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Calculus of Variations

Regularity for nonlocal problems with non-standard growth

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Abstract
We study robust regularity estimates for local minimizers of nonlocal functionals with non-standard growth of (p, q)-type and for weak solutions to a related class of nonlocal equations. The main results of this paper are local boundedness and Hölder continuity of minimizers and weak solutions. Our approach is based on the study of corresponding De Giorgi classes.

Mathematics Subject Classification 35B65 · 47G20 · 35D30 · 35B45 · 35A15

1 Introduction

The aim of this paper is to prove regularity properties of local minimizers and weak solutions to a class of nonlocal problems with non-standard growth.

Let $s \in (0, 1)$, $\Lambda \geq 1$ and $\Omega \subset \mathbb{R}^d$ be an open set. We study energy functionals of the form

$$u \mapsto \mathcal{I}_f(u) = (1-s) \iint_{(\Omega^c \times \Omega^c)^c} f\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{k(x,y)}{|x-y|^d} \,\mathrm{d}y\,\mathrm{d}x, \tag{1.1}$$

where $f : [0, \infty) \to [0, \infty)$ is a convex increasing function and $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a measurable function satisfying

$$k(x, y) = k(y, x) \quad \text{and} \quad \Lambda^{-1} \leq k(x, y) \leq \Lambda \quad \text{for a.e. } x, y \in \mathbb{R}^d. \tag{k}$$

When $f(t) = t^p$ with $p > 1$ and $\Lambda = 1$, the functional (1.1) becomes the standard fractional p-functional whose corresponding operator is the fractional p-Laplacian. The regularity theory for this case is well established, see [17, 19, 20].

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Apparently the functional (1.1) is governed by the function f, which controls the growth behavior for large and small values of $|u(x)-u(y)||x-y|^{-s}$. To establish the regularity theory, we need some growth conditions on f.

Let $1 \le p \le q$. We say that f satisfies (f_p^q) if f is differentiable and satisfies for all $t \ge 0$

$$pf(t) \le tf'(t), \qquad (f_p)$$
$$tf'(t) \le qf(t). \qquad (f^q)$$

Condition (f_p^q) can be interpreted as a (p,q)-*growth condition* since it implies that

$$f(1)(t^p \wedge t^q) \le f(t) \le f(1)(t^p \vee t^q). \tag{1.2}$$

For a detailed discussion of (f_p^q) we refer to Sect. 2.

To study local boundedness of minimizers, we work under the assumption that there exists a constant $c_0 > 0$ such that for all $t \ge 0$

$$c_0 t^p \le f(t). \qquad (f \gtrsim t^p)$$

Throughout the paper, we will assume without loss of generality that $f(0) = 0$ and $f(1) = 1$. Note that the assumption $f(0) = 0$ is required in order for $\mathcal{I}_f(0) < \infty$, while the second assumption is not restrictive since u minimizes $\mathcal{I}_f$ if and only if u minimizes $\mathcal{I}_{f/f(1)}$.

In the following, we present the first main result of this paper. It is concerned with Hölder estimates and local boundedness for local minimizers of (1.1).

Theorem 1.1 (Local minimizers) *Let $s_0 \in (0,1)$, $1 < p \le q$, $\Lambda \ge 1$, $c_0 > 0$ and assume $s \in [s_0, 1)$. Let $f : [0,\infty) \to [0,\infty)$ be a convex increasing function satisfying (f^q) and let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function satisfying (k). Let $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ be a local minimizer of (1.1).*

(i) Assume that f satisfies (f_p). Then, there exist $\alpha \in (0,1)$ and $C > 0$, depending on d, s_0, p, q and Λ, such that for any $B_{8R}(x_0) \subset \Omega$

$$R^\alpha [u]_{C^\alpha(\overline{B_R(x_0)})} \le C\|u\|_{L^\infty(B_{4R}(x_0))} + \mathrm{Tail}_{f'}(u; x_0, 4R). \tag{1.3}$$

(ii) Assume that $sp < d$, $q < p^* := dp/(d-sp)$ and that f satisfies $(f \gtrsim t^p)$. Then, $u \in L^\infty_{\mathrm{loc}}(\Omega)$. Moreover, for each $B_{2R}(x_0) \subset \Omega$ there exists $C > 0$, depending on d, s_0, p, q, $p^* - q$, Λ, c_0 and R, such that for every $\delta \in (0,1)$

$$\sup_{B_R(x_0)} |u| \le \delta \mathrm{Tail}_{f'}(u; x_0, R)$$
$$+ C\delta^{-(q-1)\frac{p^*}{p}\frac{1}{p^*-q}} \left(\fint_{B_{2R}(x_0)} |u(x)|^q \,\mathrm{d}x \right)^{\frac{1}{p}\frac{p^*-p}{p^*-q}} + \delta^{\frac{q-1}{q}}. \tag{1.4}$$

We refer to Sect. 3 for the definition of the function space $V^{s,f}(\Omega|\mathbb{R}^d)$ and the tail term $\mathrm{Tail}_{f'}$. The proof of Theorem 1.1 and the definition of a local minimizer are given in Sect. 6.

The second main result of this paper is concerned with weak solutions to a related class of nonlocal equations. To motivate our result, we first point out that the Euler–Lagrange equation corresponding to the functional (1.1) is given by

$$(1-s)\mathrm{p.v.}\int_{\mathbb{R}^d} f'\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{u(x)-u(y)}{|u(x)-u(y)|} \frac{k(x,y)}{|x-y|^{d+s}} \,\mathrm{d}y = 0 \quad \text{in } \Omega. \tag{1.5}$$

For convex differentiable functions f, it is easy to see that weak solutions to the Euler–Lagrange equation are minimizers of the functional (1.1). In this article, we consider a more general class of equations

$$\mathcal{L}_h u = 0 \quad \text{in } \Omega \tag{1.6}$$

with nonlocal operators of the form

$$\mathcal{L}_h u(x) = (1-s)\text{p.v.} \int_{\mathbb{R}^d} h\left(x, y, \frac{u(x)-u(y)}{|x-y|^s}\right) \frac{\mathrm{d}y}{|x-y|^{d+s}}, \tag{1.7}$$

where $h : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying the structure condition

$$h(x, y, t) = h(y, x, t), \quad |h(x, y, t)| \le \Lambda f'(|t|), \quad h(x, y, t)t \ge \frac{1}{\Lambda} f'(|t|)|t| \tag{h}$$

for a.e. $x, y \in \mathbb{R}^d$ and for all $t \in \mathbb{R}$, and $f : [0, \infty) \to [0, \infty)$ is convex, increasing, differentiable and satisfies $f(0) = 0$, $f(1) = 1$.

Note that in the special case $h(x, y, t) = \text{sign}(t) f'(|t|)k(x, y)$ for some k satisfying (k), the Eqs. (1.5) and (1.6) coincide.

We are ready to state the second main result of this article, which establishes Hölder estimates and local boundedness for weak solution to (1.6).

Theorem 1.2 (Weak solutions) *Let $s_0 \in (0, 1)$, $1 < p \le q$, $\Lambda \ge 1$, $c_0 > 0$ and assume $s \in [s_0, 1)$. Let $f : [0, \infty) \to [0, \infty)$ be a convex increasing function satisfying (f^q) and let $h : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying (h). Let $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ be a weak solution to (1.6).*

(i) Assume that f satisfies (f_p). Then, there exist $\alpha \in (0, 1)$ and $C > 0$, depending on d, s_0, p, q and Λ, such that for any $B_{8R}(x_0) \subset \Omega$ the estimate (1.3) holds.

(ii) Assume that $sp < d$, $q < p^*$ and that f satisfies ($f \gtrsim t^p$). Then, $u \in L^\infty_{\text{loc}}(\Omega)$. Moreover, for each $B_{2R}(x_0) \subset \Omega$ there exists $C > 0$, depending on d, s_0, p, q, $p^* - q$, Λ, c_0 and R, such that for every $\delta \in (0, 1)$ the estimate (1.4) holds.

The proof of Theorem 1.2 and the definition of a weak solution are given in Sect. 7. Note that we can assume without loss of generality that $f(0) = 0$, $f(1) = 1$ because a function u solves $\mathcal{L}_h u = 0$ if and only if it solves $\mathcal{L}_{h/f(1)} u = 0$.

In fact, we prove Hölder estimates and local boundedness for functions in De Giorgi classes (see Sect. 3). The corresponding results, Theorems 4.2 and 5.1, are more general. Theorems 1.1 and 1.2 follow by the observation that minimizers, as well as weak solutions, belong to the corresponding De Giorgi classes, see Sects. 6 and 7.

Note that right-hand sides with suitable growth behaviors can be studied by an adequate extension of De Giorgi classes in analogy to [17].

Remark 1.3 Our results are robust in the sense that the constants C and α stay uniform as $s \to 1^-$, since they depend only on s_0, not on s.

(1) Theorem 1.1 generalizes the results in [47] to nonlocal functionals. We work under the same assumptions on the growth function f as in that article. In this sense, the assumptions on f used in our results are natural.
(2) Theorem 1.2 can be linked to the paper [40]. Some assumptions on the regularity and growth of the function f in [40] are more restrictive than our assumptions, but in return allow Lieberman to prove $C^{1,\beta}$ regularity of weak solutions to the Euler–Lagrange equations.

Our approach for studying local minimizers and weak solutions is based on so-called De Giorgi classes. We show that minimizers of (1.1) and weak solutions to (1.6) satisfy a suitable improved fractional Caccioppoli inequality, from which the definition of the De Giorgi class emerges. This inequality together with an isoperimetric-type inequality allow us to deduce the Hölder estimates for locally bounded minimizers and weak solutions following the methods from [17, 47]. We emphasize that there is no restriction on the gap between p and q for the Hölder estimates. Furthermore, we derive the local boundedness of functions in De Giorgi classes under the assumption that $1 < p \leq q < p^*$. We would like to remark that by modification of the proof of Theorem 5.1, it is possible to prove the local boundedness without any restriction on the range of p and q and without the condition ($f \gtrsim t^p$), see [16] and also [5].

In the following, we discuss related literature and describe the novelty of our results.

We first comment on related results for local operators. For this purpose, we consider functionals of the following form

$$\int_\Omega f(x, u, \nabla u)\, \mathrm{d}x,$$

where f is a non-negative function which describes the growth behavior of the functional. If the function f satisfies the so-called p-growth condition, that is

$$|\xi|^p \lesssim f(x, z, \xi) \lesssim |\xi|^p + 1 \qquad \text{for } p > 1,$$

the literature is very rich and many regularity results have been proved. We refer the reader to the classical references [31, 33, 41] and for a more comprehensive treatment to the books [34, 35].

Functionals with non-standard growth of (p, q)-type

$$|\xi|^p \lesssim f(x, z, \xi) \lesssim |\xi|^q + 1,$$

where $1 < p < q$, are naturally connected to Orlicz–spaces. The analysis of regularity of minimizers of functionals having non-standard growth of (p, q)-type was initiated by Marcellini's work [42], where he studies strictly convex C^2-functions f satisfying (p, q)-growth condition.

To the best of our knowledge, functionals with non-standard growth functions of the type (f_p^q) first appeared in the papers [40, 47] in the context of regularity results.

In the paper [40], Lieberman proves several regularity results for bounded weak solutions to a class of elliptic operators in divergence form. Furthermore, he studies quasiminimizers and proves regularity results for functions in corresponding De Giorgi classes.

In [47], Moscariello and Nania prove Hölder continuity of locally bounded minimizers for growth functions satisfying (f_p^q) and local boundedness for functions with (p, q)-growth. Their key idea is to introduce an auxiliary function which is comparable to the growth function f and to prove that any function in the De Giorgi class corresponding to the auxiliary function is Hölder continuous. We adapt this idea for the proof of Theorems 1.1 and 1.2 to the nonlocal case, see Sect. 4.

There have been many important contributions to regularity for problems with non-standard growth of (p, q)-type. Papers studying local operators with non-standard growth of (p, q)-type are, among others, [3, 6, 12, 14, 23, 26, 28, 39, 43, 44, 48, 49, 54, 55, 58]. For a more detailed picture on problems with non-standard growth, including double-phase problems, problems with variable exponents, and anisotropic problems, we refer the reader to the surveys [45, 50].

In the case of nonlocal operators, the energy functional is defined by (1.1). Local regularity results for the fractional p-Laplacian, that is $f(t) = t^p$, were first established in the papers [19, 20] by Di Castro, Kuusi and Palatucci. Another important contribution to regularity is the work [17], where he studies minimizers to nonlocal energy functionals plus a possibly discontinuous potential. The nonlocal energy has p growth for a class of symmetric kernels comparable to $(1-s)|x-y|^{-n-sp}$. Furthermore, he studies weak solutions to the Euler–Lagrange equation. He uses the nonlocality of the functional to prove an improved Caccioppoli inequality with an additional term, which disappears as the fractional order s goes to one. We follow Cozzi's ideas at several points in the present paper and also make use of some auxiliary results proved in [17] such as an isoperimetric inequality. It is worth emphasizing that the fractional De Giorgi iteration has been first employed in the papers [18, 46]. For further results on the regularity of the fractional p-Laplacian, we refer the reader to [7, 8, 53] and the references therein.

