Themistocles M. Rassias *Editor*

Approximation Theory and Analytic Inequalities

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Preface

Approximation Theory and Analytic Inequalities focuses on various important areas of Mathematics in which approximation methods play an essential role. Of course, since inequalities are of integral importance for the investigation of approximation and optimization problems, this volume also features cutting-edge research on a wide spectrum of analytic inequalities with emphasis on differential and integral inequalities in the spirit of functional analysis, operator theory, nonlinear analysis, and variational calculus, featuring a plethora of applications. In particular, in this volume the reader will be exposed to the important areas of research such as convexity theory, polynomial inequalities, extremal problems, prediction theory, fixed point theory for operators, PDEs, fractional integral inequalities, multidimensional numerical integration, Gauss–Jacobi and Hermite–Hadamard-type inequalities, Hilbert-type inequalities, as well as Ulam's stability of functional equations. This publication provides significant and up-to-date information and several research results, which could be found useful to a wide readership including graduate students and researchers working in Mathematics, Physics, Economics, Operational Research, and their interconnections. The contributed book chapters have been written by eminent researchers in their corresponding fields. The discussion of concepts, theories, problems, and methods featured in this publication makes it an invaluable reference source. It is our pleasure to express our thanks to all of the contributors in this book who participated in this collective effort. Last but not least, we would like to acknowledge the superb assistance that the Springer staff has provided for this publication.

Athens, Greece — Themistocles M. Rassias

Contents

Harmonic Hermite–Hadamard Inequalities Involving Mittag-Leffler Function

Muhammad Uzair Awan, Marcela V. Mihai, Khalida Inayat Noor, and Muhammad Aslam Noor

Abstract The main objective of this paper is to establish some new refinements of Hermite–Hadamard like inequalities via harmonic convex functions on the co-ordinates with a kernel involving generalized Mittag-Leffler function. Several special cases are also discussed as applications of our main results. The techniques of this paper may be starting point for further research in this dynamic field.

1 Introduction and Preliminaries

Inequality theory has played a fundamental and crucial part in the development of almost all the fields of pure and applied sciences and is continuing to do so. Inequalities present very active and fascinating field of research. Recently, a wide class of inequalities are being derived via different concepts of convexity. As a result of interaction between different branches of mathematical and engineering sciences, convex functions have been extended and generalized in several directions from different points of view. The ideas and techniques of convex functions are being used in a variety of diverse areas of sciences and proved to be productive and innovative. These facts have inspired and motivated the researchers to generalize and extend the concept of convexity in various directions. The development of convexity theory can be viewed as the simultaneous pursuit of two different lines of research. It reveals the fundamental facts on the qualitative behavior of the solution to important classes of problems; on the other hand, it also helps us to develop highly efficient and powerful new numerical techniques to solve complicated and complex problems. In

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fact, convexity theory provides us a sound basis for computing the approximate and analytical solutions of a large number of seemingly unrelated problems in a general and unified framework. For example, the variational inequalities, which can be regarded as a natural extension of variational principles, are related to the simple fact that the minimum of a differentiable convex function on a convex set in any normed space can be characterized by a variational inequality. However, it is remarkable and amazing that variational inequalities allow many diversified applications in ever branch of pure and applied sciences. See, for example, [18, 19, 22, 23]. On other hand, a function is a convex function, if and only if, it satisfies the Hermite–Hadamard type inequality. Convex functions have been generalized and extended in several directions using interesting and novel ideas. Several new classes of convex functions and convex sets have been introduced and investigated. Various new inequalities related to these new classes of convex functions have been derived by researchers. It is worth mentioning that the weighted arithmetic mean is used to define the convex set. Related to the arithmetic mean, we have harmonic mean, which has applications in electrical circuit theory and other branches of sciences. It is known that the total resistance of a set of parallel resisters is obtained by adding up the reciprocal of the individual resistance value and then considering the reciprocal of their total. Anderson et al.[1] have considered and studied some other properties of the harmonic convex functions. In particular, it has been shown that a function f is a harmonic convex, if and only if, it satisfies the inequality of the type

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}, \tag{1}$$

which is called the Hermite–Hadamard inequality for harmonic convex function. Noor and Noor [21] have shown that the optimality conditions of the differentiable harmonic convex functions on the harmonic convex set can be expressed by a class of variational inequalities, which is called the harmonic variational inequality. This shows that harmonic convex functions have similar properties that convex functions have. This allows us to use the analogue results of the convex functions to suggest similar numerical methods for the harmonic convex functions. This is itself an interesting problem. See also Noor et al. [22] for more details. This inequality gives us a lower and an upper estimation for the integral average of harmonic convex functions defined on compact intervals, involving the midpoint and endpoints of the domain. It is not a consequence of harmonic convexity but characterizes it as it provides us necessary and sufficient condition for a function to be harmonic convex. It plays a significant role in numerical analysis and also has applications in theory of means. For some more details, see [4, 6, 32] and the references therein. Recently, many researchers have extended Hermite–Hadamard's inequality on two dimensions utilizing co-ordinated convex function, see [2, 3, 5, 24, 28].
Fractional calculus is the branch of mathematics in which we discuss the ideas of arbitrary order differentiation and integration. Since the appearance of these ideas, there were no acceptable geometrical and physical interpretation for many years. Now we know that the geometric interpretation of fractional integration is

"Shadows on the walls" and its physical interpretation is "Shadows of the past." Recently, it experienced a rapid development due to its great many applications in different fields of pure and applied sciences since it is a good tool to describe long memory processes. For details, see [7, 11, 16]. Recently, inequalities experts have also used the ideas and techniques of fractional calculus in obtaining several fractional refinements of classical inequalities. Sarikaya et al. [29] used fractional integrals and obtained the fractional version of Hermite–Hadamard's inequality. For some recent studies and investigations, see [10, 12–15, 17, 27, 30].
The Mittag-Leffler function is a special function, which arises naturally in the solution of fractional order integral equations or fractional order differential equations. It is also involved in the study of the fractional generalization of the kinetic eqnarray, random walks, Levy flights, super diffusive transport, and in the study of complex systems. For interesting details, see [9].
We now discuss some basic concepts and results that will be helpful in obtaining main results of the paper.
In recent years, the concept of convexity has been extended and generalized in different directions. Noor et al. [20] introduced the notion of co-ordinated harmonic convex functions.

Definition 1 ([20]) Let us consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2 \setminus \{(0, 0)\}$ with $a < b, c < d$. A function $f : \Delta \to \mathbb{R}$ will be called harmonic convex on the rectangle Δ, if

$$f\left(\frac{ab}{ta + (1-t)b}, \frac{cd}{tc + (1-t)d}\right) \leq (1-t)f(a, c) + tf(b, d), \tag{2}$$

for all $(a, b), (c, d) \in \Delta$, $t \in [0, 1]$.

These co-ordinated harmonic convex functions may be defined as:

Definition 2 ([20]) Let us consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2 \setminus \{0\}$ with $a < b, c < d$. A function $f : \Delta \to \mathbb{R}$ is said to be co-ordinated harmonic convex function on the rectangle Δ, if

$$\begin{aligned} &f\left(\frac{xy}{tx + (1-t)y}, \frac{uv}{su + (1-s)v}\right) \\ &\leq (1-t)(1-s)f(x, u) + (1-s)tf(y, u) + s(1-t)f(x, v) + tsf(y, v), \end{aligned} \tag{3}$$

for all $(x, y), (u, v) \in \Delta$, $t, s \in [0, 1]$.

We would like to mention that a function $f : \Delta \subset \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is called harmonic on the co-ordinates if the partial mappings $f_y : [a, b] \to \mathbb{R}$, defined by $f_y(u) = f(u, y)$, and $f_x : [c, d] \to \mathbb{R}$, defined by $f_x(v) = f(x, v)$, are harmonic convex for all $x \in [a, b]$ and $y \in [c, d]$.

Definition 3 Let $f \in L[a, b]$, where $a \geq 0$. The Riemann–Liouville integrals $J^{\nu}_{a+}f$ and $J^{\nu}_{b-}f$, of order $\nu > 0$, are defined by

$$J_{a+}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t)\mathrm{d}t, \text{ for } x > a$$

and

$$J_{b-}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (t-x)^{\nu-1} f(t)\mathrm{d}t, \text{ for } x < b,$$

respectively. Here, $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1}\mathrm{d}t$ is the Gamma function. We also make the convention

$$J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x).$$

Definition 4 ([8, 16]) Let $f \in L(\Delta)$, $\Delta = [a, b] \times [c, d]$. The Riemann–Liouville integral

$$J_{a^+,c^+}^{\nu_1,\nu_2} f(x, y) = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_a^x \int_c^y (x-t)^{\nu_1-1}(y-s)^{\nu_2-1} f(t, s)\mathrm{d}s\mathrm{d}t, x > a, y > c$$

$$J_{a^+,d^-}^{\nu_1,\nu_2} f(x, y) = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_a^x \int_y^d (x-t)^{\nu_1-1}(s-y)^{\nu_2-1} f(t, s)\mathrm{d}s\mathrm{d}t, x > a, y < d$$

$$J_{b^-,c^+}^{\nu_1,\nu_2} f(x, y) = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_x^b \int_c^y (t-x)^{\nu_1-1}(y-s)^{\nu_2-1} f(t, s)\mathrm{d}s\mathrm{d}t, x < b, y > c$$

and

$$J_{b^-,d^-}^{\nu_1,\nu_2} f(x, y) = \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_x^b \int_y^d (t-x)^{\nu_1-1}(s-y)^{\nu_2-1} f(t, s)\mathrm{d}s\mathrm{d}t, x < b, y < d,$$

respectively.
Here, Γ is the Gamma function, $J_{a^+,c^+}^{0,0} f(x, y) = J_{a^+,d^-}^{0,0} f(x, y) = J_{b^-,c^+}^{0,0} f(x, y) = J_{b^-,d^-}^{0,0} f(x, y)$ and $J_{a^+,c^+}^{1,1} f(x, y) = \int_a^x \int_c^y f(t, s)\mathrm{d}s\mathrm{d}t$.
More details about the Riemann–Liouville fractional integrals can be found in [8].
In [26], Salim and Faraj have defined the generalized fractional integral operators containing Mittag-Leffler function:

Definition 5 Let μ, ν, k, l, γ be positive real numbers and $\omega \in \mathbb{R}$. Then the generalized fractional integral operators containing Mittag-Leffler function $\varepsilon_{\mu,\nu,l,\omega,a^+}^{\gamma,\delta,k}$ and $\varepsilon_{\mu,\nu,l,\omega,b^-}^{\gamma,\delta,k}$ for a real-valued continuous function f are defined by

$$\left(\varepsilon_{\mu,\nu,l,\omega,a^+}^{\gamma,\delta,k} f\right)(x) = \int_a^x (x-t)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega(x-t)^{\mu}\right) f(t)\mathrm{d}t, \tag{4}$$

respectively

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^-}f\right)(x)=\int_x^b(t-x)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega(t-x)^{\mu}\right)f(t)\mathrm{d}t,\tag{5}$$

where the function $E^{\gamma,\delta,k}_{\mu,\nu,l}$ is generalized Mittag-Leffler function defined as

$$E^{\gamma,\delta,k}_{\mu,\nu,l}(t)=\sum_{n=0}^{\infty}\frac{(\gamma)_{kn}}{\Gamma(\mu n+\nu)}\frac{t^n}{(\delta)_{ln}},$$

and $(a)_n$ is the Pochhammer symbol: $(a)_n=a(a+1)\cdot\ldots\cdot(a+n-1)$, $(a)_0=1$.

