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Orbital b-metric spaces and related fixed point results on advanced Nashine–Wardowski–Feng–Liu type contractions with applications

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Abstract

In this article, we prove some novel fixed-point results for a pair of multivalued dominated mappings obeying a new generalized Nashine–Wardowski–Feng–Liu-type contraction for orbitally lower semi-continuous functions in a complete orbital b-metric space. Furthermore, some new fixed-point theorems for dominated multivalued mappings are established in the scenario of ordered complete orbital b-metric spaces. Some examples are offered to demonstrate the validity of our new results' premise. To demonstrate the applicability of our findings, applications for a system of nonlinear Volterra-type integral equations and fractional differential equations are shown. These results extend the theoretical results of Nashine et al. (Nonlinear Anal., Model. Control 26(3):522–533, 2021).

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1 Introduction and basic preliminaries

Fixed point theory is a fascinating branch of mathematics that plays a critical and fundamental role in both applied and pure mathematics, including modern optimization, control theories, functional analysis, topology and geometry, economics, and modeling. Fixed point theory is a well-balanced mixture of analysis, topology, and geometry. In many fields, it is a critical investigation and detection tool. Fixed point theory is a fundamental and useful area of functional analysis. Fixed point theory has most fascinating research topics in nonlinear analysis. Fixed point theory intends to develop not only nonlinear and functional analysis, but also economics, finance, computer science, and other subjects, in deciding problems for differential, integral, and random differential equations. As a result, the theory of fixed point arose as an analytic theory. Banach [10] developed the Banach contraction principle, which is the first well-known result in fixed point theory.

This result has an important role for solving linear and nonlinear differential, integral, and functional equations in a variety of generalized spaces. There are many variations on the Banach contraction principle that involve contractive-type requirements that must be met at different distance spaces. Different multiplications of Banach's result in metric and b-metric spaces were studied by Bakhtin [9], Czerwik [13–15]. Further useful recent results on b-metric spaces can be seen in [19, 21–23, 31]. Wardowski [40] proposed a new generalization of Banach's theorem known as F-contraction, as well as some new fixed-point results. After that, Sgroi et al. [37] established the existence of new theorems for set-valued F-contraction and showed some functional and integral equation applications. Nicolae [28] demonstrated fixed-point results for Feng–Liu type contractions that perform several functions. Padcharoen et al. [30] presented periodic theorems for a novel generalized F-contraction in modular metric-like spaces. Furthermore, Rasham et al. [36] used the first of Wardowski's three F-contraction conditions to prove fixed point theorems in a complete b-like-metric space for a multivalued dominated F-contraction on a closed ball. Rasham et al. [35] went on to analyze various linked fuzzy-dominated generalized contractive maps in modular metric-like spaces, and they demonstrated applications to examine the existence of unique solution for integral and fractional differential equations. Nashine et al. [25] recently obtained fixed point theorems for set-valued maps satisfying the Wardowski–Feng–Liu-type condition for orbitally semi lower continuous functions in an orbital complete b-metric space, as well as illustrative examples to certify new results and a fractal integral equations application.

We prove some novel fixed-point theorems for a couple of multivalued dominated maps that fulfill Nashine–Wardowski–Feng–Liu-type contraction for orbitally lower semi-continuous functions in a complete orbital b-metric space proved by Nashine et al. [25]. In addition, a new existence theorem for a few multi-dominated maps meeting a new generalized Nashine–Wardowski–Feng–Liu-type rational contractive condition for orbitally lower semi continuous functions in an ordered complete orbital b-metric space is presented. To validate our findings, we give examples and new definitions. Finally, we describe applications for nonlinear Volterra-type integral and fractional differential equations to demonstrate the utility of our findings.

Definition 1.1 ([26]) A function $d_b : Y \times Y \to [0,\infty)$ satisfying the following axioms (for all $g,h,i \in Y$) is called a b-metric (d_b-metric):

i. If $d_b(g,h) = 0$, iff $g = h$;
ii. $d_b(g,h) = d_b(h,g)$;
iii. $d_b(g,h) \leq b[d_b(g,i) + d_b(i,h)]$.

The pair (Y,d_b) is called a b-metric space (abbreviated as BMS).

Example 1.2 ([26]) Let $Y = R^+ \cup \{0\}$. Define $d_b(g,h) = |g-h|^2$, $\forall g,h \in Y$. Then, (Y,d_b) is a BMS with constant $b = 2$.

Definition 1.3 ([36]) Let Q be a non-empty subset of Y and $g \in Y$. Then, $p_0 \in Q$ is said to be a best approximation in Q if

$$d_b(g,Q) = d_b(g,p_0), \quad \text{where } d_b(g,Q) = \inf_{p\in Q} d_b(g,p).$$

Here $P(Y)$ denotes the set of all closed compact subsets of Y.

Definition 1.4 ([36]) A function $H_{d_b} : P(Y) \times P(Y) \to R^+$ defined by

$$H_{d_b}(R,E) = \max\left\{\sup_{x\in R} d_b(x,E), \sup_{m\in E} d_b(R,m)\right\}$$

undergoes all the axioms of a b-metric and is known as Pompiue–Hausdorff b-metric on $P(Y)$.

Definition 1.5 ([40]) An F-contraction is a mapping $T : \mathcal{M} \to \mathcal{M}$ satisfying the following:

there exists $\tau > 0$ such that (for every $\mu, y \in \mathcal{M}$)

$$d\big(T(\mu), T(y)\big) > 0 \quad \Rightarrow \quad \tau + F\big(d\big(T(\mu), T(y)\big)\big) \le F\big(d(\mu, y)\big).$$

Here the function $F : \mathbb{R}^+ \to \mathbb{R}$ satisfies the following conditions:

($F1$) F is an strictly-increasing function;

($F2$) $\lim_{j\to+\infty} \rho_j = 0$ if and only if $\lim_{j\to+\infty} F(\rho_j) = -\infty$, for every positive sequence $\{\rho_j\}_{j=1}^{\infty}$;

(F3) for each $g \in (0,1)$, $\lim_{j\to\infty} \rho_j^{g} F(\rho_j) = 0$;

($F4$) there exists $\tau > 0$ such that for every positive sequence $\{\rho_j\}$,

$$\tau + F(b\rho_j) \le F(\rho_{j-1}) \quad \text{for all } j \in \mathbb{N}, \text{then } \tau + F\big(b^j \rho_j\big) \le \big(b^{j-1}\rho_{j-1}\big) \text{ for all } j \in \mathbb{N}.$$

Definition 1.6 ([25]) Let $b \ge 1$ be a non-negative number and ∇F_b^* represents the family of functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the conditions ($F1$) – ($F4$) and

($F5$) $F(\inf B) = \inf F(B)$ for all $B \subset (0,\infty)$ with $\inf B > 0$.

It is clear that ∇F_b^* is non-empty containing $F(g) = \ln g$ or $F(g) = g + \ln g$.

A function $f : \mathcal{H} \to \mathbb{R}$ is lower semi-continuous if for every sequence of positive numbers $\{\omega_j\}$ in $\mathcal{H}$ with $\lim_{j\to+\infty} \omega_j = \omega \in \mathcal{H}, f(\omega) \le \lim_{j\to+\infty} \inf f(\omega_j)$.

