabla f_0 + (Ze/Mc) \mathbf{v} \times \mathbf{B} \cdot \nabla_{\mathbf{v}} f_0 + (Ze/M) \langle (\mathbf{e} + c^{-1} \mathbf{v} \times \mathbf{b}) \cdot \nabla_{\mathbf{v}} f_1 \rangle_{cg} = C\{f_0\}, \tag{3.8}$$

while f_1 satisfies the linear equation

$$\partial f_1 / \partial t + \mathbf{v} \cdot \nabla f_1 + (Ze/Mc) \mathbf{v} \times \mathbf{B} \cdot \nabla_{\mathbf{v}} f_1 = -(Ze/M) (\mathbf{e} + c^{-1} \mathbf{v} \times \mathbf{b}) \cdot \nabla_{\mathbf{v}} f_0, \tag{3.9}$$

with collisions neglected since we are normally interested in wave frequencies much higher than the collision frequencies. If the sum of these last two equations is subtracted from the full kinetic equation then the kinetic equation for the difference $f - (f_0 + f_1)$ is recovered. However, this difference equation is not needed to derive the quasilinear diffusivity. Except for tokamak geometry, equations (3.8) and (3.9) are the standard starting equations for all quasilinear treatments of rf heating and current drive.

For the QL equation we use the drift kinetic limit of gyrokinetics in which we keep the magnetic moment to higher order. In these $\mathbf{r}, v, \mu, \varphi, t$ drift kinetic variables the magnetic moment is given by

$$\mu = \mu_0 + \mu_1, \tag{3.10}$$

with

$$\mu_0 = v_\perp^2 / 2B(\mathbf{r}) \tag{3.11}$$

the usual lowest-order magnetic moment. We write the velocity \mathbf{v} in the usual cylindrical velocity space variables aligned with the magnetic field

$$\mathbf{v} = \mathbf{v}_\perp + v_\parallel \mathbf{n} = v_\perp [\mathbf{e}_1(\mathbf{r}) \cos \varphi + \mathbf{e}_2(\mathbf{r}) \sin \varphi] + v_\parallel \mathbf{n}(\mathbf{r}), \tag{3.12}$$

with φ the gyrophase,

$$v = |\mathbf{v}| = \sqrt{v_\perp^2 + v_\parallel^2} \tag{3.13}$$

the magnitude of velocity of the charge (a constant of the motion for our treatment), and the three orthonormal unit vectors satisfying

$$\mathbf{e}_1(\mathbf{r}) \times \mathbf{e}_2(\mathbf{r}) = \mathbf{n}(\mathbf{r}) = \mathbf{B}(\mathbf{r})/B(\mathbf{r}). \quad (3.14)$$

Then the next-order correction to the magnetic moment is (Catto *et al.* 1981; Lee *et al.* 1983; Kagan & Catto 2008; Parra & Catto 2008)

$$\begin{aligned} \mu_1 = & -B^{-1} \mathbf{v}_\perp \cdot \mathbf{v}_d - (v_\parallel/4B\Omega)(\mathbf{v}_\perp \cdot \nabla \mathbf{n} \cdot \mathbf{v} \times \mathbf{n} + \mathbf{v} \times \mathbf{n} \cdot \nabla \mathbf{n} \cdot \mathbf{v}_\perp) \\ & - (v_\parallel v_\perp^2/2B\Omega) \mathbf{n} \cdot \nabla \times \mathbf{n}, \end{aligned} \quad (3.15)$$

with

$$\mathbf{v}_d = \Omega^{-1} \mathbf{n} \times (\mu \nabla B + v_\parallel^2 \mathbf{n} \cdot \nabla \mathbf{n}) \quad (3.16)$$

the magnetic drift velocity. The unperturbed $\mathbf{E} \times \mathbf{B}$ drift can be added to \mathbf{v}_d if the unperturbed electric field \mathbf{E} is known. In typical quasilinear treatments, including the treatment here, the perturbed drift due to the applied wave fields is considered to be small and neglected. However, these and other small fluctuating corrections to the unperturbed gyrokinetic trajectories are expected to provide the stochasticity necessary to randomize kicks as charges make successive passes through a resonance (Becoulet, Gambier & Samain 1991). While low frequency gyrokinetic treatments have been very successful in retaining the effects of low frequency fluctuating fields (Kagan & Catto 2008; Parra & Catto 2008; Parra & Calvo 2011; Calvo & Parra 2012), at present there seems to be no systematic way of treating high frequency fluctuations in high frequency gyrokinetics.

Changing to these new variables, gyroaveraging, and coarse grain averaging results in the QL equation in its drift kinetic form

$$\partial f_0/\partial t + (v_\parallel \mathbf{n} + \mathbf{v}_d) \cdot \nabla f_0 = C\{f_0\} + Q\{f_0\}, \quad (3.17)$$

with the QL operator given by

$$Q\{f_0\} = -\langle \mathbf{a} \cdot \nabla_w f_1 \rangle_{cg,\varphi} = -\left\langle \oint \frac{d\varphi}{2\pi} \mathbf{a} \cdot \nabla_w f_1 \right\rangle_{cg} = -\left\langle \oint \frac{d\varphi}{2\pi} \nabla_v \cdot (\mathbf{a} f_1) \right\rangle_{cg}, \quad (3.18)$$

and where the perturbed acceleration is defined as

$$\mathbf{a} = (Ze/M)(\mathbf{e} + c^{-1} \mathbf{v} \times \mathbf{b}). \quad (3.19)$$

There are two important points to mention here. The first point is that $Q\{f_0\}$ must be allowed to be a slow function of ϑ in a tokamak so that $(\dots)_\vartheta \neq (2\pi)^{-1} \oint d\vartheta (\dots)$. The second point is that the drift kinetic treatment requires that we average the QL term over gyrophase, as well as perform a coarse grain average. Therefore, in the unperturbed distribution function

$$f_0 = f_0(\psi, \vartheta, v, \mu, \sigma, t), \quad (3.20)$$

$\sigma = v_\parallel/|v_\parallel|$ is the sign of v_\parallel , ψ is the poloidal flux function and ϑ is the poloidal angle variable. The preceding form and/or the original full f form of the kinetic equation are convenient for forming conservation equations, as well as other moment equations. Which form is to be used depends on the accuracy required.

It is important to mention here that we will not obtain a radial spatial derivative in our quasilinear operator since we will evaluate $Q\{f_0\}$ in drift kinetic variables by assuming

$$\rho_p f_0^{-1} \nabla f_0 \ll 1, \tag{3.21}$$

with $\rho_p = \rho B/B_p$ the poloidal gyroradius and B_p the poloidal magnetic field.

To evaluate the lowest-order distribution function it is sometimes convenient to solve the transit averaged version of the preceding equation. To lowest order we then assume that streaming dominates so that

$$v_{\parallel} \mathbf{n} \cdot \nabla \bar{f}_0 = 0, \tag{3.22}$$

with the lowest-order distribution function defined to be independent of poloidal angle

$$\bar{f}_0 = \bar{f}_0(\psi, v, \mu, \sigma, t). \tag{3.23}$$

Then

$$\mathbf{v}_d \cdot \nabla \bar{f}_0 = \mathbf{v}_d \cdot \nabla \psi \frac{\partial \bar{f}_0}{\partial \psi} = v_{\parallel} \mathbf{n} \cdot \nabla \left(\frac{I v_{\parallel}}{\Omega} \frac{\partial \bar{f}_0}{\partial \psi} \right). \tag{3.24}$$

Letting

$$f_0 = \bar{f}_0 + \delta f_0, \tag{3.25}$$

with

$$\delta f_0 \ll \bar{f}_0, \tag{3.26}$$

the next-order QL equation becomes

$$\partial \bar{f}_0 / \partial t + v_{\parallel} \mathbf{n} \cdot \nabla [\delta f_0 + (I v_{\parallel} / \Omega) \partial \bar{f}_0 / \partial \psi] = C\{\bar{f}_0\} + Q\{\bar{f}_0\}. \tag{3.27}$$

It is then convenient to introduce the transit average

$$\overline{(\dots)} = \oint d\tau (\dots) / \oint d\tau = \left[\oint d\vartheta (\dots) / v_{\parallel} \mathbf{n} \cdot \nabla \vartheta \right] / \left[\oint d\vartheta / v_{\parallel} \mathbf{n} \cdot \nabla \vartheta \right], \tag{3.28}$$

with $d\tau = d\vartheta B / v_{\parallel} \mathbf{B} \cdot \nabla \vartheta$ and \bar{f}_0 the transit average of f_0 . The τ and ϑ integrations in both the numerators and denominators are over a full poloidal circuit following a charged particle. In this way v_{\parallel} and ϑ change signs together at a turning point for trapped particles, and odd functions of v_{\parallel} result in a vanishing transit average (for example, $\overline{v_{\parallel}} = 0$ and $\overline{v_{\parallel} / B} = 0$).

Equation (3.27) treats magnetic drift departures from a flux surface as being small by assuming $\delta f_0 \sim (I v_{\parallel} / \Omega) \partial \bar{f}_0 / \partial \psi$ so that poloidal gyroradius corrections are small compared to the radial scale length associated with \bar{f}_0 . This small poloidal gyroradius assumption means that finite orbit effects will be treated as being negligible.

Transit averaging the equation for δf_0 and using the periodicity of f_0 we obtain

$$\partial \bar{f}_0 / \partial t = \overline{C\{f_0\}} + \overline{Q\{f_0\}}, \tag{3.29}$$

where $\partial \bar{f}_0 / \partial t$ is often ignored as small. This transit averaged equation is the QL equation that must be solved numerically for transit averaged \bar{f}_0 once the QL term is made explicit.

To solve the linear equation we use the gyrokinetic variables \mathbf{R} , v , μ , φ and t of Lee *et al.* (1983) and Parra & Catto (2008), where

$$\mathbf{R} = \mathbf{r} + \Omega^{-1} \mathbf{v} \times \mathbf{n} + \dots, \quad (3.30)$$

and $v = |\mathbf{v}|$, $\mu = \mu_0 + \mu_1$, $\mathbf{v} = v_{\parallel} \mathbf{n} + v_{\perp} (\mathbf{e}_1 \cos \varphi + \mathbf{e}_2 \sin \varphi)$ and $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{n} = \mathbf{B}/B$ defined by (3.10) to (3.14), that is, exactly as for drift kinetics.

Higher-order terms not shown in the gyrokinetic variable \mathbf{R} are needed to remove the gyrophase dependence from $\dot{\mathbf{R}}$ to the requisite order and thereby obtain

$$\dot{\mathbf{R}} = v_{\parallel} \mathbf{n}(\mathbf{r}) + \mathbf{v}_d(\mathbf{r}), \quad (3.31)$$

where for an arbitrary scalar or vector function X ,

$$\dot{X} = dX/dt = \partial X / \partial t + \mathbf{v} \cdot \nabla X + \Omega \mathbf{v} \times \mathbf{n} \cdot \nabla_v X. \quad (3.32)$$

Similarly, the μ_1 modification of μ_0 results in

$$\dot{\mu} = 0 \quad (3.33)$$

to the requisite order. In addition, we can even retain the gyrophase to higher order,

$$\varphi = \varphi_0 + \varphi_1, \quad (3.34)$$

where to lowest order

$$\dot{\varphi}_0 = -\Omega(\mathbf{r}) = -\Omega, \quad (3.35)$$

while to next order

$$\dot{\varphi} = -\bar{\Omega} = -[\Omega(\mathbf{R}) + v_{\parallel} \mathbf{n} \cdot (\frac{1}{2} \nabla \times \mathbf{n} + \nabla \mathbf{e}_1 \cdot \mathbf{e}_2)]. \quad (3.36)$$

We assume an axisymmetric unperturbed magnetic field of the form

$$\mathbf{B} = I(\psi) \nabla \zeta + \nabla \zeta \times \nabla \psi, \quad (3.37)$$

with ζ the toroidal angle variable and $I = RB_t$ a flux function, where B_t is the toroidal magnetic field. We are free to let \mathbf{e}_1 be the unit vector in the $\nabla \psi$ direction by taking

$$\nabla \psi = RB_p \mathbf{e}_1, \quad (3.38)$$

with B_p the poloidal magnetic field and $\mathbf{e}_2 = \mathbf{n} \times \mathbf{e}_1$.