Remark If $k=l=1$ in (4), then the integral operator $\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,1,\omega,a^+}f\right)$ reduces to an integral operator $\left(\varepsilon^{\gamma,\delta,1}_{\mu,\nu,l,\omega,a^+}f\right)$ containing generalized Mittag-Leffler function $E^{\gamma,\delta,1}_{\mu,\nu,1}$ introduced by Srivastava and Tomovski in [31]. Along with $k=l=1$, in addition if $\delta=1$ then (4) reduces to an integral operator defined by Prabhaker in [25] containing Mittag-Leffler function $E^{\gamma}_{\mu,\nu}$. For $\omega=0$ in (4), the integral operator $\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}f\right)$ reduces to the Riemann–Liouville fractional integral operator [26]. In [26], the properties of the generalized integral operator and the generalized Mittag-Leffler function are studied. It is proved that $E^{\gamma,\delta,k}_{\mu,\nu,l}(t)$ is absolutely convergent for all $t\in\mathbb{R}$, where $k<l+\mu$. Since $|E^{\gamma,\delta,k}_{\mu,\nu,l}(t)|\le\sum_{n=0}^{\infty}\left|\frac{(\gamma)_{kn}}{\Gamma(\mu n+\nu)}\frac{t^n}{(\delta)_{ln}}\right|$ with $\sum_{n=0}^{\infty}\left|\frac{(\gamma)_{kn}}{\Gamma(\mu n+\nu)}\frac{t^n}{(\delta)_{ln}}\right|=S$, we have $|E^{\gamma,\delta,k}_{\mu,\nu,l}(t)|\le S$.

Inspired by Definition 5, we will give the following definition:

Definition 6 Let μ,ν,k,l,γ be positive real numbers and $\omega\in\mathbb{R}$, then

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+,c^+}f\right)(x,y)$$
$$=\int_a^x\int_c^y(x-t)^{\nu_1-1}(y-s)^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(x-t)^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(y-s)^{\mu_2}\right)f(t,s)\mathrm{d}s\mathrm{d}t,$$

$x>a$, $y>c$;

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+,d^-}f\right)(x,y)$$
$$=\int_a^x\int_y^d(x-t)^{\nu_1-1}(s-y)^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(x-t)^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(s-y)^{\mu_2}\right)f(t,s)\mathrm{d}s\mathrm{d}t,$$

$x>a$, $y<d$;

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^-,c^+}f\right)(x,y)$$
$$=\int_x^b\int_c^y(t-x)^{\nu_1-1}(y-s)^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(t-x)^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(y-s)^{\mu_2}\right)f(t,s)\mathrm{d}s\mathrm{d}t,$$

$x < b, y > d$, respectively

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b^-,d^-} f\right)(x,y)$$
$$= \int_x^b \int_y^d (t-x)^{\nu_1-1}(s-y)^{\nu_2-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(t-x)^{\mu_1}\right) E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(s-y)^{\mu_2}\right) f(t,s)\mathrm{d}s\mathrm{d}t,$$

$x < b, y < d$, where $\mu = (\mu_1,\mu_2), \nu = (\nu_1,\nu_2), \omega = (\omega_1,\omega_2), \gamma = (\gamma_1,\gamma_2), \delta = (\delta_1,\delta_2), k = (k_1,k_2), \mu,\nu,\omega,\gamma,\delta,k > (0,0)$.

Similar to Definition 6, we introduce the following fractional integrals:

Definition 7 Let μ,ν,k,l,γ be positive real numbers and $\omega \in \mathbb{R}$, then

$$\left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1,a^+}\right) f\left(x, \frac{c+d}{2}\right) = \int_a^x (x-t)^{\nu_1-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(t-x)^{\mu_1}\right) f\left(t, \frac{c+d}{2}\right)\mathrm{d}t,$$
$$\left(\varepsilon^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1,\omega_1,b^-}\right) f\left(x, \frac{c+d}{2}\right) = \int_x^b (t-x)^{\nu_1-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1(t-x)^{\mu_1}\right) f\left(t, \frac{c+d}{2}\right)\mathrm{d}t,$$
$$\left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2,c^+}\right) f\left(\frac{a+b}{2}, y\right) = \int_c^y (y-s)^{\nu_2-1} E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(y-s)^{\mu_2}\right) f\left(\frac{a+b}{2}, s\right)\mathrm{d}s,$$
$$\left(\varepsilon^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2,\omega_2,d^-}\right) f\left(\frac{a+b}{2}, y\right) = \int_y^d (s-y)^{\nu_2-1} E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2(s-y)^{\mu_2}\right) f\left(\frac{a+b}{2}, s\right)\mathrm{d}s.$$

Definition 8 A function $g : \Delta \subset \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2ab}{a+b}$ and $\frac{2cd}{c+d}$ on the co-ordinates if

$$g(x,y) = \begin{cases} g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}, \frac{1}{\frac{1}{c}+\frac{1}{d}-\frac{1}{y}}\right) \\ g\left(x, \frac{1}{\frac{1}{c}+\frac{1}{d}-\frac{1}{y}}\right) \\ g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}, y\right) \end{cases}$$

holds for all $x \in [a,b]$ and $y \in [c,d]$.

Lemma 1 *Let $p \in \mathbb{R} \setminus \{0\}$, and $g : [a,b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be integrable and p-symmetric with respect to $\frac{a^p+b^p}{2}$, then*

(i) *If $p > 0$,*

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^+} g \circ h\right)(b^p) = \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^-} g \circ h\right)(a^p)$$
$$= \frac{1}{2}\left[\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^+} g \circ h\right)(b^p) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^-} g \circ h\right)(a^p)\right],$$

with $h(x) = x^{1/p}$, $x \in [a^p, b^p]$,

(ii) *If* $p < 0$,

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^+} g \circ h\right)\left(a^p\right) = \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^-} g \circ h\right)\left(b^p\right)$$
$$= \frac{1}{2}\left[\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^+} g \circ h\right)\left(a^p\right) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\left(\frac{a^p+b^p}{2}\right)^-} g \circ h\right)\left(b^p\right)\right],$$

with $h(x) = x^{1/p}$, $x \in [b^p, a^p]$.

2 Results and Discussions

Now we are in a position to present our main results.

Lemma 2 *If the function* $g : \Delta \to \mathbb{R}$ *is nonnegative, integrable, and symmetric with respect to* $\frac{2ab}{a+b}$ *and* $\frac{2cd}{c+d}$ *on the co-ordinates, then the following equalities hold*

$$\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{2ab}{a+b}^-,\frac{2cd}{c+d}^-} g \circ h\right)\left(\frac{1}{b}, \frac{1}{d}\right) = \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{2ab}{a+b}^-,\frac{2cd}{c+d}^+} g \circ h\right)\left(\frac{1}{b}, \frac{1}{c}\right)$$
$$= \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{2ab}{a+b}^+,\frac{2cd}{c+d}^-} g \circ h\right)\left(\frac{1}{a}, \frac{1}{d}\right) = \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{2ab}{a+b}^+,\frac{2cd}{c+d}^+} g \circ h\right)\left(\frac{1}{a}, \frac{1}{c}\right)$$
$$= \frac{1}{4}\left[\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{2ab}{a+b}^-,\frac{2cd}{c+d}^-} g \circ h\right)\left(\frac{1}{b}, \frac{1}{d}\right) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{2ab}{a+b}^-,\frac{2cd}{c+d}^+} g \circ h\right)\left(\frac{1}{b}, \frac{1}{c}\right)\right.$$
$$\left. + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{2ab}{a+b}^+,\frac{2cd}{c+d}^-} g \circ h\right)\left(\frac{1}{a}, \frac{1}{d}\right) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{2ab}{a+b}^+,\frac{2cd}{c+d}^+} g \circ h\right)\left(\frac{1}{a}, \frac{1}{c}\right)\right], \quad (6)$$

where $h : \left[\frac{1}{b}, \frac{1}{a}\right] \times \left[\frac{1}{d}, \frac{1}{c}\right] \to \mathbb{R}$, $h(t, s) = \left(\frac{1}{t}, \frac{1}{s}\right)$.

Proof Since g harmonically symmetric with respect to $\frac{2ab}{a+b}$ and $\frac{2cd}{c+d}$ using Definition 8, we have

$$(g \circ h)(t, s) = g\left(\frac{1}{t}, \frac{1}{s}\right) = g\left(\frac{1}{\frac{1}{a} + \frac{1}{b} - t}, \frac{1}{\frac{1}{c} + \frac{1}{d} - s}\right),$$

for all $t \in \left[\frac{1}{b}, \frac{1}{a}\right]$, $s \in \left[\frac{1}{d}, \frac{1}{c}\right]$, where $h : \left[\frac{1}{b}, \frac{1}{a}\right] \times \left[\frac{1}{d}, \frac{1}{c}\right] \to \mathbb{R}$, $h(t, s) = \left(\frac{1}{t}, \frac{1}{s}\right)$. Hence, in the following integral setting, $x = \frac{1}{a} + \frac{1}{b} - t$, $y = \frac{1}{c} + \frac{1}{d} - s$ gives

$$
\begin{aligned}
&\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}+}g\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)\\
&=\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\int_{\frac{c+d}{2cd}}^{\frac{1}{c}}\left(\frac{1}{a}-t\right)^{\nu_1-1}\left(\frac{1}{c}-s\right)^{\nu_2-1}\\
&\times E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1\left(\frac{1}{a}-t\right)^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2\left(\frac{1}{c}-s\right)^{\mu_2}\right)g\left(\frac{1}{t},\frac{1}{s}\right)\mathrm{d}s\mathrm{d}t\\
&=\int_{\frac{a+b}{2ab}}^{\frac{1}{b}}\int_{\frac{c+d}{2cd}}^{\frac{1}{d}}\left(x-\frac{1}{b}\right)^{\nu_1-1}\left(y-\frac{1}{d}\right)^{\nu_2-1}\\
&\times E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1\left(x-\frac{1}{b}\right)^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2\left(y-\frac{1}{d}\right)^{\mu_2}\right)g\left(\frac{1}{x},\frac{1}{y}\right)(-\mathrm{d}y)(-\mathrm{d}x)\\
&=\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\int_{\frac{1}{d}}^{\frac{c+d}{2cd}}\left(x-\frac{1}{b}\right)^{\nu_1-1}\left(y-\frac{1}{d}\right)^{\nu_2-1}\\
&\times E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1\left(x-\frac{1}{b}\right)^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2\left(y-\frac{1}{d}\right)^{\mu_2}\right)g\left(\frac{1}{x},\frac{1}{y}\right)\mathrm{d}y\mathrm{d}x\\
&=\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}-}g\right)\left(\frac{1}{b},\frac{1}{d}\right).
\end{aligned}\tag{7}
$$

Similarly, we get

$$
\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}+}g\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)=\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}+}g\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right),\tag{8}
$$

respectively,

$$
\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}+}g\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)=\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}-}g\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right).\tag{9}
$$

Combining equalities (7), (8), and (9), we get equality (6) and the proof is complete. □

Remark If we take $\omega_1=\omega_2=0$, then (6) gets

$$
\begin{aligned}
&\left(J^{\nu_1,\nu_2}_{\frac{2ab}{a+b}-,\frac{2cd}{c+d}-}g\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right)=\left(J^{\nu_1,\nu_2}_{\frac{2ab}{a+b}-,\frac{2cd}{c+d}+}g\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right)\\
&=\left(J^{\nu_1,\nu_2}_{\frac{2ab}{a+b}+,\frac{2cd}{c+d}-}g\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right)=\left(J^{\nu_1,\nu_2}_{\frac{2ab}{a+b}+,\frac{2cd}{c+d}+}g\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)\\
&=\frac{1}{4}\left[\left(J^{\nu_1,\nu_2}_{\frac{2ab}{a+b}-,\frac{2cd}{c+d}-}g\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right)+\left(J^{\nu_1,\nu_2}_{\frac{2ab}{a+b}-,\frac{2cd}{c+d}+}g\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right)\right.
\end{aligned}
$$

$$+\left(J^{\nu_1,\nu_2}_{\frac{2ab}{a+b}+,\frac{2cd}{c+d}-}g\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right)+\left(J^{\nu_1,\nu_2}_{\frac{2ab}{a+b}+,\frac{2cd}{c+d}+}g\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)\Bigg].$$

The next result is the Hermite–Hadamard type inequality via harmonically convex functions on the co-ordinates containing the generalized Mittag-Leffler function.