Definition 1.7 ([32]) Let U be a non empty set, where $K \subseteq U$ and $\alpha : U \times U \to [0,+\infty)$. A mapping $W : U \to P(U)$ satisfying

$$\alpha_*(Wg, Wh) = \inf\big\{\alpha(t,z) : t \epsilon Wg, z \epsilon Wh\big\} \ge 1, \quad \text{whenever } \alpha(t,z) \ge 1, \text{ for all } t,z \in U$$

is called α_*-admissible.

A mapping $W : U \longrightarrow P(U)$ satisfying $\alpha_*(\varkappa, \mathrm{W}\varkappa) = \inf\{\alpha(\varkappa, \mathrm{h}) : \mathrm{h}\epsilon \mathrm{W}\varkappa\} \geq 1$, for all $\varkappa, h \in U$ is said to be an α_*-dominated on E.

Example 1.8 ([32]) Define $\gamma : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ and $G, R : \mathbb{R} \to P(\mathbb{R})$ by

$$\gamma(o,p) = \begin{Bmatrix} 1 & \text{if } o > p \\ \frac{1}{4} & \text{if } o \le p \end{Bmatrix},$$

and $Gs = [-4+s, -3+s]$ and $\mathrm{R}m = [-2+m, -1+m]$, respectively. Then G and R are γ_*-dominated, but they are not γ_*-admissible.

2 Main results

Let (Y, d_b) be a complete BMS and $\varkappa_0 \in Y$. Let S and T be two multi-maps from Y to $P(Y)$. Let $\varkappa_1 \in S(\varkappa_o)$, then $d_b(\varkappa_o, S(\varkappa_o)) = d_b(\varkappa_0, \varkappa_1)$. Let $\varkappa_2 \in T(\varkappa_1)$ be such that $d_b(\varkappa_1, T(\varkappa_1)) = d_b(\varkappa_1, \varkappa_2)$. Continuing this way, we attain a sequence $\{TS(\varkappa_j)\}$ in Y, where $\varkappa_{2j+1} \in S(\varkappa_{2j})$ and $\varkappa_{2j+2} \in T(\varkappa_{2j+1})$ for all $j \in \mathbb{N} \cup \{0\}$. Also, $d_b(\varkappa_{2j}, S(\varkappa_{2j})) = d_b(\varkappa_{2j}, \varkappa_{2j+1}), d_b(\varkappa_{2j+1}, T(\varkappa_{2j+1})) = d_b(\varkappa_{2j+1}, \varkappa_{2j+2})$, then $\{TS(\varkappa_j)\}$ is a sequence in Y generated by $\varkappa_0$. If, $S = T$, then we say $\{YS(\varkappa_j)\}$ instead of $\{TS(\varkappa_j)\}$.

Definition 2.1 Let S and T be two multi-maps from Y to $P(Y)$, $F \in \nabla F_b^*$ and $\eta : (0, \infty) \to (0, \infty)$. For all $\varkappa, y$ in Y with $\max\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y))\} > 0$, define a set $F_\eta^y \subseteq Y$ as:

$$F_\eta^y = \left\{ \begin{array}{c} y \in S(\varkappa), z \in T(y) : F(d_b(y, z)) \\ \leq F(b[\max\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y)), \frac{d_b(\varkappa, S(\varkappa)).d_b(y, T(y))}{1+d_b(\varkappa, y)}\}]) + \eta(d_b(y, z)) \end{array} \right\}.$$

Let $S, T : Y \to Y$. For any $_{y0} \in Y$, $O(_{y0}) = \{_{y0}, S(y_0), T(y_1), \dots\}$ denotes the orbit of $_{y0}$. A mapping $f : Y \to \mathbb{R}$ is said (S, T)-orbitally lower semi continuous if $f(y) < \lim_{n\to\infty} \inf f(_{y0})$ for all sequences $\{_{ST(yn)}\} \subset O(_{y0})$ with $\lim_{n\to\infty}\{_{ST(yn)}\} = y \in Y$.

Definition 2.2 Let $S, T : Y \to P(Y)$ be a couple of multi-maps on (Y, d_b). An orbit for a pair (S, T) in a point $_{y0} \in Y$ denoted by $O(_{y0})$ is a sequence defined as $\{_{yn}:_{yn} \in S_{T(yn-1)})\}$.

Definition 2.3 Let $S, T : Y \to P(Y)$ be a couple of multi-maps on (Y, d_b). If a Cauchy sequence $\{_{yn}:_{yn} \in S_{T(yn-1)})\}$ converges in b-metric space Y, then Y is said to be (S, T)-orbitally complete.

It is noted that an orbitally complete BMS may not be complete. Now we begin our main theorem.

Theorem 2.4 *Let (Y, d_b) be an orbitally complete BMS. Let $y_0 \in Y, \alpha : Y \times Y \to [0, \infty)$ and $S, T : Y \to P(Y)$ be two α_*-dominated multi-maps and $F \in \nabla F_b^*$. Assume the following properties hold*:

i. *The mapping* $z \mapsto \max\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y))\}$ *is orbitally lower semi continuous*;
ii. *There exist functions* $\tau, \eta : (0, \infty) \to (0, \infty)$ *such that for all* $t \geq 0$;

$$\tau(t) > \eta(t), \quad \lim_{s\to t^+} \inf \tau(t) > \lim_{s\to t^+} \inf \eta(t);$$

iii. *For all* $\varkappa, y \in \{ST(\varkappa_j)\}$ *with* $\alpha(x, y) \geq 1$ *and* $\max\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y))\} > 0$, *there exist* $\varkappa, y \in F_\eta^\varkappa$ *satisfying*;

$$\tau\bigl(d_b(\varkappa, y)\bigr) + F\left(b\left[\max\left\{d_b\bigl(\varkappa, S(\varkappa)\bigr), d_b\bigl(y, T(y)\bigr), \frac{d_b(\varkappa, S(\varkappa)).d_b(y, T(y))}{1 + d_b(\varkappa, y)}\right\}\right]\right) \leq F\bigl(d_b(\varkappa, y)\bigr). \tag{2.1}$$

If (2.1) holds, then S and T have a common fixed point q in Y.