Using these higher-order gyrokinetic variables the perturbed kinetic equation becomes the high frequency gyrokinetic equation (Lee *et al.* 1983)

$$\frac{\partial f_1}{\partial t} - \bar{\Omega} \frac{\partial f_1}{\partial \varphi} + (v_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \nabla_{\mathbf{R}} f_1 = -\mathbf{a} \cdot \nabla_v f_0 = - \left[\frac{\mathbf{a} \cdot \mathbf{v}}{v} \frac{\partial f_0}{\partial v} + \mathbf{a} \cdot \nabla_v \mu \frac{\partial f_0}{\partial \mu} \right], \quad (3.39)$$

to the requisite order. To obtain the preceding it is not necessary to gyroaverage the perturbed kinetic equation so that f_1 depends on guiding centre location \mathbf{R} ,

$$f_1 = f_1(\mathbf{R}, v, \mu, \varphi, \sigma, t), \quad (3.40)$$

but the rf electric (and magnetic) field depends only on particle location \mathbf{r} ,

$$\mathbf{e} = \mathbf{e}(\mathbf{r}, t). \quad (3.41)$$

This distinction between the guiding centre location \mathbf{R} and the particle location \mathbf{r} gives rise to magnetic drift and Bessel function modifications in (3.39).

4. Gyrokinetic solution of the linear equation

There is an important subtlety we must address before proceeding any further. As we need to perform the gyrokinetic change of variables on the perturbed equation to higher order than usual it is necessary to depart from the more straightforward treatment by solving explicitly for certain pieces of the perturbed distribution function. To do so we first use Faraday’s law,

$$c\nabla \times \mathbf{e} = -\partial \mathbf{b} / \partial t, \tag{4.1}$$

to write

$$\frac{\partial \mathbf{a}}{\partial t} = \frac{Ze}{M} \left[\frac{\partial \mathbf{e}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{e}) \right] = \frac{Ze}{M} \left(\frac{\partial \mathbf{e}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{e} - \nabla \mathbf{e} \cdot \mathbf{v} \right), \tag{4.2}$$

and then use

$$\nabla_v \mu_0 = B^{-1} \mathbf{v}_\perp \tag{4.3}$$

to write

$$\frac{\partial \mathbf{a}}{\partial t} \cdot \nabla_v \mu_0 = \frac{Ze}{MB} \left[\frac{\partial \mathbf{e}}{\partial t} \cdot \mathbf{v} + v_\parallel \mathbf{n} \cdot \nabla \mathbf{e} \cdot \mathbf{v} - v_\parallel \left(\frac{\partial \mathbf{e}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{e} \right) \cdot \mathbf{n} \right]. \tag{4.4}$$

We begin by ignoring the poloidal gyroradius over unperturbed scale length corrections by using the approximation

$$\left(\frac{\partial \mathbf{e}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{e} \right) \cdot (n v_\parallel B^{-1} \partial f_0 / \partial \mu) \simeq \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \Omega \mathbf{v} \times \mathbf{n} \cdot \nabla_v \right) (\mathbf{e} \cdot n v_\parallel B^{-1} \partial f_0 / \partial \mu). \tag{4.5}$$

The preceding term requires special handling since it must be treated in the same way we treat the adiabatic portion of the $\mathbf{v} \cdot \nabla f_1$ term in standard gyrokinetics.

To deal with time derivatives without Fourier (or Laplace) transforming in time we form the time derivative of the perturbed equation

$$\partial^2 f_1 / \partial t^2 + \mathbf{v} \cdot \nabla (\partial f_1 / \partial t) + (Ze/Mc) \mathbf{v} \times \mathbf{B} \cdot \nabla_v (\partial f_1 / \partial t) = -(\partial \mathbf{a} / \partial t) \cdot \nabla_v f_0. \tag{4.6}$$

The next step is unconventional, but is one of two steps that allow us to obtain results with drifts included that would not otherwise be possible. We extract the first non-resonant portion of the perturbed distribution function by defining h via

$$\frac{\partial h}{\partial t} = \frac{\partial f_1}{\partial t} - \frac{Zev_\parallel}{MB} \mathbf{e} \cdot \mathbf{n} \frac{\partial f_0}{\partial \mu}, \tag{4.7}$$

to obtain the alternate and convenient form

$$\begin{aligned} & \frac{\partial^2 h}{\partial t^2} + \mathbf{v} \cdot \nabla \frac{\partial h}{\partial t} + (Ze/Mc) \mathbf{v} \times \mathbf{B} \cdot \nabla_v \frac{\partial h}{\partial t} \\ &= -\frac{\partial \mathbf{e}}{\partial t} \cdot \mathbf{v} \left(\frac{1}{v} \frac{\partial f_0}{\partial v} + \frac{1}{B} \frac{\partial f_0}{\partial \mu} \right) - \mathbf{n} \cdot \nabla \mathbf{e} \cdot \mathbf{v} \frac{v_\parallel}{B} \frac{\partial f_0}{\partial \mu} \\ & \quad - \left(\frac{\partial \mathbf{e}}{\partial t} \cdot \nabla_v \mu_1 + \mathbf{v} \cdot \nabla \mathbf{e} \cdot \nabla_v \mu_1 - \nabla_v \mu_1 \cdot \nabla \mathbf{e} \cdot \mathbf{v} \right) \frac{\partial f_0}{\partial \mu}, \end{aligned} \tag{4.8}$$

where we define

$$\boldsymbol{\varepsilon} = \frac{Ze}{M} \mathbf{e}. \tag{4.9}$$

In the preceding equation for h we use

$$\nabla_v \mu = \nabla_v \mu_0 + \nabla_v \mu_1, \tag{4.10}$$

where

$$\begin{aligned} B \nabla_v \mu_1 = & -\mathbf{v}_d + \frac{v_{\parallel}}{4\Omega} [(\overleftrightarrow{I} - \mathbf{nn}) \cdot \nabla \mathbf{n} \times \mathbf{n} \cdot \mathbf{v} + \mathbf{v}_{\perp} \cdot \nabla \mathbf{n} \times \mathbf{n} - \mathbf{n} \times \nabla \mathbf{n} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{n} \times \nabla \mathbf{n} \\ & - 4\mathbf{v}_{\perp} \mathbf{n} \cdot \nabla \times \mathbf{n}] + \mathbf{n} \frac{1}{4\Omega} [\mathbf{v}_{\perp} \cdot \nabla \mathbf{n} \times \mathbf{n} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{n} \times \nabla \mathbf{n} \cdot \mathbf{v} \\ & - 2v_{\perp}^2 \mathbf{n} \cdot \nabla \times \mathbf{n}] - \nabla_v \mathbf{v}_d \cdot \mathbf{v}_{\perp}, \end{aligned} \tag{4.11}$$

with

$$\nabla_v \mathbf{v}_d \cdot \mathbf{v}_{\perp} = \Omega^{-1} [B^{-1} \mathbf{v}_{\perp} \mathbf{v}_{\perp} \cdot \mathbf{n} \times \nabla B + 2v_{\parallel} \mathbf{n} \mathbf{v}_{\perp} \cdot \mathbf{n} \times (\mathbf{n} \cdot \nabla \mathbf{n})]. \tag{4.12}$$

Next, we use the second unconventional step to extract a second term by using

$$\left(\frac{\partial \boldsymbol{\varepsilon}}{\partial t} + \mathbf{v} \cdot \nabla \boldsymbol{\varepsilon} \right) \cdot \mathbf{v}_d \simeq \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \Omega \mathbf{v} \times \mathbf{n} \cdot \nabla_v \right) (\boldsymbol{\varepsilon} \cdot \mathbf{v}_d), \tag{4.13}$$

where we neglect $k_{\perp}^{-1} \nabla \ell n B \sim 1/k_{\perp} R \ll 1$ corrections. Then, we extract the second non-resonant piece of the perturbed distribution function by defining g as

$$\frac{\partial g}{\partial t} = \frac{\partial h}{\partial t} - \frac{Ze}{MB} \mathbf{e} \cdot \mathbf{v}_d \frac{\partial f_0}{\partial \mu} = \frac{\partial f_1}{\partial t} - \frac{Ze}{MB} \mathbf{e} \cdot (v_{\parallel} \mathbf{n} + \mathbf{v}_d) \frac{\partial f_0}{\partial \mu}, \tag{4.14}$$

to obtain the more convenient form

$$\begin{aligned} & \frac{\partial^2 g}{\partial t^2} + \mathbf{v} \cdot \nabla \frac{\partial g}{\partial t} + (Ze/Mc) \mathbf{v} \times \mathbf{B} \cdot \nabla_v \frac{\partial g}{\partial t} \\ & = -\frac{\partial \boldsymbol{\varepsilon}}{\partial t} \cdot \frac{\mathbf{v}}{v} \frac{\partial f_0}{\partial v} - \left[\frac{\partial \boldsymbol{\varepsilon}}{\partial t} + (v_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \nabla \boldsymbol{\varepsilon} \right] \cdot \frac{\mathbf{v}}{B} \frac{\partial f_0}{\partial \mu} \\ & \quad - \left(\frac{\partial \boldsymbol{\varepsilon}}{\partial t} \cdot \mathbf{G} + \mathbf{v} \cdot \nabla \boldsymbol{\varepsilon} \cdot \mathbf{G} - \mathbf{G} \cdot \nabla \boldsymbol{\varepsilon} \cdot \mathbf{v} \right) \frac{1}{B} \frac{\partial f_0}{\partial \mu}, \end{aligned} \tag{4.15}$$

with

$$\mathbf{G} \equiv B \nabla_v \mu_1 + \mathbf{v}_d \sim \mathbf{v}_d. \tag{4.16}$$

Three of the terms depending on $\mathbf{G} \sim \mathbf{v}_d$ can be seen to be small since

$$\frac{\partial \boldsymbol{\varepsilon}}{\partial t} \cdot \mathbf{v} \gg \frac{\partial \boldsymbol{\varepsilon}}{\partial t} \cdot \mathbf{G} \sim \frac{\partial \boldsymbol{\varepsilon}}{\partial t} \cdot \mathbf{v}_d, \tag{4.17}$$

$$v_{\parallel} \mathbf{n} \cdot \nabla \boldsymbol{\varepsilon} \cdot \mathbf{v} \gg v_{\parallel} \mathbf{n} \cdot \nabla \boldsymbol{\varepsilon} \cdot \mathbf{G} \sim v_{\parallel} \mathbf{n} \cdot \nabla \boldsymbol{\varepsilon} \cdot \mathbf{v}_d, \tag{4.18}$$

and

$$v_{\parallel} \mathbf{n} \cdot \nabla \boldsymbol{\varepsilon} \cdot \mathbf{v} \gg \mathbf{n} \cdot \mathbf{G} \mathbf{n} \cdot \nabla \boldsymbol{\varepsilon} \cdot \mathbf{v} \sim |v_d| \mathbf{n} \cdot \nabla \boldsymbol{\varepsilon} \cdot \mathbf{v}. \tag{4.19}$$

Consequently, we are left to consider

$$\begin{aligned} & \frac{\partial^2 g}{\partial t^2} + \mathbf{v} \cdot \nabla \frac{\partial g}{\partial t} + (Ze/Mc) \mathbf{v} \times \mathbf{B} \cdot \nabla_v \frac{\partial g}{\partial t} \\ &= - \frac{\partial \boldsymbol{\varepsilon}}{\partial t} \cdot \frac{\mathbf{v}}{v} \frac{\partial f_0}{\partial v} - \left[\frac{\partial \boldsymbol{\varepsilon}}{\partial t} + (v_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \nabla \boldsymbol{\varepsilon} \right] \cdot \frac{\mathbf{v}}{B} \frac{\partial f_0}{\partial \mu} \\ & \quad - (v_{\perp} \cdot \nabla \boldsymbol{\varepsilon} \cdot \mathbf{G} - \mathbf{G}_{\perp} \cdot \nabla \boldsymbol{\varepsilon} \cdot \mathbf{v}) \frac{1}{B} \frac{\partial f_0}{\partial \mu}. \end{aligned} \tag{4.20}$$

We can Fourier decompose the applied electric field $\boldsymbol{\varepsilon}$ poloidally and toroidally. We also Fourier decompose in time to permit more than a single wave frequency as might be the case for lower hybrid and ion cyclotron heating. In addition, we allow the local fine scale radial eikonal variation to have multiple roots (that must be summed over) due to mode conversion. Therefore, we take

$$\boldsymbol{\varepsilon} = \sum_{m,n,\omega,\kappa} \boldsymbol{\varepsilon}_m \exp(-i\omega t - in\zeta + iS(\psi) + im\vartheta) = \sum_{m,n,\omega,\kappa} \boldsymbol{\varepsilon}_m \exp(-i\omega t + im\vartheta + i\Upsilon(\mathbf{r})). \tag{4.21}$$