Theorem 1 *Let $f:\Delta\to\mathbb{R}$ be harmonically convex on the co-ordinates on $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}^2\setminus\{(0,0)\}$ and $f\in L[\Delta]$. Then one has the inequalities:*

$$\begin{aligned}
&f\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)\left[\begin{array}{l}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}+}1\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}-}1\right)\left(\frac{1}{a},\frac{1}{d}\right)\\+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}+}1\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}-}1\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right]\\
&\le\left[\begin{array}{l}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}+}f\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}-}f\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right)\\+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}+}f\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}-}f\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right]\\
&\le\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}\\
&\quad\times\left[\begin{array}{l}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}+}1\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}-}1\right)\left(\frac{1}{a},\frac{1}{d}\right)\\+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}+}1\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}-}1\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right],
\end{aligned}\tag{10}$$

where $\mu=(\mu_1,\mu_2)$, $\nu=(\nu_1,\nu_2)$, $\omega'=\left(\omega_1',\omega_2'\right)$, $\gamma=(\gamma_1,\gamma_2)$, $\delta=(\delta_1,\delta_2)$, $k=(k_1,k_2)$, $\mu,\nu,\omega',\gamma,\delta,k>(0,0)$ with $\omega_1'=\frac{\omega_1}{(b-a)^{\mu_1}}$, $\omega_2'=\frac{\omega_2}{(d-c)^{\mu_2}}$ and $h:\left[\frac{1}{b},\frac{1}{a}\right]\times\left[\frac{1}{d},\frac{1}{c}\right]\to\mathbb{R}$, $h(t,s)=\left(\frac{1}{t},\frac{1}{s}\right)$.

Proof If we take $t=s=\frac{1}{2}$ in (3), we get

$$f\left(\frac{2xy}{x+y},\frac{2uv}{u+v}\right)\le\frac{f(x,u)+f(x,v)+f(y,u)+f(y,v)}{4}.\tag{11}$$

Using the substitutions $x=\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b}$, $y=\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a}$, $u=\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}$, and $v=\frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}$ inequality (11) gets

$$f\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)\le\frac{1}{4}\left[\begin{array}{l}f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)+f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right)\\+f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)+f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a},\frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right)\end{array}\right].\tag{12}$$

Thus, multiplying both sides of (12) by $t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1t^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2s^{\mu_2}\right)$ and then by integrating with respect to (t,s) on $[0,1]\times[0,1]$, we obtain

$$f\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right)\mathrm{d}s\mathrm{d}t$$

$$\leq\frac{1}{4}\left[\begin{array}{c}\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\\ f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)\mathrm{d}s\mathrm{d}t\\ +\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\\ f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right)\mathrm{d}s\mathrm{d}t\\ +\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\\ f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)\mathrm{d}s\mathrm{d}t\\ +\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\\ f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a},\frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right)\mathrm{d}s\mathrm{d}t\end{array}\right]. \tag{13}$$

Using substitutions $x=\frac{1}{ab}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)$, $y=\frac{1}{cd}\left(\frac{s}{2}c+\frac{2-s}{2}d\right)$, we have

$$\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right)\mathrm{d}s\mathrm{d}t$$

$$=\left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{c+d}\right)^{\nu_2}\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\int_{\frac{c+d}{2cd}}^{\frac{1}{c}}\left(\frac{1}{a}-x\right)^{\nu_1-1}\left(\frac{1}{c}-y\right)^{\nu_2-1}$$

$$\times E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'\left(\frac{1}{a}-x\right)^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'\left(\frac{1}{c}-y\right)^{\mu_2}\right)\mathrm{d}y\mathrm{d}x$$

$$=\left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{d-c}\right)^{\nu_2}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+,\frac{c+d}{2cd}+}1\right)\left(\frac{1}{a},\frac{1}{c}\right), \tag{14}$$

$$\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right)f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)\mathrm{d}s\mathrm{d}t$$

$$=\left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{c+d}\right)^{\nu_2}\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\int_{\frac{c+d}{2cd}}^{\frac{1}{c}}\left(\frac{1}{a}-x\right)^{\nu_1-1}\left(\frac{1}{c}-y\right)^{\nu_2-1}$$

$$\times E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'\left(\frac{1}{a}-x\right)^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'\left(\frac{1}{c}-y\right)^{\mu_2}\right)f\left(\frac{1}{x},\frac{1}{y}\right)\mathrm{d}y\mathrm{d}x$$

$$=\left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{d-c}\right)^{\nu_2}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+,\frac{c+d}{2cd}+}f\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right). \tag{15}$$

Analogously, we obtain

$$\int_0^1 \int_0^1 t^{\nu_1-1} s^{\nu_2-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right) E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right) f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b}, \frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right) \mathrm{d}s\mathrm{d}t$$
$$= \left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{d-c}\right)^{\nu_2}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+,\frac{c+d}{2cd}-} f\circ h\right)\left(\frac{1}{a}, \frac{1}{d}\right), \tag{16}$$

$$\int_0^1 \int_0^1 t^{\nu_1-1} s^{\nu_2-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right) E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right) f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a}, \frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right) \mathrm{d}s\mathrm{d}t$$
$$= \left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{d-c}\right)^{\nu_2}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-,\frac{c+d}{2cd}+} f\circ h\right)\left(\frac{1}{b}, \frac{1}{c}\right), \tag{17}$$

$$\int_0^1 \int_0^1 t^{\nu_1-1} s^{\nu_2-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right) E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right) f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a}, \frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right) \mathrm{d}s\mathrm{d}t$$
$$= \left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{d-c}\right)^{\nu_2}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-,\frac{c+d}{2cd}-} f\circ h\right)\left(\frac{1}{b}, \frac{1}{d}\right). \tag{18}$$

Introducing relationships (14)–(18) in (13), we get, after multiplying with $\left(\frac{b-a}{2ab}\right)^{\nu_1}\left(\frac{d-c}{2cd}\right)^{\nu_2}$ and using Lemma 2

$$f\left(\frac{2ab}{a+b}, \frac{2cd}{c+d}\right)\left[\begin{array}{c}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}+}1\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}-}1\right)\left(\frac{1}{a},\frac{1}{d}\right)\\ +\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}+}1\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}-}1\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right]$$
$$\leq \left[\begin{array}{c}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}+}f\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}-}f\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right)\\ +\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}+}f\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}-}f\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right],$$

with which the first inequality of (10) is proved.

For the proof of the second inequality in (10), we first note that f is a harmonic convex function on the co-ordinates on Δ, and then, by using (3) it yields

$$f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b}, \frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)$$
$$\leq \frac{2-t}{2}\cdot\frac{2-s}{2}f(a,c)+\frac{2-s}{2}\cdot\frac{t}{2}f(b,c)+\frac{2-t}{2}\cdot\frac{s}{2}f(a,d)+\frac{t}{2}\cdot\frac{s}{2}f(b,d),$$
$$f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b}, \frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right)$$

$$\leq \frac{2-t}{2}\cdot\frac{2-s}{2}f(a,d)+\frac{2-s}{2}\cdot\frac{t}{2}f(b,d)+\frac{2-t}{2}\cdot\frac{s}{2}f(a,c)+\frac{t}{2}\cdot\frac{s}{2}f(b,c),$$

$$f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)$$

$$\leq \frac{2-t}{2}\cdot\frac{2-s}{2}f(b,c)+\frac{2-s}{2}\cdot\frac{t}{2}f(a,c)+\frac{2-t}{2}\cdot\frac{s}{2}f(b,d)+\frac{t}{2}\cdot\frac{s}{2}f(a,d),$$

$$f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a},\frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right)$$

$$\leq \frac{2-t}{2}\cdot\frac{2-s}{2}f(b,d)+\frac{2-s}{2}\cdot\frac{t}{2}f(a,d)+\frac{2-t}{2}\cdot\frac{s}{2}f(b,c)+\frac{t}{2}\cdot\frac{s}{2}f(a,c).$$

By adding these inequalities, we have

$$\begin{aligned}&f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)+f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right)\\&+f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)+f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a},\frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right)\\&\leq f(a,c)+f(b,c)+f(a,d)+f(b,d).\end{aligned}\tag{19}$$

Then, multiplying both sides of (19) by $\frac{1}{4}t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})$ and integrating with respect to (t,s) on $[0,1]\times[0,1]$, we get

$$\begin{aligned}&\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)\mathrm{d}s\mathrm{d}t\\&+\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right)\mathrm{d}s\mathrm{d}t\\&+\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)\mathrm{d}s\mathrm{d}t\\&+\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\,f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a},\frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right)\mathrm{d}s\mathrm{d}t\\&\leq\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}(\omega_1 t^{\mu_1})\,E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}(\omega_2 s^{\mu_2})\begin{bmatrix}f(a,c)+f(b,c)\\+f(a,d)+f(b,d)\end{bmatrix}\mathrm{d}s\mathrm{d}t.\end{aligned}$$

So, after multiplying with $\left(\frac{b-a}{2ab}\right)^{\nu_1}\left(\frac{d-c}{2dc}\right)^{\nu_2}$ and using Lemma 2, we have

$$\begin{bmatrix} \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^+} f\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^-} f\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right) \\ + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^+} f\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-} f\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right) \end{bmatrix}$$
$$\leq \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}$$
$$\times \begin{bmatrix} \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^+} 1\right)\left(\frac{1}{a},\frac{1}{c}\right) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^-} 1\right)\left(\frac{1}{a},\frac{1}{d}\right) \\ + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^+} 1\right)\left(\frac{1}{b},\frac{1}{c}\right) + \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-} 1\right)\left(\frac{1}{b},\frac{1}{d}\right) \end{bmatrix},$$

which finishes the proof. □

Remark For $\omega_1 = \omega_2 = 0$, Theorem 1 is transformed into a new theorem with integrals of Riemann–Liouville type:

Theorem 2 *Let $f : \Delta \to \mathbb{R}$ be harmonically convex on the co-ordinates on $\Delta = [a,b]\times[c,d]$ in $\mathbb{R}^2 \setminus \{(0,0)\}$ and $f \in L[\Delta]$. Then one has the inequalities:*

$$f\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)\begin{bmatrix} \left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^+} 1\right)\left(\frac{1}{a},\frac{1}{c}\right) + \left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^-} 1\right)\left(\frac{1}{a},\frac{1}{d}\right) \\ + \left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^+} 1\right)\left(\frac{1}{b},\frac{1}{c}\right) + \left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-} 1\right)\left(\frac{1}{b},\frac{1}{d}\right) \end{bmatrix}$$
$$\leq \begin{bmatrix} \left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^+} f\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right) + \left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^-} f\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right) \\ + \left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^+} f\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right) + \left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-} f\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right) \end{bmatrix}$$
$$\leq \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}$$
$$\times \begin{bmatrix} \left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^+} 1\right)\left(\frac{1}{a},\frac{1}{c}\right) + \left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^-} 1\right)\left(\frac{1}{a},\frac{1}{d}\right) \\ + \left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^+} 1\right)\left(\frac{1}{b},\frac{1}{c}\right) + \left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-} 1\right)\left(\frac{1}{b},\frac{1}{d}\right) \end{bmatrix},$$

where $h : \left[\frac{1}{b},\frac{1}{a}\right]\times\left[\frac{1}{d},\frac{1}{c}\right] \to \mathbb{R}$, $h(t,s) = \left(\frac{1}{t},\frac{1}{s}\right)$.

The following theorem establishes Hermite–Hadamard–Fejer type inequalities for co-ordinated harmonic convex functions containing the generalized Mittag-Leffler function.

Theorem 3 *Let $f : \Delta \to \mathbb{R}$ be harmonically convex on the co-ordinates on $\Delta = [a,b]\times[c,d]$ in $\mathbb{R}^2\setminus\{(0,0)\}$, $f \in L[\Delta]$, and the function $g : \Delta \to \mathbb{R}$ is nonnegative, integrable, and symmetric with respect to $\frac{2ab}{a+b}$ and $\frac{2cd}{c+d}$ on the co-ordinates, then one has the inequalities:*

$$f\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)\left[\begin{array}{l}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}+}g\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)\\+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}-}g\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right)\\+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}+}g\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right)\\+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}-}g\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right]$$

$$\leq\left[\begin{array}{l}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}+}fg\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}-}fg\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right)\\+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}+}fg\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}-}fg\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right]$$

$$\leq\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}$$

$$\times\left[\begin{array}{l}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}+}g\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}+,\frac{c+d}{2cd}-}g\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right)\\+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}+}g\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}-,\frac{c+d}{2cd}-}g\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right],\quad(20)$$

where $\mu=(\mu_1,\mu_2)$, $\nu=(\nu_1,\nu_2)$, $\omega'=(\omega_1',\omega_2')$, $\gamma=(\gamma_1,\gamma_2)$, $\delta=(\delta_1,\delta_2)$, $k=(k_1,k_2)$, $\mu,\nu,\omega',\gamma,\delta,k>(0,0)$ *with* $\omega_1'=\frac{\omega_1}{(b-a)^{\mu_1}}$, $\omega_2'=\frac{\omega_2}{(d-c)^{\mu_2}}$ *and* $h:\left[\frac{1}{b},\frac{1}{a}\right]\times\left[\frac{1}{d},\frac{1}{c}\right]\to\mathbb{R}$, $h(t,s)=\left(\frac{1}{t},\frac{1}{s}\right)$.