Lately, the interest in the analysis of nonlocal problems with non-standard growth has increased. For instance, regularity results for nonlocal double phase equations and nonlocal equation with variable exponents are proved in [10, 21, 32, 57], respectively [15, 56]. However, we would like to note that both, double phase equations and equations with variable exponents, do not fall into our setup. See also [36–38] for further regularity results concerning nonlocal operators with non-standard growth.

As far as we know, first regularity results for fractional order Orlicz–Sobolev spaces have been proved in [13] by Fernández Bonder, Salort and Vivas. The authors establish regularity results for weak solutions to the Dirichlet problem for the fractional g-Laplacian. They prove interior and up to the boundary Hölder regularity to the corresponding Dirichlet problem.

See also [52], where qualitative properties of solutions such as a Liouville type theorem and symmetry results are proved.

The present paper is substantially different from [13]. On the one hand, we do not only study weak solutions but also local minimizers for the functional $\mathcal{I}_f$. The present paper also allows for $p > 1$ and is not restricted to the case $p \geq 2$. Furthermore, we use a completely different approach via De Giorgi classes.

Notation

We write c and C for strictly positive constants whose exact values are not important and might change from line to line. Furthermore, we use the notations $c = c(\cdot)$ and $C = C(\cdot)$ if we want to highlight all quantities the constant depends on.

Outline

The paper consists of seven sections and is organized as follows. In Sect. 2.1, we introduce an auxiliary growth function and prove several properties for convex functions satisfying (f_p^q). Moreover, in Sect. 2.2 we recall some functional inequalities. The fractional De Giorgi classes with general convex functions with non-standard growth are introduced in Sect. 3. Furthermore, we introduce fractional Orlicz–Sobolev spaces. In Sect. 4 we prove Hölder continuity and in Sect. 5 local boundedness for functions in fractional De Giorgi classes. Finally, in Sect. 6 resp. Section 7, we show that minimizers resp. weak solutions belong to the fractional De Giorgi classes and prove Theorems 1.1 and 1.2.

2 Preliminaries

In this section, we study properties of the growth functions f under consideration and collect some functional inequalities.

2.1 Auxiliary results on growth functions

Let us collect several results in order to illustrate the assumption (f_p^q). The first two lemmas provide equivalent conditions for the upper and lower bounds in (f_p^q), respectively.

Lemma 2.1 *Let $q \geq 1$ and $f : [0,\infty) \to [0,\infty)$ be a differentiable function. Then the following are equivalent:*

(i) (f^q),
(ii) $t \mapsto t^{-q} f(t)$ *is decreasing,*
(iii) $f(\lambda t) \leq \lambda^q f(t)$ *for all* $\lambda \geq 1$,
(iv) $\lambda^q f(t) \leq f(\lambda t)$ *for all* $\lambda \leq 1$.

Proof (i) $\Leftrightarrow$ (ii) follows from the observation that $\frac{\mathrm{d}}{\mathrm{d}t}(t^{-q} f(t)) = t^{-q-1}(tf'(t) - qf(t))$. (i) $\Leftrightarrow$ (iii) follows from the observation that (iii) can be rewritten as

$$\frac{f(\lambda t)}{(\lambda t)^q} \leq \frac{f(t)}{t^q}, \quad \text{for all } \lambda \geq 1. \tag{2.1}$$

Since also (iv) can be rewritten as (2.1), the equivalence (iii) $\Leftrightarrow$ (iv) is trivial. □

Lemma 2.2 *Let $p \geq 1$ and $f : [0,\infty) \to [0,\infty)$ be a differentiable function. Then the following are equivalent:*

(i) (f_p),
(ii) $t \mapsto t^{-p} f(t)$ *is increasing,*
(iii) $\lambda^p f(t) \leq f(\lambda t)$ *for all* $\lambda \geq 1$,
(iv) $f(\lambda t) \leq \lambda^p f(t)$ *for all* $\lambda \leq 1$.

Proof The proof works exactly like the proof of Lemma 2.1. □

The following lemma provides a useful property of convex functions.

Lemma 2.3 *Let $f : [0,\infty) \to [0,\infty)$ be convex and $f(0) = 0$. Then, the function $t \mapsto f(t)/t$ is increasing. If f is differentiable, then f satisfies (f_p) with $p = 1$.*

Proof The assertions follow from $f(\lambda t) = f(\lambda t + (1-\lambda)0) \leq \lambda f(t) + (1-\lambda) f(0)$ and Lemma 2.2 with $p = 1$. □

As a consequence we obtain some doubling-type inequalities for f'. These inequalities play an important role for the tail estimates in the upcoming regularity theory.

Corollary 2.4 *Let $1 \leq p \leq q$ and let $f : [0,\infty) \to [0,\infty)$ be a function satisfying (f_p^q). Then,*

$$\frac{p}{q}\lambda^{p-1} f'(t) \leq f'(\lambda t) \leq \frac{q}{p}\lambda^{q-1} f'(t) \quad \text{for all } \lambda \geq 1, \tag{2.2}$$

$$\frac{p}{q}\lambda^{q-1} f'(t) \leq f'(\lambda t) \leq \frac{q}{p}\lambda^{p-1} f'(t) \quad \text{for all } \lambda \leq 1, \tag{2.3}$$

and

$$\frac{1}{2}f'(t)+\frac{1}{2}f'(s)\le f'(t+s)\le \frac{q}{p}2^{q-1}(f'(t)+f'(s)) \tag{2.4}$$

for all $t,s\ge 0$.

Proof For the second inequality in (2.2), we compute using (f_p^q) and Lemma 2.1

$$f'(\lambda t)\le q\frac{f(\lambda t)}{\lambda t}\le q\lambda^{q-1}\frac{f(t)}{t}\le \frac{q}{p}\lambda^{q-1}f'(t).$$

The first inequality in (2.2) and (2.3) can be proved in the same way. The first estimate in (2.4) is a direct consequence of monotonicity of f'. For the second estimate in (2.4), we may assume that $t\le s$. Then, we obtain

$$f'(t+s)\le q\frac{f(t+s)}{(t+s)^p}(t+s)^{p-1}\le q\frac{f(2s)}{2s}\le q2^{q-1}\frac{f(s)}{s}\le \frac{q}{p}2^{q-1}(f'(t)+f'(s))$$

by using (f_p^q), Lemmas 2.1 and 2.2. □

Another useful property of convex functions is the following:

Lemma 2.5 *Let* $f:[0,\infty)\to[0,\infty)$ *be convex and* $f(0)=0$. *Let* $c>1$ *and assume that for some* $t,s>0$ *it holds that* $f(t)\le cf(s)$. *Then* $t\le cs$.

Proof Let $t,s>0$ be such that $f(t)\le cf(s)$. We assume that $t>cs$. Then by Lemmas 2.3 and 2.2 with $p=1$, we have

$$\frac{f(s)}{s}\le\frac{f(cs)}{cs}\le\frac{f(t)}{t}\le\frac{cf(s)}{t}<\frac{cf(s)}{cs}=\frac{f(s)}{s}.$$

This is a contradiction, so it must hold that $t\le cs$, as desired. □

One of the key ideas of proving Hölder regularity in [47] is to construct F, which is a convex increasing function satisfying some growth conditions and the comparability of $g(t):=F(t^p)$ and $f(t)$. These properties are important in our framework as well but we also need the comparability of the derivatives of these functions for the regularity estimates.

Proposition 2.6 (c.f. [30]) *Let* $1\le p\le q$ *and* $f:[0,\infty)\to[0,\infty)$ *be a convex, increasing, and differentiable function satisfying* $f(0)=0$. *Define* $F,g:[0,\infty)\to[0,\infty)$ *by*

$$F(t)=\int_0^{t^{1/p}}\frac{f(s)}{s}\,\mathrm{d}s\quad\text{and}\quad g(t)=F(t^p). \tag{2.5}$$

If f *satisfies* (f_p^q), *then* F *is a convex increasing function satisfying*

$$F(t)\le tF'(t)\le\frac{q}{p}F(t) \tag{2.6}$$

and g *is an increasing function satisfying*

$$\frac{1}{q}f(t)\le g(t)\le\frac{1}{p}f(t) \tag{2.7}$$

$$\frac{1}{q}f'(t)\le g'(t)\le\frac{1}{p}f'(t). \tag{2.8}$$

Proof First of all, the function F is well-defined by Lemma 2.3. The functions F and g are increasing by definition. Moreover, F is convex since

$$F''(t) = \frac{1}{p^2} t^{\frac{1}{p}-2} f'(t^{\frac{1}{p}}) - \frac{1}{p} t^{-2} f(t^{\frac{1}{p}}) \geq \frac{1}{p} t^{-2} f(t^{\frac{1}{p}}) - \frac{1}{p} t^{-2} f(t^{\frac{1}{p}}) = 0$$

by (f_p). Thus, the first inequality in (2.6) follows from Lemma 2.3. The second inequality follows from

$$F(\lambda t) = \int_0^{(\lambda t)^{1/p}} \frac{f(s)}{s} \, \mathrm{d}s = \int_0^{t^{1/p}} \frac{f(\lambda^{1/p} s)}{s} \, \mathrm{d}s \leq \lambda^{q/p} \int_0^{t^{1/p}} \frac{f(s)}{s} \, \mathrm{d}s = \lambda^{q/p} F(t)$$

for $\lambda \geq 1$, where we used Lemma 2.1. By (f_p^q) we have

$$p \frac{f(s)}{s} \leq f'(s) \leq q \frac{f(s)}{s},$$

and after integrating from 0 to t and using that $f(0) = 0$, we deduce (2.7). Finally, we compute

$$pg'(t) = p \frac{f(t)}{t} \leq f'(t) \leq q \frac{f(t)}{t} = qg'(t),$$

using (f_p^q) from where (2.8) follows. □

We close this subsection with two estimates for convex functions. Lemmas 2.7 and 2.8 are generalizations of [17, Lemma 4.1 and 4.2].

Lemma 2.7 *Let* $f : [0, \infty) \to [0, \infty)$ *be convex, differentiable and* $f(0) = 0$*. Then, for any* $\theta \in [0, 1]$ *and* $a, b \geq 0$*:*

$$f(a + b) - f(a) \geq \theta f'(a)b + (1 - \theta) f(b).$$

Proof The result is clear for $\theta = 0$ by the superadditivity of convex functions with $f(0) = 0$. For $\theta = 1$, we compute

$$f(a + b) - f(a) = \int_a^{a+b} f'(\tau) \, \mathrm{d}\tau \geq f'(a)b,$$

where we used the fact that $t \mapsto f'(t)$ is increasing since f is convex. The result for $\theta \in (0, 1)$ follows by interpolation. □

Lemma 2.8 *Let* $f : [0, \infty) \to [0, \infty)$ *be convex, increasing and differentiable. Then, for every* $\mu \in [0, 1]$ *and* $a, b \geq 0$*:*

$$f(|\mu a - b|) - f(|a - b|) \leq f'(b)a.$$

Proof Let us first assume that $b \geq a$. Then

$$f(|\mu a - b|) - f(|a - b|) = \int_{b-a}^{b-\mu a} f'(\tau) \, \mathrm{d}\tau \leq f'(b - \mu a)a(1 - \mu) \leq f'(b)a.$$

If on the other hand $\mu a \leq b < a$, then

$$f(|\mu a - b|) - f(|a - b|) = \int_{a-b}^{b-\mu a} f'(\tau) \, \mathrm{d}\tau \leq f'(b - \mu a)(2b - (1 + \mu)a) \leq f'(b)a.$$

It remains to consider the case $b < \mu a$, but then $f(|\mu a - b|) - f(|a - b|) \leq 0$, so there is nothing to prove. □

2.2 Functional inequalities

In this section, we collect some well-known functional inequalities which are useful for the application of De Giorgi's methods for nonlocal operators. While the first three results are embeddings for fractional Sobolev spaces, the last proposition is a fractional isoperimetric inequality.