To solve for f_1 we seek solutions of the form

$$\begin{aligned} g &= \sum_{p,n,\omega,\kappa} g_p(\Theta) \exp(-i\omega t - inZ + iS(\Psi) - ip\varphi) \\ &= \sum_{p,\omega,n,\kappa} g_p(\Theta) \exp(-i\omega t + i\Upsilon(\mathbf{R}) - ip\varphi), \end{aligned} \tag{4.22}$$

$$\begin{aligned} h &= \sum_{p,n,\omega,\kappa} h_p(\Theta) \exp(-i\omega t - inZ + iS(\Psi) - ip\varphi) \\ &= \sum_{p,\omega,n,\kappa} h_p(\Theta) e^{-i\omega t + i\Upsilon(\mathbf{R}) - ip\varphi}, \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} f_1 &= \sum_{p,n,\omega,\kappa} f_p(\Theta) \exp(-i\omega t - inZ + iS(\Psi) - ip\varphi) \\ &= \sum_{p,\omega,n,\kappa} f_p(\Theta) \exp(-i\omega t + i\Upsilon(\mathbf{R}) - ip\varphi), \end{aligned} \tag{4.24}$$

with

$$\Upsilon(\mathbf{r}) = S(\psi) - n\zeta, \tag{4.25}$$

$$S(\Psi) = S(\psi) + (\Psi - \psi) \partial S / \partial \psi + \dots \simeq S(\psi) + \Omega^{-1} \mathbf{v} \times \mathbf{n} \cdot \boldsymbol{\kappa} + \dots, \tag{4.26}$$

and

$$\Upsilon(\mathbf{R}) = \Upsilon(\mathbf{r}) + (\mathbf{R} - \mathbf{r}) \cdot \nabla \Upsilon + \dots = \Upsilon(\mathbf{r}) + \Omega^{-1} \mathbf{v} \times \mathbf{n} \cdot (\boldsymbol{\kappa} - n \nabla \zeta) + \dots, \tag{4.27}$$

and where we define

$$\Psi = \psi + \Omega^{-1} \mathbf{v} \times \mathbf{n} \cdot \nabla \psi + \dots, \quad \Theta = \vartheta + \Omega^{-1} \mathbf{v} \times \mathbf{n} \cdot \nabla \vartheta + \dots, \quad (4.28a,b)$$

$$\mathbf{Z} = \zeta + \Omega^{-1} \mathbf{v} \times \mathbf{n} \cdot \nabla \zeta + \dots, \quad (4.29)$$

and

$$\boldsymbol{\kappa} = \nabla S = \nabla \psi \partial S / \partial \psi = \mathbf{e}_1 R B_p \partial S / \partial \psi = \kappa \mathbf{e}_1. \quad (4.30)$$

The sums are over all wavenumbers and frequencies for the poloidally periodic applied wave fields. However, because of the presence of trapped particles, g, h and f_1 are not Fourier decomposed poloidally. The extra sum over p in g, h and f_1 is for the Fourier decomposition in gyrophase. The toroidal and poloidal wave numbers are taken to be n and m , respectively, and ω denotes wave frequencies. The vector

$$\boldsymbol{\kappa} = \kappa \mathbf{e}_1 \quad (4.31)$$

is the radial wave vector.

The distinction between drift kinetic and gyrokinetic variables is important and must be treated carefully in exponentials. In particular, we need to use the preceding expressions to write the right-hand side of the g equation in terms of the gyrokinetic variables $\Psi, \Theta, \mathbf{Z}, v, \mu, \varphi$. Before doing so it is useful to note that f_p and g_p are related by

$$\begin{aligned} & \sum_{p,n,\omega,\kappa} [f_p(\Theta) - g_p(\Theta)] \exp(-i\omega t - inZ + iS(\Psi) - ip\varphi) \\ &= -\frac{Ze}{MB} \frac{\partial f_0}{\partial \mu} (v_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \sum_{m,\omega,n,\kappa} \frac{\mathbf{e}_m}{i\omega} \exp(-i\omega t - in\zeta + iS(\psi) + im\vartheta). \end{aligned} \quad (4.32)$$

Using the preceding expressions, multiplying by $\exp(ip'\varphi)$, integrating over ϕ from 0 to 2π , defining,

$$\mathbf{k} = \boldsymbol{\kappa} + m\nabla\vartheta - n\nabla\zeta, \quad (4.33)$$

$$\mathbf{k}_{\perp} = k_{\perp}(\mathbf{e}_1 \cos \alpha + \mathbf{e}_2 \sin \alpha), \quad (4.34)$$

and

$$L \equiv \Omega^{-1} \mathbf{v} \times \mathbf{n} \cdot \mathbf{k} = (k_{\perp} v_{\perp} / \Omega) \sin(\varphi - \alpha), \quad (4.35)$$

recalling the ‘Bessel generating function’

$$e^{-iL} = e^{-iw \sin(\varphi - \alpha)} = \sum_p e^{-ip(\varphi - \alpha)} J_p(w), \quad (4.36)$$

and performing the gyrophase integrals using

$$U_0 \equiv \oint \frac{d\varphi}{2\pi} e^{ip\varphi - iL} = \oint \frac{d\varphi}{2\pi} e^{ip\varphi - iw \sin(\varphi - \alpha)} = \sum_{p'} e^{-ip'(\varphi - \alpha) + ip\varphi} J_p(w) = e^{ip\alpha} J_p(w), \quad (4.37)$$

we obtain in gyrokinetic variables that

$$f_p(\Theta) = g_p(\Theta) - \frac{Ze}{i\omega MB} \frac{\partial f_0}{\partial \mu} (\mathbf{v}_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \sum_m \mathbf{e}_m J_p(w) e^{im\Theta}, \tag{4.38}$$

where we define

$$w \equiv k_{\perp} v_{\perp} / \Omega, \tag{4.39}$$

and we have ignored small poloidal gyroradius corrections that arise from Taylor expanding $B^{-1} v_{\parallel} \partial f_0 / \partial \mu$ about Ψ . The distinction between drift and gyrokinetic variables on the right- and left-hand sides of the equation results in the term $L = \Omega^{-1} \mathbf{v} \times \mathbf{n} \cdot \mathbf{k} \sim k_{\perp} \rho$.

We are now ready to Fourier transform the kinetic equation for g in φ to obtain

$$\begin{aligned} & \sum_p e^{-ip\varphi} \{ (\mathbf{v}_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \nabla \vartheta \frac{\partial g_p}{\partial \Theta} - i[\omega - p\bar{\Omega} - \boldsymbol{\kappa} \cdot \mathbf{v}_d + n(\mathbf{v}_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \nabla \zeta] g_p \} \\ &= - \sum_m e^{im\Theta - iL} \left\{ (\boldsymbol{\varepsilon}_m \cdot \mathbf{n} v_{\parallel} + \boldsymbol{\varepsilon}_m \cdot \mathbf{v}_{\perp}) \left[\frac{1}{v} \frac{\partial f_0}{\partial v} + \frac{(\omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d)}{\omega B} \frac{\partial f_0}{\partial \mu} \right] - \frac{\Delta_m}{\omega B} \frac{\partial f_0}{\partial \mu} \right\}, \end{aligned} \tag{4.40}$$

where we define

$$\bar{\Omega} \simeq \Omega(\mathbf{R}), \tag{4.41}$$

$$k_{\parallel} = (m - qn)I / qR^2 B \sim (m - qn) / qR, \tag{4.42}$$

$$q = \mathbf{B} \cdot \nabla \zeta / \mathbf{B} \cdot \nabla \vartheta, \tag{4.43}$$

and

$$\Delta_m \equiv \mathbf{k}_{\perp} \cdot \mathbf{v}_{\perp} \boldsymbol{\varepsilon}_m \cdot \mathbf{G} - \mathbf{k}_{\perp} \cdot \mathbf{G}_{\perp} \boldsymbol{\varepsilon}_m \cdot \mathbf{v}. \tag{4.44}$$

In writing down this form of the kinetic equation for g_p we have taken the liberty of using the eikonal approximation for the radial variation to streamline the solution. We could avoid doing this by performing a more complicated trajectory integral in Ψ as well as Θ , rather than just in Θ .

We again multiply by $\exp(ip'\varphi)$ and integrate over φ , but this time we use the additional results

$$\begin{aligned} \oint \frac{d\varphi}{2\pi} \exp(i(p \pm 1)\varphi - iw \sin(\varphi - \alpha)) &= \sum_{p'} \exp(-ip'(\varphi - \alpha) + i(p \pm 1)\varphi) J_p(w) \\ &= e^{i(p \pm 1)\alpha} J_{p \pm 1}(w), \end{aligned} \tag{4.45}$$

and

$$U_1 \equiv \oint \frac{d\varphi}{2\pi v_{\perp}} \mathbf{v}_{\perp} e^{ip\varphi - iL} = \oint \frac{d\varphi}{2\pi v_{\perp}} \mathbf{v}_{\perp} \exp(ip\varphi - iw \sin(\varphi - \alpha)) = e^{ip\alpha} \mathbf{e}_{\perp p}, \tag{4.46}$$

where we define \mathbf{e}_{\pm} as

$$\mathbf{e}_{\pm} = \mathbf{e}_1 \pm i\mathbf{e}_2 \tag{4.47}$$

and $\mathbf{e}_{\perp p}$ as

$$\mathbf{e}_{\perp p} = \frac{1}{2}[\mathbf{e}_- e^{i\alpha} J_{p+1}(w) + \mathbf{e}_+ e^{-i\alpha} J_{p-1}(w)] = \frac{P}{k_{\perp} w} J_p(w) \mathbf{k}_{\perp} + \frac{i}{k_{\perp}} J'_p(w) \mathbf{n} \times \mathbf{k}, \tag{4.48}$$

with

$$\mathbf{k}_{\perp} = (k_{\perp}/2)(\mathbf{e}_- e^{i\alpha} + \mathbf{e}_+ e^{-i\alpha}). \tag{4.49}$$

The alternate form of $\mathbf{e}_{\perp p}$ is found by using the Bessel recurrence relations.

Using the preceding we obtain

$$\begin{aligned} & (v_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \nabla \vartheta \frac{\partial g_p}{\partial \Theta} - i[\omega - p\bar{\Omega} - \boldsymbol{\kappa} \cdot \mathbf{v}_d + n(v_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \nabla \zeta] g_p \\ &= - \sum_m e^{im\Theta + ip\alpha} \boldsymbol{\epsilon}_m \cdot \left[\mathbf{n} J_p(w) \frac{v_{\parallel}}{v} + \mathbf{e}_{\perp p} \frac{v_{\perp}}{v} \right] \frac{\partial f_0}{\partial v} \\ & \quad - \sum_m e^{im\Theta} \left\{ e^{ip\alpha} \boldsymbol{\epsilon}_m \cdot \left[\mathbf{n} J_p(w) \frac{v_{\parallel}}{v} + \mathbf{e}_{\perp p} \frac{v_{\perp}}{v} \right] \left[\frac{\partial f_0}{\partial v} + \frac{(\omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d) v}{\omega B} \frac{\partial f_0}{\partial \mu} \right] \right. \\ & \quad \left. - \frac{\Delta_m e^{ip\varphi - iL}}{\omega B} \frac{\partial f_0}{\partial \mu} \right\}. \end{aligned} \tag{4.50}$$

The approximate sign in (4.50) is a reminder of the approximations (4.5) and (4.13) used to remove the non-resonant terms. An alternate form follows using $q\mathbf{B} \cdot \nabla \vartheta = \mathbf{B} \cdot \nabla \zeta = I/R^2$ with $q = q(\psi)$ the safety factor,

$$\begin{aligned} & \left(\frac{I v_{\parallel}}{q R^2 B} + \mathbf{v}_d \cdot \nabla \vartheta \right) \frac{\partial g_p}{\partial \Theta} - i \left[(\omega - ip\bar{\Omega}) + \frac{n I v_{\parallel}}{R^2 B} - (\boldsymbol{\kappa} - n \nabla \zeta) \cdot \mathbf{v}_d \right] g_p \\ &= - \sum_m e^{im\Theta} (W_{p,m} - \delta_{p,m}), \end{aligned} \tag{4.51}$$

where

$$W_{p,m} = e^{ip\alpha} \boldsymbol{\epsilon}_m \cdot \left[\mathbf{n} v_{\parallel} J_p(w) + \mathbf{e}_{\perp p} v_{\perp} \right] \left[\frac{1}{v} \frac{\partial f_0}{\partial v} + \frac{(\omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d)}{\omega B} \frac{\partial f_0}{\partial \mu} \right] \tag{4.52}$$

and

$$\delta_{p,m} = \oint \frac{d\varphi}{2\pi} e^{ip\varphi - iL} \frac{\Delta_m}{\omega B} \frac{\partial f_0}{\partial \mu}. \tag{4.53}$$

To see that it does not vanish, the integrals needed to evaluate $\delta_{p,m}$ can be performed as sketched in the appendix. However, there is actually no need to do so since the $\delta_{p,m}$ term can be neglected for the following two reasons. First, to derive the quasilinear operator we will show that only the residual associated with $\omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d = p\Omega$ is

needed for $p \neq 0$. Second, for $p = 0$ we need only assume $\Delta_m \ll \omega$. The first condition means we can neglect the $\delta_{p,m}$ for $p \neq 0$ as long as we assume

$$\frac{\Delta_m}{\omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d} \sim \frac{k_{\perp} |\mathbf{G}|}{p\Omega} \sim \frac{k_{\perp} |\mathbf{v}_d|}{p\Omega} \sim k_{\perp} \rho \frac{\rho}{pR} \ll 1. \tag{4.54}$$

The second condition for neglecting $\delta_{p,m}$ requires the even less stringent assumption

$$\frac{\Delta_m}{\omega} \sim \frac{k_{\perp} |\mathbf{G}|}{\omega} \sim \frac{k_{\perp} |\mathbf{v}_d|}{\omega} \sim \frac{\omega_* L_n}{\omega R} \ll 1 \tag{4.55}$$

for $p=0$, where $\omega_* \sim k_{\perp} \rho^2 \Omega / L_n$ is the diamagnetic drift frequency with L_n the density scale length.