Proof Since f is a harmonically convex function on Δ, we have inequality (12). Multiplying both sides of this inequality with

$$t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1t^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2s^{\mu_2}\right)g\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)$$

and integrating with respect to (t,s) on $[0,1]\times[0,1]$, we obtain

$$f\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)$$

$$\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1t^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2s^{\mu_2}\right)g\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)\mathrm{d}s\mathrm{d}t$$

$$\leq \frac{1}{4}\left[\begin{array}{c} \int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right) \\ \times f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)g\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)\mathrm{d}s\mathrm{d}t \\ +\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right) \\ \times f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right)g\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)\mathrm{d}s\mathrm{d}t \\ +\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right) \\ \times f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)g\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)\mathrm{d}s\mathrm{d}t \\ +\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right) \\ \times f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a},\frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right)g\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)\mathrm{d}s\mathrm{d}t \end{array}\right]. \tag{21}$$

Using substitutions $x=\frac{1}{ab}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)$, $y=\frac{1}{cd}\left(\frac{s}{2}c+\frac{2-s}{2}d\right)$, we have

$$\begin{aligned}
&\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right)g\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)\mathrm{d}s\mathrm{d}t \\
&=\left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{c+d}\right)^{\nu_2}\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\int_{\frac{c+d}{2cd}}^{\frac{1}{c}}\left(\frac{1}{a}-x\right)^{\nu_1-1}\left(\frac{1}{c}-y\right)^{\nu_2-1} \\
&\times E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'\left(\frac{1}{a}-x\right)^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'\left(\frac{1}{c}-y\right)^{\mu_2}\right)g\left(\frac{1}{x},\frac{1}{y}\right)\mathrm{d}y\mathrm{d}x \\
&=\left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{c+d}\right)^{\nu_2}\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\int_{\frac{c+d}{2cd}}^{\frac{1}{c}}\left(\frac{1}{a}-x\right)^{\nu_1-1}\left(\frac{1}{c}-y\right)^{\nu_2-1} \\
&\times E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'\left(\frac{1}{a}-x\right)^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'\left(\frac{1}{c}-y\right)^{\mu_2}\right)(g\circ h)(x,y)\,\mathrm{d}y\mathrm{d}x \\
&=\left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{d-c}\right)^{\nu_2}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}^{+},\frac{c+d}{2cd}^{+}}g\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right),
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\int_0^1\int_0^1 t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right) \\
&\times f\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)g\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b},\frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)\mathrm{d}s\mathrm{d}t \\
&=\left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{c+d}\right)^{\nu_2}\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\int_{\frac{c+d}{2cd}}^{\frac{1}{c}}\left(\frac{1}{a}-x\right)^{\nu_1-1}\left(\frac{1}{c}-y\right)^{\nu_2-1} \\
&\times E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'\left(\frac{1}{a}-x\right)^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'\left(\frac{1}{c}-y\right)^{\mu_2}\right)f\left(\frac{1}{x},\frac{1}{y}\right)g\left(\frac{1}{x},\frac{1}{y}\right)\mathrm{d}y\mathrm{d}x \\
&=\left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{d-c}\right)^{\nu_2}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}^{+},\frac{c+d}{2cd}^{+}}fg\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right).
\end{aligned} \tag{23}$$

For the next three inequalities, we use previous substitutions and substitutions $u = \frac{1}{a} + \frac{1}{b} - x$, $v = \frac{1}{c} + \frac{1}{d} - y$, respectively, the harmonically symmetric with respect to $\frac{2ab}{a+b}$ and $\frac{2cd}{c+d}$ of g.

$$\int_0^1 \int_0^1 t^{\nu_1-1} s^{\nu_2-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right) E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right)$$
$$\times f\left(\frac{ab}{\frac{t}{2}a + \frac{2-t}{2}b}, \frac{cd}{\frac{s}{2}d + \frac{2-s}{2}c}\right) g\left(\frac{ab}{\frac{t}{2}a + \frac{2-t}{2}b}, \frac{cd}{\frac{s}{2}c + \frac{2-s}{2}d}\right) \mathrm{d}s\mathrm{d}t$$
$$= \left(\frac{2ab}{b-a}\right)^{\nu_1} \left(\frac{2cd}{d-c}\right)^{\nu_2} \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \int_{\frac{c+d}{2cd}}^{\frac{1}{c}} \left(\frac{1}{a} - x\right)^{\nu_1-1} \left(\frac{1}{c} - y\right)^{\nu_2-1}$$
$$\times E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'\left(\frac{1}{a} - x\right)^{\mu_1}\right) E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'\left(\frac{1}{c} - y\right)^{\mu_2}\right)$$
$$f\left(\frac{1}{x}, \frac{1}{\frac{1}{c} + \frac{1}{d} - y}\right) g\left(\frac{1}{x}, \frac{1}{y}\right) \mathrm{d}y\mathrm{d}x$$
$$= \left(\frac{2ab}{b-a}\right)^{\nu_1} \left(\frac{2cd}{d-c}\right)^{\nu_2} \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \int_{\frac{c+d}{2cd}}^{\frac{1}{c}} \left(\frac{1}{a} - x\right)^{\nu_1-1} \left(\frac{1}{c} - y\right)^{\nu_2-1}$$
$$\times E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'\left(\frac{1}{a} - x\right)^{\mu_1}\right) E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'\left(\frac{1}{c} - y\right)^{\mu_2}\right)$$
$$f\left(\frac{1}{x}, \frac{1}{\frac{1}{c} + \frac{1}{d} - y}\right) g\left(\frac{1}{x}, \frac{1}{\frac{1}{c} + \frac{1}{d} - y}\right) \mathrm{d}y\mathrm{d}x$$
$$= \left(\frac{2ab}{b-a}\right)^{\nu_1} \left(\frac{2cd}{d-c}\right)^{\nu_2} \int_{\frac{a+b}{2ab}}^{\frac{1}{a}} \int_{\frac{1}{d}}^{\frac{c+d}{2cd}} \left(\frac{1}{a} - x\right)^{\nu_1-1} \left(v - \frac{1}{d}\right)^{\nu_2-1}$$
$$\times E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1'\left(\frac{1}{a} - x\right)^{\mu_1}\right) E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2'\left(v - \frac{1}{d}\right)^{\mu_2}\right) f\left(\frac{1}{x}, \frac{1}{v}\right) g\left(\frac{1}{x}, \frac{1}{v}\right) \mathrm{d}v\mathrm{d}x$$
$$= \left(\frac{2ab}{b-a}\right)^{\nu_1} \left(\frac{2cd}{d-c}\right)^{\nu_2} \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}+,\frac{c+d}{2cd}-} fg \circ h\right)\left(\frac{1}{a}, \frac{1}{d}\right), \tag{24}$$

$$\int_0^1 \int_0^1 t^{\nu_1-1} s^{\nu_2-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right) E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right)$$
$$\times f\left(\frac{ab}{\frac{t}{2}b + \frac{2-t}{2}a}, \frac{cd}{\frac{s}{2}c + \frac{2-s}{2}d}\right) g\left(\frac{ab}{\frac{t}{2}a + \frac{2-t}{2}b}, \frac{cd}{\frac{s}{2}c + \frac{2-s}{2}d}\right) \mathrm{d}s\mathrm{d}t$$
$$= \left(\frac{2ab}{b-a}\right)^{\nu_1} \left(\frac{2cd}{d-c}\right)^{\nu_2} \left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}-,\frac{c+d}{2cd}+} fg \circ h\right)\left(\frac{1}{b}, \frac{1}{c}\right), \tag{25}$$

$$\int_0^1 \int_0^1 t^{\nu_1-1} s^{\nu_2-1} E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right) E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right)$$
$$\times f\left(\frac{ab}{\frac{t}{2}b+\frac{2-t}{2}a}, \frac{cd}{\frac{s}{2}d+\frac{2-s}{2}c}\right) g\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b}, \frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right) \mathrm{d}s\mathrm{d}t$$
$$= \left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{d-c}\right)^{\nu_2}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-} fg\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right). \tag{26}$$

Introducing relationships (22)–(26) in (13), we get, after multiplying with $\left(\frac{b-a}{2ab}\right)^{\nu_1}\left(\frac{d-c}{2cd}\right)^{\nu_2}$ and using Lemma 2

$$f\left(\frac{2ab}{a+b}, \frac{2cd}{c+d}\right)\left[\begin{array}{l}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^+} g\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^-} g\right)\left(\frac{1}{a},\frac{1}{d}\right)\\ +\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^+} g\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-} g\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right]$$
$$\leq \left[\begin{array}{l}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^+} fg\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^-} fg\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right)\\ +\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^+} fg\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-} fg\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right],$$

with this the first inequality of (20) is proved.

For the proof of the second inequality in (20), we multiply inequality (19) with $t^{\nu_1-1}s^{\nu_2-1}E^{\gamma_1,\delta_1,k_1}_{\mu_1,\nu_1,l_1}\left(\omega_1 t^{\mu_1}\right)E^{\gamma_2,\delta_2,k_2}_{\mu_2,\nu_2,l_2}\left(\omega_2 s^{\mu_2}\right) g\left(\frac{ab}{\frac{t}{2}a+\frac{2-t}{2}b}, \frac{cd}{\frac{s}{2}c+\frac{2-s}{2}d}\right)$ and integrating with respect to (t, s) on $[0, 1]\times[0, 1]$. By computing, we get

$$\left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{d-c}\right)^{\nu_2}\left[\begin{array}{l}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^+} f\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)\\ +\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^-} f\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right)\\ +\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^+} f\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right)\\ +\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-} f\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right]$$
$$\leq \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}\cdot\left(\frac{2ab}{b-a}\right)^{\nu_1}\left(\frac{2cd}{d-c}\right)^{\nu_2}$$
$$\times\left[\begin{array}{l}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^+} 1\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^-} 1\right)\left(\frac{1}{a},\frac{1}{d}\right)\\ +\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^+} 1\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-} 1\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right].$$

So, after multiplying with $\left(\frac{b-a}{2ab}\right)^{\nu_1}\left(\frac{d-c}{2dc}\right)^{\nu_2}$ and using Lemma 2, we have

$$\begin{aligned}
&\left[\begin{array}{l}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^+}f\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^-}f\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right)\\ +\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^+}f\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-}f\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right]\\
&\leq\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}\\
&\times\left[\begin{array}{l}\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^+}1\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^-}1\right)\left(\frac{1}{a},\frac{1}{d}\right)\\ +\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^+}1\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(\varepsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-}1\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right],
\end{aligned}$$

which finishes the proof. □

Remark For $\omega_1=\omega_2=0$, Theorem 3 becomes a new theorem with fractional integrals of Riemann–Liouville type:

Theorem 4 *Let $f:\Delta\to\mathbb{R}$ be harmonically convex on the co-ordinates on $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}^2\backslash\{(0,0)\}$, $f\in L[\Delta]$, and the function $g:\Delta\to\mathbb{R}$ is nonnegative, integrable, and symmetric with respect to $\frac{2ab}{a+b}$ and $\frac{2cd}{c+d}$ on the co-ordinates, then one has the inequalities:*

$$\begin{aligned}
&f\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)\left[\begin{array}{l}\left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^+}g\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^-}g\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right)\\ +\left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^+}g\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-}g\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right]\\
&\leq\left[\begin{array}{l}\left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^+}fg\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^-}fg\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right)\\ +\left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^+}fg\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-}fg\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right]\\
&\leq\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}\\
&\times\left[\begin{array}{l}\left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^+}g\circ h\right)\left(\frac{1}{a},\frac{1}{c}\right)+\left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^+,\frac{c+d}{2cd}^-}g\circ h\right)\left(\frac{1}{a},\frac{1}{d}\right)\\ +\left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^+}g\circ h\right)\left(\frac{1}{b},\frac{1}{c}\right)+\left(J^{\nu_1,\nu_2}_{\frac{a+b}{2ab}^-,\frac{c+d}{2cd}^-}g\circ h\right)\left(\frac{1}{b},\frac{1}{d}\right)\end{array}\right],
\end{aligned}$$

where $h:\left[\frac{1}{b},\frac{1}{a}\right]\times\left[\frac{1}{d},\frac{1}{c}\right]\to\mathbb{R}$, $h(t,s)=\left(\frac{1}{t},\frac{1}{s}\right)$.