Lemma 2.9 [17, Lemma 4.6] *Let* $0 < \tilde{\sigma} < \sigma < 1$ *and* $1 \leq \tilde{p} < p$. *Let* $\Omega' \subset \Omega \subset \mathbb{R}^d$ *be two bounded measurable sets, then for any* $u \in W^{\sigma,p}(\Omega)$

$$\left(\int_{\Omega'}\int_{\Omega} \frac{|u(x)-u(y)|^{\tilde{p}}}{|x-y|^{d+\tilde{\sigma}\tilde{p}}}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/\tilde{p}} \leq C|\Omega'|^{\frac{p-\tilde{p}}{p\tilde{p}}}\operatorname{diam}(\Omega)^{\sigma-\tilde{\sigma}}\left(\int_{\Omega'}\int_{\Omega} \frac{|u(x)-u(y)|^{p}}{|x-y|^{d+\sigma p}}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/p},$$

where

$$C = \left(\frac{d(p-\tilde{p})}{(\sigma-\tilde{\sigma})p\tilde{p}}|B_1|\right)^{\frac{p-\tilde{p}}{p\tilde{p}}}.$$

Theorem 2.10 [17, Corollary 4.9] *Let* $0 < s_0 \leq s < 1$ *and* $p \geq 1$ *be such that* $sp < d$. *Let* $u \in W_0^{s,p}(B_R)$ *and assume* $u = 0$ *on a set* $\Omega \subset B_R$ *with* $|\Omega| \geq \gamma|B_R|$ *for some* $\gamma \in (0, 1]$. *Then, there exists a constant* $C > 0$, *depending on* d, s_0, p *and* γ, *such that*

$$\|u\|^p_{L^{p^*}(B_R)} \leq C\frac{1-s}{(d-sp)^{p-1}}\int_{B_R}\int_{B_R}\frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}y\,\mathrm{d}x.$$

Theorem 2.11 [4, 25, 51] *Let* $0 < s_0 \leq s < 1$ *and* $p \geq 1$ *be such that* $sp < d$. *Let* $u \in W^{s,p}(B_R)$. *Then, there exists a constant* $C > 0$, *depending on* d, s_0 *and* p, *such that*

$$\|u\|^p_{L^{p^*}(B_R)} \leq C\frac{1-s}{(d-sp)^{p-1}}\int_{B_R}\int_{B_R}\frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}y\,\mathrm{d}x + CR^{-sp}\|u\|^p_{L^p(B_R)}.$$

Proposition 2.12 [17, Proposition 5.1] *Let* $p > 1$, $C_0 > 0$ *and* $\gamma, \gamma_0 \in (0, 1)$. *Then, there exist constants* $\bar{s} \in (0, 1)$ *and* $C > 0$, *depending on* d, p, γ, γ_0 *and* C_0, *such that if* $s \in [\bar{s}, 1)$ *and if* $u \in W^{s,p}(B_R)$ *satisfies*

$$|B_R \cap \{u \leq h\}| \geq \gamma|B_R|, \quad |B_R \cap \{u \geq k\}| \geq \gamma_0|B_R| \quad \text{and}$$
$$\|u\|^p_{L^p(B_R)} + (1-s)R^{sp}[u]^p_{W^{s,p}(B_R)} \leq C_0 R^d (k-h)^p \quad \text{for } k > h,$$

then

$$(k-h)\Big(|B_R \cap \{u \leq h\}||B_R \cap \{u \geq k\}|\Big)^{\frac{d-1}{d}} \leq CR^{d-2+s}(1-s)^{1/p}[u]_{W^{s,p}(B_R)}|B_R \cap \{h < u < k\}|^{\frac{p-1}{p}}.$$

3 De Giorgi classes

In this section, we introduce fractional order Orlicz–Sobolev spaces and define fractional De Giorgi classes governed by convex functions having non-standard growth.

Fractional order Orlicz–Sobolev spaces have been introduced in [29] by Fernández Bonder and Salort. The authors prove several properties of the spaces including that the fractional order Orlicz–Sobolev space approximates some Orlicz–Sobolev space as the fractional parameter goes to 1. For further results concerning fractional Orlicz–Sobolev spaces, we refer the reader to [1, 2, 9, 11, 24] and the references therein.

Let $f : [0, \infty) \to [0, \infty)$ be a convex increasing function satisfying (f^q) and $f(0) = 0$. Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^d$ be open. We define the *Orlicz* and *Orlicz–Sobolev spaces* by

$$\begin{aligned} L^f(\Omega) &= \{u : \Omega \to \mathbb{R} \text{ measurable} : \Phi_{L^f(\Omega)}(u) < \infty\}, \\ W^{s,f}(\Omega) &= \{u \in L^f(\Omega) : \Phi_{W^{s,f}(\Omega)}(u) < \infty\}, \\ V^{s,f}(\Omega|\mathbb{R}^d) &= \{u \in L^f(\Omega) : \Phi_{V^{s,f}(\Omega)}(u) < \infty\}, \end{aligned}$$

where $\Phi_{L^f(\Omega)}$, $\Phi_{W^{s,f}(\Omega)}$ and $\Phi_{V^{s,f}(\Omega|\mathbb{R}^d)}$ are *modulars* defined by

$$\begin{aligned} \Phi_{L^f(\Omega)}(u) &= \int_\Omega f(|u(x)|)\,\mathrm{d}x, \\ \Phi_{W^{s,f}(\Omega)}(u) &= (1-s)\int_\Omega\int_\Omega f\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}, \\ \Phi_{V^{s,f}(\Omega|\mathbb{R}^d)}(u) &= (1-s)\iint_{(\Omega^c\times\Omega^c)^c} f\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}. \end{aligned}$$

Under more restrictive assumptions on f (for instance f being an Orlicz function), $L^f(\Omega)$, $W^{s,f}(\Omega)$, and $V^{s,f}(\Omega|\mathbb{R}^d)$ are Banach spaces endowed with the norms

$$\begin{aligned} \|u\|_{L^f(\Omega)} &= \inf\{\lambda > 0 : \Phi_{L^f(\Omega)}(u/\lambda) \le 1\}, \\ \|u\|_{W^{s,f}(\Omega)} &= \|u\|_{L^f(\Omega)} + [u]_{W^{s,f}(\Omega)} = \|u\|_{L^f(\Omega)} + \inf\{\lambda > 0 : \Phi_{W^{s,f}(\Omega)}(u/\lambda) \le 1\}, \\ \|u\|_{V^{s,f}(\Omega|\mathbb{R}^d)} &= \|u\|_{L^f(\Omega)} + [u]_{V^{s,f}(\Omega|\mathbb{R}^d)} = \|u\|_{L^f(\Omega)} + \inf\{\lambda > 0 : \Phi_{V^{s,f}(\Omega|\mathbb{R}^d)}(u/\lambda) \le 1\}. \end{aligned}$$

For details, see [1, 9, 29].

Let us next define nonlocal tails, which capture the behavior of functions $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ at large scales. For this purpose, we consider a convex and differentiable function f and a generalized inverse function of its derivative f'. There are several definitions of generalized inverse functions in the literature, see [22, 27], but we use the following definition for a generalized inverse of f':

$$(f')^{-1}(y) = \inf\{t : f'(t) \ge y\}. \tag{3.1}$$

The advantage of this definition is that (3.1) enjoys the following properties, which play an important role in the proof of regularity estimates.

Proposition 3.1 *Let $f : [0, \infty) \to [0, \infty)$ be a convex and differentiable function. Then*

$$(f' \circ (f')^{-1})(y) \ge y \quad \textit{for all } y \ge 0, \tag{3.2}$$

$$((f')^{-1} \circ f')(t) \le t \quad \textit{for all } t \ge 0. \tag{3.3}$$

Proof To prove (3.2), let $y \ge 0$ and $t = (f')^{-1}(y)$. Then, there exists a sequence $(t_n)_{n\ge1}$ such that $t_n \ge t$, $t_n \to t$, and $f'(t_n) \ge y$. Since f' is continuous by Darboux's theorem, we obtain $f'(t) \ge y$ by taking the limit $n \to \infty$. Assertion (3.3) is obvious by definition of $(f')^{-1}$. □

We define the nonlocal f'-Tail by

$$\mathrm{Tail}_{f'}(u; x_0, R) = R^s (f')^{-1}\left((1-s)R^s \int_{\mathbb{R}^d \setminus B_R(x_0)} f'\left(\frac{|u(y)|}{|y-x_0|^s}\right) \frac{\mathrm{d}y}{|y-x_0|^{d+s}}\right). \tag{3.4}$$

Note that $\mathrm{Tail}_{f'}$ coincides with the standard tail considered in [17, 19, 20] in the special case $f(t) \asymp t^p$. Moreover, it is natural in the sense that the following scaling property is satisfied:

$$\mathrm{Tail}_{f'}(u; x_0, R) = \mathrm{Tail}_{f'(\cdot/R^s)}(u(R\cdot); x_0/R, 1).$$

We claim that the nonlocal f'-Tail is well-defined for functions in the fractional Orlicz–Sobolev space $V^{s,f}(\Omega|\mathbb{R}^d)$. To this end, we define the *Legendre transform* $f^* : [0,\infty) \to [0,\infty)$ by $f^*(s) = \sup_{t\in[0,\infty)}(st - f(t))$. It is well known that

$$(f^*(f'))(t) = f'(t)t - f(t) \tag{3.5}$$

and that

$$st \le f(t) + f^*(s), \quad \text{for all } t, s \ge 0. \tag{3.6}$$

Inequality (3.6) is called *Fenchel's inequality*.

Proposition 3.2 *Let $q > 1$, $s \in (0,1)$ and $\Omega \subset \mathbb{R}^d$ be open. Let $f : [0,\infty) \to [0,\infty)$ be a convex increasing function satisfying (f^q) and $f(0) = 0$. If $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ and $B_R(x_0) \subset \Omega$, then* $\mathrm{Tail}_{f'}(u; x_0, R) < \infty$.

Proof Let $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ and $B_R(x_0) \subset \Omega$. We may assume that $x_0 = 0$. In order to prove finiteness of the tail, it is sufficient to show that

$$\int_{\mathbb{R}^d \setminus B_R} f'\left(\frac{|u(y)|}{|y|^s}\right) \frac{\mathrm{d}y}{|y|^{d+s}} < \infty. \tag{3.7}$$

Since $B_R \subset \Omega$ and $|x-y| \le 2|y|$ for $x \in B_R$, $y \in B_R^c$, we have

$$\begin{aligned} \infty &> \Phi_{L^f(\Omega)}(u) + \Phi_{V^{s,f}(\Omega|\mathbb{R}^d)}(u) \\ &\ge \int_{B_R} f(|u(x)|)\,\mathrm{d}x + C(1-s) \iint_{B_R \times B_R^c} f\left(\frac{|u(x)-u(y)|}{|y|^s}\right) \frac{\mathrm{d}y\,\mathrm{d}x}{|y|^d} \end{aligned} \tag{3.8}$$

by using (f^q) and Lemma 2.1. Our aim is to estimate the right-hand side of (3.8) from below by the expression in (3.7). Note that by Lemmas 2.3, 2.2 and (f^q): $f(|u(x)|)(|y|^{-s} \vee |y|^{-sq}) \ge f(|y|^{-s}|u(x)|)$. Therefore,

$$\begin{aligned} f(|u(x)|)(|y|^{-s} \vee |y|^{-sq}) + f\left(\frac{|u(x)-u(y)|}{|y|^s}\right) &\ge Cf\left(\frac{|u(x)| + |u(x)-u(y)|}{|y|^s}\right) \\ &\ge Cf\left(\frac{|u(y)|}{|y|^s}\right), \end{aligned}$$

where we also used $f(t+s) \le 2^q(f(t)+f(s))$. This is a direct consequence of monotonicity and (f^q). Since $\int_{B_R^c} |y|^{-d}(|y|^{-s} \vee |y|^{-sq})\,\mathrm{d}y < \infty$, we estimate the right-hand side of (3.8) by

$$\iint_{B_R\times B_R^c} f(|u(x)|)\frac{\mathrm{d}y\,\mathrm{d}x}{|y|^d(|y|^s\wedge|y|^{sq})} + (1-s)\iint_{B_R\times B_R^c} f\left(\frac{|u(x)-u(y)|}{|y|^s}\right)\frac{\mathrm{d}y\,\mathrm{d}x}{|y|^d}$$
$$\geq C\iint_{B_R\times B_R^c} f\left(\frac{|u(y)|}{|y|^s}\right)\frac{\mathrm{d}y\,\mathrm{d}x}{|y|^d} \geq C\int_{B_R^c} f\left(\frac{|u(y)|}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^d}.$$

Altogether,

$$\int_{B_R^c} f\left(\frac{|u(y)|}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^d} < \infty. \tag{3.9}$$

Note that we have $(f^*(f'))(t) \leq (q-1)f(t)$ by (3.5) and (f^q). Therefore, we obtain

$$\int_{B_R^c} f'\left(\frac{|u(y)|}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}} \leq \int_{B_R^c} f^*\left(f'\left(\frac{|u(y)|}{|y|^s}\right)\right)\frac{\mathrm{d}y}{|y|^d} + \int_{B_R^c} f\left(\frac{1}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^d}$$
$$\leq C\int_{B_R^c} f\left(\frac{|u(y)|}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^d} + C,$$

where we used Fenchel's inequality (3.6). Combining the previous estimate with (3.9) finishes the proof. □

Having defined the fractional order Orlicz–Sobolev spaces, we are ready to introduce De Giorgi classes that are suitable to our setting.

Definition 3.3 Let $q > 1$, $c > 0$, $s \in (0,1)$, and let Ω be open. Let $f : [0,\infty) \mapsto [0,\infty)$ be convex and differentiable with $f(0) = 0$, $f(1) = 1$. We say that $u \in G_+(\Omega; q, c, s, f)$ if $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ and if for every $x_0 \in \Omega$, $0 < r < R \leq d(x_0, \partial\Omega)$, $k \in \mathbb{R}$, it holds

$$\begin{aligned}
&\Phi_{W^{s,f}(B_r(x_0))}(w_+) + (1-s)\int_{B_r(x_0)}\int_{A_k^-} f'\left(\frac{w_-(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}\\
&\leq c\left(\frac{R}{R-r}\right)^q \Phi_{L^f(B_R(x_0))}\left(\frac{w_+}{R^s}\right)\\
&\quad + c(1-s)\left(\frac{R}{R-r}\right)^{d+sq}\|w_+\|_{L^1(B_R(x_0))}\int_{\mathbb{R}^d\setminus B_r(x_0)} f'\left(\frac{w_+(y)}{|y-x_0|^s}\right)\frac{\mathrm{d}y}{|y-x_0|^{d+s}},
\end{aligned} \tag{3.10}$$

where $w_\pm(x) = (u(x)-k)_\pm$ and $A_k^- = \{y \in \mathbb{R}^d : u(y) < k\}$. We say that $u \in G_-(\Omega; q, c, s, f)$ if (3.10) holds true with w_+, w_- and A_k^- replaced by w_-, w_+ and $A_k^+ = \{y \in \mathbb{R}^d : u(y) > k\}$, respectively. Moreover, we denote by $G(\Omega; q, c, s, f) = G_+(\Omega; q, c, s, f) \cap G_-(\Omega; q, c, s, f)$.