Proof Suppose S and T have no fixed point. Then for each $\varkappa, y \in Y$ we have $\max\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y))\} > 0$. Since $S(\varkappa), T(y) \in P(Y)$ for every $\varkappa, y \in Y$ and $F \in \nabla F_b^*$, it is simple to prove that F_η^x is a non-empty set for all $\varkappa, y \in Y$ (proof will follow in the first line of [20]). As $S, T : Y \to P(Y)$ are two α_*-dominated multi-maps on $\{TS(x_j)\}$, by definition we have $\alpha_*(\varkappa_{2j}, S(\varkappa_{2j})) \geq 1$ and $\alpha_*(\varkappa_{2i+1}, T(\varkappa_{2j+1})) \geq 1$ for all $j \in \mathbb{N}$. As $\alpha_*(\varkappa_{2j}, S(\varkappa_{2j})) \geq 1$, this implies that $\inf\{\alpha(x_{2j}, b) : b \in S(\varkappa_{2j})\} \geq 1$ and therefore, $\alpha(x_{2j}, x_{2j+1}) \geq 1$. If $\varkappa_0 \in Y$ is any initial point then $\varkappa_{2j}, \varkappa_{2j+1} \in F_\eta^{\varkappa_0}$ and using (2.1), we have

$$\begin{aligned}
&\tau\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) + F\Bigg(b\Bigg[\max\Bigg\{d_b\big(\varkappa_{2j}, S(\varkappa_{2j})\big), d_b\big(\varkappa_{2j+1}, T(\varkappa_{2j+1})\big),\\
&\quad \frac{d_b(\varkappa_{2j}, S(\varkappa_{2j})).d_b(\varkappa_{2j+1}, T(\varkappa_{2j+1}))}{1 + d_b(\varkappa_{2j}, \varkappa_{2j+1})}\Bigg\}\Bigg]\Bigg)\\
&\quad \leq F\big(d_b(\varkappa_{2j}, \varkappa_{2j+1})\big),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\tau\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) + F\Bigg(b\Bigg[\max\Bigg\{d_b(\varkappa_{2j}, \varkappa_{2j+1}), d_b(\varkappa_{2j+1}, \varkappa_{2j+2}),\\
&\quad \frac{d_b(\varkappa_{2j}, \varkappa_{2j+1}).d_b(\varkappa_{2j+1}, \varkappa_{2j+2})}{1 + d_b(\varkappa_{2j}, \varkappa_{2j+1})}\Bigg\}\Bigg]\Bigg)\\
&\quad \leq F\big(d_b(\varkappa_{2j}, \varkappa_{2j+1})\big),
\end{aligned}$$

that is

$$\begin{aligned}
&\tau\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) + F\big(b\big[\max\big\{d_b(\varkappa_{2j}, \varkappa_{2j+1}), d_b(\varkappa_{2j+1}, \varkappa_{2j+2}), d_b(\varkappa_{2j+1}, \varkappa_{2j+2})\big\}\big]\big)\\
&\quad \leq F\big(d_b(\varkappa_{2j}, \varkappa_{2j+1})\big).
\end{aligned}$$

This implies that,

$$\begin{aligned}
&\tau\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) + F\big(b\big[\max\big\{d_b(\varkappa_{2j}, \varkappa_{2j+1}), d_b(\varkappa_{2j+1}, \varkappa_{2j+2})\big\}\big]\big)\\
&\quad \leq F\big(d_b(\varkappa_{2j}, \varkappa_{2j+1})\big).
\end{aligned} \tag{2.2}$$

If $\max\{d_b(\varkappa_{2j}, \varkappa_{2j+1}), d_b(\varkappa_{2j+1}, \varkappa_{2j+2})\} = d_b(\varkappa_{2j}, \varkappa_{2j+1})$, then from (2.2), we have

$$\tau\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) + F\big(b\big[d_b(\varkappa_{2j}, \varkappa_{2j+1})\big]\big) \leq F\big(d_b(\varkappa_{2j}, \varkappa_{2j+1})\big),$$

which is wrong due to $(F1)$, which mentions that F is strictly increasing. Therefore,

$$\max\big\{d_b(\varkappa_{2j}, \varkappa_{2j+1}), d_b(\varkappa_{2j+1}, \varkappa_{2j+2})\big\} = d_b(\varkappa_{2j+1}, \varkappa_{2j+2}),$$

for each $j \in \mathbb{N} \cap \{0\}$. So, we have

$$\tau\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) + F\big(b\big[d_b(\varkappa_{2j+1}, \varkappa_{2j+2})\big]\big) \leq F\big(d_b(\varkappa_{2j}, \varkappa_{2j+1})\big), \tag{2.3}$$

for all $j \in \mathbb{N} \cap \{0\}$. From (2.3) and applying (F4) we get,

$$\tau\left(d_b(\varkappa_{2j}, \varkappa_{2j+1})\right) + F\big(b^{2j+1}\big[d_b(\varkappa_{2j+1}, \varkappa_{2j+2})\big]\big) \leq F\big(b^{2j} d_b(\varkappa_{2j}, \varkappa_{2j+1})\big). \tag{2.4}$$

Since, $\varkappa_{j+1} \in F^{y}_{\eta}$ then by the definition of F^{y}_{η} we have,

$$F\big(d_b(\varkappa_{2j}, \varkappa_{2j+1})\big) \le F\big(d_b(\varkappa_{2j}, T\varkappa_{2j})\big) + \eta d_b(\varkappa_{2j}, \varkappa_{2j+1}),$$

which implies that,

$$F\big(b^{2j} d_b(\varkappa_{2j}, \varkappa_{2j+1})\big) \le F\big(b^{2j} d_b(\varkappa_{2j}, T\varkappa_{2j})\big) + \eta d_b(\varkappa_{2j}, \varkappa_{2j+1}). \tag{2.5}$$

From (2.4) and (2.5), we have,

$$\begin{aligned} &F\big(b^{2j+1}\big[d_b(\varkappa_{2j+1}, \varkappa_{2j+2})\big]\big) \\ &\quad\le F\big(b^{2j} d_b(\varkappa_{2j}, \varkappa_{2j+1})\big) + \eta d_b(\varkappa_{2j}, \varkappa_{2j+1}) - \tau\big(d_b(\varkappa_{2j}, \varkappa_{2j+1})\big). \end{aligned} \tag{2.6}$$

Similarly, for each $j \in \mathbb{N} \cap \{0\}$ we have,

$$\begin{aligned} &F\big(b^{2j}\big[d_b(\varkappa_{2j}, \varkappa_{2j+1})\big]\big) \\ &\quad\le F\big(b^{2j-1} d_b(\varkappa_{2j-1}, \varkappa_{2j})\big) + \eta d_b(\varkappa_{2j-1}, \varkappa_{2j}) - \tau\big(d_b(\varkappa_{2j-1}, \varkappa_{2j})\big). \end{aligned} \tag{2.7}$$

Using (2.7) in (2.5) we have,

$$\begin{aligned} &F\big(b^{2j+1}\big[d_b(\varkappa_{2j+1}, \varkappa_{2j+2})\big]\big) \\ &\quad\le F\big(b^{2j-1} d_b(\varkappa_{2j-1}, \varkappa_{2j})\big) + \eta d_b(\varkappa_{2j}, \varkappa_{2j+1}) + \eta d_b(\varkappa_{2j-1}, \varkappa_{2j}) \\ &\qquad - \tau\big(d_b(\varkappa_{2j-1}, \varkappa_{2j})\big) - \tau\big(d_b(\varkappa_{2j}, \varkappa_{2j+1})\big). \end{aligned} \tag{2.8}$$