In addition, we are continuing to assume that on the right-hand side of the kinetic equation for g_p the distinction between gyrokinetic and drift kinetic variables is unimportant for $W_{p,m}$ as long as poloidal gyroradius over unperturbed radial scale lengths are small as is normally assumed in the core. Consequently, on the right we may assume $\mathbf{R} \simeq \mathbf{r}$ to the order required except in $\exp(im\Theta)$. This $\rho_p f_0^{-1} \nabla f_0 \ll 1$ assumption may fail in a high confinement pedestal or for the very energetic ions produced by minority heating, but such considerations are beyond the scope of the present treatment.

To ignore the ρ/R terms in $\bar{\Omega}$ on the left-hand side of (4.51) for g_p , we need only recall that the poloidal variation of the magnetic field is much more important because $B \propto R^{-1} \propto [1 + (r/R) \cos \vartheta]^{-1}$. As a result, we need only assume

$$r \gg \rho_i, \tag{4.56}$$

with ρ_i the ion gyroradius. As this assumption is easily satisfied from here on we use

$$\bar{\Omega} \simeq \Omega(\mathbf{R}) \simeq \Omega(\mathbf{r}) \simeq \Omega. \tag{4.57}$$

The preceding assumptions mean we need only solve

$$\left(\frac{I v_{\parallel}}{q R^2 B} + \mathbf{v}_d \cdot \nabla \vartheta \right) \frac{\partial g_p}{\partial \Theta} - i \left[(\omega - ip\Omega) + \frac{n I v_{\parallel}}{R^2 B} - (\boldsymbol{\kappa} - n \nabla \zeta) \cdot \mathbf{v}_d \right] g_p = - \sum_m e^{im\Theta} W_{p,m}. \tag{4.58}$$

To integrate the g_p equation we introduce the trajectory time variable τ via

$$\frac{d\Theta(\tau)}{d\tau} = (v_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \nabla \Theta \simeq (v_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \nabla \vartheta = \frac{I v_{\parallel}}{q R^2 B} + \mathbf{v}_d \cdot \nabla \vartheta, \tag{4.59}$$

with

$$\Theta(\tau = 0) = \Theta. \tag{4.60}$$

Defining

$$\chi = \int_0^{\tau} dt' \{ \omega - p \bar{\Omega}[\Theta(\tau')] - \boldsymbol{\kappa} \cdot \mathbf{v}_d + n(v_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \nabla \zeta \}, \tag{4.61}$$

with $\boldsymbol{\kappa} \cdot \mathbf{v}_d$ and $(v_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \nabla \zeta$ slow functions of space, gives upon integrating along the trajectory

$$\frac{d}{d\tau} \{g_p[\Theta(\tau)]e^{-i\chi(\tau)}\} = - \sum_m e^{im\Theta(\tau)-i\chi(\tau)} W_{p,m}(\tau), \tag{4.62}$$

where the weak $\Theta(\tau)$ and/or τ dependence of the coefficient $W_{p,m} \exp(ip\alpha)$ will turn out to be unimportant because of our approximation in the next paragraph. Integrating from $\tau \rightarrow -\infty$ to $\tau = 0$, and assuming any memory of the initial state is lost so we can take $g_p[\Theta(\tau \rightarrow -\infty)] = 0$ gives the general form

$$g_p(\Theta) = - \sum_m e^{im\Theta} \int_{-\infty}^0 d\tau \exp(-i\chi(\tau) + im[\Theta(\tau) - \Theta]) W_{p,m}(\tau). \tag{4.63}$$

The usual derivations of the QL operator always neglect any memory of the initial condition of the charges. The use of unperturbed trajectories might seem to imply a long memory of the past, in which case even collisions might matter. However, in practice the applied rf fields actually perturb these trajectories and thereby introduce stochasticity (Becoulet *et al.* 1991). As a result, for a QL treatment the unperturbed trajectories are only approximately valid between successive passes through resonance.

To simplify this expression we integrate by parts by taking advantage of the rapid variation of the exponential compared to the slow variation of $W_{p,m}$. To do so we use

$$\begin{aligned} & \exp(-i\chi(\tau) + im[\Theta(\tau) - \Theta]) W_{p,m}(\tau) \\ &= \frac{-W_{p,m}(\tau)}{i(\omega - p\Omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d)} \frac{d}{d\tau} \exp(-i\chi(\tau) + im[\Theta(\tau) - \Theta]) \\ &= - \frac{d}{d\tau} \left\{ \frac{W_{p,m}(\tau) \exp(-i\chi(\tau) + im[\Theta(\tau) - \Theta])}{i(\omega - p\Omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d)} \right\} \\ &+ \frac{\exp(-i\chi(\tau) + im[\Theta(\tau) - \Theta])}{i(\omega - p\Omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d)} \frac{dW_{p,m}(\tau)}{d\tau}. \end{aligned} \tag{4.64}$$

The

$$\mathbf{k} \cdot \mathbf{v}_d = (\nabla S + m\nabla\vartheta - n\nabla\zeta) \cdot \mathbf{v}_d \tag{4.65}$$

term includes the following contributions:

$$\mathbf{v}_d \cdot \nabla \psi = v_{\parallel} \mathbf{n} \cdot \nabla (I v_{\parallel} / \Omega) = - \frac{\kappa(v_{\parallel}^2 + \mu B) I^2}{qR^3 B^2 \Omega B_p} \frac{\partial B}{\partial \vartheta}, \tag{4.66}$$

$$\mathbf{v}_d \cdot \nabla \vartheta = \left[(v_{\parallel}^2 + \mu B) \frac{\partial B}{\partial \psi} + \frac{4\pi v_{\parallel}^2}{B} \frac{\partial P}{\partial \psi} \right] \frac{I \mathbf{B} \cdot \nabla \vartheta}{\Omega B^2}, \tag{4.67}$$

and

$$\mathbf{v}_d \cdot \nabla \zeta = \frac{v_{\parallel}^2 + \mu B}{\Omega B^2 R^2} \nabla \psi \cdot \nabla B - \frac{4\pi v_{\parallel}^2 B_p^2}{\Omega B^3} \frac{\partial P}{\partial \psi}, \tag{4.68}$$

with P the total plasma pressure.

We could also integrate (4.63) over the periodic poloidal motion of the trapped and passing particles by assuming that the unperturbed trajectories remained valid for many complete poloidal periods. This procedure could imply weaker stochasticity since successive passes through the resonance are correlated, leading to bounce and poloidal transit resonances appearing in the resonant denominator. Such behaviour is inherent in transit averaged descriptions which make an implicit assumption of weaker stochasticity.

The coefficient $W_{p,m}$ is expected to be slowly varying compared to the variation in the vicinity of the denominator since $W_{p,m}$ is constant in a constant magnetic field. Assuming the denominator changes slowly as the charged particle passes through resonance,

$$\frac{W_{p,m}^{-1}(\tau) dW_{p,m}(\tau)/d\tau}{(\omega - p\Omega - k_{\parallel}v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d)} \ll 1, \tag{4.69}$$

then an additional integration by parts yields the form convenient for full wave treatments:

$$g_p(\Theta) = \sum_m \frac{e^{im\Theta} W_{p,m}}{i(\omega - p\Omega - k_{\parallel}v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d)} + \dots, \tag{4.70}$$

where we neglect the small corrections. For unmagnetized plasmas or low frequency waves ($\omega \ll \Omega$) only the $p = 0$ term matters since all other terms will be small by ω/Ω . For higher frequencies such that $\omega \sim p\Omega + k_{\parallel}v_{\parallel} + \mathbf{k} \cdot \mathbf{v}_d$ only the p term in the sum is significant since all other terms will be small by $|\omega - p\Omega - k_{\parallel}v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d|/p\Omega \ll 1$. For higher frequencies, the $k_{\parallel}v_{\parallel}$ and $\mathbf{k} \cdot \mathbf{v}_d$ terms are important because they determine the location of the wave damping – and this can be quite different than where $\omega = p\Omega$. This form is used to evaluate the density and current in full wave treatment since we can write g in drift kinetic variables as

$$\begin{aligned} g &= \sum_{p,n,\omega,\kappa} g_p(\Theta) \exp(-i\omega t - inZ + iS(\Psi) - ip\varphi) \\ &= \sum_{p,n,\omega,\kappa} g_p(\Theta) \exp(-i\omega t - in\zeta + iS(\psi) - ip\varphi - in\Omega^{-1}\mathbf{v} \times \mathbf{n} \cdot \nabla\zeta + i\Omega^{-1}\mathbf{v} \times \mathbf{n} \cdot \nabla S) \\ &= \sum_{p,n,\omega,\kappa} g_p(\Theta) \exp(im(\vartheta - \Theta) - i\omega t - in\zeta + iS(\psi) - ip\varphi + iL) \\ &\simeq \sum_{m,p,n,\omega,\kappa} \frac{W_{p,m} \exp(-i\omega t + im\vartheta - in\zeta + iS(\psi) - ip\varphi + iL)}{i(\omega - p\Omega - k_{\parallel}v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d)} + \dots \end{aligned} \tag{4.71}$$

This form is not a strict Fourier representation in poloidal angle since the coefficients depend on poloidal angle. To see that this is a compact and efficient means of treating the poloidal variation of the resonant part of the distribution function, assume that there is only a single poloidal mode for the applied rf electric field of (4.21). Then there is only a single m term in our non-standard representation for the distribution function g of (4.71). Due to the resonant denominator its coefficient varies strongly with poloidal angle ϑ . As a result, a standard Fourier representation would require an infinite sum of poloidal modes.

The preceding form (4.71) is convenient to use when forming the density and species current,

$$\int d^3v g \simeq \text{Re} \sum_{m,p,n,\omega,\kappa} \exp(-i\omega t + im\vartheta - in\zeta + iS(\psi)) \int d^3v \frac{e^{-ip\alpha} W_{p,m} J_p(w)}{i(\omega - p\Omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d)} \tag{4.72}$$

and

$$\int d^3v v g \simeq \text{Re} \sum_{m,p,n,\omega,\kappa} \exp(-i\omega t + im\vartheta - in\zeta + iS(\psi)) \times \int d^3v \frac{\exp(-ip\alpha) W_{p,m} [\mathbf{n} v_{\parallel} J_p(w) + \mathbf{e}_{\perp p} v_{\perp}]}{i(\omega - p\Omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d)}, \tag{4.73}$$

with

$$\int d^3v v \frac{\partial}{\partial t} (f_1 - g) = \frac{Ze}{MB} \int d^3v \frac{\partial f_0}{\partial \mu} (v_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \mathbf{e} \tag{4.74}$$

and

$$\int d^3v v \frac{\partial}{\partial t} (f_1 - g) = \frac{Ze}{MB} \mathbf{n} \int d^3v \frac{\partial f_0}{\partial \mu} v_{\parallel} (v_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \mathbf{e}, \tag{4.75}$$

where Re denotes the real part. As we did not Fourier decompose g_p in poloidal angle, our full wave results are quite general and compact, with all poloidal angle coupling due to magnetic field variation retained. We have allowed

$$k_{\parallel} v_{\parallel} / \mathbf{k} \cdot \mathbf{v}_{di} \sim k_{\parallel} R / k_{\perp} \rho_{pi} \sim 1, \tag{4.76}$$

so that drift effects can become significant for the ions (denoted by a subscript i) when $k_{\perp} \rho_i \sim 1$ for $B_p \ll B$. Here $\rho_{pi} = \rho_i B / B_p$ is the ion poloidal gyroradius and B_p the poloidal magnetic field. These moment expressions ignore any drift corrections from $\delta_{p,m}$ implying that $\omega \gg \Delta_m$ for $p = 0$ and $\omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d \sim p\Omega \gg \Delta_m$ for $p \neq 0$.