The De Giorgi classes under consideration will contain minimizers of (1.1) and weak solutions for (1.7) under suitable additional assumptions on f, see Theorems 6.2 and 7.3.

The following proposition allows us to infer Hölder regularity of minimizers of $\mathcal{I}_f$ from regularity of functions in $G(\Omega; q, c, s, F(\cdot^p))$, where F is as in Proposition 2.6.

Proposition 3.4 *Let $1 < p \leq q$, $s \in (0,1)$ and $\Omega \subset \mathbb{R}^d$ be open. Let $f : [0,\infty) \mapsto [0,\infty)$ be convex and increasing with (f_p^q). Then for every $c_1 > 0$, there exists $c_2 = c_2(c_1, p, q) > 0$ such that $G_\pm(\Omega; q, c_1, s, f) \subset G_\pm(\Omega; q, c_2, s, F(\cdot^p))$.*

Proof The proof follows directly from (2.7) and (2.8). □

4 Hölder estimate

In this section, we prove Hölder estimates for functions in the De Giorgi class $G(\Omega; q, c, s, g)$, where g is the function given by (2.5). Let us first prove a growth lemma for functions in $G_-(\Omega; q, c, s, g)$.

Theorem 4.1 *Let $1 < p \le q$, $c, H > 0$, $R > 0$, $s_0 \in (0,1)$ and assume $s \in [s_0, 1)$. Let $f : [0,\infty) \to [0,\infty)$ be a convex increasing function satisfying (f_p^q). Suppose that $B_{4R} = B_{4R}(x_0) \subset \Omega$. Let $u \in G_-(\Omega; q, c, s, g)$ satisfy $u \ge 0$ in B_{4R} and*

$$|B_{2R} \cap \{u \ge H\}| \ge \gamma |B_{2R}| \tag{4.1}$$

for some $\gamma \in (0,1)$. There exists $\delta \in (0,1)$ such that if

$$\mathrm{Tail}_{f'}(u_-; x_0, 4R) \le \delta H, \tag{4.2}$$

then

$$u \ge \delta H \quad in\ B_R. \tag{4.3}$$

The constant δ depends only on d, s_0, p, q, c and γ.

Proof Within the proof we use $C > 0$ to denote a constant depending on d, s_0, p, q, c and γ and whose value might change from line to line. We may assume that $x_0 = 0$.

Let us assume

$$|B_{2R} \cap \{u < 2\delta H\}| \le \gamma_0 |B_{2R}| \tag{4.4}$$

for some $\gamma_0 \in (0, 2^{-d-1}]$. We first prove the assertion of the lemma under the assumption (4.4) and then verify (4.4) using (4.1).

Let $0 < \tilde\sigma := \max\{s_0/4, 2s-1\} < \sigma := \max\{s_0/2, (3s-1)/2\} < s$. Then, we have

$$1 - \tilde\sigma \le 2(1-s), \quad 1 - \sigma \le \frac{3}{2}(1-s) \quad \text{and} \quad \sigma - \tilde\sigma \ge C(1-s) \tag{4.5}$$

for some $C = C(s_0) > 0$. Indeed, for the last inequality in (4.5), we observe that

$$s_0/4 \ge 2s - 1 \implies \sigma - \tilde\sigma \ge \frac{s_0}{4} \ge \frac{s_0}{4(1-s_0)}(1-s) \quad \text{and}$$
$$s_0/4 < 2s - 1 \implies \sigma - \tilde\sigma \ge \frac{3s-1}{2} - (2s-1) = \frac{1}{2}(1-s).$$

Let $\delta \in (0, 1/8)$ to be determined later. Let $\delta H \le h < k \le 2\delta H$, $R \le \rho < \tau \le 2R$ and define $w_\pm = (u-k)_\pm$, $A^-_{k,R} = \{x \in B_R : u(x) < k\} = \mathrm{supp}(w_-) \cap B_R$. By (4.4) we have

$$\begin{aligned}|B_\rho \cap \{w_- = 0\}| &\ge |B_\rho| - |B_{2R} \cap \{u < 2\delta H\}| \ge |B_\rho| - \gamma_0 |B_{2R}| \\ &= \left(1 - \gamma_0 \left(\frac{2R}{\rho}\right)^d\right) |B_\rho| \ge \frac{1}{2}|B_\rho|.\end{aligned}$$

Thus, we apply Theorem 2.10 to w to obtain

$$(k-h)|A^-_{h,\rho}|^{\frac{d-\tilde\sigma}{d}} \le \left(\int_{B_\rho} w_-(x)^{\frac{d}{d-\tilde\sigma}}\,\mathrm{d}x\right)^{\frac{d-\tilde\sigma}{d}} \le C(1-\tilde\sigma)\int_{B_\rho}\int_{B_\rho} \frac{|w_-(x) - w_-(y)|}{|x-y|^{d+\tilde\sigma}}\,\mathrm{d}y\,\mathrm{d}x.$$

Moreover, by (4.5) and Lemma 2.9 we have

$$(k-h)|A^-_{h,\rho}|^{\frac{d-\tilde\sigma}{d}} \le C(1-s)\int_{A^-_{k,\rho}}\int_{B_\rho}\frac{|w_-(x)-w_-(y)|}{|x-y|^{d+\tilde\sigma}}\,\mathrm{d}y\,\mathrm{d}x$$
$$\le C\rho^{\sigma-\tilde\sigma}|A^-_{k,\tau}|^{\frac{p-1}{p}}\left((1-s)\int_{A^-_{k,\rho}}\int_{B_\rho}\frac{|w_-(x)-w_-(y)|^p}{|x-y|^{d+\sigma p}}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/p},$$

or equivalently,

$$\frac{(k-h)^p|A^-_{h,\rho}|^{\frac{d-\tilde\sigma}{d}p}}{C|A^-_{k,\tau}|^{p-1}\rho^{(\sigma-\tilde\sigma)p}\mu(A^-_{k,\rho}\times B_\rho)} \le \fint_{A^-_{k,\rho}\times B_\rho}\frac{|w_-(x)-w_-(y)|^p}{|x-y|^{sp}}\mu(\mathrm{d}X),$$

where $\mu(\mathrm{d}X)=(1-s)|x-y|^{-d+(s-\sigma)p}\,\mathrm{d}y\,\mathrm{d}x$. Since F is increasing and convex, Jensen's inequality yields

$$F\left(\frac{(k-h)^p|A^-_{h,\rho}|^{\frac{d-\tilde\sigma}{d}p}}{C|A^-_{k,\tau}|^{p-1}\rho^{(\sigma-\tilde\sigma)p}\mu(A^-_{k,\rho}\times B_\rho)}\right) \le \fint_{A^-_{k,\rho}\times B_\rho}F\left(\frac{|w_-(x)-w_-(y)|^p}{|x-y|^{sp}}\right)\mu(\mathrm{d}X)$$
$$\le \frac{C\rho^{(s-\sigma)p}}{\mu(A^-_{k,\rho}\times B_\rho)}\Phi_{W^{s,g}(B_\rho)}(w_-). \tag{4.6}$$

By definition of $G_-(\Omega; q, c, s, g)$, we have

$$\Phi_{W^{s,g}(B_\rho)}(w_-)+(1-s)\int_{B_\rho}\int_{A^+_k}g'\left(\frac{w_+(y)}{|x-y|^s}\right)\frac{w_-(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}$$
$$\le c\left(\frac{\tau}{\tau-\rho}\right)^q\Phi_{L^g(B_\tau)}\left(\frac{w_-}{\tau^s}\right)$$
$$+c(1-s)\left(\frac{\tau}{\tau-\rho}\right)^{d+sq}\|w_-\|_{L^1(B_\tau)}\int_{\mathbb{R}^d\setminus B_\rho}g'\left(\frac{w_-(y)}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}},$$

where $A^+_k=\{y\in\mathbb{R}^d : u(y)<k\}$. Let us estimate the right-hand side of this inequality. Using the assumptions that $u\ge 0$ in B_{4R} and the fact that F is increasing, we estimate $\|w_-\|_{L^1(B_\tau)}\le C|A^-_{k,\tau}|k$ and

$$\Phi_{L^g(B_\tau)}\left(\frac{w_-}{\tau^s}\right)\le C|A^-_{k,\tau}|F\left(\left(\frac{k}{R^s}\right)^p\right).$$

Moreover, using (2.8) and (2.4), we obtain

$$(1-s)\int_{\mathbb{R}^d\setminus B_\rho}g'\left(\frac{w_-(y)}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}}\le C(1-s)\int_{\mathbb{R}^d\setminus B_\rho}\left(f'\left(\frac{k}{R^s}\right)+f'\left(\frac{u_-(y)}{|y|^s}\right)\right)\frac{\mathrm{d}y}{|y|^{d+s}}.$$

The first term is controlled by $CR^{-s}f'(k/R^s)$. For the second term, we use (4.2), (3.2) and $\delta H<k$ to obtain

$$C(1-s)\int_{\mathbb{R}^d\setminus B_\rho}f'\left(\frac{u_-(y)}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}}\le\frac{C}{R^s}f'\left(\frac{\delta H}{R^s}\right)\le\frac{C}{R^s}f'\left(\frac{k}{R^s}\right). \tag{4.7}$$

Note that the integral over $B_{4R}\setminus B_\rho$ vanishes since $u_-=0$ in B_{4R} and $f'(0)=0$ by (f_p). Furthermore, we have

$$\frac{1}{R^s}f'\left(\frac{k}{R^s}\right)\le\frac{C}{k}f\left(\frac{k}{R^s}\right)\le\frac{C}{k}F\left(\left(\frac{k}{R^s}\right)^p\right)$$

by (f^q) and (2.7). Therefore, we have estimated

$$\begin{aligned} &\Phi_{W^{s,g}(B_\rho)}(w_-) + (1-s)\int_{B_\rho}\int_{A_k^+} g'\left(\frac{w_+(y)}{|x-y|^s}\right)\frac{w_-(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d} \\ &\le C\left(\frac{\tau}{\tau-\rho}\right)^{d+q}|A_{k,\tau}^-|F\left(\left(\frac{k}{R^s}\right)^p\right). \end{aligned} \tag{4.8}$$

Combining (4.6) and (4.8), we can find a constant $C>0$, depending on d, s_0, p, q, c and γ, such that

$$F\left(\frac{(k-h)^p|A_{h,\rho}^-|^{\frac{d-\tilde\sigma}{d}p}}{C|A_{k,\tau}^-|^{p-1}\rho^{(\sigma-\tilde\sigma)p}\mu(A_{k,\rho}^-\times B_\rho)}\right) \le \frac{C\rho^{(s-\sigma)p}|A_{k,\tau}^-|}{\mu(A_{k,\rho}^-\times B_\rho)}\left(\frac{\tau}{\tau-\rho}\right)^{d+q}F\left(\left(\frac{k}{R^s}\right)^p\right).$$

Using Lemma 2.5, we deduce that

$$(k-h)|A_{h,\rho}^-|^{\frac{d-\tilde\sigma}{d}} \le C\left(\frac{\tau}{\tau-\rho}\right)^{\frac{d+q}{p}}\frac{k}{R^{\tilde\sigma}}|A_{k,\tau}^-|. \tag{4.9}$$

Note that Lemma 2.5 is applicable because $\mu(A_{k,\rho}^-\times B_\rho)\le C|A_{k,\rho}^-|\rho^{(s-\sigma)p}$ and therefore

$$\frac{C\rho^{(s-\sigma)p}|A_{k,\tau}^-|}{\mu(A_{k,\rho}^-\times B_\rho)}\left(\frac{\tau}{\tau-\rho}\right)^{d+q} \ge \frac{C|A_{k,\tau}^-|}{|A_{k,\rho}^-|}\ge C.$$

We iterate inequality (4.9) with $k=k_j$, $h=k_{j+1}$, $\rho=R_{j+1}$, and $\tau=R_j$, where

$$R_j=(1+2^{-j})R \quad\text{and}\quad k_j=(1+2^{-j})\delta H$$

for $j\in\mathbb{N}\cup\{0\}$. Let $y_j=|A_{k_j,R_j}^-|/|B_{R_j}|$, then

$$\frac{\delta H}{2^{j+1}}\left(y_{j+1}|B_{R_{j+1}}|\right)^{\frac{d-\tilde\sigma}{d}} \le C2^{\frac{d+q}{p}j}\frac{\delta H}{R^{\tilde\sigma}}y_j|B_{R_j}|.$$

In other words, we have $y_{j+1}\le Cb^jy_j^{d/(d-\tilde\sigma)}\le Cb^j\max\{y_j^{1+\beta_1},y_j^{1+\beta_2}\}$, where

$$\beta_1=\frac{1}{d-1},\quad \beta_2=\frac{s_0/4}{d-s_0/4},\quad\text{and}\quad b=2^{(\frac{d+q}{p}+1)\frac{d}{d-1}}.$$

Thus, $y_j\to 0$ as $j\to\infty$, provided that $y_0\le\min\{C^{-1/\beta_2}b^{-1/\beta_2^2},C^{-1/\beta_1}b^{-1/\beta_1^2}\}$. See [15, Lemma 4.4]. By taking $\gamma_0\le\min\{C^{-1/\beta_2}b^{-1/\beta_2^2},C^{-1/\beta_1}b^{-1/\beta_1^2}\}$, we conclude (4.3) from (4.4).