Now put $2j + 1 = g$ and also $d_b(\varkappa_{2j}, \varkappa_{2j+1}) = \partial_g$ for all $\in \mathbb{N} \cap \{0\}$. As $\partial_g > 0$ and from (2.8) $\{\partial_g\}$ is decreasing. Therefore, there exists $\varepsilon > 0$ such that $\lim_{g\to\infty} \partial_g = \varepsilon$. As $\varepsilon > 0$, let $\gamma(t) = \tau(t) - \eta(t) \ge 0$, when $t \to s^{+}$. Then, using (2.6), we get

$$\begin{aligned} &F\big(b^{g}\partial_g\big) \le F\big(b^{g-1}\partial_g\big) - \gamma(\partial_g) \\ &\le F\big(b^{g-1}\partial_{g-1}\big) - \gamma(\partial_g) - \gamma(\partial_{g-1}) \\ &\cdots \\ &\le F(\partial_0) - \gamma(\partial_g) - \gamma(\partial_{g-1}) - \cdots - \gamma(\partial_0). \end{aligned} \tag{2.9}$$

Let l_g be the greatest number in $\{0, 1, 2, 3, \dots, g-1\}$ such that

$$\gamma(\partial_{l_g}) = \min\big\{\gamma(\partial_0), \gamma(\partial_1), \gamma(\partial_2), \dots, \gamma(\partial_g)\big\}.$$

For every $g \in \mathbb{N}$, so the sequence $\{\partial_g\}$ is non-decreasing. Now, from equation (2.9) we get,

$$F\big(b^{g}\partial_g\big) \le F(\partial_0) - g\gamma(\partial_{l_g}).$$

Similarly, from (2.6) we can obtain

$$F\big(b^{g+1} d_b\big(\varkappa_{g+1}, T(\varkappa_{g+1})\big)\big) \le F\big(b^{g} d_b(\varkappa_0, \varkappa_1)\big) - g\gamma(\partial_{l_g}). \tag{2.10}$$

Now, for the sequence $\{\gamma(\partial_{l_g})\}$, we have two cases.

Case I: For each $g \in \mathbb{N}$ there exists $e > g$ such that $\gamma(\partial_{l_e}) > \gamma(\partial_{l_g})$. Then, we obtain a subsequence $\{\gamma(\partial_{l_{g_k}})\}$ of $\{\partial_{l_g}\}$ with $\gamma(\partial_{l_{g_k}}) > \gamma(\partial_{l_{g_{k+1}}})$ for all k. Since $\partial_{l_{g_k}} \to \delta^+$, we deduce that,

$$\lim_{t\to\delta^+} \inf \gamma(\partial_{l_{g_k}}) > 0.$$

Hence,

$$F\big(b^{g_k}\partial_{g_k}\big) \leq F(\partial_0) - g^k\gamma(\partial_{l_{g_k}}),$$

For every k. Consequently, $\lim_{k\to\infty} F(b^{g_k}\partial_{g_k}) = -\infty$ and by $(F2)$ $\lim_{k\to+\infty} b^{g_k}\partial_{g_k} = 0$, which is not true that $\lim_{k\to+\infty}\partial_{g_k} > 0$ for $b > 1$.

Case II: There exists $g_0 \in \mathbb{N}$ so that $\gamma(\partial_{l_{g_0}}) > \gamma(\partial_{l_e})$ for every $e > g_0$. Then

$$F(\partial_e) \leq F(\partial_0) - e\gamma(\partial_{l_e}), \quad \text{for all } e > g_0.$$

Hence, $\lim_{e\to+\infty} F(\partial_e) = -\infty$ and by $(F2)$ $\lim_{e\to+\infty}\partial_e = 0$, which contradicts the fact that $\lim_{e\to+\infty}\partial_e > 0$. Thus, $\lim_{e\to+\infty}\partial_e = 0$. From (F3). there exists $k \in (0,1)$ such that $\lim_{g\to+\infty}(b^g\partial_g)^k F(b^g\partial_g) = -\infty$ and, from inequality (2.10), the following holds for all $g \in \mathbb{N}$:

$$\begin{aligned}\big(b^g\partial_g\big)^k F\big(b^g\partial_g\big) - \big(b^g\partial_g\big)^k F(\partial_0) &\leq \big(b^g\partial_g\big)^k\big(F(\partial_0) - g\gamma(\partial_{l_g})\big) - \big(b^g\partial_g\big)^k F(\partial_0) \\ &= -g\big(b^g\partial_g\big)^k\gamma(\partial_{l_g}) \leq 0. \end{aligned}\tag{2.11}$$

Passing to the limit as $g \to +\infty$ in (2.11), we obtain

$$\lim_{g\to+\infty} g\big(b^g\partial_g\big)^k\gamma(\partial_{l_g}) = 0.$$

Since $\zeta = \lim_{g\to+\infty}\gamma(\partial_{l_g}) > 0$, there exists $g_0 \in \mathbb{N}$ such that $\gamma(\partial_{l_g}) > \frac{\zeta}{2}$ for all $g \neq g_0$. Thus,

$$g\big(b^g\partial_g\big)^k\frac{\zeta}{2} < g\big(b^g\partial_g\big)^k\gamma(\partial_{l_g}),\tag{2.12}$$

for each $g > g_0$. Letting $g \to +\infty$ in (2.12), we have,

$$0 \leq \lim_{g\to+\infty} g\big(b^g\partial_g\big)^k\frac{\zeta}{2} < \lim_{g\to+\infty} g\big(b^g\partial_g\big)^k\gamma(\partial_{l_g}) = 0.$$

That is,

$$\lim_{g\to+\infty} g\big(b^g\partial_g\big)^k = 0,\tag{2.13}$$

from (2.13) there exists $g_1 \in \mathbb{N}$ such that $g(b^g\partial_g)^k \leq 1$ for all $g > g_1$

$$\big(b^g\partial_g\big)^k \leq \frac{1}{g},$$

which concludes that,

$$\partial_g \leq \frac{1}{b^g g^{1/k}}, \quad \text{for all } g > g_{1.}$$

Now, taking limit as $g \to +\infty$ then series $\sum_{g=1}^{\infty} b^g \partial_g$ becomes convergent, and the sequence $\{\varkappa_g\}$ is a Cauchy in Y. Since Y is a complete orbitally BMS, there exists $q \in O(y_0)$ so that $\varkappa_g \to q$ when $g \to +\infty$. By (2.10) and $(F2)$, we find $\lim_{g\to+\infty} d_b(\varkappa_g, T(\varkappa_g)) = 0$. Since $q \to d_b(\varkappa, T(\varkappa))$ is orbitally lower semi-continuous and $\alpha(\varkappa_g, T(\varkappa_g) > 1$, we have,

$$\begin{aligned} 0 \leq d_b\big(q, T(q)\big) &\leq \lim_{g\to+\infty} \inf d_b\big(\varkappa_g, T(\varkappa_g)\big) \leq \lim_{j\to+\infty} \inf d_b(\varkappa_g, \varkappa_{g+1}), \\ &\leq \lim_{g\to+\infty} b\big[d_b(\varkappa_g, \varkappa) + d_b(\varkappa, \varkappa_{g+1})\big] = 0. \end{aligned}$$

Hence, $q \in T(q)$.