3 Conclusion

We have derived several new integral inequalities of Hermite–Hadamard type via the functions having harmonic convexity property on the co-ordinates. These inequalities involve a kernel containing generalized Mittag-Leffler function. We have also discussed some new special cases of the main results. It is expected that the results obtained in the paper may inspire the researchers of this field.

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Two-Dimensional Trapezium Inequalities via pq-Convex Functions

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Abstract We establish some new two-dimensional trapezium-like inequalities involving partial differentiable pq-convex functions on rectangle. The concept of pq-convex functions also includes the harmonic convex functions and convex functions as special cases. These results represent refinement and improvement of the known results. Some cases are discussed, which can be obtained as applications of the results. The ideas and techniques of this chapter may be a starting point for further research.

1 Introduction

In recent years, the classical theory of convexity has experienced rapid development due to its great many applications in different fields of pure and applied sciences. Recently, the classical concept of convexity has been extended and generalized in different directions. For more information, see [1–4, 7–9, 16–18, 26]. Power means [9] can be viewed as a natural extension of the arithmetic means and have been used to introduce the concept of p-convex functions. Zhang et al. [26] studied various properties of the p-convex functions. Obviously, the p-convex functions include the convex functions and harmonic convex functions as special cases. Several Hermite–Hadamard-type inequalities have been obtained for the p-convex functions in recent years. Noor et al. [17] extended the class of p-convex functions to two-dimensional pq-convex function and derived some new and novel integral inequalities.

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It is a known fact that convexity has a close relationship with the theory of inequalities. Many inequalities can be obtained directly using the definition of convex functions. Hermite–Hadamard inequality is one of the most studied results, which can be obtained using convex functions. This result provides us the necessary and sufficient condition for a function to be convex. It reads as: Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2},$$

and conversely. This result is called Hermite–Hadamard's inequality. For some recent studies, see [1, 3–7, 16, 17, 19–25].

In this chapter, we consider the class of pq-convex functions on a rectangle. We establish some new trapezium-like inequalities using pq-convex functions on rectangle. Several special cases of results are obtained as applications. Our results can viewed as significant refinement and improvement of the previous known results. It is an interesting problem to consider the applications of two-dimensional inequalities in numerical analysis and approximation theory.

2 Preliminary Results

In this section, we recall some previously known concepts. For more details, see an excellent book [9].

Definition 1 ([26]) A set K_p is said to be a p-convex set, if

$$[tx^p + (1-t)y^p]^{\frac{1}{p}} \in K_p, \quad \forall x, y \in I, t \in [0, 1], p \neq 0. \tag{1}$$

It is worth mentioning that for $p = 1$, the set K_p becomes the convex set K and for $p = -1$, the p-convex set K_p reduces to the harmonic convex set K_h, respectively. This shows that the p-convex set is quite general and includes the convex set and harmonic convex set as special cases.

Definition 2 ([26]) A function $f : K_p \to \mathbb{R}$ is said to be p-convex function, if

$$f(tx^p + (1-t)y^p)^{\frac{1}{p}} \leq tf(x) + (1-t)f(y), \ \forall x, y \in K_p, t \in [0, 1].$$

Also note that for $t = \frac{1}{2}$, Definition 2, becomes

$$f\left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}} \leq \frac{f(x)+f(y)}{2}, \ \forall x, y \in K_p, t \in [0, 1].$$

The function f is called the Jensen p-convex function.

We now discuss some important special cases of p-convex functions,

I. If $p = 1$, then p-convex functions reduce to:

Definition 3 A function $f : K \to \mathbb{R}$ is said to be a convex function on the convex set K, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),\ \forall x, y \in K, t \in [0, 1].$$

It is known that the minimum $u \in K$ is a minimum of a differentiable convex function f on a convex set K, if and only if, $u \in K$ satisfies the inequality

$$\langle f'(u), v - u\rangle \geq 0, \forall v \in K,$$

which is called the variational inequality. The variational inequalities can be viewed as the natural extension and generalization of the variational principles, the origin of which can be traced back to Euler, Lagrange and Bernoulli's brothers. Variational inequalities have appeared to be a powerful tool to study a wide class of unrelated problems in a unified framework. For the applications, formulation, numerical results, dynamical systems and other aspects of the variational inequalities, see [9–15] and the references therein.

II. If $p = -1$, then p-convex functions reduce to:

Definition 4 A function $f : K_h \to \mathbb{R}$ is said to be a harmonic convex function, if

$$f\left(\frac{xy}{(1-t)x + ty}\right) \leq tf(x) + (1-t)f(y),\ \forall x, y \in K_h, t \in [0, 1].$$

It has been shown by Noor and Noor [13] that $u \in K_h$ is the minimum of a differentiable harmonic convex function, if and only if, $u \in K_h$ satisfies the inequality

$$\left\langle f'(u), \frac{uv}{u - v}\right\rangle \geq 0, \forall v \in K_h,$$

which is called harmonic variational inequality. It is an interesting problem to study the applications and numerical aspects of harmonic variational inequalities. For further details, see [13, 14].

We now consider two-dimensional integral inequalities for pq-convex functions, which is the main focus of this paper. Let us consider a bidimensional interval $\Delta = [a, b] \times [c, d] \subset \mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta \to \mathbb{R}$ is said to be convex function on Δ if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x, y) + (1-t)f(z, w),$$

holds, for all $(x, y), (z, w) \in \Delta$ and $t \in [0, 1]$. This is definition is mainly due to Dragomir [4].

A function $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the partial functions $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex for all $x \in [a, b]$ and $y \in [c, d]$.

Definition 5 ([4]) Let $\Delta = [a, b] \times [c, d] \subset \mathbb{R}^2$ be a rectangle. A function $f : \Delta \to \mathbb{R}$ is said to be (coordinated) convex function on rectangle, if

$$\begin{aligned} &f(tx + (1-t)y, ru + (1-r)w) \\ &\leq trf(x, u) + t(1-r)f(x, w) + r(1-t)f(y, u) + (1-t)(1-r)f(y, w), \end{aligned}$$

whenever $x, y \in [a, b]$, $u, w \in [c, d]$ and $t, r \in [0, 1]$.

Definition 6 ([17]) Let $\Delta = [a, b] \times [c, d] \subset \mathbb{R}^2$ be a rectangle. A function $f : \Delta \to \mathbb{R}$ is said to be pq-convex function on rectangle, if

$$\begin{aligned} &f(M_p(x_1, x_2; t), M_q(y_1, y_2; r)) \\ &\leq trf(x_1, y_1) + t(1-r)f(x_1, y_2) + r(1-t)f(x_2, y_1) \\ &\qquad + (1-t)(1-r)f(x_2, y_2), \end{aligned} \tag{2}$$

whenever $x_1, x_2 \in [a, b]$, $y_1, y_2 \in [c, d]$ and $t, r \in [0, 1]$.

We now discuss some special cases of Definition 6.

I. If $p = q$, then, we have

Definition 7 Let $\Delta = [a, b] \times [c, d] \subset \mathbb{R}^2$ be a rectangle. A function $f : \Delta \to \mathbb{R}$ is said to be p-convex function on rectangle, if

$$\begin{aligned} &f(M_p(x_1, x_2; t), M_p(y_1, y_2; r)) \\ &\leq trf(x_1, y_1) + t(1-r)f(x_1, y_2) + r(1-t)f(x_2, y_1) + (1-t)(1-r)f(x_2, y_2). \end{aligned}$$

II. If $p = 1 = q$, then Definition 6 and Definition 7 reduce to Definition 5.

This shows that the concept of pq-convex functions on rectangle is quite flexible and unifying one.

For some recent investigations on pq-convex functions, see [17].

For the reader's convenience, we recall here the definitions of the Gamma function

$$\Gamma(x) = \int_0^\infty e^{-x} t^{x-1} \mathrm{d}t$$

and the Beta function

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

It holds

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The integral form of the hypergeometric function is

$${}_2F_1(x, y; c; z) = \frac{1}{\mathrm{B}(y, c-y)} \int_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}\mathrm{d}t$$

for $|z| < 1, c > y > 0$.

3 Results and Discussions

In this section, we discuss our main results. For this purpose, we need the following auxiliary result.

Lemma 1 *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a partial differentiable function on $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial t \partial r} \in L_1(\Delta)$, then*

$$\begin{aligned}
&R_f(t, r; p, q; \Delta) \\
&= \frac{(b^p - a^p)(d^q - c^q)}{4pq} \\
&\quad \times \int_0^1 \int_0^1 \left(\frac{1-2t}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}}\right)\left(\frac{1-2r}{[ra^p + (1-r)b^p]^{1-\frac{1}{p}}}\right) \\
&\quad \times \frac{\partial^2 f}{\partial t \partial r}\big(M_p(a, b; t), M_q(c, d; r)\big)\mathrm{d}t\mathrm{d}r,
\end{aligned}$$

where

$$\begin{aligned}
&R_f(t, r; p, q; \Delta) \\
&= \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{4} \\
&\quad - \frac{p}{2(b^p - a^p)}\left[\int_a^b x^{p-1} f(x, c)\mathrm{d}x + \int_a^b x^{p-1} f(x, d)\mathrm{d}x\right]
\end{aligned}$$

$$-\frac{q}{2(d^q-c^q)}\left[\int_c^d y^{q-1}f(a,y)\mathrm{d}y+\int_c^d y^{q-1}f(b,y)\mathrm{d}y\right]$$
$$+\frac{pq}{(b^p-a^p)(d^q-c^q)}\int_a^b\int_c^d x^{p-1}y^{q-1}f(x,y)\mathrm{d}x\mathrm{d}y.$$

Proof Consider

$$\int_0^1\int_0^1\left(\frac{1-2t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right)\left(\frac{1-2r}{[ra^p+(1-r)b^p]^{1-\frac{1}{p}}}\right)$$
$$\frac{\partial^2 f}{\partial t\partial r}\big(M_p(a,b;t),M_q(c,d;r)\big)\mathrm{d}t\mathrm{d}r.$$

This implies

$$I=\int_0^1\left(\frac{1-2t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right)$$
$$\left\{\underbrace{\int_0^1\left(\frac{1-2r}{[ra^p+(1-r)b^p]^{1-\frac{1}{p}}}\right)\frac{\partial^2 f}{\partial t\partial r}\big(M_p(a,b;t),M_q(c,d;r)\big)\mathrm{d}r}_{I_1}\right\}\mathrm{d}t. \quad (3)$$

Integrating by parts, we have

$$I_1=\int_0^1\left(\frac{1-2r}{[ra^p+(1-r)b^p]^{1-\frac{1}{p}}}\right)\frac{\partial^2 f}{\partial t\partial r}\big(M_p(a,b;t),M_q(c,d;r)\big)\mathrm{d}r$$
$$=\frac{q}{d^q-c^q}\frac{\partial f}{\partial t}([ta^p+(1-t)b^p]^{\frac{1}{p}},c)+\frac{q}{d^q-c^q}\frac{\partial f}{\partial t}([ta^p+(1-t)b^p]^{\frac{1}{p}},d)$$
$$+\frac{2q}{d^q-c^q}\int_0^1\frac{\partial f}{\partial t}([ta^p+(1-t)b^p]^{\frac{1}{p}},[rc^q+(1-r)d^q]^{\frac{1}{q}})\mathrm{d}r. \quad (4)$$