Let us next prove (4.4) by contradiction using the assumption (4.1). Suppose that (4.4) does not hold, i.e.,

$$|B_{2R}\cap\{u<2\delta H\}|>\gamma_0|B_{2R}|. \tag{4.10}$$

Let $\bar s$ be the constant given in Proposition 2.12. We distinguish two cases $s\in[\bar s,1)$ and $s\in(0,\bar s)$. For the first case, we let l be the unique integer such that $2^{-l-1}\le\delta<2^{-l}$ and set $k_i=2^{-i}H$ for $i=0,1,\dots,l-1$. To apply Proposition 2.12 to $(u-k_{i-1})_-$ with $h=k_{i-1}-k_i$ and $k=k_{i-1}-k_{i+1}$, we check the following conditions: By (4.1) and (4.10)

$$|B_{2R}\cap\{(u-k_{i-1})_-\le h\}|=|B_{2R}\cap\{u\ge k_i\}|\ge|B_{2R}\cap\{u\ge H\}|\ge\gamma|B_{2R}| \tag{4.11}$$

and

$$|B_{2R}\cap\{(u-k_{i-1})_-\ge k\}|=|B_{2R}\cap\{u\le k_{i+1}\}|\ge|B_{2R}\cap\{u<2\delta H\}|>\gamma_0|B_{2R}|$$

for $i=1,\dots,l-2$. Moreover, we prove that there is a constant $C>0$ such that

$$\|(u-k_{i-1})_-\|^p_{L^p(B_R)}+(1-\sigma)R^{\sigma p}[(u-k_{i-1})_-]^p_{W^{\sigma,p}(B_R)}\le CR^d(k_i-k_{i+1})^p.$$

Indeed, it follows from $u\ge 0$ in B_{4R} that

$$\|(u-k_{i-1})_-\|^p_{L^p(B_R)}\le CR^dk^p_{i-1}.$$

The estimate

$$(1-\sigma)R^{\sigma p}[(u-k_{i-1})_-]^p_{W^{\sigma,p}(B_R)}\le CR^dk^p_{i-1} \tag{4.12}$$

follows from (4.5) and the computation

$$\begin{aligned}
&F\left(\fint_{B_R\times B_R}\frac{|(u(x)-k_{i-1})_--(u(y)-k_{i-1})_-|^p}{|x-y|^{sp}}\,\mathrm{d}\mu(X)\right)\\
&\le\frac{R^{(s-\sigma)p}}{\mu(B_R\times B_R)}(1-s)\int_{B_R}\int_{B_R}F\left(\frac{|(u(x)-k_{i-1})_--(u(y)-k_{i-1})_-|^p}{|x-y|^{sp}}\right)\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}\\
&\le C\frac{R^{(s-\sigma)p}}{\mu(B_R\times B_R)}|B_{2R}\cap\{u\le k_{i-1}\}|F\left(\left(\frac{k_{i-1}}{R^s}\right)^p\right),
\end{aligned} \tag{4.13}$$

which can be obtained along the lines of the first part of this proof. Estimating $|B_{2R}\cap\{u\le k_{i-1}\}|\le CR^d$ and applying Lemma 2.5, we obtain (4.12), as desired.

Therefore, by applying Proposition 2.12 and using (4.11) we have

$$(k_i-k_{i+1})^p|B_{2R}\cap\{u\le k_{i+1}\}|^{\frac{d-1}{d}p}\le C(1-\sigma)R^{(-1+\sigma)p}[(u-k_{i-1})_-]^p_{W^{\sigma,p}(B_{2R})}|D_i|^{p-1},$$

where $D_i=B_{2R}\cap\{h\le(u-k_{i-1})_-<k\}=B_{2R}\cap\{k_{i+1}<u\le k_i\}$. Using (4.12) and $k_{i+1}\ge 2\delta H$, we obtain

$$|B_{2R}\cap\{u\le 2\delta H\}|^{\frac{d-1}{d}\frac{p}{p-1}}\le CR^{\frac{d-p}{p-1}}|D_i|.$$

We sum up this inequality over $i=1,\dots,l-2$ to derive

$$(l-2)|B_{2R}\cap\{u\le 2\delta H\}|^{\frac{d-1}{d}\frac{p}{p-1}}\le CR^{\frac{d-p}{p-1}}R^d,$$

from which we conclude by definition of l

$$|B_{2R}\cap\{u\le 2\delta H\}|\le C|B_{2R}||\log\delta|^{-\frac{d}{d-1}\frac{p-1}{p}}.$$

Therefore, we arrive at a contradiction by using (4.10) and taking δ sufficiently small. The proof for the case $s\in[\bar{s},1)$ is finished.

For the case $s\in(0,\bar{s})$, we use the estimate (4.8) with $k=4\delta H$ to obtain

$$\begin{aligned}
CR^dF\left(\left(\frac{4\delta H}{R^s}\right)^p\right)&\ge(1-\bar{s})\int_{B_{2R}}\int_{A^+_{4\delta H,2R}}g'\left(\frac{w_+(y)}{|x-y|^s}\right)\frac{w_-(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}\\
&\ge\frac{C}{R^{d+s}}\left(\int_{B_{2R}\cap\{u\ge H\}}g'\left(\frac{u(y)-4\delta H}{(2R)^s}\right)\mathrm{d}y\right)\left(\int_{B_{2R}\cap\{u<2\delta H\}}(4\delta H-u(x))\,\mathrm{d}x\right)\\
&\ge\frac{C}{R^d}2\delta\frac{H}{4R^s}g'\left(\frac{H}{4R^s}\right)|B_{2R}\cap\{u\ge H\}||B_{2R}\cap\{u<2\delta H\}|.
\end{aligned}$$

By (4.1) and (4.10), we obtain

$$\delta F\left(\left(\frac{H}{4R^s}\right)^p\right) \le CF\left(\left(\frac{4\delta H}{R^s}\right)^p\right),$$

where we also used (2.8), (f_p) and (2.7). Therefore, by Lemma 2.5 we obtain $\delta \le C\delta^p$. Since $p > 1$, we arrive at a contradiction by taking δ sufficiently small. □

Using Theorem 4.1, we prove Hölder estimates for functions in $G(\Omega; q, c, s, g)$.

Theorem 4.2 *Let $1 < p \le q$, $c > 0$, $s_0 \in (0, 1)$, and assume $s \in [s_0, 1)$. Let $f : [0, \infty) \to [0, \infty)$ be a convex increasing function satisfying (f_p^q). Then, there exist $\alpha \in (0, 1)$ and $C > 0$, depending on d, s_0, p, q and c, such that for every $u \in G(\Omega; q, c, s, g)$ and any $B_{8R}(x_0) \subset \Omega$,*

$$R^\alpha [u]_{C^\alpha(\overline{B_R(x_0)})} \le C\|u\|_{L^\infty(B_{4R}(x_0))} + \mathrm{Tail}_{f'}(u; x_0, 4R).$$

Proof Let $B_{8R}(x_0) \subset \Omega$. We may assume that $x_0 = 0$ and that $\|u\|_{L^\infty(B_{4R})} < \infty$. The idea of the proof is to find a small constant $\alpha \in (0, 1)$ and to construct a non-increasing sequence (M_j) and a non-decreasing sequence (m_j) satisfying

$$m_j \le u \le M_j \quad \text{in } B_{4R_j} \quad \text{and} \quad M_j - m_j = L4^{-\alpha j}, \tag{4.14}$$

for all $j \ge 0$, where $R_j = 4^{-j}R$ and

$$L = C_0\|u\|_{L^\infty(B_{4R})} + \mathrm{Tail}_{f'}(u; 0, 4R)$$

for some $C_0 > 0$. Once we construct such sequences, the desired result follows by a standard argument.

We set $M_j = 4^{-\alpha j}L/2$ and $m_j = -4^{-\alpha j}L/2$ for $j = 0, 1, \dots, j_0$ for some $j_0 \in \mathbb{N}$ to be determined later. Moreover, we take C_0 sufficiently large so that $C_0 \ge 2 \cdot 4^{\alpha j_0}$. This ensures that M_j and m_j satisfy (4.14) up to j_0. Let us now fix $j \ge j_0$ and suppose that the sequences (M_j) and (m_j) have been constructed up to j. It is enough to construct M_{j+1} and m_{j+1} satisfying (4.14). We first assume

$$|B_{2R_j} \cap \{u \ge m_j + (M_j - m_j)/2\}| \ge \frac{1}{2}|B_{2R_j}|. \tag{4.15}$$

In this case, we define $v = u - m_j$ and set $H = (M_j - m_j)/2 = 4^{-\alpha j}L/2$. Then, $0 \le v \le 2H$ in B_{4R_j} and

$$|B_{2R_j} \cap \{v \ge H\}| \ge \frac{1}{2}|B_{2R_j}|.$$

To apply Theorem 4.1 to v, we let δ be the constant in Theorem 4.1 and verify (4.2). Indeed, it is easy to see that

$$v(y) \ge -2H\left(\left(\frac{|y|}{R_j}\right)^\alpha - 1\right)$$

for $y \in B_{4R} \setminus B_{4R_j}$ and $v(y) \ge -|u(y)| - L/2$ for $y \in \mathbb{R}^d \setminus B_{4R}$. Thus, using (2.4)

$$\begin{aligned}
&\int_{\mathbb{R}^d \setminus B_{4R_j}} f'\left(\frac{v_-(y)}{|y|^s}\right)|y|^{-d-s}\,\mathrm{d}y \\
&\le C\int_{B_{4R}\setminus B_{4R_j}} f'\left(\frac{2H((|y|/R_j)^\alpha - 1)}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}} + C\int_{\mathbb{R}^d\setminus B_{4R}} f'\left(\frac{|u(y)| + L/2}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}} \\
&=: J_1 + J_2.
\end{aligned}$$

Using the change of variables, we obtain

$$J_1 \le \frac{C}{R_j^s}\int_{\mathbb{R}^d\setminus B_4} f'\left(\frac{2H(|y|^\alpha-1)}{R_j^s|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}} \le \frac{C}{R_j^s}\int_4^\infty f'\left(\frac{2H(\rho^\alpha-1)}{R_j^s\rho^s}\right)\frac{\mathrm{d}\rho}{\rho^{1+s}}.$$

By (2.2) and (2.3) we have

$$f'\left(\frac{2H(\rho^\alpha-1)}{R_j^s\rho^s}\right) \le \frac{q}{p}\max\left\{\left(\frac{8(\rho^\alpha-1)}{\delta\rho^s}\right)^{q-1}, \left(\frac{8(\rho^\alpha-1)}{\delta\rho^s}\right)^{p-1}\right\} f'\left(\frac{\delta H}{(4R_j)^s}\right),$$

and hence

$$J_1 \le \frac{C}{(4R_j)^s}\left(\int_4^\infty \frac{(\rho^\alpha-1)^{q-1}}{\rho^{1+s_0p}}\,\mathrm{d}\rho\right) f'\left(\frac{\delta H}{(4R_j)^s}\right).$$

Taking $\alpha = \alpha(d, s_0, p, q) \in (0,1)$ sufficiently small so that

$$C\int_4^\infty \frac{(\rho^\alpha-1)^{q-1}}{\rho^{1+s_0p}}\,\mathrm{d}\rho \le \frac{1}{2}, \tag{4.16}$$

we obtain

$$J_1 \le \frac{1}{2}(4R_j)^{-s} f'\left(\frac{\delta H}{(4R_j)^s}\right).$$

For J_2, we use (2.4) to deduce

$$J_2 \le C\left(\int_{\mathbb{R}^d\setminus B_{4R}} f'\left(\frac{|u(y)|}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}} + \int_{\mathbb{R}^d\setminus B_{4R}} f'\left(\frac{L/2}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}}\right).$$

Since $L \ge \mathrm{Tail}_{f'}(u; 0, 4R)$, it follows from (3.2) and the definition of H

$$\int_{\mathbb{R}^d\setminus B_{4R}} f'\left(\frac{|u(y)|}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}} \le R^{-s} f'\left(\frac{L}{R^s}\right) = R^{-s} f'\left(\frac{2H4^{\alpha j}}{R^s}\right).$$

Choosing j_0 sufficiently large so that $8\cdot 4^{(\alpha-s_0)j_0} \le \widetilde{\delta}$, for some $\widetilde{\delta} < \delta$ to be determined later, we have

$$R^{-s} f'\left(\frac{2H4^{\alpha j}}{R^s}\right) \le (4R_j)^{-s} f'\left(\frac{\widetilde{\delta} H}{(4R_j)^s}\right).$$

Similarly, by (2.3)

$$\begin{aligned}\int_{\mathbb{R}^d\setminus B_{4R}} f'\left(\frac{L/2}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}} &= \frac{1}{R^s}\int_{\mathbb{R}^d\setminus B_4} f'\left(\frac{L/2}{R^s|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}}\\ &\le C_1\left(\int_{\mathbb{R}^d\setminus B_4} |y|^{-s(p-1)}\frac{\mathrm{d}y}{|y|^{d+s}}\right)\frac{1}{R^s} f'\left(\frac{L}{R^s}\right)\\ &\le C_2(4R_j)^{-s} f'\left(\frac{\widetilde{\delta} H}{(4R_j)^s}\right)\end{aligned}$$

for some $C_1, C_2 \ge 1$ depending on d, s_0 and p. We now choose $\widetilde{\delta} = (2^{q+1}C_2)^{-\frac{1}{p-1}}\delta > 0$, and obtain:

$$(1-s)\int_{\mathbb{R}^d\setminus B_{4R_j}} f'\left(\frac{v_-(y)}{|y|^s}\right)|y|^{-d-s}\,\mathrm{d}y \le J_1 + J_2 \le (4R_j)^{-s} f'\left(\frac{\delta H}{(4R_j)^s}\right).$$

This inequality together with (3.3) verify (4.2) and allow us to apply Theorem 4.1 to v. Therefore, we obtain $v \geq \delta H$ in B_{R_j}, which implies

$$u \geq m_j + \delta H = m_j + 4^{-\alpha j}\frac{\delta}{2}L \geq m_j + 4^{-\alpha j}(1-4^{-\alpha})L \quad \text{in } B_{R_j},$$

upon choosing $\alpha \in (0,1)$ so small that it satisfies (4.16) and $\alpha < \log_4\left(\frac{2}{2-\delta}\right)$.