As the series $\sum_{g=1}^{\infty} b^g \partial_g$ is convergent, and the sequence $\{y_g\}$ is Cauchy in Y and Y is a complete orbitally BMS, there exists $q \in O(y_0)$ such that $y_g \to q$ as $g \to +\infty$. Using (2.10) and $(F2)$, we get $\lim_{g\to+\infty} d_b(y_g, S(y_g)) = 0$. Since $q \to d_b(y, S(y))$ is orbitally lower semi-continuous and also $\alpha(y_g, S(y_g)) > 1$, we have,

$$\begin{aligned} 0 \leq d_b\big(q, S(q)\big) &\leq \lim_{g\to+\infty} \inf d_b\big(y_g, S(y_g)\big) \leq \lim_{j\to+\infty} \inf d_b(y_g, y_{g+1}), \\ &\leq \lim_{g\to+\infty} b\big[d_b(y_g, y) + d_b(y, y_{g+1})\big] = 0. \end{aligned}$$

Therefore, $q \in S(q)$. Hence, S has a fixed point. So, T and S both have a common fixed point in Y.

Recall that $u \preccurlyeq A$ means there is $b \in A$ such that $u \preccurlyeq b$. The function $S : Y \to P(Y)$ is multi $\preccurlyeq$-dominated on A if $u \preccurlyeq Su$ for any $u \in Y$.

We prove upcoming theorem for $\preccurlyeq$-dominated multivalued mappings on $\{TS(c_j)\}$ in a complete orbitally BMS. For each $x, y \in Y$ with $\max\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y))\} > 0$, define a set $F^y_{\eta,\preccurlyeq} \subseteq Y$ as

$$F^y_{\eta,\preccurlyeq} = \left\{ \begin{array}{c} y \in S(\varkappa), z \in T(y) : F(d_b(y,z)) \\ \leq F(b[\max\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y)), \frac{d_b(\varkappa,S(\varkappa)).d_b(y,T(y))}{1+d_b(\varkappa,y)}\}]) \\ +\eta(d_b(y,z)), x \preccurlyeq y \text{ and } y \preccurlyeq z \end{array} \right\}.$$

□

Theorem 2.5 *Let $(Y, \preccurlyeq, d_b)$ be a complete orbitally ordered BMS. Let $y_0 \in Y, \alpha : Y \times Y \to [0,\infty)$ and $S, T : Y \to P(Y)$ be two α_*-dominated multi-maps and $F \in \xi F_b^*$. Assume that following properties hold*:

i. *The mapping $z \mapsto \max\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y))\}$ is orbitally lower semi continuous*;
ii. *There exists functions $\tau, \eta : (0,\infty) \to (0,\infty)$ such that for all $t \geq 0$*;

$$\tau(t) > \eta(t) \quad \textit{yeilds that } \lim_{s\to t^+} \inf \tau(t) > \lim_{s\to t^+} \inf \eta(t);$$

iii. *For all $\varkappa, y \in \{ST(f_j)\}$ with either $x \preccurlyeq y$ or $y \preccurlyeq x$ and max $\{d_b(\varkappa, S(\varkappa)), d_b(y, T(y))\} > 0$, also $\{ST(f_j)\} \to f^*$, there exist $\varkappa, y$ in $F^y_{\eta,\preccurlyeq}$ satisfying*;

$$\tau\big(d_b(\varkappa, y)\big) + F\left(b\left[\max\left\{d_b\big(\varkappa, S(\varkappa)\big), d_b\big(y, T(y)\big), \frac{d_b(\varkappa, S(\varkappa)).d_b(y, T(y))}{1 + d_b(\varkappa, y)}\right\}\right]\right)$$

$$\leq F\big(d_b(\varkappa, y)\big). \tag{2.14}$$

Also, if (2.14) *holds for* $f^*, f^* \preccurlyeq f_j$ *or* $f_j \preccurlyeq f^*$ *where* $j = \{0,1,2,3,\ldots.\}$, *then both* S *and* T *have a common fixed point* f^* *in* Y.

Proof Let $\alpha : Y \times Y \to [0,+\infty)$ be a function defined by $\alpha(f,q) = 1$ for each $f \in Y$ with $f \preccurlyeq q$, and $\alpha(f,q) = 0$ for each incomparable elements $f, q \in Y$. As S and T are two multi dominated maps on Y, so $f \preccurlyeq S(\varkappa)$ and $f \preccurlyeq T(y)$ for all $f \in Y$. This implies that $f \preccurlyeq u$ for every $u \in S(\varkappa)$ and $f \preccurlyeq u$ for each $f \in T(y)$. So, $\alpha(f,u) = 1$ for every $u \in S(\varkappa)$ and $\alpha(f,u) = 1$ for each $f \in T(y)$. This implies that $\inf\{\alpha(f,q) : q \in S(\varkappa)\} = 1$, and $\inf\{\alpha(f,q) : q \in T(y)\} = 1$.

Hence, $\alpha_*(f, S(\varkappa)) = 1$, $\alpha_*(f, T(y)) = 1$ for each $f \in Y$. So, $S, T : Y \to P(Y)$ are α_*-dominated multi mappings on Y. Moreover, inequality (2.14) can be written as

$$\begin{aligned} &\tau\big(d_b(\varkappa, y)\big) + F\left(b\left[\max\left\{d_b\big(\varkappa, S(\varkappa)\big), d_b\big(y, T(y)\big), \frac{d_b(\varkappa, S(\varkappa)).d_b(y, T(y))}{1 + d_b(\varkappa, y)}, \right\}\right]\right) \\ &\quad \leq F\big(d_b(\varkappa, y)\big), \end{aligned}$$

for each f, q in $\{TS(f_j)\}$, with either $\alpha(f,q) \geq 1$ or $\alpha(q,f) \geq 1$. Then, by Theorem 2.4, the sequence $\{TS(f_j)\}$ is convergent in Y that is $\{TS(f_j)\} \to f^* \in Y$. Now, $f_j, f^* \in Y$ and either $f_j \preccurlyeq f^*$, or $f^* \preccurlyeq f_j$ implies that the either $\alpha(f_j, f^*)$, or $\alpha(f^*, f_j) \geq 1$. Hence, all the conditions of Theorem 2.6 are satisfied. So, from Theorem 2.5, both S and T have a multi fixed point f^* in Y and $d_b(f^*, f^*) = 0$. □

Example 2.6 Let $Y = [0,\infty)$, and consider a relation $\preccurlyeq$ on Y by $x \preccurlyeq y$ if and only if x divides y. Then it is easy to verify that $(Y, \preccurlyeq)$ is a partially ordered set. Now we define

$$d_b : Y \times Y \to \mathbb{R} \text{ by} \quad d_b(p,q) = \begin{cases} 0 & \text{if } x = y; \\ 3(\frac{1}{n} + \frac{1}{m}) & \text{if } x = n, y = m \text{ and } n \neq m; \\ \frac{1}{n} & \text{if } x = n, y = 0, \text{or } x = 0, y = n; \end{cases}$$

for each $p, q \in Y$. Then the function d_b becomes an ordered b-metric on Y with $b = 3$. Also, we define mappings $S, T : Y \to P(Y)$ by

$$S(p) = \{p, 3p, 5p\} \quad \text{and} \quad T(q) = \{2q, 4q, 6q\},$$

and $\alpha : Y \times Y \to R$ as

$$\alpha(p,q) = \begin{cases} 1 & \text{if } p > q, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Now, for all $p, q \in \{TS(x_j)\}$ either $\alpha(p,q) \geq 1$ or $\alpha(q,p) \geq 1$. Define a function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(p) = \ln(p)$ for all $p \in \mathbb{R}^+$ and $\tau(t) = \frac{1}{8}$ and $\eta(t) = \frac{1}{10}$ for each $t \in (0,\infty)$. Then, clearly $F \in \xi F_b^*$ and $\tau(t) > \eta(t)$, $\lim_{s\to t^+} \inf \tau(t) > \lim_{s\to t^+} \inf \eta(t)$. As $p, q \in Y$, then

$$S(p) = \begin{cases} 0 & \text{if } p = 0, \\ \frac{14}{3p} & \text{if } p \neq 0, \end{cases} \quad \text{and} \quad T(q) = \begin{cases} 0 & \text{if } q = 0, \\ \frac{13}{3q} & \text{if } q \neq 0. \end{cases}$$