From (3) and (4), we have

$$I_2 = \frac{q}{d^q - c^q} \int_0^1 \left(\frac{1-2t}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right) \frac{\partial f}{\partial t}([ta^p + (1-t)b^p]^{\frac{1}{p}}, c)\mathrm{d}t$$

$$= \frac{q}{d^q - c^q} \left[\frac{p}{b^p - a^a} f(a, c) + \frac{p}{b^p - a^a} f(b, c) \right.$$

$$\left. - \frac{2p^2}{(b^p - a^p)^2} \int_a^b x^{p-1} f(x, c)\mathrm{d}x \right]. \quad (5)$$

Similarly from (3) and (4), we have

$$I_3 = \frac{q}{d^q - c^q} \int_0^1 \left(\frac{1-2t}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right) \frac{\partial f}{\partial t}([ta^p + (1-t)b^p]^{\frac{1}{p}}, d)\mathrm{d}t$$

$$= \frac{q}{d^q - c^q} \left[\frac{p}{b^p - a^a} f(a, d) + \frac{p}{b^p - a^a} f(b, d) \right.$$

$$\left. - \frac{2p^2}{(b^p - a^p)^2} \int_a^b x^{p-1} f(x, d)\mathrm{d}x \right]. \quad (6)$$

Also,

$$I_4 = \frac{2q}{d^q - c^q} \int_0^1 \int_0^1 \left(\frac{1-2t}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right) \frac{\partial f}{\partial t}([ta^p + (1-t)b^p]^{\frac{1}{p}},$$

$$[rc^q + (1-r)d^q]^{\frac{1}{q}})\mathrm{d}r\mathrm{d}t$$

$$= \frac{2pq^2}{(b^p - a^p)(d^q - c^q)^2} \int_c^d y^{q-1} f(a, y)\mathrm{d}y$$

$$+ \frac{2pq^2}{(b^p - a^p)(d^q - c^q)^2} \int_c^d y^{q-1} f(b, y)\mathrm{d}y$$

$$- \frac{4p^2q^2}{(b^p - a^p)^2(d^q - c^q)^q} \int_a^b \int_c^d x^{p-1} y^{q-1} f(x, y)\mathrm{d}x\mathrm{d}y. \quad (7)$$

On summation of (3), (4), (5), (6) and (7) and multiplying by $\frac{(b^p-a^p)(d^q-c^q)}{4pq}$ completes the proof. □

Now using Lemma 1, we derive our coming results.

Theorem 1 *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a partial differentiable function on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$ and $\frac{\partial^2 f}{\partial t \partial r} \in L_1(\Delta)$. If $\left|\frac{\partial^2 f}{\partial r \partial t}\right|$ is pq-convex function on rectangle, then*

$$
\begin{aligned}
&|R_f(t,r;p,q;\Delta)| \\
&\leq \frac{(b^p-a^p)(d^q-c^q)}{4pq} \\
&\quad \times \left[K_1\left|\frac{\partial^2 f}{\partial t \partial r}(a,c)\right| + K_2\left|\frac{\partial^2 f}{\partial t \partial r}(b,c)\right| + K_3\left|\frac{\partial^2 f}{\partial t \partial r}(a,d)\right| + K_4\left|\frac{\partial^2 f}{\partial t \partial r}(b,d)\right|\right],
\end{aligned}
$$

where

$$
\begin{aligned}
K_1 &= \int_0^1\int_0^1 \left(\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right)\left(\frac{|1-2r|}{[ra^p+(1-r)b^p]^{1-\frac{1}{p}}}\right) tr\mathrm{d}r\mathrm{d}t \\
&= b^{1-p}\left[\begin{array}{c}\frac{2}{3}\cdot{}_2F_1\left(1-\frac{1}{p},3;4;1-\frac{a^p}{b^p}\right)-\frac{1}{2}\cdot{}_2F_1\left(1-\frac{1}{p},2;3;1-\frac{a^p}{b^p}\right)\\ +\frac{1}{12}\cdot{}_2F_1\left(1-\frac{1}{p},2;4;\frac{1}{2}\left(1-\frac{a^p}{b^p}\right)\right)\end{array}\right] \\
&\quad \times d^{1-p}\left[\begin{array}{c}\frac{2}{3}\cdot{}_2F_1\left(1-\frac{1}{p},3;4;1-\frac{c^p}{d^p}\right)-\frac{1}{2}\cdot{}_2F_1\left(1-\frac{1}{p},2;3;1-\frac{c^p}{d^p}\right)\\ +\frac{1}{12}\cdot{}_2F_1\left(1-\frac{1}{p},2;4;\frac{1}{2}\left(1-\frac{c^p}{d^p}\right)\right)\end{array}\right];
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
K_2 &= \int_0^1\int_0^1 \left(\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right)\left(\frac{|1-2r|}{[ra^p+(1-r)b^p]^{1-\frac{1}{p}}}\right)(1-t)r\mathrm{d}r\mathrm{d}t \\
&= b^{1-p}\left[\begin{array}{c}\frac{1}{3}\cdot{}_2F_1\left(1-\frac{1}{p},2;4;1-\frac{a^p}{b^p}\right)\\ -\frac{1}{2}\cdot{}_2F_1\left(1-\frac{1}{p},1;3;1-\frac{a^p}{b^p}\right)\\ +\frac{1}{2}\cdot{}_2F_1\left(1-\frac{1}{p},1;3;\frac{1}{2}\left(1-\frac{a^p}{b^p}\right)\right)\\ -\frac{1}{22}\cdot{}_2F_1\left(1-\frac{1}{p},2;4;\frac{1}{2}\left(1-\frac{a^p}{b^p}\right)\right)\end{array}\right] \\
&\quad \times d^{1-p}\left[\begin{array}{c}\frac{2}{3}\cdot{}_2F_1\left(1-\frac{1}{p},3;4;1-\frac{c^p}{d^p}\right)\\ -\frac{1}{2}\cdot{}_2F_1\left(1-\frac{1}{p},2;3;1-\frac{c^p}{d^p}\right)\\ +\frac{1}{12}\cdot{}_2F_1\left(1-\frac{1}{p},2;4;\frac{1}{2}\left(1-\frac{c^p}{d^p}\right)\right)\end{array}\right];
\end{aligned} \tag{9}
$$

$$K_3 = \int_0^1 \int_0^1 \left(\frac{|1-2t|}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right) \left(\frac{|1-2r|}{[ra^p + (1-r)b^p]^{1-\frac{1}{p}}} \right) t(1-r)\mathrm{d}r\mathrm{d}t$$

$$= b^{1-p} \begin{bmatrix} \frac{2}{3} \cdot {}_2F_1\left(1-\frac{1}{p}, 3; 4; 1-\frac{a^p}{b^p}\right) \\ -\frac{1}{2} \cdot {}_2F_1\left(1-\frac{1}{p}, 2; 3; 1-\frac{a^p}{b^p}\right) \\ +\frac{1}{12} \cdot {}_2F_1\left(1-\frac{1}{p}, 2; 4; \frac{1}{2}\left(1-\frac{a^p}{b^p}\right)\right) \end{bmatrix}$$

$$\times d^{1-p} \begin{bmatrix} \frac{1}{3} \cdot {}_2F_1\left(1-\frac{1}{p}, 2; 4; 1-\frac{c^p}{d^p}\right) \\ -\frac{1}{2} \cdot {}_2F_1\left(1-\frac{1}{p}, 1; 3; 1-\frac{c^p}{d^p}\right) \\ +\frac{1}{2} \cdot {}_2F_1\left(1-\frac{1}{p}, 1; 3; \frac{1}{2}\left(1-\frac{c^p}{d^p}\right)\right) \\ -\frac{1}{22} \cdot {}_2F_1\left(1-\frac{1}{p}, 2; 4; \frac{1}{2}\left(1-\frac{c^p}{d^p}\right)\right) \end{bmatrix}, \tag{10}$$

and

$$K_4 = \int_0^1 \int_0^1 \left(\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}} \right) \left(\frac{|1-2r|}{[ra^p+(1-r)b^p]^{1-\frac{1}{p}}} \right) (1-t)(1-r)\mathrm{d}r\mathrm{d}t$$

$$= b^{1-p} \begin{bmatrix} \frac{1}{3} \cdot {}_2F_1\left(1-\frac{1}{p}, 2; 4; 1-\frac{a^p}{b^p}\right) \\ -\frac{1}{2} \cdot {}_2F_1\left(1-\frac{1}{p}, 1; 3; 1-\frac{a^p}{b^p}\right) \\ +\frac{1}{2} \cdot {}_2F_1\left(1-\frac{1}{p}, 1; 3; \frac{1}{2}\left(1-\frac{a^p}{b^p}\right)\right) \\ -\frac{1}{22} \cdot {}_2F_1\left(1-\frac{1}{p}, 2; 4; \frac{1}{2}\left(1-\frac{a^p}{b^p}\right)\right) \end{bmatrix}$$

$$\times d^{1-p} \begin{bmatrix} \frac{1}{3} \cdot {}_2F_1\left(1-\frac{1}{p}, 2; 4; 1-\frac{c^p}{d^p}\right) \\ -\frac{1}{2} \cdot {}_2F_1\left(1-\frac{1}{p}, 1; 3; 1-\frac{c^p}{d^p}\right) \\ +\frac{1}{2} \cdot {}_2F_1\left(1-\frac{1}{p}, 1; 3; \frac{1}{2}\left(1-\frac{c^p}{d^p}\right)\right) \\ -\frac{1}{22} \cdot {}_2F_1\left(1-\frac{1}{p}, 2; 4; \frac{1}{2}\left(1-\frac{c^p}{d^p}\right)\right) \end{bmatrix}. \tag{11}$$

Proof Using Lemma 1, property of modulus and the fact that $|\frac{\partial^2 f}{\partial t \partial r}|$ is pq-convex on rectangle, we have

$$|R_f(t, r; p, q; \Delta)|$$
$$= \left| \frac{(b^p - a^p)(d^q - c^q)}{4pq} \right.$$

$$\times \int_0^1 \int_0^1 \left(\frac{1-2t}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right) \left(\frac{1-2r}{[ra^p + (1-r)b^p]^{1-\frac{1}{p}}} \right)$$

$$\times \frac{\partial^2 f}{\partial t \partial r} \big(M_p(a, b; t), M_q(c, d; r) \big) \mathrm{d}t \mathrm{d}r \Bigg|$$

$$\leq \frac{(b^p - a^p)(d^q - c^q)}{4pq}$$

$$\times \int_0^1 \int_0^1 \left(\frac{|1-2t|}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right) \left(\frac{|1-2r|}{[ra^p + (1-r)b^p]^{1-\frac{1}{p}}} \right)$$

$$\times \left| \frac{\partial^2 f}{\partial t \partial r} \big(M_p(a, b; t), M_q(c, d; r) \big) \right| \mathrm{d}t \mathrm{d}r$$

$$\leq \frac{(b^p - a^p)(d^q - c^q)}{4pq} \int_0^1 \int_0^1 \left(\frac{|1-2t|}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right) \left(\frac{|1-2r|}{[ra^p + (1-r)b^p]^{1-\frac{1}{p}}} \right)$$

$$\times \left\{ \begin{array}{c} tr\left|\frac{\partial^2 f}{\partial t \partial r}(a, c)\right| + (1-t)r\left|\frac{\partial^2 f}{\partial t \partial r}(b, c)\right| + t(1-r)\left|\frac{\partial^2 f}{\partial t \partial r}(a, d)\right| \\ +(1-t)(1-r)\left|\frac{\partial^2 f}{\partial t \partial r}(b, d)\right| \end{array} \right\} \mathrm{d}t \mathrm{d}r$$

$$= \frac{(b^p - a^p)(d^q - c^q)}{4pq}$$

$$\times \left[K_1 \left| \frac{\partial^2 f}{\partial t \partial r}(a, c) \right| + K_2 \left| \frac{\partial^2 f}{\partial t \partial r}(b, c) \right| + K_3 \left| \frac{\partial^2 f}{\partial t \partial r}(a, d) \right| + K_4 \left| \frac{\partial^2 f}{\partial t \partial r}(b, d) \right| \right].$$