Therefore, we define $M_{j+1} = M_j$ and $m_{j+1} = m_j + 4^{-\alpha j}(1-4^{-\alpha})L$ in the case (4.15). The other case can be proved in a similar way. □

5 Local boundedness

The goal of this section is to prove local boundedness of functions $u \in G(\Omega; q, c, s, f)$. More precisely, we prove that a function $u \in G_+(\Omega; q, c, s, f)$ is locally bounded from above. Similarly, one can prove that functions $u \in G_-(\Omega; q, c, s, f)$ are locally bounded from below by considering $-u$.

Theorem 5.1 *Let $1 \leq p \leq q < p^*$, $s_0 \in (0,1)$, $c_0, c_1 > 0$ and assume $s \in [s_0, 1)$. Let $f : [0,\infty) \to [0,\infty)$ be a convex increasing function satisfying (f^q) and $(f \gtrsim t^p)$. If $u \in G_+(\Omega; q, c_1, s, f)$, then u is locally bounded from above. Moreover, for each $B_{2R}(x_0) \subset \Omega$, there exists $C > 0$ such that for every $\delta \in (0,1)$*

$$\sup_{B_R(x_0)} u \leq \delta \mathrm{Tail}_{f'}(u_+; x_0, R) + C\delta^{-(q-1)\frac{p^*}{p}\frac{1}{p^*-q}}\left(\fint_{B_{2R}(x_0)} u_+^q(x)\,\mathrm{d}x\right)^{\frac{1}{p}\frac{p^*-p}{p^*-q}} + \delta^{\frac{q-1}{q}}.$$

The constant C depends on d, s_0, p, q, p^*-q, c_0, c_1 and R.

Note that $p=1$ is allowed in the proof of local boundedness.

Proof Let $x_0 \in \Omega$ and $R > 0$ be such that $B_{2R}(x_0) \subset \Omega$. We assume without loss of generality that $x_0 = 0$. For $j = 0, 1, \dots$, let

$$R_j = (1+2^{-j})R, \quad k_j = (1-2^{-j})k, \quad \text{and} \quad \tilde{k}_j = (k_j + k_{j+1})/2,$$

where k is an arbitrary positive number that will be determined later. We define $w_j = (u-k_j)_+$, $\tilde{w}_j = (u-\tilde{k}_j)_+$, $A^+_{k,R} = \{x \in B_R : u(x) > k\}$, and

$$Y_j = \fint_{B_{R_j}} w_j^q(x)\,\mathrm{d}x.$$

Since $u \in G_+(\Omega; q, c_1, s, f)$, using the assumptions (f^q) and $(f \gtrsim t^p)$ we have

$$\begin{aligned}(1-s)[\tilde{w}_j]^p_{W^{s,p}(B_{R_{j+1}})} &\leq C2^{qj}\left(\int_{A^+_{\tilde{k}_j,R_j}}\left(\frac{\tilde{w}_j(x)}{R^s}\right)^q \mathrm{d}x + |A^+_{\tilde{k}_j,R_j}|\right) \\ &\quad + C(1-s)2^{(d+sq)j}\|\tilde{w}_j\|_{L^1(B_{R_j})}\int_{\mathbb{R}^d\setminus B_{R_{j+1}}} f'\left(\frac{\tilde{w}_j(y)}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}} \\ &=: J_1 + J_2\end{aligned}$$

for some $C = C(d, q, c_0, c_1) > 0$. Since

$$|A^+_{\tilde{k}_j,R_j}| \le \frac{1}{(\tilde{k}_j - k_j)^q} \int_{A^+_{\tilde{k}_j,R_j}} w_j^q(x)\,\mathrm{d}x \le C\left(\frac{2^j}{k}\right)^q Y_j$$

and $\tilde{w}_j \le w_j$, by assuming $k \ge \delta^{\frac{q-1}{q}}$ we have

$$J_1 \le C\delta^{-(q-1)} 2^{2qj} Y_j$$

for some $C = C(d, q, c_0, c_1, R) > 0$. For J_2, we observe

$$(2^{-j-2}k)^{q-1}\tilde{w}_j = (\tilde{k}_j - k_j)^{q-1}\tilde{w}_j \le w_j^q$$

and $\mathrm{Tail}_{f'}(\tilde{w}_j; 0, R_{j+1}) \le \mathrm{Tail}_{f'}(u_+; 0, R)$. Fix $\delta \in (0, 1)$ and assume

$$k \ge \delta \mathrm{Tail}_{f'}(u_+; 0, R) + \delta^{\frac{q-1}{q}}. \tag{5.1}$$

Then, using (5.1), (3.2), (1.2) and $f(1) = 1$, we deduce

$$\begin{aligned}
J_2 &\le C2^{(d+sq)j} \frac{R_j^d}{(2^{-j-2}k)^{q-1}} Y_j \frac{1}{R^s_{j+1}} f'\left(\frac{k}{\delta R^s_{j+1}}\right) \\
&\le C2^{(d+sq)j} \frac{2^{(q-1)j}}{k^{q-1}} \frac{(1 + (\frac{k}{\delta R^s})^q)}{\frac{k}{\delta R^s}} Y_j \le C2^{(d+2q)j}\delta^{-(q-1)} Y_j
\end{aligned}$$

for some constant $C = C(d, q, c_0, c_1, s_0) > 0$. Combining the estimates of J_1 and J_2, we arrive at

$$(1 - s)[\tilde{w}_j]^p_{W^{s,p}(B_{R_{j+1}})} \le C2^{(d+2q)j}\delta^{-(q-1)} Y_j.$$

Using Theorem 2.11 and the inequality $\tilde{w}_j^{p^*} \ge (k_{j+1} - \tilde{k}_j)^{p^*-q} w^q_{j+1}$, we deduce

$$\begin{aligned}
(k_{j+1} - \tilde{k}_j)^{(p^*-q)p/p^*} Y_{j+1}^{p/p^*} &\le C\|\tilde{w}_j\|^p_{L^{p^*}(B_{R_{j+1}})} \\
&\le C\left(\|\tilde{w}_j\|^p_{L^p(B_{R_{j+1}})} + (1 - s)[\tilde{w}_j]^p_{W^{s,p}(B_{R_{j+1}})}\right) \\
&\le C2^{(d+2q)j}\delta^{-(q-1)} Y_j,
\end{aligned}$$

or

$$Y_{j+1} \le Ck^{q-p^*}\delta^{-(q-1)p^*/p} b^j Y_j^{1+\beta},$$

where $b = 2^{p^*-q+(d+2q)p^*/p}$ and $\beta = \frac{p^*}{p} - 1$. If $Y_0 \le (Ck^{q-p^*}\delta^{-(q-1)p^*/p})^{-1/\beta} b^{-1/\beta^2}$, then $Y_j \to 0$ as $j \to \infty$. Thus, if we assume

$$k^{p^*-q} \ge C\delta^{-(q-1)\frac{p^*}{p}} b^{1/\beta} Y_0^\beta, \tag{5.2}$$

then

$$\sup_{B_R} u \le k.$$

We now take

$$k = \delta \mathrm{Tail}_{f'}(u_+; 0, R) + C_0\delta^{-(q-1)\frac{p^*}{p}\frac{1}{p^*-q}} \left(\fint_{B_{2R}} u^q_+(x)\,\mathrm{d}x\right)^{\frac{1}{p}\frac{p^*-p}{p^*-q}} + \delta^{\frac{q-1}{q}}$$

with $C_0 = (Cb^{1/\beta})^{1/(p^*-q)}$, which is in accordance with (5.1) and (5.2). □

6 Application to minimizers

In this section, we prove Theorem 1.1 by showing that local minimizers of (1.1) belong to the De Giorgi class $G(\Omega; q, c, s, f)$ under some assumptions on f and using several results from previous sections.

Let us first define local minimizers of (1.1). We assume that f is a convex increasing function and k satisfies (k), i.e.,

$$k(x, y) = k(y, x) \quad \text{and} \quad \Lambda^{-1} \le k(x, y) \le \Lambda \quad \text{for a.e. } x, y \in \mathbb{R}^d. \tag{k}$$

Definition 6.1 [minimizer] We say that $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ is a *local subminimizer* (*superminimizer*) of (1.1) if for every measurable function $v : \mathbb{R}^d \to \mathbb{R}$ with $v = u$ a.e. in $\mathbb{R}^d \setminus \Omega$ and $v \le u$ ($v \ge u$) a.e. in Ω, it holds that $\mathcal{I}_f(u) \le \mathcal{I}_f(v)$. We call $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ a *local minimizer* of (1.1) if it is a subminimizer and a superminimizer.

Recall that we always assume $f(0) = 0$ and $f(1) = 1$.

Theorem 6.2 *Let $s \in (0, 1)$, $q \ge 1$ and $\Lambda \ge 1$. Let $f : [0, \infty) \to [0, \infty)$ be a convex increasing function satisfying (f^q) and let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function satisfying (K). Let $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ be a local subminimizer of (1.1). Then, $u \in G_+(\Omega; q, c, s, f)$ for some $c = c(d, q, \Lambda) > 0$.*

It is worth emphasizing that the result of the theorem is true in the case $q = 1$.

Proof We follow the strategy carried out in [17, Proposition 7.5]. Let $x_0 \in \Omega$, $0 < r < R \le d(x_0, \partial\Omega)$ and $k \in \mathbb{R}$. Without loss of generality, we take $x_0 = 0$. Let $r \le \rho < \tau \le R$ and let $\eta \in C_c^\infty(\mathbb{R}^d)$ be a cutoff function with $0 \le \eta \le 1$, $\operatorname{supp}(\eta) = B_{\frac{\tau+\rho}{2}}$, $\eta \equiv 1$ in B_ρ, and $\|\nabla\eta\|_\infty \le \frac{4}{\tau-\rho}$. Let $w_\pm(x)$ and $A_k^\pm$ be as in Definition 3.3. We define $A_{k,R}^+ = \{x \in B_R : u(x) > k\}$ and $A_{k,R}^- = \{x \in B_R : u(x) < k\}$.

We set $v := u - \eta^q w_+$, then $u \equiv v$ in $\mathbb{R}^d \setminus B_\tau$ and $u \ge v$ a.e. in $\mathbb{R}^d$. Since u is a local subminimizer of (1.1), it holds that $\mathcal{I}_f(u) \le \mathcal{I}_f(v)$, i.e.,

$$0 \le (1-s) \int_{B_\tau} \int_{\mathbb{R}^d} A(x, y) \frac{k(x, y)}{|x-y|^d} \, dy \, dx, \tag{6.1}$$

where

$$A(x, y) = f\left(\frac{|v(x) - v(y)|}{|x-y|^s}\right) - f\left(\frac{|u(x) - u(y)|}{|x-y|^s}\right).$$

To estimate $A(x, y)$, we distinguish four different cases and prove the following.

If $x \in A_k^-$ or $y \in A_k^-$, then

$$A(x, y) \le 0. \tag{6.2}$$

Furthermore, if $x, y \in A_{k,\rho}^+$, then

$$A(x, y) = -f\left(\frac{|w_+(x) - w_+(y)|}{|x-y|^s}\right). \tag{6.3}$$

If $x \in A^+_{k,\rho}$ and $y \in A^-_k$, then

$$A(x,y) \le -\frac{1}{2}\left[f'\left(\frac{w_-(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s} + f\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)\right]. \tag{6.4}$$

Finally, if $x, y \in A^+_k$, then we have

$$A(x,y) \le f\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right) + cf\left(\frac{|\eta(x)-\eta(y)|(w_+(x)\vee w_+(y))}{|x-y|^s}\right) \tag{6.5}$$

for some $c = c(q) > 0$, where $a \vee b := \max\{a, b\}$. In the following, we prove (6.2)–(6.5). The proof of (6.2) is a direct consequence of monotonicity of f. Namely, if $x \notin A^+_k$ and $y \in A^+_k$, then

$$|v(x)-v(y)| = |(1-\eta^q(y))w_+(y) + w_-(x)| \le |w_+(y) + w_-(x)| = |u(x)-u(y)|.$$

By symmetry, one can treat the case $x \in A^+_k$ and $y \notin A^+_k$. The remaining case is trivial.