Therefore, $q \to \max\{d_b(p, S(p)), d_b(q, T(q))\}$ is orbitally lower semicontinuous. For $q \in Y$, we have $q = 4p \in F^{\varkappa}_{\eta,\preccurlyeq}$, and for this q, we have,

$$\begin{aligned}
&\tau\big(d_b(p,q)\big) + F\left(b\left[\max\left\{d_b\big(p,S(p)\big), d_b\big(q,T(q)\big), \frac{d_b(p,S(p)).d_b(q,T(q))}{1+d_b(p,q)}\right\}\right]\right) \\
&= \frac{1}{8} + \ln\left[3\max\left\{\frac{14}{12p}, \frac{13}{12q}, \frac{7\times 13}{9pq}\right\}\right] \\
&= \frac{1}{8} + \ln\left(\frac{39}{12q}\right) \\
&\le \ln\left(\frac{4}{3}\cdot\frac{39}{12q}\right) \le \ln\left(\frac{39}{12q}\right), \\
&= F\big(d_b(p,q)\big).
\end{aligned}$$

Hence, all the requirements of our Theorem 2.5 hold for $F(p) = \ln(p)$, where $p > 0$.

3 Application to integral equation

Using distinct generalized contractions in different settings of generalized metric spaces, a significant number of writers showed necessary and sufficient conditions for different types of linear and nonlinear (Volterra and Fredholm) type integrals in fixed point theory. Rasham et al., [34] proved existence of new fixed-point results for two families of multivalued mappings, and their main result was utilized to examine necessary conditions for the solution of nonlinear integral equations. For more recent fixed-point results incorporating integral inclusions can be seen in [5, 26, 33, 37].

Let $X = (\mathbb{C}[0,1], \mathbb{R}_+)$ be the set of continuous functions on $[0,1]$ endowed with the metric $d_b : X \times X \to \mathbb{R}$ defined by $d_b(f,g) = \sup|f(t) - g(t)|^2 \forall f, g \in (\mathbb{C}[0,1], \mathbb{R})$ and $t \in [0,1]$. Define $\alpha : X \times X \to \mathbb{R}$ as

$$\alpha(p,q) = \begin{cases} 1 & \text{if } p(t) \le q(t), \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Take an integral equation:

$$l(t) = \int_0^t \mathcal{K}\big(t, s, \varkappa(s)\big)\, ds; \tag{3.1}$$

where $\mathcal{K} : [0,1] \times [0,1] \times \mathcal{X} \to \mathbb{R}$ and l are continuous for all $s, t \in [0,1]$. Our aim is to show the existence of the solution to Eq. (3.1) by applying Theorem 2.4.

Theorem 3.1 *Let $X = (\mathbb{C}[0,1], \mathbb{R})$ and $S, T : X \to X$ be Volterra integral operator defined as follows*:

$$\begin{aligned}
(S\varkappa)(t) &= \int_0^t \mathcal{K}\big(t, s, \varkappa(s)\big)\, ds, \quad \text{and} \\
(Ty)(t) &= \int_0^t \mathcal{K}\big(t, s, y(s)\big)\, ds, \quad \text{where } \varkappa, y \in \mathbb{C}[0,1].
\end{aligned}$$

Also, $\mathcal{K} : [0,1] \times [0,1] \times \mathcal{X} \to \mathbb{R}$ is continuous on $\mathbb{R}$ for all $\varkappa, t \in [0,1]$ and the following holds:

i. *There exists a continuous function $\mu : [0,1] \to \mathbb{R}_+$ such that $(\int_0^t \mu(s)\,ds)^2 \leq \frac{e^{-\tau(t)}}{2}; t > 0$ and satisfying, for all $\varkappa, y \in \mathcal{X}$ such that $\varkappa(t) \leq y(t)$,*

$$\left|\mathcal{K}\big(t,s,\varkappa(s)\big) - \mathcal{K}\big(t,s,y(s)\big)\right| \leq \mu(s)\left|\varkappa(s) - y(s)\right|.$$

ii. *For all $\varkappa, y \in X$ such that $\varkappa(t) \leq y(t)$, we have*

$$\max\left\{d_b\big(\varkappa, S(\varkappa)\big), d_b\big(y, T(y)\big), \frac{d_b(\varkappa, S(\varkappa)).d_b(y, T(y))}{1 + d_b(\varkappa, y)}\right\} \leq \left|(Sy)(t) - (Ty)(t)\right|^2.$$

Then, (3.1) *has a unique solution*.

Proof We give an assurance here that both the multi-maps S and T hold all the necessary requirements of our main Theorem 2.4 for single-valued mappings. Let $\varkappa \in X = (\mathbb{C}[0,1], \mathbb{R}_+)$ and

$$\begin{aligned}\left|(Sy)(t) - (Ty)(t)\right|^2 &\leq \left(\int_0^t \left|\mathcal{K}\big(t,s,\varkappa(s)\big) - \mathcal{K}\big(t,s,y(s)\big)\right| ds\right)^2 \\ &\leq \left(\int_0^t \mu(s)\left|\varkappa(s) - y(s)\right| ds\right)^2 \\ &\leq d_b(\varkappa, y)\left(\int_0^t \mu(s)\,ds\right)^2 \quad \text{for all } t, s \in [0,1].\end{aligned}$$

Which means that,

$$\left|(Sy)(t) - (Ty)(t)\right|^2 \leq \frac{d_b(\varkappa, y)e^{-\tau(t)}}{2}.$$

Thus, by (ii), we have,

$$2e^{\tau(t)}\max\left\{d_b\big(\varkappa, S(\varkappa)\big), d_b\big(y, T(y)\big), \frac{d_b(\varkappa, S(\varkappa)).d_b(y, T(y))}{1 + d_b(\varkappa, y)}\right\} \leq d_b(\varkappa, y).$$

Take $b = 2$ and ln on both sides,

$$\ln e^{\tau(t)} + \ln \mathrm{b}\max\left\{d_b\big(\varkappa, S(\varkappa)\big), d_b\big(y, T(y)\big), \frac{d_b(\varkappa, S(\varkappa)).d_b(y, T(y))}{1 + d_b(\varkappa, y)}\right\} \leq \ln\big(d_b(\varkappa, y)\big).$$

Define a function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(p) = \ln(p)$ for all $p \in \mathbb{R}^+$ and $\tau(t) = \frac{1}{8}$ and $\eta(t) = \frac{1}{10}$ for each $t \in (0, \infty)$. Then, clearly $F \in \nabla F_b^*$ and $\tau(t) > \eta(t), \lim_{s \to t^+} \inf \tau(t) > \lim_{s \to t^+} \inf \eta(t)$. So, all the conditions of our main Theorem 2.4 for single-valued mappings are satisfied. Hence, the integral equation (3.1) has a unique common solution. □

4 Application to fractional differential equation

Many relevant aspects of fractional differentials were introduced and shown by Lacroix (1819). Later, a wide number of academics proved a variety of important fixed-point theorems in many types of metric spaces using various generalized contractions as well as applications to fractional differential equations (see [25, 27, 32]). A variety of new models connected to the Caputo–Fabrizio derivative (CFD) have recently been constructed and demonstrated, see [17, 39]. In this section, we will show that one of these types of models exists in b-metric spaces.