To form the quasilinear (QL) operator, the principle value part of the integral

$$\int_{-\infty}^0 d\tau \exp(-i\chi(\tau) + im[\Theta(\tau) - \Theta]) W_{p,m} \simeq P \int_{-\infty}^0 d\tau \exp(-i\chi(\tau) + im[\Theta(\tau) - \Theta]) W_{p,m} + \frac{W_{p,m}}{i(\omega - p\Omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d)} \tag{4.77}$$

is neglected. The preceding expression uses the δ function form found from a causal Laplace transform with a positive imaginary part to ω , $\text{Im}\omega > 0$,

$$\frac{1}{i(\omega - p\Omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d)} \rightarrow \pi \delta(\omega - p\Omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d). \tag{4.78}$$

Using this result gives the approximation needed for the QL operator to be

$$g_p(\Theta) \simeq \pi \sum_m e^{im\Theta} W_{p,m} \delta(\omega - p\Omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d) + \dots \tag{4.79}$$

For Kennel & Engelmann (1966) to be valid we need to assume $k_{\parallel}R \gg k_{\perp}\rho$.

To examine the validity of the preceding expressions we use

$$W_{p,m}^{-1}(\tau) dW_{p,m}(\tau)/d\tau \simeq [(v_{\parallel}\mathbf{n} + \mathbf{v}_d) \cdot \nabla\vartheta] W_{p,m}^{-1} \partial W_{p,m} / \partial\vartheta. \tag{4.80}$$

Then the width of the resonance is determined by the broadening so we can estimate

$$|\omega - p\Omega - k_{\parallel}v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d| \sim k_{\parallel}v_{\parallel} + \mathbf{k} \cdot \mathbf{v}_d. \tag{4.81}$$

As a result, we expect our results to be valid when

$$\frac{W_{p,m}^{-1}(\tau) dW_{p,m}(\tau)/d\tau}{(\omega - p\Omega - k_{\parallel}v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d)} \sim \frac{(v_{\parallel}\mathbf{n} + \mathbf{v}_d) \cdot \nabla\vartheta}{(k_{\parallel}v_{\parallel} + \mathbf{k} \cdot \mathbf{v}_d)W_{p,m}} \frac{\partial W_{p,m}}{\partial\vartheta} \sim \frac{1}{W_{p,m}} \frac{\partial W_{p,m}}{\partial\vartheta} \ll \frac{r}{R}, \tag{4.82}$$

since poloidal angle variation of $W_{p,m}$ varies slowly through the narrow resonance region that is much less than the minor radius r appearing in the coefficient of the poloidal varying portion of the magnetic field $B \propto R^{-1} \propto [1 + (r/R) \cos\vartheta]^{-1}$. However, trapped particles reflecting ($v_{\parallel}=0$) in the resonance layer $\omega \simeq p\Omega$ are known to violate (4.82) when $\mathbf{k} \cdot \mathbf{v}_d \rightarrow 0$ (Catto & Myra 1992; Belikov & Kolesnichenko 1994; Catto, Myra & Russell 1994). Whether this continues to be the case when $\mathbf{k} \cdot \mathbf{v}_d$ is retained is less clear since $\mathbf{v}_d \neq 0$ when $v_{\parallel} = 0$.

The new term we obtain in the resonant denominator or argument of the delta function is the expected Doppler broadening of the resonance due to the shift $\mathbf{k} \cdot \mathbf{v}_d$ caused by magnetic drift effects. As a result of this drift motion the actual resonance is shifted to a different location on the flux surface with the resonance occurring at a slightly different poloidal location in velocity space experiencing different wave fields. The distinction between the gyrofrequency of the guiding centre and charge does not matter since only $k_{\perp}\rho_i \gg 1$ are of interest. The damping associated with the $\mathbf{k} \cdot \mathbf{v}_d$ magnetic drift is illustrated in Bajaj & Krall (1972) and Lee *et al.* (1983) for ion drift cyclotron waves. Although unimportant for our purposes, there is also Landau damping associated with the distinction between the guiding centre and charge location as illustrated in Antonsen & Manheimer (1978) and Lee *et al.* (1983) for ion Bernstein modes. All these new broadening terms are expected to matter more for ions than electrons.

5. Quasilinear operator

To form the QL operator we need g in drift kinetic variables. Expanding the exponential we obtain

$$\begin{aligned} g &= \sum_{p,n,\omega,\kappa} g_p(\Theta) \exp(-i\omega t - inZ + iS(\Psi) - ip\varphi) \\ &= \sum_{p,n,\omega,\kappa} g_p(\Theta) \exp(-i\omega t - in\zeta + iS(\psi) - ip\varphi - in\Omega^{-1}\mathbf{v} \times \mathbf{n} \cdot \nabla\zeta + i\Omega^{-1}\mathbf{v} \times \mathbf{n} \cdot \nabla S). \end{aligned} \tag{5.1}$$

Then, expanding the exponential in (4.79)

$$g_p(\Theta) \simeq \pi \sum_m \exp(im\vartheta + im\Omega^{-1} \mathbf{v} \times \mathbf{n} \cdot \nabla\vartheta) \times W_{p,m} \delta(\omega - p\Omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d) + \dots \tag{5.2}$$

gives

$$g = -\pi \sum_{p,m,n,\omega,\kappa} W_{p,m} \delta(\omega - p\Omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d) \times \exp(-i\omega t + im\vartheta - in\zeta + iS(\psi) - ip\varphi + iL), \tag{5.3}$$

where we have carefully kept the distinction between drift kinetic and gyrokinetic variables in the exponential to get the L factor. Consistent with our orderings we ignore the distinction between gyrokinetic and drift kinetic variables in the gyrofrequency.

In the QL operator we can use the delta function to simplify $W_{p,m}$ by letting

$$W_{p,m} \rightarrow e^{ip\alpha} \boldsymbol{\epsilon}_m \cdot [\mathbf{n} v_{\parallel} J_p(w) + \mathbf{e}_{\perp p} v_{\perp}] \left[\frac{1}{v} \frac{\partial f_0}{\partial v} + \frac{p\Omega}{\omega B} \frac{\partial f_0}{\partial \mu} \right]. \tag{5.4}$$

Moreover, we need to account for the difference between f_1 and g ,

$$f_1 = g - \frac{Ze}{MB} \frac{\partial f_0}{\partial \mu} (\mathbf{v}_{\parallel} \mathbf{n} + \mathbf{v}_d) \cdot \sum_{\omega,m,n,\kappa} \frac{\boldsymbol{\epsilon}_m}{i\omega} \exp(-i\omega t + im\vartheta - in\zeta + iS(\psi)). \tag{5.5}$$

The difference $f_1 - g$ is gyrophase independent, while $\langle \mathbf{v}_{\perp} \rangle_{\varphi} = 0$, leaving

$$\left\langle \oint \frac{d\varphi}{2\pi} \nabla_v \cdot [\mathbf{a}(f_1 - g)] \right\rangle_{cg} = \frac{v_{\parallel} \partial}{v \partial v} \left\langle \frac{(f_1 - g) \boldsymbol{\epsilon} \cdot \mathbf{n} v_{\parallel}}{v_{\parallel}} \right\rangle_{cg} + \frac{v_{\parallel} \partial}{v \partial \mu} \left\langle \frac{v(f_1 - g)}{v_{\parallel}} \oint \frac{d\varphi \mathbf{a} \cdot \nabla_v \mu}{2\pi} \right\rangle_{cg}, \tag{5.6}$$

where we use $\nabla_v \varphi = v_{\perp}^{-2} \mathbf{n} \times \mathbf{v}$ to obtain the lowest-order result $\nabla_v v \times \nabla_v \mu_0 \cdot \nabla_v \varphi = 1/vB$.

We must again consider μ to higher order to retain drift effects

$$\oint \frac{d\varphi}{2\pi} B \nabla_v \mu_1 = -\mathbf{v}_d - \frac{\mu}{\Omega} \mathbf{n} \times \nabla B \tag{5.7}$$

and

$$\oint \frac{d\varphi}{2\pi} (B \nabla_v \mu_1) \mathbf{v}_{\perp} = -\frac{v_{\parallel} v_{\perp}^2}{\Omega} \mathbf{n} \mathbf{n} \times (\mathbf{n} \cdot \nabla \mathbf{n}) \tag{5.8}$$

giving

$$\begin{aligned} & \oint \frac{d\varphi \mathbf{a} \cdot \nabla_v \mu_1}{2\pi} \\ &= -\frac{Ze}{M} \left[\left(\mathbf{e} + \frac{v_{\parallel}}{c} \mathbf{n} \times \mathbf{b} \right) \cdot \left(\mathbf{v}_d + \frac{\mu}{\Omega} \mathbf{n} \times \nabla B \right) + \frac{v_{\parallel} v_{\perp}^2}{c\Omega} \mathbf{n} \times (\mathbf{n} \cdot \nabla \mathbf{n}) \cdot \mathbf{n} \times \mathbf{b} \right]. \end{aligned} \tag{5.9}$$

To simplify further for harmonic fields $\boldsymbol{\varepsilon}$, $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$ we use

$$\langle \boldsymbol{\varepsilon} \boldsymbol{\varepsilon} \rangle_{cg} = \frac{1}{4} \sum_{\omega, m, n, \kappa} (\boldsymbol{\varepsilon}_m \boldsymbol{\varepsilon}_m^* + \boldsymbol{\varepsilon}_m^* \boldsymbol{\varepsilon}_m) \tag{5.10}$$

and

$$\langle \boldsymbol{\zeta} \boldsymbol{\chi} \rangle_{cg} = \frac{1}{4} \sum_{\omega, m, n, \kappa} (\boldsymbol{\zeta}_m \boldsymbol{\chi}_m^* + \boldsymbol{\zeta}_m^* \boldsymbol{\chi}_m), \tag{5.11}$$

so if $\boldsymbol{\zeta}_m = \boldsymbol{e}_m$ and $\boldsymbol{\chi}_m = i\boldsymbol{e}_m$ we get zero. The $\langle \dots \rangle_{\vartheta}$ in $\langle \dots \rangle_{cg}$ replaces the double poloidal angle sum over m and m' by a single sum over m by introducing the Kronecker delta function $\delta_{m'm}$. By using (5.10) and (5.11) we are not allowing poloidal mode numbers that differ by order unity to contribute to the quasilinear operator. This property of $\langle \dots \rangle_{\vartheta}$ is necessary to maintain the correct entropy production for our QL operator and recover the Kennel–Engelmann (1966) operator. The preceding averages are performed in real physical space, unlike the elegant Hamiltonian action-angle treatment of Kaufman (1972) and Eriksson & Helander (1994) that employ similar averages in phase space. Their Hamiltonian treatments lead to transit average quasilinear operators in action-angle variables, while we will find the non-transit averaged form in configuration space variables.

Recalling that for $\text{Im } \omega > 0$

$$\int_{-\infty}^t dt e^{-i\omega t} = \frac{-1}{i\omega} e^{-i\omega t} \tag{5.12}$$

we find

$$\left\langle \oint \frac{d\varphi}{2\pi} \nabla_v \cdot [\boldsymbol{a}(f_1 - g)] \right\rangle_{cg} = 0. \tag{5.13}$$

Therefore, we need only consider

$$Q\{f_0\} = -\frac{v_{\parallel}}{v} \frac{\partial}{\partial v} \left[\left\langle \oint \frac{d\varphi}{2\pi v_{\parallel}} g \boldsymbol{v} \cdot \boldsymbol{\varepsilon} \right\rangle_{cg} - \frac{v_{\parallel}}{v} \frac{\partial}{\partial \mu} \left\langle \frac{v}{v_{\parallel}} \oint \frac{d\varphi g \boldsymbol{a} \cdot \nabla_v \boldsymbol{\mu}}{2\pi} \right\rangle_{cg} \right]. \tag{5.14}$$

Using the preceding and

$$\oint \frac{d\varphi}{2\pi} e^{iL - ip(\varphi - \alpha)} = J_p(w), \tag{5.15}$$

and

$$\oint \frac{d\varphi}{2\pi v_{\perp}} \boldsymbol{v}_{\perp} e^{iL - ip(\varphi - \alpha)} = \frac{1}{2} [\boldsymbol{e}_{-} e^{i\alpha} J_{p-1}(w) + \boldsymbol{e}_{+} e^{-i\alpha} J_{p+1}(w)] = \boldsymbol{e}_{\perp p}^*, \tag{5.16}$$

gives

$$\begin{aligned} \left\langle \oint \frac{d\varphi}{2\pi} g \boldsymbol{v} \cdot \boldsymbol{\varepsilon} \right\rangle_{cg} &= -\frac{\pi}{2v} \sum_{\omega, m, n, \kappa, p} \delta(\omega - p\Omega - k_{\parallel} v_{\parallel} - \boldsymbol{k} \cdot \boldsymbol{v}_d) |\boldsymbol{\varepsilon}_m \cdot [\boldsymbol{n} v_{\parallel} J_p(w) + \boldsymbol{e}_{\perp p} v_{\perp}]|^2 \\ &\times \left(\frac{\partial f_0}{\partial v} + \frac{p\Omega v}{\omega B} \frac{\partial f_0}{\partial \mu} \right). \end{aligned} \tag{5.17}$$