To see (6.3), observe that for $x, y \in A^+_{k,\rho}$ it holds that $\eta(x) = \eta(y) = 1$, and therefore

$$|v(x)-v(y)| = |u(x)-u(y)-w_+(x)+w_+(y)| = 0.$$

Let us prove (6.4). For $x \in A^+_{k,\rho}$ and $y \notin A^+_k$ it holds that

$$|v(x)-v(y)| = |(1-\eta^q(x))w_+(x) + w_-(y)| = w_-(y),$$
$$|u(x)-u(y)| = w_+(x) + w_-(y).$$

By application of Lemma 2.7 with $\theta = \frac{1}{2}$, $a = \frac{w_-(y)}{|x-y|^s}$ and $b = \frac{w_+(x)}{|x-y|^s}$, we obtain

$$A(x,y) \le -\frac{1}{2}\left[f'\left(\frac{w_-(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s} + f\left(\frac{w_+(x)}{|x-y|^s}\right)\right],$$

which implies (6.4) since $w_+(y) = 0$.

To prove (6.5) let us take $x, y \in A^+_k$. We compute

$$\begin{aligned}|v(x)-v(y)| &= |(1-\eta^q(x))w_+(x) - (1-\eta^q(y))w_+(y)| \\ &= |(1-\eta^q(x))(w_+(x)-w_+(y)) + (\eta^q(y)-\eta^q(x))w_+(y)|.\end{aligned}$$

Let us assume without loss of generality that $\eta(x) \ge \eta(y)$. Then, we have $|\eta^q(y)-\eta^q(x)| \le q\eta^{q-1}(x)|\eta(y)-\eta(x)|$. We estimate, using monotonicity and convexity of f, as well as (f^q)

$$\begin{aligned}f\left(\frac{|v(x)-v(y)|}{|x-y|^s}\right) &\le f\left((1-\eta^q(x))\frac{|w_+(x)-w_+(y)|}{|x-y|^s} + \eta^q(x)\frac{qw_+(y)|\eta(y)-\eta(x)|}{\eta(x)|x-y|^s}\right) \\ &\le (1-\eta^q(x))f\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right) + \eta^q(x)f\left(\frac{qw_+(y)|\eta(y)-\eta(x)|}{\eta(x)|x-y|^s}\right) \\ &\le f\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right) + q^q f\left(\frac{|\eta(y)-\eta(x)|}{|x-y|^s}w_+(y)\right),\end{aligned}$$

which implies (6.5).

By putting together the information from (6.2)–(6.4) and using assumptions on k, we deduce

$$\begin{aligned}&(1-s)\int_{B_\rho}\int_{B_\rho} A(x,y)\frac{k(x,y)}{|x-y|^d}\,\mathrm{d}y\,\mathrm{d}x \\ &\le -\frac{1}{2\Lambda}\Phi_{W^{s,f}(B_\rho)}(w_+) - \frac{1}{2\Lambda}(1-s)\int_{B_\rho}\int_{A^-_{\rho,k}} f'\left(\frac{w_-(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}.\end{aligned} \tag{6.6}$$

Moreover, from (6.2), (6.4) and (6.5) we obtain

$$\begin{aligned}
&(1-s)\iint_{B_\tau^2\setminus B_\rho^2} A(x,y)\frac{k(x,y)}{|x-y|^d}\,\mathrm{d}y\,\mathrm{d}x \\
&\le \Lambda(1-s)\iint_{B_\tau^2\setminus B_\rho^2} f\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d} \\
&\quad + c\Lambda(1-s)\iint_{B_\tau^2\setminus B_\rho^2} f\left(\frac{|\eta(x)-\eta(y)|(w_+(y)\vee w_+(x))}{|x-y|^s}\right)\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d} \\
&\quad - \frac{1}{2\Lambda}(1-s)\int_{B_\rho}\int_{(B_\tau\setminus B_\rho)\cap A_k^-} f'\left(\frac{w_-(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}.
\end{aligned} \tag{6.7}$$

Note that by monotonicity of f and Lemma 2.1, we have

$$\begin{aligned}
&(1-s)\int_{B_R}\int_{B_R} f\left(\frac{|\eta(x)-\eta(y)|}{|x-y|^s}w_+(x)\right)|x-y|^{-d}\,\mathrm{d}y\,\mathrm{d}x \\
&\le (1-s)\int_{B_R}\int_{B_{2R}(x)} f\left(\frac{4|x-y|^{1-s}}{\tau-\rho}w_+(x)\right)|x-y|^{-d}\,\mathrm{d}y\,\mathrm{d}x \\
&\le (1-s)\left(\frac{4R}{\tau-\rho}\right)^q\int_{B_R}\sum_{k=0}^{\infty}\int_{B_{2^{-k+1}R}(x)\setminus B_{2^{-k}R}(x)} f\left((2^{-k+1})^{1-s}\frac{w_+(x)}{R^s}\right)(2^{-k}R)^{-d}\,\mathrm{d}y\,\mathrm{d}x \\
&\le c(1-s)\left(\frac{R}{\tau-\rho}\right)^q\int_{B_R}\sum_{k=0}^{\infty} f\left(2^{-k(1-s)}\frac{w_+(x)}{R^s}\right)\mathrm{d}x
\end{aligned}$$

for some $c=c(d,q)>0$. We use Lemmas 2.3 and 2.2 to obtain

$$\begin{aligned}
(1-s)\sum_{k=0}^{\infty} f\left(2^{-k(1-s)}\frac{w_+(x)}{R^s}\right) &\le (1-s)\sum_{k=0}^{\infty}2^{-k(1-s)} f\left(\frac{w_+(x)}{R^s}\right) \\
&\le \frac{1-s}{1-2^{-(1-s)}} f\left(\frac{w_+(x)}{R^s}\right).
\end{aligned}$$

Since the map $s\mapsto (1-s)/(1-2^{-(1-s)})$ is bounded on $(0,1)$ from above, we have

$$\begin{aligned}
&(1-s)\int_{B_R}\int_{B_R} f\left(\frac{|\eta(x)-\eta(y)|}{|x-y|^s}w_+(x)\right)|x-y|^{-d}\,\mathrm{d}y\,\mathrm{d}x \\
&\quad \le c\left(\frac{R}{\tau-\rho}\right)^q \Phi_{L^f(B_R)}\left(\frac{w_+}{R^s}\right)
\end{aligned} \tag{6.8}$$

for some $c=c(d,q)>0$.

By combination of the estimates (6.6), (6.7) and (6.8), we get:

$$\begin{aligned}
&(1-s)\int_{B_\tau}\int_{B_\tau} A(x,y)\frac{k(x,y)}{|x-y|^d}\,\mathrm{d}y\,\mathrm{d}x \\
&\le c(1-s)\iint_{B_\tau^2\setminus B_\rho^2} f\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d} + c\left(\frac{R}{\tau-\rho}\right)^q\Phi_{L^f(B_R)}\left(\frac{w_+}{R^s}\right) \\
&\quad - c\Phi_{W^{s,f}(B_\rho)}(w_+) - c(1-s)\int_{B_\rho}\int_{A_{\tau,k}^-} f'\left(\frac{w_-(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}.
\end{aligned} \tag{6.9}$$

Let us deduce two more estimates for $A(x,y)$:

$$A(x,y)=0 \quad \text{for } x,y\notin A^+_{k,\frac{\tau+\rho}{2}}, \tag{6.10}$$

$$A(x,y)\le f'\left(\frac{w_+(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s} \quad \text{for } x\in A^+_k \text{ and } y\in A^+_k\setminus B_\tau. \tag{6.11}$$

The proof of (6.10) is trivial since $\operatorname{supp}(\eta)=B_{\frac{\tau+\rho}{2}}$ and $w_+=0$ on A^-_k. To see (6.11), we compute

$$A(x,y)=f\left(\frac{|(1-\eta^q(x))w_+(x)-w_+(y)|}{|x-y|^s}\right)-f\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)$$

and apply Lemma 2.8 with $\mu=1-\eta^q(x)$, $a=\frac{w_+(x)}{|x-y|^s}$ and $b=\frac{w_+(y)}{|x-y|^s}$. Consequently, by (6.2), (6.4), (6.10) and (6.11), it holds that

$$\begin{aligned}
&(1-s)\int_{B_\tau}\int_{\mathbb{R}^d\setminus B_\tau}A(x,y)\frac{k(x,y)}{|x-y|^d}\,\mathrm{d}y\,\mathrm{d}x\\
&\le \Lambda(1-s)\int_{B_{\frac{\tau+\rho}{2}}}\int_{\mathbb{R}^d\setminus B_\tau}f'\left(\frac{w_+(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}\\
&\quad-\frac{1}{2\Lambda}(1-s)\int_{B_\rho}\int_{(\mathbb{R}^d\setminus B_\tau)\cap A^-_k}f'\left(\frac{w_-(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}.
\end{aligned} \tag{6.12}$$

Moreover, using Lemma 2.3 and (2.2) with $p=1$ we observe that

$$\begin{aligned}
&(1-s)\int_{B_{\frac{\tau+\rho}{2}}}\int_{\mathbb{R}^d\setminus B_\tau}f'\left(\frac{w_+(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}\\
&\le q(1-s)\left(\frac{2R}{\tau-\rho}\right)^{d+sq}\int_{B_{\frac{\tau+\rho}{2}}}\int_{\mathbb{R}^d\setminus B_\tau}f'\left(\frac{w_+(y)}{|y|^s}\right)\frac{w_+(x)}{|y|^s}|y|^{-d}\,\mathrm{d}y\,\mathrm{d}x\\
&\le c(1-s)\left(\frac{R}{\tau-\rho}\right)^{d+sq}\|w_+\|_{L^1(B_R)}\int_{\mathbb{R}^d\setminus B_r}f'\left(\frac{w_+(y)}{|y|^s}\right)|y|^{-d-s}\,\mathrm{d}y.
\end{aligned} \tag{6.13}$$

By combining estimates (6.1), (6.9), (6.12), and (6.13), we derive

$$\begin{aligned}
&\Phi_{W^{s,f}(B_\rho)}(w_+)+(1-s)\int_{B_\rho}\int_{A^-_k}f'\left(\frac{w_-(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}\\
&\le c\left(\Phi_{W^{s,f}(B_\tau)}(w_+)-\Phi_{W^{s,f}(B_\rho)}(w_+)\right)+c\left(\frac{R}{\tau-\rho}\right)^q\Phi_{L^f(B_R)}\left(\frac{w_+}{R^s}\right)\\
&\quad+c(1-s)\left(\frac{R}{\tau-\rho}\right)^{d+sq}\|w_+\|_{L^1(B_R)}\int_{\mathbb{R}^d\setminus B_r}f'\left(\frac{w_+(y)}{|y|^s}\right)|y|^{-d-s}\,\mathrm{d}y
\end{aligned}$$

for some $c=c(d,q,\Lambda)>0$. By setting

$$\phi(\rho)=\Phi_{W^{s,f}(B_\rho)}(w_+)+(1-s)\int_{B_\rho}\int_{A^-_k}f'\left(\frac{w_-(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d},$$

we can deduce from the above line that

$$\begin{aligned}\phi(\rho) &\le c(\phi(\tau)-\phi(\rho)) + c\left(\frac{R}{\tau-\rho}\right)^q \Phi_{L^f(B_R)}\left(\frac{w_+}{R^s}\right)\\ &\quad + c(1-s)\left(\frac{R}{\tau-\rho}\right)^{d+sq} \|w_+\|_{L^1(B_R)} \int_{\mathbb{R}^d\setminus B_r} f'\left(\frac{w_+(y)}{|y|^s}\right)|y|^{-d-s}\,\mathrm{d}y.\end{aligned}$$

We “fill the hole" by adding $c\phi(\rho)$ to both sides. After dividing by $1+c$, we get that

$$\begin{aligned}\phi(\rho) &\le \gamma\phi(\tau) + c\left(\frac{R}{\tau-\rho}\right)^q \Phi_{L^f(B_R)}\left(\frac{w_+}{R^s}\right)\\ &\quad + c(1-s)\left(\frac{R}{\tau-\rho}\right)^{d+sq} \|w_+\|_{L^1(B_R)} \int_{\mathbb{R}^d\setminus B_r} f'\left(\frac{w_+(y)}{|y|^s}\right)|y|^{-d-s}\,\mathrm{d}y,\end{aligned} \tag{6.14}$$

where $\gamma \in (0,1)$ and $c = c(d,q,\Lambda) > 0$. The desired result follows now from a standard iteration argument, see Lemma 4.11 in [17]. □

Remark 6.3 Similar to the proof of Theorem 6.2, it is possible to show that local superminimizer (minimizers, respectively) $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ satisfies $u \in G_-(\Omega; q, c, s, f)$ ($u \in G(\Omega; q, c, s, f)$, respectively) for some $c = c(d,q,\Lambda) > 0$.