Let $I = [0,1]$ and $\mathtt{C}(I,\mathbb{R})$ be the space of continuous functions defined on I. Define the metric $d_b : \mathtt{C}(I,\mathbb{R}) \times \mathtt{C}(I,\mathbb{R}) \to [0.\infty)$ by $d_b(u,v) = \|u-v\|_\infty^2 = \max_{l\in[0,L]} |u(l)-v(l)|^2$, for all $u,v \in \mathtt{C}(I,\mathbb{R})$. Then $(\mathtt{C}(I,\mathbb{R}),d_b)$ is a complete orbital b-metric space. Define $\alpha : X \times X \to \mathbb{R}$ as

$$\alpha(p,q) = \begin{cases} 1 & \text{if } p(t) \leq q(t), \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Let $\mathcal{K}_1, \mathcal{K}_2 : I \times \mathbb{R} \to \mathbb{R}$ be two mappings such that $\mathcal{K}_1(l,g(e)), \mathcal{K}_2(l,h(e)) \geq 0$ for all $l \in I$ and $g,h \in \mathtt{C}(I,\mathbb{R})$. We will investigate the following system of fractional differential equation:

$$^{\mathtt{C}}D^\nu g(l) = \mathcal{K}_1\big(l,g(l)\big); \quad g \in \mathtt{C}(I,\mathbb{R}), \tag{4.1}$$

$$^{\mathtt{C}}D^\nu h(l) = \mathcal{K}_2\big(l,h(l)\big); \quad h \in \mathtt{C}(I,\mathbb{R}) \tag{4.2}$$

with boundary conditions $g(0) = 0, Ig(1) = g'(0), h(0) = 0, Ih(1) = h'(0)$.

Here, $^{\mathtt{C}}D^\nu$ represents the CFD of order ν given as

$$^{\mathtt{C}}D^\nu g(l) = \frac{1}{\gamma(p-\nu)} \int_0^l \big((l-e)^{p-\nu-1} g(e)\big)\, de,$$

where $p-1 < \nu < p$ and $p = [n]+1$, and $I^\nu g$ is defined by:

$$I^\nu g(l) = \frac{1}{\gamma(\nu)} \int_0^l \big((l-e)^{\nu-1} g(e)\big)\, de, \quad \text{with } \nu > 0.$$

Then the Eqs. (4.1) and (4.2) can be modified to

$$g(l) = \frac{1}{\gamma(\nu)} \int_0^l (l-e)^{\nu-1} \mathcal{K}_1\big(e,g(e)\big)\, de + \frac{2l}{\gamma(\nu)} \int_0^L \int_0^e (e-z)^{\nu-1} \mathcal{K}_1\big(z,g(z)\big)\, dz\, de,$$
$$h(l) = \frac{1}{\gamma(\nu)} \int_0^l (l-e)^{\nu-1} \mathcal{K}_2\big(e,h(e)\big)\, de + \frac{2l}{\gamma(\nu)} \int_0^L \int_0^e (e-z)^{\nu-1} \mathcal{K}_2\big(z,h(z)\big)\, dz\, de.$$

Suppose that,

(a) there exists $\tau > 0$ such that,

$$\big|\mathcal{K}_1\big(l,g(e)\big) - \mathcal{K}_2\big(l,h(e)\big)\big| \leq \frac{e^{-\tau}\gamma(\nu+1)}{4} \big|g(e)-h(e)\big|,$$

for all $e \in I$.

(b) There exists $f_0 \in \mathtt{C}(I,\mathbb{R})$ so that for any $l \in I$,

$$g_0(l) \leq \frac{1}{\gamma(\nu)} \int_0^l (l-e)^{\nu-1} \mathcal{K}_1\big(e,u_0(e)\big)\, de$$

$$+\frac{2l}{\gamma(\nu)}\int_0^L\int_0^e (e-z)^{\nu-1}\mathcal{K}_1\big(z,u_0(z)\big)\,dz\,de,$$

$$\begin{aligned} h_0(l) \le{}& \frac{1}{\gamma(\nu)}\int_0^l (l-e)^{\nu-1}\mathcal{K}_2\big(e,j_0(e)\big)\,de \\ &+\frac{2l}{\gamma(\nu)}\int_0^L\int_0^e (e-z)^{\nu-1}\mathcal{K}_2\big(z,j_0(z)\big)\,dz\,de \end{aligned}$$

(c) Let $X=\{u\in \mathsf{C}(I,\mathbb{R}) : u(l)\geq 0 \text{ for all } l\in I\}$ and define the operator $\mathcal{R}_1,\mathcal{R}_2 : X\to Xby$:

$$\begin{aligned} (\mathcal{R}_1 q)(l) ={}& \frac{1}{\gamma(\nu)}\int_0^l (l-e)^{\nu-1}\mathcal{K}_1\big(e,q(e)\big)\,de \\ &+\frac{2l}{\gamma(\nu)}\int_0^L\int_0^e (e-z)^{\nu-1}\mathcal{K}_1\big(z,q(z)\big)\,dz\,de \quad \text{and} \\ (\mathcal{R}_2 h)(l) ={}& \frac{1}{\gamma(\nu)}\int_0^l (l-e)^{\nu-1}\mathcal{K}_2\big(e,h(e)\big)\,de \\ &+\frac{2l}{\gamma(\nu)}\int_0^L\int_0^e (e-z)^{\nu-1}\mathcal{K}_2\big(z,h(z)\big)\,dz\,de \end{aligned}$$

satisfying

$$\begin{aligned} &\max\left\{d_b\big(\varkappa,\mathcal{R}_1(\varkappa)\big), d_b\big(y,\mathcal{R}_2(y)\big), \frac{d_b(\varkappa,\mathcal{R}_1(\varkappa)).d_b(y,\mathcal{R}_2(y))}{1+d_b(\varkappa,y)}\right\} \\ &\quad\le \big|(\mathcal{R}_1 y)(t)-(\mathcal{R}_2 y)(t)\big|^2. \end{aligned}$$

Theorem 4.1 *The Eqs.* (4.1) *and* (4.2) *admit a common solution in* $\mathsf{C}(I,\mathbb{R})$ *if the conditions* (*a*)–(*c*) *are satisfied.*