In addition, we use

$$\left\langle B \oint \frac{d\varphi g \mathbf{a} \cdot \nabla_{\mathbf{v}} \mu}{2\pi} \right\rangle_{cg} = \left\langle \oint \frac{d\varphi}{2\pi} g \mathbf{a} \cdot (\mathbf{v}_{\perp} - \mathbf{v}_d) \right\rangle_{cg} + \left\langle \oint \frac{d\varphi}{2\pi} g \mathbf{a} \cdot \mathbf{G} \right\rangle_{cg}, \quad (5.18)$$

with

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{v}_{\perp} - \mathbf{v}_d) &= \sum_{\omega, m, n, \kappa} \left[\left(1 - \frac{k_{\parallel} v_{\parallel}}{\omega} - \frac{\mathbf{k} \cdot \mathbf{v}_d}{\omega} \right) \boldsymbol{\epsilon}_m \cdot \mathbf{v}_{\perp} + \left(\frac{\mathbf{k} \cdot \mathbf{v}_{\perp}}{\omega} - \frac{\mathbf{k} \cdot \mathbf{v}_d}{\omega} \right) \boldsymbol{\epsilon}_m \cdot \mathbf{n} v_{\parallel} \right. \\ &\quad \left. - \left(1 - \frac{\mathbf{k} \cdot \mathbf{v}}{\omega} \right) \boldsymbol{\epsilon}_m \cdot \mathbf{v}_d \right] \exp(-i\omega t + im\vartheta - in\zeta + iS(\psi)) \\ &\simeq \sum_{\omega, m, n, \kappa} \left[\left(1 - \frac{k_{\parallel} v_{\parallel}}{\omega} - \frac{\mathbf{k} \cdot \mathbf{v}_d}{\omega} \right) \boldsymbol{\epsilon}_m \cdot \mathbf{v}_{\perp} + \frac{\mathbf{k}_{\perp} \cdot \mathbf{v}_{\perp}}{\omega} \boldsymbol{\epsilon}_m \cdot \mathbf{n} v_{\parallel} \right] \\ &\quad \times \exp(-i\omega t + im\vartheta - in\zeta + iS(\psi)), \end{aligned} \quad (5.19)$$

where in the last form we have neglected unimportant drift corrections. Using this result along with earlier results, and $\mathbf{k}_{\perp} \cdot \mathbf{e}_{\perp p} v_{\perp} = p\Omega J_p(\omega)$ and $\omega - p\Omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d = 0$ gives

$$\begin{aligned} \left\langle \oint \frac{d\varphi}{2\pi} g \mathbf{a} \cdot (\mathbf{v}_{\perp} - \mathbf{v}_d) \right\rangle_{cg} &= -\frac{\pi}{2v} \sum_{\omega, m, n, \kappa, p} \delta(\omega - p\Omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d) \frac{p\Omega}{\omega} \\ &\quad \times |\boldsymbol{\epsilon}_m \cdot [\mathbf{n} v_{\parallel} J_p(\omega) + \mathbf{e}_{\perp p} v_{\perp}]|^2 \left(\frac{\partial f_0}{\partial v} + \frac{p\Omega v}{\omega B} \frac{\partial f_0}{\partial \mu} \right). \end{aligned} \quad (5.20)$$

Then we need only note that all the terms in

$$\left\langle \oint \frac{d\varphi}{2\pi} g \mathbf{a} \cdot \mathbf{G} \right\rangle_{cg} \quad (5.21)$$

are small compared to the terms just evaluated.

Combining the preceding results, we find our final form for the QL operator to be

$$Q\{f_0\} = \sum_{\omega, p} \frac{v_{\parallel}}{v} \left(\frac{\partial}{\partial v} + \frac{p\Omega v}{\omega B} \frac{\partial}{\partial \mu} \right) \left[D \frac{v}{v_{\parallel}} \left(\frac{\partial f_0}{\partial v} + \frac{p\Omega v}{\omega B} \frac{\partial f_0}{\partial \mu} \right) \right], \quad (5.22)$$

with

$$D = \frac{\pi Z^2 e^2}{2M^2 v^2} \sum_{m, n, \kappa} \delta(\omega - p\Omega - k_{\parallel} v_{\parallel} - \mathbf{k} \cdot \mathbf{v}_d) |\boldsymbol{\epsilon}_m \cdot [\mathbf{n} v_{\parallel} J_p(\omega) + \mathbf{e}_{\perp p} v_{\perp}]|^2, \quad (5.23)$$

manifestly positive. Consequently, we have managed to carefully derive the desired tokamak form for the QL operator that is the same as the Kennel–Englemann (1966) form except for the simple $\mathbf{k} \cdot \mathbf{v}_d$ modification to the argument of the delta function. Our QL ordering $Q \sim v f_0$ with $D \sim Z^2 e^2 |\mathbf{e}_m|^2 / M^2 \omega \sim \omega v_{\text{quiv}}^2$ and $v_{\text{quiv}} \sim Ze |\mathbf{e}_m| / M\omega$, then requires small quiver speeds v_{quiv} compared to the thermal speed v_{thermal} such that $v_{\text{quiv}}^2 / v_{\text{thermal}}^2 \sim v / \omega \ll 1$.

The preceding form satisfies a QL entropy principle. Multiplying Q by $\ell n f_0$ and integrating over all velocity space using $d^3v \rightarrow 2\pi v B dv d\mu/|v_{\parallel}|$ gives the entropy production to be negative definite as required,

$$\int d^3v \ell n f_0 Q\{f_0\} = - \int d^3v \sum_{\omega,p} \frac{D}{f_0} \left| \frac{\partial f_0}{\partial v} + \frac{p\Omega v}{\omega B} \frac{\partial f_0}{\partial \mu} \right|^2 \leq 0. \tag{5.24}$$

By changing to the cylindrical velocity space variables v_{\perp} , v_{\parallel} , and φ we can rewrite the quasilinear operator in vector form as

$$Q\{f_0\} = \nabla_v \cdot (\vec{D} \cdot \nabla_v f_0), \tag{5.25}$$

with

$$\vec{D} = D_{\parallel} \mathbf{nn} + v_{\perp}^{-2} D_{\perp} \mathbf{v}_{\perp} \mathbf{v}_{\perp} + v_{\perp}^{-1} D_{\times} (\mathbf{nv}_{\perp} + \mathbf{v}_{\perp} \mathbf{n}), \tag{5.26}$$

where the various diffusivities are defined as

$$D_{\parallel} = \sum_{\omega,p} \left(1 - \frac{p\Omega}{\omega}\right)^2 \frac{v^2}{v_{\parallel}^2} D, \tag{5.27}$$

$$D_{\perp} = \sum_{\omega,p} \left(\frac{p\Omega v}{\omega v_{\perp}}\right)^2 D, \tag{5.28}$$

and

$$D_{\times} = \sum_{\omega,p} \frac{p\Omega}{\omega} \left(1 - \frac{p\Omega}{\omega}\right) \frac{v^2}{v_{\parallel} v_{\perp}} D. \tag{5.29}$$

For simulations the delta function can be approximated as the limiting form of a Gaussian. Also, using form (5.25), we can easily evaluate the energy moment of $Q\{f_0\}$ as

$$\begin{aligned} \frac{1}{2} \int d^3v v^2 Q\{f_0\} &= - \int d^3v \mathbf{v} \cdot \vec{D} \cdot \nabla_v f_0 \\ &= - \sum_{\omega,p} \int d^3v v^2 \frac{D}{v_{\parallel}} \left[\left(1 - \frac{p\Omega}{\omega}\right) \frac{\partial f_0}{\partial v_{\parallel}} + \frac{p\Omega v_{\parallel}}{\omega v_{\perp}} \frac{\partial f_0}{\partial v_{\perp}} \right]. \end{aligned} \tag{5.30}$$

For completeness we also note that the transit averaged QL operator is

$$\overline{Q\{f_0\}} = \frac{1}{v \langle \oint d\tau \rangle} \sum_{\omega,p} \left(\frac{\partial}{\partial v} + \frac{p\Omega v}{\omega B} \frac{\partial}{\partial \mu} \right) \left[v \overline{D} \left(\oint d\tau \right) \left(\frac{\partial \overline{f_0}}{\partial v} + \frac{p\Omega v}{\omega B} \frac{\partial \overline{f_0}}{\partial \mu} \right) \right], \tag{5.31}$$

with \overline{D} the transit average of the full diffusivity

$$\overline{D} = \oint d\tau D / \oint d\tau = \left[\oint d\vartheta D / v_{\parallel} \mathbf{n} \cdot \nabla \vartheta \right] / \left[\oint d\vartheta / v_{\parallel} \mathbf{n} \cdot \nabla \vartheta \right]. \tag{5.32}$$

Sometimes a pitch angle variable is used in place of μ . The transit average of D is most easily performed explicitly by using the argument of the delta function to select the poloidal angle at resonance.

Our expression for the transit averaged quasilinear operator $\overline{Q\{f_0\}}$ is equivalent to the usual Kennel & Engelmann (1966) and Eriksson & Helander (1994) result written in an explicit and compact form in drift kinetic variables, but without the radial diffusion terms. It is consistent with the transit averaged form of Belikov & Kolesnichenko (1994) that extended the result of Catto & Myra (1992) to allow finite k_{\parallel} and retain radial diffusion. Both references explicitly treat the distinction between trapped and passing particles, but without the drift modifications considered here.

In heating and current drive applications the wave fields for the QL diffusivities are assumed to be known from full wave codes such as TORIC (Brambilla 1999; Wright *et al.* 2004) and AORSA (Jaeger *et al.* 2002). The density and currents needed for a full wave code have already been partially evaluated by performing the required gyroaverages in (4.72) to (4.75). The remaining integrals can be performed once f_0 is found numerically by solving the QL equation. Coarse grain averaging is unnecessary for the full wave representation of the linear wave fields. Otherwise they are derived with the same approximations as used to obtain the QL operator and so provide a consistent evaluation for a full wave code in a representation that is expected to require keeping fewer poloidal modes. They can be converted to v_{\parallel} and μ variables so the non-transit averaged form of the kinetic equation for f_0 can be employed to keep streaming effects that can sometimes play a role (Helander & Catto 2001).

6. Summary

We have derived the modification of the standard Kennel–Engelmann (1966) QL operator that incorporates magnetic drift effects in tokamak geometry in a non-transit averaged form and continues to have non-negative entropy production. The only modification is in the argument of delta function that now includes a broadening term due to the total magnetic drift as well as usual Landau and cyclotron broadening of the resonance. To carefully derive this drift modified form for the quasilinear operator is not as straightforward as might be expected because the higher-order magnetic moment must be retained. The need to go to higher order requires the unconventional steps of solving explicitly for two non-resonant terms in the perturbed distribution function that do not alter the QL operator and directly lead to its most compact form. Our QL form also has the advantages that it is explicitly in drift kinetic variables and not transit averaged.

The perturbed distribution function need not be periodic in poloidal angle because of the presence of trapped particles. Therefore, it is not Fourier decomposed in poloidal angle like the periodic wave fields. Consequently, a compact modified Fourier representation for the perturbed distribution function is found having poloidally angle dependent coefficients. This form is expected to be useful for full wave treatments.