This completes the proof. □

Theorem 2 *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a partial differentiable function on $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$ and $\frac{\partial^2 f}{\partial t \partial r} \in L_1(\Delta)$. If $\left| \frac{\partial^2 f}{\partial r \partial t} \right|^\beta$ is pq-convex function on rectangle, where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $\alpha, \beta > 1$, then*

$$|R_f(t, r; p, q; \Delta)|$$

$$\leq \frac{(b^p - a^p)(d^q - c^q)}{4pq} C^{1-\frac{1}{\beta}}$$

$$\times \left[K_1 \left| \frac{\partial^2 f}{\partial t \partial r}(a, c) \right|^\beta + K_2 \left| \frac{\partial^2 f}{\partial t \partial r}(b, c) \right|^\beta + K_3 \left| \frac{\partial^2 f}{\partial t \partial r}(a, d) \right|^\beta + K_4 \left| \frac{\partial^2 f}{\partial t \partial r}(b, d) \right|^\beta \right]^{\frac{1}{\beta}},$$

where K_1, K_2, K_3 and K_4 are given by (8), (9), (10) and (11), respectively, and

$$
\begin{aligned}
C = & \int_0^1 \int_0^1 \left(\frac{|1-2t|}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right) \left(\frac{|1-2r|}{[ra^p + (1-r)b^p]^{1-\frac{1}{p}}} \right) \mathrm{d}r \mathrm{d}t \\
= & \ {}_2F_1\left(1 - \frac{1}{p}, 2; 3; 1 - \frac{a^p}{b^p}\right) - {}_2F_1\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{a^p}{b^p}\right) \\
& + {}_2F_1\left(1 - \frac{1}{p}, 1; 3; \frac{1}{2}\left(1 - \frac{a^p}{b^p}\right)\right) \\
& \times {}_2F_1\left(1 - \frac{1}{p}, 2; 3; 1 - \frac{c^p}{d^p}\right) - {}_2F_1\left(1 - \frac{1}{p}, 1; 2; 1 - \frac{c^p}{d^p}\right) \\
& + {}_2F_1\left(1 - \frac{1}{p}, 1; 3; \frac{1}{2}\left(1 - \frac{c^p}{d^p}\right)\right).
\end{aligned}
$$

Proof Using Lemma 1, property of modulus, Holder's inequality and the fact that $|\frac{\partial^2 f}{\partial r \partial t}|^\beta$ is pq-convex function on rectangle, we have

$$
\begin{aligned}
& |R_f(t, r; p, q; \Delta)| \\
= & \Bigg| \frac{(b^p - a^p)(d^q - c^q)}{4pq} \\
& \times \int_0^1 \int_0^1 \left(\frac{1-2t}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right) \left(\frac{1-2r}{[ra^p + (1-r)b^p]^{1-\frac{1}{p}}} \right) \\
& \qquad \times \frac{\partial^2 f}{\partial t \partial r}\big(M_p(a, b; t), M_q(c, d; r)\big) \mathrm{d}t \mathrm{d}r \Bigg| \\
\le & \frac{(b^p - a^p)(d^q - c^q)}{4pq} \\
& \times \int_0^1 \int_0^1 \left(\frac{|1-2t|}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right) \left(\frac{|1-2r|}{[ra^p + (1-r)b^p]^{1-\frac{1}{p}}} \right) \Bigg| \\
& \qquad \frac{\partial^2 f}{\partial t \partial r}\big(M_p(a, b; t), M_q(c, d; r)\big) \Bigg| \mathrm{d}t \mathrm{d}r \\
\le & \frac{(b^p - a^p)(d^q - c^q)}{4pq} \left(\int_0^1 \int_0^1 \left(\frac{|1-2t|}{[ta^p + (1-t)b^p]^{1-\frac{1}{p}}} \right) \left(\frac{|1-2r|}{[ra^p + (1-r)b^p]^{1-\frac{1}{p}}} \right) \mathrm{d}t \mathrm{d}r \right)^{1-\frac{1}{\beta}} \\
& \times \left(\begin{array}{c} \int_0^1 \int_0^1 \left(\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}} \right) \left(\frac{|1-2r|}{[ra^p+(1-r)b^p]^{1-\frac{1}{p}}} \right) \\ \times \left| \frac{\partial^2 f}{\partial t \partial r}\left([ta^p + (1-t)b^p]^{\frac{1}{p}}, [rc^q + (1-r)d^q]^{\frac{1}{q}}\right) \right|^\beta \mathrm{d}t \mathrm{d}r \end{array} \right)^{\frac{1}{\beta}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(b^p-a^p)(d^q-c^q)}{4pq}\left(\int_0^1\int_0^1\left(\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right)\left(\frac{|1-2r|}{[ra^p+(1-r)b^p]^{1-\frac{1}{p}}}\right)\mathrm{d}t\mathrm{d}r\right)^{1-\frac{1}{\beta}}\\
&\times\left(\int_0^1\int_0^1\left(\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right)\left(\frac{|1-2r|}{[ra^p+(1-r)b^p]^{1-\frac{1}{p}}}\right)\right.\\
&\times\left\{tr\left|\frac{\partial^2 f}{\partial t\partial r}(a,c)\right|^\beta+(1-t)r\left|\frac{\partial^2 f}{\partial t\partial r}(b,c)\right|^\beta\right.\\
&\qquad\left.\left.+t(1-r)\left|\frac{\partial^2 f}{\partial t\partial r}(a,d)\right|^\beta+(1-t)(1-r)\left|\frac{\partial^2 f}{\partial t\partial r}(b,d)\right|^\beta\right\}\mathrm{d}t\mathrm{d}r\right)^{\frac{1}{\beta}}\\
&\leq \frac{(b^p-a^p)(d^q-c^q)}{4pq}C^{1-\frac{1}{\beta}}\\
&\times\left[K_1\left|\frac{\partial^2 f}{\partial t\partial r}(a,c)\right|^\beta+K_2\left|\frac{\partial^2 f}{\partial t\partial r}(b,c)\right|^\beta+K_3\left|\frac{\partial^2 f}{\partial t\partial r}(a,d)\right|^\beta+K_4\left|\frac{\partial^2 f}{\partial t\partial r}(b,d)\right|^\beta\right]^{\frac{1}{\beta}}.
\end{aligned}
$$

This completes the proof. □

Remark 1 It is worth to mention here that for $p = q$ in the above results, we have the results for p-convex functions on rectangle, which to the best of our knowledge are new in the literature. If $p = 1 = q$, then the above results reduce to the results for convex functions on rectangle. Note that in particular if $p = -1$, then our results collapse to the results for harmonically convex functions on rectangle, see [16].

4 Conclusion

A new integral identity for partial differentiable functions has been derived. Utilizing this new auxiliary result, we have established several new trapezoidal-like inequalities via pq-convex functions. It has been observed that under suitable values of p and q we obtain several new and known results. Interested readers may explore the applications of these new inequalities in engineering, mathematical sciences, numerical analysis and optimization.

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New k-Conformable Fractional Integral Inequalities

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Abstract A new integral identity using the concepts of k-conformable fractional calculus is obtained. Utilizing the preinvexity property of the functions associated upper bounds is also obtained. Some special cases of the obtained results are also discussed.

1 Introduction

Theory of convexity can be regarded as mathematical foundation for minimax theory, Lagrange multiplier theory, and duality. Convex functions played a very significant role in the theory of inequalities. A set $\mathscr{K} \subset \mathbb{R}$ is said to be convex, if

$$(1-t)x + ty \in \mathscr{X}, \quad \forall x, y \in \mathscr{X}, t \in [0, 1].$$

Similarly, convex functions are defined as A function $f : \mathscr{K} \to \mathbb{R}$ is said to be convex, if

$$(1-t)f(x) + tf(y) \geq f((1-t)x + ty)$$

holds for all $x, y \in \mathscr{K}$ and $t \in [0, 1]$.

Due to its great many utilities in different fields of pure as well as in applied sciences, it received full attention by the researchers. In recent decades, the classical

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concept of convexity has been generalized and extended according to the need of the problems. A very significant extension of convexity that is differentiable invex functions in optimization theory was given by Henson [6], but he has not used the term invex. It was Craven [2] who used the terminology invex for this class of functions. Mititelu [10] described invex sets as A set $\mathscr{X} \in \mathbb{R}$ is said to be invex with respect to bifunction $\zeta(.,.)$, if

$$x + t\zeta(y, x) \in \mathscr{X}, \quad \forall x, y \in \mathscr{X}, t \in [0, 1].$$

Note that convexity can be recaptured from invexity by taking $\zeta(y, x) = y - x$. This shows that every convex set is an invex with respect to $\zeta(y, x) = y - x$, but the converse is not true in general.

Weir and Mond [19] introduced the class of preinvex functions (a generalization of convex functions) as A function $\mathscr{F} : \mathscr{X} \to \mathbb{R}$ is said to be preinvex with respect to bifunction $\zeta(.,.)$, if

$$\mathscr{F}(x + t\zeta(y, x)) \leq (1 - t)\mathscr{F}(x) + t\mathscr{F}(y), \quad \forall x, y \in \mathscr{X}, t \in [0, 1].$$

If $\zeta(y, x) = y - x$, the class of preinvex functions reduces to the class of convex functions.

The relationship between theory of convexity and theory of inequalities has attracted many researchers. Many inequalities known to us in the literature can easily be obtained using the functions having convexity property. For example, a very famous result in this regard is of Hermite and Hadamard commonly known as Hermite–Hadamard's inequality. This result reads as

Theorem 1 *Let $\mathscr{F} : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function, then*

$$\mathscr{F}\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a}\int_a^b \mathscr{F}(x)\mathrm{d}x \leq \frac{\mathscr{F}(a) + \mathscr{F}(b)}{2}.$$

This double inequality provides us necessary and sufficient condition for a function to be convex. Noor [11] obtained a new general version of Hermite–Hadamard's inequality using the class of preinvex functions. It reads as

Theorem 2 *Let $\mathscr{F} : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a preinvex function. If $\zeta(.,.)$ satisfies condition C, then*

$$\mathscr{F}\left(\frac{2a + \zeta(b, a)}{2}\right) \leq \frac{1}{\zeta(b, a)}\int_a^{a+\zeta(b,a)} \mathscr{F}(x)\mathrm{d}x \leq \frac{\mathscr{F}(a) + \mathscr{F}(b)}{2}.$$

Noor et al. [12] further generalized this result using the class of h-preinvex functions. For some recent developments on Hermite–Hadamard's inequality and its applications, see [4, 13].

Fractional calculus also known as non-integer calculus has emerged as interdisciplinary subject. It grows out of the long established definitions of the ordinary calculus integral and derivative operators. It experienced a rapid development in past 100 years; however, the birthday of the fractional calculus is regarded as 30 September 1695. In the start, it was reserved to few mathematicians, but latter on many researchers started working on it. One of the most classical definitions in fractional calculus was that of Riemann–Liouville definition presented in the nineteenth century. Since then, fractional calculus helped many applied mathematicians in solving different physical problems. The definition of Riemann–Liouville integrals is given as

Definition 1 ([9]) Let $\mathscr{F} \in L_1[a,b]$. Then the Riemann–Liouville integrals $\mathfrak{J}^{\alpha}_{a^+}\mathscr{F}$ and $\mathfrak{J}^{\alpha}_{b^-}\mathscr{F}$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$\mathfrak{J}^{\alpha}_{a^+}\mathscr{F}(x) = \frac{1}{\Gamma(\alpha)}\int\limits_a^x (x-t)^{\alpha-1}\mathscr{F}(t)\mathrm{d}t, \quad x > a,$$

and

$$\mathfrak{J}^{\alpha}_{b^-}\mathscr{F}(x) = \frac{1}{\Gamma(\alpha)}\int\limits_x^b (t-x)^{\alpha-1}\mathscr{F}(t)\mathrm{d}t, \quad x < b,$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}\mathrm{d}t,$$

is the well known Gamma function.