Proof Consider,

$$\big|(\mathcal{R}_1 g)(l)-(\mathcal{R}_2 u)(l)\big| = \left|\begin{array}{l} \frac{1}{\gamma(\nu)}\int_0^l (l-e)^{\nu-1}\mathcal{K}_1\big(e,g(e)\big)\,de \\ -\frac{1}{\gamma(\nu)}\int_0^l (l-e)^{\nu-1}\mathcal{K}_2\big(e,u(e)\big)\,de \\ +\frac{2l}{\gamma(\nu)}\int_0^L\int_0^z (z-w)^{\nu-1}\mathcal{K}_1\big(w,g(w)\big)\,dw\,dz \\ -\frac{2l}{\gamma(\nu)}\int_0^L\int_0^z (z-w)^{\nu-1}\mathcal{K}_2\big(w,u(w)\big)\,dw\,dz \end{array}\right|,$$

which implies that,

$$\begin{aligned} &\big|(\mathcal{R}_1 g)(l)-(\mathcal{R}_2 u)(l)\big| \\ &\quad\le \left|\int_0^l \left(\frac{1}{\gamma(\nu)}(l-e)^{\nu-1}\mathcal{K}_1\big(e,g(e)\big)-\frac{1}{\gamma(\nu)}(l-e)^{\nu-1}\mathcal{K}_2\big(e,u(e)\big)\right)de\right| \\ &\qquad+\left|\int_0^L\int_0^z \left(\frac{2}{\gamma(\nu)}(z-w)^{\nu-1}\mathcal{K}_1\big(w,g(w)\big)-\frac{2}{\gamma(\nu)}(z-w)^{\nu-1}\mathcal{K}_2\big(w,u(w)\big)\right)dw\,dz\right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\gamma(\nu)} \frac{e^{-\tau(t)}\gamma(\nu+1)}{4} \int_0^l (l-e)^{\nu-1}\big(\big|g(e)-u(e)\big|\big)\,de \\
&\quad + \frac{2}{\gamma(\nu)} \frac{e^{-\tau(t)}\gamma(\nu+1)}{4} \int_0^L \int_0^z (z-w)^{\nu-1}\big(\big|g(w)-u(w)\big|\big)\,dw\,dz \\
&\leq \frac{1}{\gamma(\nu)} \frac{e^{-\tau(t)}\gamma(\nu+1)}{4} .\big|g(e)-u(e)\big|. \int_0^l (l-e)^{\nu-1}\,de \\
&\quad + \frac{2}{\gamma(\nu)} \frac{e^{-\tau(t)}\gamma(\nu).\gamma(\nu+1)}{4(\nu).\gamma(\nu+1)} .\big|g(e)-u(e)\big|. \int_0^L \int_0^z (z-w)^{\nu-1}\,dw\,dz \\
&\leq \left(\frac{e^{-\tau(t)}\gamma(\nu).\gamma(\nu+1)}{4\gamma(\nu).\gamma(\nu+1)}\right).\big|g(e)-u(e)\big| + 2e^{-\tau(t)}B(\nu+1,1)\frac{\gamma(\nu).\gamma(\nu+1)}{4\gamma(\nu)\gamma.(\nu+1)}.\big|g(e)-u(e)\big| \\
&\leq \frac{e^{-\tau(t)}}{4}\big|g(e)-u(e)\big| + \frac{e^{-\tau(t)}}{2}\big|g(e)-u(e)\big| < \frac{e^{-\tau(t)}}{4}\big|g(e)-u(e)\big| < \frac{e^{-\tau(t)}}{2}\big|g(e)-u(e)\big|.
\end{aligned}$$

This implies that,

$$\big|(\mathcal{R}_1 g)(l) - (\mathcal{R}_2 u)(l)\big| \leq \frac{e^{-\tau(t)}}{2}\big|g(e)-u(e)\big|. \tag{4.3}$$

On taking square both sides of inequality (4.3), we deduce that,

$$\big|(\mathcal{R}_1 g)(l) - (\mathcal{R}_2 u)(l)\big|^2 \leq \frac{e^{-2\tau(t)}}{4}\big|g(e)-u(e)\big|^2 \leq \frac{e^{-\tau(t)}}{4}\big|g(e)-u(e)\big|^2, \tag{4.4}$$

where B is the beta function. By using assumption (c), the final inequality (4.4) is written as

$$\begin{aligned}
&4\max\left\{d_b\big(\varkappa,\mathcal{R}_1(\varkappa)\big), d_b\big(y,\mathcal{R}_2(y)\big), \frac{d_b(\varkappa,\mathcal{R}_1(\varkappa)).d_b(y,\mathcal{R}_2(y))}{1+d_b(\varkappa,y)}\right\} \\
&\quad \leq e^{-\tau(t)} d_b(\varkappa,y); \varkappa, y \in \complement(I,\mathbb{R}), t > 0.
\end{aligned} \tag{4.5}$$

Define $F(q(l)) = \ln(q(l))$ for all $q \in \complement(I,\mathbb{R}^+)$, and $\tau(t) = \frac{1}{8}$ and $\eta(t) = \frac{1}{10}$ for each $t \in (0,\infty)$. Then, clearly $F \in \nabla F_b^*$ and $\tau(t) > \eta(t), \lim_{s\to t^+}\inf\tau(t) > \lim_{s\to t^+}\inf\eta(t)$ then the inequality (4.5) can be written as

$$\begin{aligned}
&\tau\big(d_b(g,u)\big) + \ln\left(4\max\left\{d_b\big(\varkappa,\mathcal{R}_1(\varkappa)\big), d_b\big(y,\mathcal{R}_2(y)\big), \frac{d_b(\varkappa,\mathcal{R}_1(\varkappa)).d_b(y,\mathcal{R}_2(y))}{1+d_b(\varkappa,y)}\right\}\right) \\
&\quad \leq \ln\big(d_b(g,u)\big).
\end{aligned}$$

All the conditions of Theorem 2.4 for a single-valued mapping are verified and the mappings $\mathcal{R}_1$ and $\mathcal{R}_2$ admit a fixed point. Hence, Eqs. (4.1) and (4.2) have a unique common solution. □

5 Conclusion

In this paper, we prove some new fixed point theorems for coupled dominated multivalued mappings on a complete orbital b-metric space that generate a novel extended Nashine–Wardawski–Feng–Liu-type-contraction via orbitally lower semi continuous functions. Furthermore, the presence of some new fixed point outcomes for a pair of dominated multivalued mappings are established in the setting of ordered complete orbital b-metric spaces. A few instances are provided to support our new findings. To highlight the originality of our findings, applications for a system of nonlinear integral equations and fractional differential equations are offered. In addition, we improve and generalize the findings of Nashine et al. [25] and Rasham et al. [32, 33, 36], as well as many others [1–3, 6–8, 18, 26, 28, 40]. We can broaden our horizons to include fuzzy mappings, L-fuzzy mappings, intuitionistic fuzzy mappings, and bipolar fuzzy mappings.

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
T. R and M. S. S together discussed the problem and prepared this research article. M. N analyzed all multivalued hybrid pair results for dominated mappings and made necessary improvements in applications section. A. M; write the original draft of this paper, C. P writing—review and editing, T. R.; project administration. All authors have read and approved final version of the manuscript.

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