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Appendix A

To see that the integral (4.52) does not vanish we present a partial evaluation of

$$\oint \frac{d\varphi}{2\pi} e^{ip\varphi - iL} \Delta_m = \oint \frac{d\varphi}{2\pi} e^{ip\varphi - iL} (\mathbf{k}_\perp \cdot \mathbf{v}_\perp \boldsymbol{\varepsilon}_m \cdot \mathbf{G} - \mathbf{k}_\perp \cdot \mathbf{G} \boldsymbol{\varepsilon}_m \cdot \mathbf{v}). \tag{A 1}$$

We first write

$$\boldsymbol{\varepsilon}_m = \boldsymbol{\varepsilon}_m^\perp k_\perp^{-1} \mathbf{k}_\perp + \boldsymbol{\varepsilon}_m^\times k_\perp^{-1} \mathbf{n} \times \mathbf{k} + \boldsymbol{\varepsilon}_m^\parallel \mathbf{n} \tag{A 2}$$

and note that the \mathbf{k}_\perp component $\boldsymbol{\varepsilon}_m^\perp$ does not contribute. Then, defining

$$\mathbf{G} \equiv B \nabla_v \mu_1 + \mathbf{v}_d = G_\parallel \mathbf{n} + v_\perp^{-1} G_\perp \mathbf{v}_\perp + v_\perp^{-1} G_\times \mathbf{v} \times \mathbf{n}, \tag{A 3}$$

we see that we need only consider

$$\begin{aligned} \Delta_m &= \boldsymbol{\varepsilon}_m^\parallel \left[\left(G_\parallel - \frac{v_\parallel}{v_\perp} G_\perp \right) \mathbf{k}_\perp \cdot \mathbf{v}_\perp - \frac{v_\parallel}{v_\perp} G_\times \mathbf{k} \cdot \mathbf{v} \times \mathbf{n} \right] + \boldsymbol{\varepsilon}_m^\times k_\perp v_\perp G_\times \\ &= \Omega \boldsymbol{\varepsilon}_m \cdot \mathbf{n} \left[\left(G_\parallel - \frac{v_\parallel}{v_\perp} G_\perp \right) \frac{\partial L}{\partial \varphi} - \frac{v_\parallel}{v_\perp} G_\times \mathbf{k} \cdot \nabla_k L \right] + \boldsymbol{\varepsilon}_m \times \mathbf{n} \cdot \mathbf{k} v_\perp G_\times, \end{aligned} \tag{A 4}$$

where

$$\mathbf{n} \times \mathbf{v} = \partial \mathbf{v}_\perp / \partial \vartheta, \tag{A 5}$$

$$\mathbf{k}_\perp \cdot \mathbf{v}_\perp = \Omega \partial L / \partial \varphi, \tag{A 6}$$

and

$$\mathbf{v} \times \mathbf{n} = \Omega \nabla_k L. \tag{A 7}$$

In the preceding we define

$$\begin{aligned} G_\parallel &= -\frac{1}{4\Omega} [\mathbf{v}_\perp \cdot \nabla \mathbf{n} \cdot \mathbf{v} \times \mathbf{n} + \mathbf{v} \times \mathbf{n} \cdot \nabla \mathbf{n} \cdot \mathbf{v}_\perp + 2v_\perp^2 \mathbf{n} \cdot \nabla \times \mathbf{n}] \\ &\quad - \frac{2v_\parallel}{\Omega} \mathbf{v} \times \mathbf{n} \cdot (\mathbf{n} \cdot \nabla \mathbf{n}), \end{aligned} \tag{A 8}$$

$$v_\perp G_\times = \mathbf{v} \times \mathbf{n} \cdot \mathbf{G} = \frac{v_\parallel}{2\Omega} [\mathbf{v}_\perp \cdot \nabla \mathbf{n} \cdot \mathbf{v}_\perp - \mathbf{v} \times \mathbf{n} \cdot \nabla \mathbf{n} \cdot \mathbf{v} \times \mathbf{n}], \tag{A 9}$$

and

$$\begin{aligned} v_\perp G_\perp &= \mathbf{v}_\perp \cdot \mathbf{G} \\ &= -\frac{v_\parallel}{2\Omega} [\mathbf{v}_\perp \cdot \nabla \mathbf{n} \cdot \mathbf{v} \times \mathbf{n} + \mathbf{v} \times \mathbf{n} \cdot \nabla \mathbf{n} \cdot \mathbf{v}_\perp + 2v_\perp^2 \mathbf{n} \cdot \nabla \times \mathbf{n}] \\ &\quad - \frac{2\mu}{\Omega} \mathbf{v} \times \mathbf{n} \cdot \nabla B, \end{aligned} \tag{A 10}$$

and note

$$v_\perp G_\perp + \frac{2\mu}{\Omega} \mathbf{v} \times \mathbf{n} \cdot \nabla B = 2v_\parallel G_\parallel + \frac{4v_\parallel^2}{\Omega} \mathbf{v} \times \mathbf{n} \cdot (\mathbf{n} \cdot \nabla \mathbf{n}), \tag{A 11}$$

$$\frac{\partial}{\partial \varphi} (v_\perp G_\times) = -\frac{v_\parallel}{\Omega} [\mathbf{v} \times \mathbf{n} \cdot \nabla \mathbf{n} \cdot \mathbf{v}_\perp + \mathbf{v}_\perp \cdot \nabla \mathbf{n} \cdot \mathbf{v} \times \mathbf{n}], \tag{A 12}$$

$$\frac{\partial^2}{\partial \varphi^2}(v_{\perp} G_{\times}) = -4v_{\perp} G_{\times}, \tag{A 13}$$

$$v_{\perp} G_{\perp} = \frac{\partial}{\partial \varphi} \left(\frac{v_{\perp} G_{\times}}{2} + \frac{v_{\perp}^2}{\Omega B} \mathbf{v}_{\perp} \cdot \nabla B \right) - \frac{v_{\parallel} v_{\perp}^2}{\Omega} \mathbf{n} \cdot \nabla \times \mathbf{n}, \tag{A 14}$$

and

$$v_{\parallel} G_{\parallel} = \frac{\partial}{\partial \varphi} \left[\frac{v_{\perp} G_{\times}}{4} + \frac{2v_{\parallel}^2}{\Omega} \mathbf{v}_{\perp} \cdot (\mathbf{n} \cdot \nabla \mathbf{n}) \right] - \frac{v_{\parallel} v_{\perp}^2}{2\Omega} \mathbf{n} \cdot \nabla \times \mathbf{n}. \tag{A 15}$$

To evaluate the required integrals we also use

$$\nabla_k \alpha = k_{\perp}^{-2} \mathbf{n} \times \mathbf{k} = (i/2k_{\perp})(e_{-} e^{i\alpha} - e_{+} e^{-i\alpha}). \tag{A 16}$$

Then, we find as before that

$$U_1 = i v_{\perp}^{-1} \Omega \mathbf{n} \times \nabla_k U_0 = i v_{\perp}^{-1} \Omega \mathbf{n} \times \nabla_k [e^{i p \alpha} J_p(w)] = e^{i p \alpha} e_{\perp p}. \tag{A 17}$$

In addition, we can evaluate

$$\begin{aligned} \vec{U}_2 &\equiv e^{i p \alpha} \vec{e}_{\perp \perp p} \equiv \oint \frac{d\varphi}{2\pi v_{\perp}^2} \mathbf{v}_{\perp} \mathbf{v}_{\perp} e^{i p \varphi - i L} = i v_{\perp}^{-1} \Omega \mathbf{n} \times \nabla_k U_1 = i v_{\perp}^{-1} \Omega \mathbf{n} \times \nabla_k (e^{i p \alpha} e_{\perp p}) \\ &= \frac{e^{i p \alpha}}{4} [e_{-} e_{-} e^{i 2 \alpha} J_{p+2}(w) + e_{+} e_{+} e^{-i 2 \alpha} J_{p-2}(w) + (e_{-} e_{+} + e_{+} e_{-}) J_p(w)], \\ &= \frac{e^{i p \alpha}}{4 k_{\perp}^2} [J_{p+2}(w)(\mathbf{k}_{\perp} - \mathbf{i n} \times \mathbf{k})(\mathbf{k}_{\perp} - \mathbf{i n} \times \mathbf{k}) + J_{p-2}(w)(\mathbf{k}_{\perp} + \mathbf{i n} \times \mathbf{k})(\mathbf{k}_{\perp} + \mathbf{i n} \times \mathbf{k}) \\ &\quad + 2 J_p(w)(\mathbf{k}_{\perp} \mathbf{k}_{\perp} + \mathbf{n} \times \mathbf{k n} \times \mathbf{k})], \end{aligned} \tag{A 18}$$

where

$$\vec{U}_2 : \vec{I} = \vec{I} : \vec{U}_2 = U_0, \tag{A 19}$$

$$e_{-} e_{+} + e_{+} e_{-} = 2 k_{\perp}^{-2} (\mathbf{k}_{\perp} \mathbf{k}_{\perp} + \mathbf{n} \times \mathbf{k n} \times \mathbf{k}), \tag{A 20}$$

$$e_{-} e_{-} e^{i 2 \alpha} = k_{\perp}^{-2} (\mathbf{k}_{\perp} - \mathbf{i n} \times \mathbf{k})(\mathbf{k}_{\perp} - \mathbf{i n} \times \mathbf{k}), \tag{A 21}$$

and

$$e_{+} e_{+} e^{-i 2 \alpha} = k_{\perp}^{-2} (\mathbf{k}_{\perp} + \mathbf{i n} \times \mathbf{k})(\mathbf{k}_{\perp} + \mathbf{i n} \times \mathbf{k}). \tag{A 22}$$

Then, integrating by parts allows us to evaluate

$$\begin{aligned} \oint \frac{d\varphi}{2\pi} e^{i p \varphi - i L} \Delta_m &= i \Omega \boldsymbol{\varepsilon}_m \cdot \mathbf{n} \oint \frac{d\varphi}{2\pi} e^{i p \varphi} \left(G_{\parallel} - \frac{v_{\parallel}}{v_{\perp}} G_{\perp} \right) \frac{\partial}{\partial \varphi} e^{-i L} \\ &\quad + \left[\boldsymbol{\varepsilon}_m \times \mathbf{n} \cdot \mathbf{k} v_{\perp} - i \frac{v_{\parallel}}{v_{\perp}} \Omega \boldsymbol{\varepsilon}_m \cdot \mathbf{n} \mathbf{k}_{\perp} \cdot \nabla_k \right] \left[\oint \frac{d\varphi}{2\pi} e^{i p \varphi - i L} G_{\times} \right] \\ &= p \Omega \boldsymbol{\varepsilon}_m \cdot \mathbf{n} \oint \frac{d\varphi}{2\pi} e^{i p \varphi - i L} \left(G_{\parallel} - \frac{v_{\parallel}}{v_{\perp}} G_{\perp} \right) \\ &\quad - i \Omega \boldsymbol{\varepsilon}_m \cdot \mathbf{n} \oint \frac{d\varphi}{2\pi} e^{i p \varphi - i L} \left(\frac{\partial G_{\parallel}}{\partial \varphi} - \frac{v_{\parallel}}{v_{\perp}} \frac{\partial G_{\perp}}{\partial \varphi} \right) \\ &\quad + \left[\boldsymbol{\varepsilon}_m \times \mathbf{n} \cdot \mathbf{k} v_{\perp} - i \frac{v_{\parallel}}{v_{\perp}} \Omega \boldsymbol{\varepsilon}_m \cdot \mathbf{n} \mathbf{k}_{\perp} \cdot \nabla_k \right] \left[\oint \frac{d\varphi}{2\pi} e^{i p \varphi - i L} G_{\times} \right] \end{aligned} \tag{A 23}$$

by using only U_0 , U_1 and \vec{U}_2 .

The G_{\times} integral is

$$\begin{aligned} \oint \frac{d\varphi}{2\pi v_{\perp}} e^{i p \varphi - i L} G_{\times} &= \frac{v_{\parallel}}{2\Omega} \oint \frac{d\varphi}{2\pi v_{\perp}^2} e^{i p \varphi - i L} \mathbf{v}_{\perp} \mathbf{v}_{\perp} : (\nabla \mathbf{n} + \mathbf{n} \times \nabla \mathbf{n} \times \mathbf{n}) \\ &= \frac{v_{\parallel}}{2\Omega} e^{i p \alpha} \mathbf{e}_{\perp \perp p} : (\nabla \mathbf{n} + \mathbf{n} \times \nabla \mathbf{n} \times \mathbf{n}) \\ &= \frac{v_{\parallel} e^{i p \alpha}}{4\Omega} [J_{p+2}(w) K_{-} + J_{p-2}(w) K_{+}] \equiv I_p^{+} e^{i p \alpha}, \end{aligned} \tag{A 24}$$

where we also define

$$\begin{aligned} K_{\mp} &\equiv k_{\perp}^{-2} (\mathbf{k}_{\perp} \cdot \nabla \mathbf{n} \cdot \mathbf{k}_{\perp} \mp i \mathbf{k}_{\perp} \cdot \nabla \mathbf{n} \cdot \mathbf{n} \times \mathbf{k} \mp i \mathbf{n} \times \mathbf{k} \cdot \nabla \mathbf{n} \cdot \mathbf{k}_{\perp} - \mathbf{n} \times \mathbf{k} \cdot \nabla \mathbf{n} \cdot \mathbf{n} \times \mathbf{k}) \\ &= k_{\perp}^{-2} (\mathbf{k}_{\perp} \mathbf{k}_{\perp} \mp i \mathbf{k}_{\perp} \mathbf{n} \times \mathbf{k} \mp i \mathbf{n} \times \mathbf{k} \mathbf{k}_{\perp} - \mathbf{n} \times \mathbf{k} \mathbf{n} \times \mathbf{k}) : \nabla \mathbf{n}. \end{aligned} \tag{A 25}$$

Using $\mathbf{k}_{\perp} \cdot \nabla_k K_{\mp} = 0$ and $\mathbf{k}_{\perp} \cdot \nabla_k w = w$, the preceding gives

$$\mathbf{k}_{\perp} \cdot \nabla_k \left[\oint \frac{d\varphi}{2\pi v_{\perp}} e^{i p \varphi - i L} G_{\times} \right] = \frac{v_{\parallel} w e^{i p \alpha}}{4\Omega} [J'_{p+2}(w) K_{-} + J'_{p-2}(w) K_{+}]. \tag{A 26}$$