Proof *(Proof of Theorem 1.1)* By Theorem 6.2, it follows that $u \in G(\Omega; q, c_1, s, f)$ for some $c_1 = c_1(d,q,\Lambda) > 0$. According to Proposition 3.4, it holds that $u \in G(\Omega; q, c_2, s, g)$ for some $c_2 = c_2(d,p,q,\Lambda) > 0$. From Theorems 4.2 and 5.1 we deduce the desired result. □

7 Application to weak solutions

In this section we aim to study weak solutions to nonlocal equations (1.6) and prove Theorem 1.2. Throughout this section we assume that f is a convex increasing function satisfying (f^q) and h is a measurable function satisfying the structure condition (h), i.e.,

$$h(x,y,t) = h(y,x,t), \quad |h(x,y,t)| \le \Lambda f'(|t|), \quad h(x,y,t)t \ge \frac{1}{\Lambda} f'(|t|)|t| \tag{h}$$

for a.e. $x, y \in \mathbb{R}^d$ and for all $t \in \mathbb{R}$. We define weak solutions to (1.6) as follows:

Definition 7.1 (*Weak solution*) We say that $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ is a *weak subsolution* to (1.6) if for every $\phi \in V^{s,f}(\Omega|\mathbb{R}^d)$ with $\phi = 0$ a.e. in $\mathbb{R}^d \setminus \Omega$ and $\phi \ge 0$ a.e. in Ω, it holds that

$$(1-s)\iint_{(\Omega^c\times\Omega^c)^c} h\left(x, y, \frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\phi(x)-\phi(y)}{|x-y|^{d+s}}\,\mathrm{d}y\,\mathrm{d}x \le 0. \tag{7.1}$$

We say that $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ is a *weak supersolution* if $-u$ is a weak subsolution. A function $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ is called a *weak solution* if it is a weak subsolution and a weak supersolution.

Recall that we always assume $f(0) = 0$ and $f(1) = 1$. This assumption can be made without loss of generality since u solves $\mathcal{L}_h u = 0$ if and only if u solves $\mathcal{L}_{h/f(1)} u = 0$ and one can always choose $f(0) = 0$.

Remark 7.2 Let us prove that the weak formulation (7.1) is well-defined. Let $u, \phi \in V^{s,f}(\Omega|\mathbb{R}^d)$. Then, by (h) and Fenchel's inequality (3.6), we have

$$
\begin{aligned}
&(1-s)\iint_{(\Omega^c\times\Omega^c)^c}\left|h\left(x,y,\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{\phi(x)-\phi(y)}{|x-y|^{d+s}}\right|\,\mathrm{d}y\,\mathrm{d}x\\
&\quad\le \Lambda(1-s)\iint_{(\Omega^c\times\Omega^c)^c} f'\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{|\phi(x)-\phi(y)|}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}\\
&\quad\le \Lambda(1-s)\iint_{(\Omega^c\times\Omega^c)^c}\left[f^*\left(f'\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\right)+f\left(\frac{|\phi(x)-\phi(y)|}{|x-y|^s}\right)\right]\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}\\
&\quad\le \Lambda(q-1)\Phi_{V^{s,f}(\Omega|\mathbb{R}^d)}(u)+\Lambda\Phi_{V^{s,f}(\Omega|\mathbb{R}^d)}(\phi)<\infty,
\end{aligned}
$$

where we used that by (3.5) and (f^q): $f^*(f'(t)) \le (q-1)f(t)$.

The following theorem yields that weak solutions to (1.6) belong to the De Giorgi classes introduced in Sect. 3.

Theorem 7.3 *Let $s \in (0,1)$, $q \ge 1$ and $\Lambda \ge 1$. Let $f : [0,\infty) \to [0,\infty)$ be a convex increasing function satisfying (f^q) and let $h : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying (h). Let $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ be a weak subsolution to (1.6). Then, $u \in G_+(\Omega; q, c, s, f)$ for some $c > 0$ depending on d, q and Λ.*

Note that the result remains true for $q = 1$.

Proof The desired result follows from a similar argument as in the proof of Theorem 6.2. Let $x_0 \in \Omega$, $0 < r < R \le d(x_0, \partial\Omega)$, and $k \in \mathbb{R}$. We may assume without loss of generality that $x_0 = 0$. Let $r \le \rho < \tau \le R$ and let $\eta \in C_c^\infty(\mathbb{R}^d)$ be a cutoff function with $0 \le \eta \le 1$, $\operatorname{supp}(\eta) = B_{\frac{\tau+\rho}{2}}$, $\eta \equiv 1$ in B_ρ, and $\|\nabla\eta\|_\infty \le \frac{4}{\tau-\rho}$. We define $w_\pm(x)$, $A_k^\pm$ and $A_{k,R}^\pm$ as in Theorem 6.2. We set $v = \eta^q w_+$. Since u is a weak subsolution to (1.6), we have

$$
0 \ge (1-s)\iint_{(\Omega^c\times\Omega^c)^c} B(x,y)\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}, \tag{7.2}
$$

where

$$
B(x,y) = h\left(x,y,\frac{u(x)-u(y)}{|x-y|^s}\right)\frac{v(x)-v(y)}{|x-y|^s}.
$$

Let us estimate $B(x,y)$. If $x, y \in A_k^-$, then

$$
B(x,y) = 0. \tag{7.3}
$$

If $x \in A_k^+$ and $y \in A_k^-$, then by (h)

$$
h\left(x,y,\frac{u(x)-u(y)}{|x-y|^s}\right) = h\left(x,y,\frac{w_+(x)+w_-(y)}{|x-y|^s}\right) \ge \frac{1}{\Lambda}f'\left(\frac{w_+(x)+w_-(y)}{|x-y|^s}\right).
$$

Thus, we obtain

$$
\begin{aligned}
B(x,y) &\ge \frac{1}{\Lambda}f'\left(\frac{|w_+(x)+w_-(y)|}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\eta^q(x)\\
&\ge \frac{1}{2\Lambda}\left[f'\left(\frac{w_+(x)}{|x-y|^s}\right)+f'\left(\frac{w_-(y)}{|x-y|^s}\right)\right]\frac{w_+(x)}{|x-y|^s}\eta^q(x)\\
&\ge \frac{1}{2\Lambda}f\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)\eta^q(x)+\frac{1}{2\Lambda}f'\left(\frac{w_-(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\eta^q(x),
\end{aligned} \tag{7.4}
$$

where we used (2.4) and Lemma 2.3.

If $x, y \in A_k^+$, we prove

$$B(x,y) \geq \frac{1}{\Lambda} f\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right)(\eta^q(x) \vee \eta^q(y)) - \varepsilon\Lambda(q-1) f\left(\frac{|w_+(x)-w_+(y)|}{|x-y|^s}\right) \\ -c\Lambda f\left(\frac{w_+(x) \vee w_+(y)}{|x-y|^s}|\eta(x)-\eta(y)|\right) \tag{7.5}$$

for any $\varepsilon \in (0,1)$, where $c = c(q,\varepsilon) > 0$. It is enough to prove (7.5) for the case $w_+(x) \geq w_+(y)$. If $\eta(x) \geq \eta(y)$, then (7.5) follows from

$$\begin{aligned} B(x,y) &\geq h\left(x, y, \frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\frac{w_+(x)-w_+(y)}{|x-y|^s}\eta^q(x) \\ &\geq \frac{1}{\Lambda} f\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\eta^q(x), \end{aligned}$$

where we used (h) and Lemma 2.3. When $\eta(y) \geq \eta(x)$, we observe that

$$\begin{aligned} B(x,y) &= h\left(x, y, \frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\eta^q(y) - \frac{w_+(x)}{|x-y|^s}(\eta^q(y)-\eta^q(x))\right) \\ &\geq \frac{1}{\Lambda} f\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\eta^q(y) - \Lambda f'\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}(\eta^q(y)-\eta^q(x)), \end{aligned} \tag{7.6}$$

where we used Lemma 2.3 again. Note that

$$\eta^q(y) - \eta^q(x) \leq q\eta^{q-1}(y)(\eta(y)-\eta(x)) \leq q(\eta(y)-\eta(x)).$$

Thus, for $\varepsilon \in (0,1)$ we use Lemma 2.1 and the Fenchel's inequality (3.6) to obtain

$$\begin{aligned} & f'\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}(\eta^q(y)-\eta^q(x)) \\ &\leq q\varepsilon^{-q} f'\left(\varepsilon\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}(\eta(y)-\eta(x)) \\ &\leq f^*\left(f'\left(\varepsilon\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\right) + f\left(q\varepsilon^{-q}\frac{\eta(y)-\eta(x)}{|x-y|^s}w_+(x)\right). \end{aligned}$$

By (3.5), (f^q) and Lemma 2.2 (iv) with $p = 1$, we deduce that

$$\begin{aligned} & f'\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}(\eta^q(y)-\eta^q(x)) \\ &\leq \varepsilon(q-1) f\left(\frac{w_+(x)-w_+(y)}{|x-y|^s}\right) + cf\left(\frac{\eta(y)-\eta(x)}{|x-y|^s}w_+(x)\right) \end{aligned} \tag{7.7}$$

for some $c = c(q,\varepsilon) > 0$. Therefore, (7.5) follows from (7.6) and (7.7).

Combining (7.3), (7.4) and (7.5), we have

$$\begin{aligned} &(1-s)\int_{B_\tau}\int_{B_\tau} B(x,y)\frac{dy\,dx}{|x-y|^d} \\ &\geq \frac{1}{2\Lambda}\Phi_{W^{s,f}(B_\rho)}(w_+) + \frac{1}{2\Lambda}(1-s)\int_{B_\rho}\int_{A_{k,\tau}^-} f'\left(\frac{w_-(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\frac{dy\,dx}{|x-y|^d} \\ &\quad - \varepsilon\Lambda(q-1)\Phi_{W^{s,f}(B_\tau)}(w_+) - c\Lambda(1-s)\int_{B_\tau}\int_{B_\tau} f\left(\frac{|\eta(x)-\eta(y)|}{|x-y|^s}w_+(x)\right)\frac{dy\,dx}{|x-y|^d}. \end{aligned} \tag{7.8}$$

Let us now take into account the pairs $(x, y) \in (\mathbb{R}^d \times \mathbb{R}^d) \setminus (B_\tau \times B_\tau)$. Using (h) we compute

$$\begin{aligned}
&(1-s)\int_{B_\tau}\int_{\mathbb{R}^d\setminus B_\tau} B(x,y)\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}\\
&\geq \frac{1}{\Lambda}(1-s)\int_{A^+_{k,\rho}}\int_{\{u(x)>u(y)\}\setminus B_\tau} f'\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}\\
&\quad -\frac{1}{\Lambda}(1-s)\int_{A^+_{k,\frac{\tau+\rho}{2}}} w_+(x)\int_{\{u(y)>u(x)\}\setminus B_\tau} f'\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^{d+s}} =: I_1 - I_2.
\end{aligned} \tag{7.9}$$

By monotonicity of f', we have that

$$I_1 \geq \frac{1}{2\Lambda}(1-s)\int_{B_\rho}\int_{(\mathbb{R}^d\setminus B_\tau)\cap A^-_k} f'\left(\frac{w_-(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}. \tag{7.10}$$

Moreover, by (6.13) we obtain

$$I_2 \leq c(1-s)\left(\frac{R}{\tau-\rho}\right)^{d+sq}\|w_+\|_{L^1(B_R)}\int_{\mathbb{R}^d\setminus B_\tau} f'\left(\frac{w_+(y)}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}}. \tag{7.11}$$

Therefore, it follows from (7.2), (7.8)–(7.11) and (6.8) that

$$\begin{aligned}
&\frac{1}{2\Lambda}\Phi_{W^{s,f}(B_\rho)}(w_+) + \frac{1}{2\Lambda}(1-s)\int_{B_\rho}\int_{A^-_k} f'\left(\frac{w_-(y)}{|x-y|^s}\right)\frac{w_+(x)}{|x-y|^s}\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^d}\\
&\leq \varepsilon\Lambda(q-1)\Phi_{W^{s,f}(B_\tau)}(w_+) + c\left(\frac{R}{\tau-\rho}\right)^q\Phi_{L^f(B_R)}\left(\frac{w_+}{R^s}\right)\\
&\quad + c(1-s)\left(\frac{R}{\tau-\rho}\right)^{d+sq}\|w_+\|_{L^1(B_R)}\int_{\mathbb{R}^d\setminus B_r} f'\left(\frac{w_+(y)}{|y|^s}\right)\frac{\mathrm{d}y}{|y|^{d+s}},
\end{aligned} \tag{7.12}$$

where $c = c(d, q, \Lambda, \varepsilon) > 0$. We take $\varepsilon = (4\Lambda^2(q-1))^{-1} \wedge 2^{-1} \in (0, 1)$ so that (7.12) boils down to (6.14) with $\gamma = 1/2$ and $c = c(d, q, \Lambda) > 0$. This finishes the proof. □

Remark 7.4 Similar to the proof of Theorem 7.3, it is possible to show that weak solutions $u \in V^{s,f}(\Omega|\mathbb{R}^d)$ to $\mathcal{L}_h u = 0$ in Ω satisfy $u \in G(\Omega; q, c, s, f)$ for some $c > 0$ depending on d, q and Λ.

Proof *(Proof of Theorem 1.2)* By Theorem 7.3, it follows that $u \in G(\Omega; q, c_1, s, f)$ for some $c_1 > 0$ depending on d, q and Λ. According to Proposition 3.4, it holds that $u \in G(\Omega; q, c_2, s, g)$ for some $c_2 = c_2(d, p, q, \Lambda) > 0$. From Theorems 4.2 and 5.1 we deduce the desired result. □

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Declarations

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