The second of the remaining two integrals gives

$$\begin{aligned} &\oint \frac{d\varphi}{2\pi v_{\perp}} e^{i p \varphi - i L} \left(\frac{\partial G_{\parallel}}{\partial \varphi} - \frac{v_{\parallel}}{v_{\perp}} \frac{\partial G_{\perp}}{\partial \varphi} \right) \\ &= \left(\frac{2v_{\parallel}}{v_{\perp}} - \frac{v_{\perp}}{v_{\parallel}} \right) \oint \frac{d\varphi}{2\pi v_{\perp}} e^{i p \varphi - i L} G_{\times} + \frac{v_{\parallel}}{\Omega} e^{i p \alpha} \mathbf{e}_{\perp p} \cdot (\nabla \ell n B - 2\mathbf{n} \cdot \nabla \mathbf{n}) \\ &= \left(\frac{2v_{\parallel}}{v_{\perp}} - \frac{v_{\perp}}{v_{\parallel}} \right) I_p^{+} e^{i p \alpha} + u_p e^{i p \alpha}, \end{aligned} \tag{A 27}$$

where

$$\begin{aligned} u_p e^{i p \alpha} &\equiv \frac{v_{\parallel}}{\Omega} \mathbf{e}_{\perp p} \cdot (\nabla \ell n B - 2\mathbf{n} \cdot \nabla \mathbf{n}) \\ &= \frac{v_{\parallel} e^{i p \alpha}}{k_{\perp} \Omega} \left[\frac{p}{w} J_p(w) \mathbf{k}_{\perp} + i J'_p(w) \mathbf{n} \times \mathbf{k} \right] \cdot (\nabla \ell n B - 2\mathbf{n} \cdot \nabla \mathbf{n}). \end{aligned} \tag{A 28}$$

The remaining integral is

$$\begin{aligned} &\oint \frac{d\varphi}{2\pi v_{\perp}} e^{i p \varphi - i L} \left(G_{\parallel} - \frac{v_{\parallel}}{v_{\perp}} G_{\perp} \right) \\ &= \frac{1}{4\Omega} \left(\frac{2v_{\parallel}^2}{v_{\perp}^2} - 1 \right) \oint \frac{d\varphi}{2\pi v_{\perp}} e^{i p \varphi - i L} (\mathbf{v}_{\perp} \mathbf{v}_{\perp} \times \mathbf{n} - \mathbf{n} \times \mathbf{v}_{\perp} \mathbf{v}_{\perp}) : \nabla \mathbf{n} \\ &\quad + \frac{v_{\perp}}{2\Omega} \left(\frac{v_{\parallel}^2}{v_{\perp}^2} - \frac{1}{2} \right) e^{i p \alpha} J_p(w) \mathbf{n} \cdot \nabla \times \mathbf{n} + \frac{v_{\parallel}}{\Omega} e^{i p \alpha} \mathbf{e}_{\perp p} \cdot \mathbf{n} \times (\nabla \ell n B - 2\mathbf{n} \cdot \nabla \mathbf{n}) \\ &= \frac{v_{\perp}}{2\Omega} \left(\frac{v_{\parallel}^2}{v_{\perp}^2} - \frac{1}{2} \right) e^{i p \alpha} (\mathbf{e}_{\perp \perp p} \times \mathbf{n} - \mathbf{n} \times \mathbf{e}_{\perp \perp p}) : \nabla \mathbf{n} \end{aligned}$$

$$\begin{aligned}
& + \frac{v_{\perp}}{2\Omega} \left(\frac{v_{\parallel}^2}{v_{\perp}^2} - \frac{1}{2} \right) e^{ip\alpha} J_p(w) \mathbf{n} \cdot \nabla \times \mathbf{n} + s_p e^{ip\alpha} \\
& = -i \left(\frac{v_{\parallel}}{v_{\perp}} - \frac{v_{\perp}}{2v_{\parallel}} \right) e^{ip\alpha} I_p^- + \frac{v_{\perp}}{2\Omega} \left(\frac{v_{\parallel}^2}{v_{\perp}^2} - \frac{1}{2} \right) e^{ip\alpha} J_p(w) \mathbf{n} \cdot \nabla \times \mathbf{n} + s_p e^{ip\alpha}, \quad (\text{A } 29)
\end{aligned}$$

where

$$\begin{aligned}
s_p e^{ip\alpha} & \equiv \frac{v_{\parallel}}{\Omega} \mathbf{e}_{\perp p} \cdot \mathbf{n} \times (\nabla \ln B - 2\mathbf{n} \cdot \nabla \mathbf{n}) \\
& = \frac{v_{\parallel} e^{ip\alpha}}{k_{\perp} \Omega} \left[\frac{p}{w} J_p(w) \mathbf{k} \times \mathbf{n} + i J_p'(w) \mathbf{k}_{\perp} \right] \cdot (\nabla \ln B - 2\mathbf{n} \cdot \nabla \mathbf{n}) \quad (\text{A } 30)
\end{aligned}$$

and

$$I_p^{\pm} \equiv \frac{v_{\parallel}}{4\Omega} [J_{p+2}(w) K_{-} \pm J_{p-2}(w) K_{+}], \quad (\text{A } 31)$$

with K_{-} and K_{+} complex conjugates, $K_{\mp}^* = K_{\pm}$.

REFERENCES

- ANTONSEN, T. M. & MANHEIMER, W. M. 1978 Electromagnetic wave propagation in inhomogeneous plasmas. *Phys. Fluids* **21**, 2295–2305.
- BAJAJ, N. K. & KRALL, N. A. 1972 Effect of finite β on the drift cyclotron instability. *Phys. Fluids* **15**, 657–661.
- BECOULET, A., GAMBIER, D. J & SAMAIN, A. 1991 Hamiltonian theory of the ion cyclotron minority heating dynamics in tokamak plasmas. *Phys. Fluids B* **3**, 137–150.
- BELIKOV, V. S. & KOLESNICHENKO, Y. I. 1994 Quasilinear theory for a tokamak plasma in the presence of cyclotron resonance. *Plasma Phys. Control. Fusion* **36**, 1703–1718.
- BRAMBILLA, M. 1999 Numerical simulation of ion cyclotron waves in tokamak plasmas. *Plasma Phys. Control. Fusion* **41**, 1–34.
- CALVO, I. & PARRA, F. I. 2012 Long-wavelength limit of gyrokinetics in a turbulent tokamak and its intrinsic ambipolarity. *Plasma Phys. Control. Fusion* **54**, 115007.
- CATTO, P. J. 1978 Linearized gyrokinetics. *Plasma Phys.* **20**, 719–722.
- CATTO, P. J., LASHMORE-DAVIES, C. N. & MARTIN, T. J. 1993 A kinetic model of fast wave propagation in the vicinity of the minority ion cyclotron resonance in a toroidal magnetic field. *Phys. Fluids B* **5**, 2909–2921.
- CATTO, P. J. & MYRA, J. R. 1992 A quasilinear description for fast wave minority heating permitting off magnetic axis heating in a tokamak. *Phys. Fluids B* **4**, 187–199.
- CATTO, P. J., MYRA, J. R. & RUSSELL, D. A. 1994 A quasilinear, Fokker–Planck description of fast wave minority heating permitting off-axis tangency interactions. *Phys. Plasmas* **1**, 52–63.
- CATTO, P. J., TANG, W. M. & BALDWIN, D. E. 1981 Generalized gyrokinetics. *Plasma Phys.* **23**, 639–650.
- CHAPMAN, I. T., GRAVES, J. P., LENNHOLM, M., FAUSTIN, J., LERCHE, E., JOHNSON, T. & THOLERUS, S. 2015 The merits of ion cyclotron resonance heating schemes for sawtooth control in tokamak plasmas. *J. Plasma Phys.* **81**, 365810601.
- ERIKSSON, L.-G. & HELANDER, P. 1994 Monte Carlo operators for orbit-averaged Fokker–Planck equations. *Phys. Plasmas* **1**, 308–314.
- ERIKSSON, L.-G., MANTSINEN, M., BORBA, D., FASOLI, A., HEETER, R., SHARAPOV, S., START, D. F. H., CARLSSON, J., GONDHALEKAR, A., HELLSTEN, T. *et al.* 1998 Evidence for a wave-induced particle pinch in the presence of toroidally asymmetric ICRF waves. *Phys. Rev. Lett.* **81**, 1231–1234.

- FAULCONER, D. W. 1987 Echo-like nonlocal effect due to parallel magnetic inhomogeneity. *Plasma Phys. Control. Fusion* **29**, 433–435.
- HELANDER, P. & CATTO, P. J. 2001 Neoclassical current drive by waves with a symmetric spectrum. *Phys. Plasmas* **8**, 1988–1994.
- HELLSTEN, T., CARLSSON, J. & ERIKSSON, L.-G. 1995 Minority ion cyclotron current drive in tokamaks. *Phys. Rev. Lett.* **74**, 3612–3615.
- JAEGER, E. F., BERRY, L. A., DAZEVEDO, E., BATCHELOR, D. B., CARTER, M. D. & WHITE, K. F. 2002 Advances in full-wave modeling of radio frequency heated, multidimensional plasmas. *Phys. Plasmas* **9**, 1873–1881.
- JAEGER, E. F., BERRY, L. A., MYRA, J. R., BATCHELOR, D. B., DAZEVEDO, E., BONOLI, P. T., PHILLIPS, C. K., SMITHE, D. N., D'IPPOLITO, D. A., CARTER, M. D. *et al.* 2003 Sheared poloidal flow driven by mode conversion in tokamak plasmas. *Phys. Rev. Lett.* **90**, 195001–195004.
- KAGAN, G. & CATTO, P. J. 2008 Arbitrary poloidal gyroradius effects in tokamak pedestals and transport barriers. *Plasma Phys. Control. Fusion* **50**, 085010–085025.
- KAPPER, G., KASILOV, S. V., KERNBICHLER, W., MARTITSCH, A. F., HEYN, M. F., MARUSHCHENKO, N. B. & TURKIN, Y. 2016 Electron cyclotron current drive simulations for finite collisionality plasmas in Wendelstein 7-X using the full linearized collision model. *Phys. Plasmas* **23**, 112511–112514.
- KAUFMAN, A. N. 1972 Quasilinear diffusion of an axisymmetric toroidal plasma. *Phys. Fluids* **15**, 1063–1069.
- KENNEL, C. F. & ENGELMANN, F. 1966 Velocity space diffusion from weak plasma turbulence in a magnetic field. *Phys. Fluids* **9**, 2377–2388.
- LEE, X. S., MYRA, J. R. & CATTO, P. J. 1983 General frequency gyrokinetics. *Phys. Fluids* **26**, 223–229.
- LIN, Y., WUKITCH, S. J., BONOLI, P. T., MARMAR, E., MOSSESIAN, E. D., NELSON-MELBY, E., PHILLIPS, P., PORKOLAB, M., SCHILLING, G., WOLFE, S. *et al.* 2003 Ion cyclotron range of frequencies mode conversion electron heating in deuterium–hydrogen plasmas in the Alcator C-Mod tokamak. *Plasma Phys. Control. Fusion* **45**, 1013–1026.
- MANTSINEN, M. J., INGESSON, L. C., JOHNSON, T., KIPTILY, V. G., MAYORAL, M.-L., SHARAPOV, S., ALPER, B., BERTALOT, L., CONROY, S., ERIKSSON, L.-G. *et al.* 2002 Controlling the profile of ion-cyclotron-resonant ions in JET with the wave-induced pinch effect. *Phys. Rev. Lett.* **89**, 115004.
- PARRA, F. I. & CALVO, I. 2011 Phase-space Lagrangian derivation of electrostatic gyrokinetics in general geometry. *Plasma Phys. Control. Fusion* **53**, 045001, 49 pp.
- PARRA, F. I. & CATTO, P. J. 2008 Limitations of gyrokinetics on transport time scales. *Plasma Phys. Control. Fusion* **50**, 065014, 23 pp.
- PERKINS, F. W. 1977 Heating tokamaks via the ion-cyclotron and ion–ion hybrid resonances. *Nucl. Fusion* **17**, 1197–1224.
- SMITHE, D., COLESTOCK, P., KAMMASH, T. & KASHUBA, R. 1988 Effect of parallel magnetic field gradients on absorption and mode conversion in the ion-cyclotron range of frequencies. *Phys. Rev. Lett.* **60**, 801–804.
- WRIGHT, J. C., BONOLI, P. T., BRAMBILLA, M., MEO, F., DAZEVEDO, E., BATCHELOR, D. B., JAEGER, E. F., BERRY, L. A., PHILLIPS, C. K. & PLETZER, A. 2004 Full wave simulations of fast wave mode conversion and lower hybrid wave propagation in tokamaks. *Phys. Plasmas* **11**, 2473–2479.