Sarikaya et al. [17] utilized the concepts of Riemann–Liouville fractional integrals and obtained a fractional analogue of Hermite–Hadamard's inequality. Since the appearance of this article, a number of new and novel fractional analogues of Hermite–Hadamard's inequality, see [5, 15, 18]. In recent years, the classical concepts of fractional calculus have been extended and generalized in different directions using novel and innovative ideas. For example, Sarikaya et al. [16] introduced the notion of k-Riemann–Liouville fractional integrals and discussed some of its interesting aspects and applications.

Definition 2 ([16]) Let $\mathscr{F} \in L_1[a,b]$. Then the k- Riemann–Liouville integrals ${}_k\mathfrak{J}^{\alpha}_{a^+}\mathscr{F}$ and ${}_k\mathfrak{J}^{\alpha}_{b^-}\mathscr{F}$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$${}_k\mathfrak{J}^{\alpha}_{a^+}\mathscr{F}(x) = \frac{1}{k\Gamma_k(\alpha)}\int\limits_a^x (x-t)^{\frac{\alpha}{k}-1}\mathscr{F}(t)\mathrm{d}t, \quad x > a, k > 0,$$

and

$$ {}_k\mathfrak{J}_{b^-}^{\alpha}\mathscr{F}(x) = \frac{1}{k\Gamma_k(\alpha)} \int\limits_x^b (t-x)^{\frac{\alpha}{k}-1}\mathscr{F}(t)\mathrm{d}t, \quad x < b, k > 0. $$

Here, $\Gamma_k(x) = \int\limits_0^{\infty} t^{x-1}e^{-\frac{t^k}{k}}\mathrm{d}t$, $\Re(x) > 0$ is the one parameter deformation of classical gamma function called as k-gamma function and was introduced by Diaz et al. [3]. Γ_k is based on the repeated appearance of the expression of: $\phi(\phi+k)(\phi+2k)(\phi+3k)\dots(\phi+(n-1)k)$. Diaz et al. [3] also introduced the notion of k-Beta function as

$$ \begin{aligned} B_k(x,y) &= \frac{1}{k}\int\limits_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\mathrm{d}t \\ &= \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad \Re(x) > 0, \Re(y) > 0. \end{aligned} $$

For more information on k-analogues of special functions, see [3].

Roughly, we can say that the core idea behind fractional calculus depends upon two approaches: one that of Riemann–Liouville approach, and the other one is Grunwald–Letnikov approach. However, utilizing these approaches, the obtained results seem to be very complicated and lose some basic properties of the classical concepts. Taking this into account in [8], the authors introduced a simple, well-behaved fractional derivative called as conformable fractional derivative. This definition reads as: for a function $f : (0, \infty) \to \mathbb{R}$, the conformable fractional derivative is defined as

$$ I_\alpha f(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}, $$

where $0 < \alpha \le 1, t > 0$.

Abdeljawad [1] defined the left and right conformable fractional derivatives as

Definition 3 The left conformable fractional derivative starting from a of function $f : [a, \infty) \to \mathbb{R}$ of order $0 < \alpha \le 1$ is given as

$$ I_\alpha^a f(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon(t-a)^{1-\alpha})}{\epsilon}, $$

and the right conformable fractional derivative terminating at b is given as

$$ I_\alpha^b f(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon(b-t)^{1-\alpha})}{\epsilon}. $$

Abdeljawad [1] also defined the left and right conformable fractional integrals of any order $\alpha > 0$ as

Definition 4 ([1]) Let $\alpha \in (n, n+1]$ and $\beta = \alpha - n$. Then the left and right conformable fractional integrals starting at a of order α are defined by

$$I_\alpha^a f(t) = \frac{1}{n!} \int_a^t (t-u)^n (u-a)^{\beta-1} f(u)\mathrm{d}u,$$

and

$$I_\alpha^b f(\lambda) = \frac{1}{n!} \int_t^b (u-t)^n (b-u)^{\beta-1} f(u)\mathrm{d}u.$$

Note that if $\alpha = n+1$ then $\beta = 1$ where $n = 0, 1, 2, \ldots$.

Recently, Jarad et al. [7] introduced new left and right conformable fractional integrals as

Definition 5 ([7]) Let $\beta \in \mathbb{C}$, $\Re(\beta) > 0$ and $\alpha \in \mathbb{R} \setminus 0$, and then the left and right conformable fractional integrals are defined as

$${}_a^\beta \mathscr{J}^\alpha f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} \mathrm{d}t,$$

and

$${}^\beta \mathscr{J}_b^\alpha f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \left(\frac{(b-x)^\alpha - (b-t)^\alpha}{\alpha} \right)^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}} \mathrm{d}t.$$

Qi et al. [14] extended the definition of conformable fractional integrals introduced by Jarad et al. [7] using the concept of k-calculus. They defined new general conformable fractional integrals as

Definition 6 ([14]) Let $\beta \in \mathbb{C}$, $\Re(\beta) > 0$, $k > 0$ and $\alpha \in \mathbb{R} \setminus 0$, and then the left and right conformable fractional integrals are defined as

$${}_a^\beta \mathscr{J}^\alpha f(x) = \frac{1}{k\Gamma_k(\beta)} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\frac{\beta}{k}-1} \frac{f(t)}{(t-a)^{1-\alpha}} \mathrm{d}t,$$

and

$$ {}^{\beta}\mathscr{J}_{b}^{\alpha} f(x) = \frac{1}{k\Gamma_k(\beta)} \int_x^b \left(\frac{(b-x)^{\alpha} - (b-t)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}-1} \frac{f(t)}{(b-t)^{1-\alpha}} \mathrm{d}t. $$

The aim of this article is to obtain some new k-analogues of trapezium like inequalities involving the class of preinvex functions. In order to obtain the main results of the paper, we derive a new conformable fractional integral identity that will serve as an auxiliary result. This is the main motivation of this article.

2 Results and Discussions

In this section, we discuss our main results.

Lemma 1 *Let $\mathscr{T} : [a, a + \zeta(b, a)] \to \mathbb{R}$ be a differentiable function on $(a, a + \zeta(b, a))$ with $\zeta(b, a) > 0$ and $\mathscr{T}' \in L[a, a+\zeta(b, a)]$. Also let $\alpha, \beta \in \mathbb{R}^{+}$. Then the following equality for k-fractional conformable integrals holds for $k > 0$:*

$$ \begin{aligned} &\frac{\zeta^{\frac{\alpha\beta}{k}}(x, a)\mathscr{T}(a) + \zeta^{\frac{\alpha\beta}{k}}(x, b)\mathscr{T}(b)}{\alpha^{\beta}\zeta(b, a)} \\ &\quad - \frac{\Gamma_k(\beta+k)}{\zeta(b, a)} \left[{}^{\beta}_{k}\mathscr{H}^{\alpha}_{[a+\zeta(x,a)]^{-}} \mathscr{T}(a) - {}^{\beta}_{k}\mathscr{H}^{\alpha}_{b^{+}} \mathscr{T}(b+\zeta(x, b)) \right] \\ &= \frac{\zeta^{\frac{\alpha\beta}{k}+1}(x, a)}{\zeta(b, a)} \int_0^1 \left[\left(\frac{1-(1-t)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} - \frac{1}{\alpha^{\frac{\beta}{k}}} \right] \mathscr{T}'(a + t\zeta(x, a))\mathrm{d}t \\ &\quad - \frac{\zeta^{\frac{\alpha\beta}{k}+1}(x, b)}{\zeta(b, a)} \int_0^1 \left[\frac{1}{\alpha^{\frac{\beta}{k}}} - \left(\frac{1-(1-t)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} \right] \mathscr{T}'(b + t\zeta(x, b))\mathrm{d}t. \end{aligned} $$

Proof Integrating by parts, we have

$$ \begin{aligned} J_1 &= \int_0^1 \left[\left(\frac{1-(1-t)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} - \frac{1}{\alpha^{\frac{\beta}{k}}} \right] \mathscr{T}'(a + t\zeta(x, a))\mathrm{d}t \\ &= \int_0^1 \left(\frac{1-(1-t)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} \mathscr{T}'(a + t\zeta(x, a))\mathrm{d}t - \frac{1}{\alpha^{\frac{\beta}{k}}} \int_0^1 \mathscr{T}'(a + t\zeta(x, a))\mathrm{d}t \\ &= \left(\frac{1-(1-t)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} \frac{\mathscr{T}(a + t\zeta(x, a))}{\zeta(x, a)} \Bigg|_0^1 \end{aligned} $$

$$
-\frac{\beta}{k\zeta(x,a)}\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{\mathscr{T}(a+t\zeta(x,a))}{(1-t)^{1-\alpha}}\mathrm{d}t-\frac{1}{\alpha^{\frac{\beta}{k}}}\frac{\mathscr{T}(a+t\zeta(x,a))}{\zeta(x,a)}\Bigg|_0^1
$$

$$
=\frac{\mathscr{T}(a)}{\alpha^{\frac{\beta}{k}}\zeta(x,a)}-\frac{\Gamma_k(\beta+k)}{\zeta^{\frac{\alpha\beta}{k}+1}(x,a)}\,{}^{\beta}_{k}\mathscr{H}^{\alpha}_{[a+\zeta(x,a)]^-}\mathscr{T}(a), \tag{1}
$$

and

$$
\begin{aligned}
J_2 &=\int_0^1\left[\frac{1}{\alpha^{\frac{\beta}{k}}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right]\mathscr{T}'(b+t\zeta(x,b))\mathrm{d}t\\
&=\frac{1}{\alpha^{\frac{\beta}{k}}}\int_0^1\mathscr{T}'(b+t\zeta(x,b))\mathrm{d}t-\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\mathscr{T}'(b+t\zeta(x,b))\mathrm{d}t\\
&=\frac{1}{\alpha^{\frac{\beta}{k}}}\frac{\mathscr{T}(b+t\zeta(x,b))}{\zeta(x,b)}\Bigg|_0^1-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\frac{\mathscr{T}(b+t\zeta(x,b))}{\zeta(x,b)}\Bigg|_0^1\\
&\quad+\frac{\beta}{k\zeta(x,b)}\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}\frac{\mathscr{T}(b+t\zeta(x,b))}{(1-t)^{1-\alpha}}\mathrm{d}t\\
&=-\frac{\mathscr{T}(b)}{\alpha^{\frac{\beta}{k}}\zeta(x,b)}+\frac{\Gamma_k(\beta+k)}{\zeta^{\frac{\alpha\beta}{k}+1}(x,b)}\,{}^{\beta}_{k}\mathscr{H}^{\alpha}_{b^+}\mathscr{T}(b+\zeta(x,b)).
\end{aligned} \tag{2}
$$

Multiplying equality (1) by $\frac{\zeta^{\frac{\alpha\beta}{k}+1}(x,a)}{\zeta(b,a)}$ and equality (2) by $\frac{\zeta^{\frac{\alpha\beta}{k}+1}(x,b)}{\zeta(b,a)}$ and then subtracting the resulting equalities, we obtained the required result.

Now using Lemma 1, we derive our next results.

Theorem 3 *Let $\mathscr{T}:[a,a+\zeta(b,a)]\to\mathbb{R}$ be a differentiable function on $(a,a+\zeta(b,a))$ with $\zeta(b,a)>0$ and $\mathscr{T}'\in L[a,a+\zeta(b,a)]$. Also let $|\mathscr{T}'|$ be a preinvex function on $[a,a+\zeta(b,a)]$ and $\alpha,\beta\in\mathbb{R}^+$. Then the following inequality for k-fractional conformable integrals holds for $k>0$:*

$$
\begin{aligned}
&\Bigg|\frac{\zeta^{\frac{\alpha\beta}{k}}(x,a)\mathscr{T}(a)+\zeta^{\frac{\alpha\beta}{k}}(x,b)\mathscr{T}(b)}{\alpha^{\frac{\beta}{k}}\zeta(b,a)}\\
&\qquad-\frac{\Gamma_k(\beta+k)}{\zeta(b,a)}\left[{}^{\beta}_{k}\mathscr{H}^{\alpha}_{[a+\zeta(x,a)]^-}\mathscr{T}(a)-{}^{\beta}_{k}\mathscr{H}^{\alpha}_{b^+}\mathscr{T}(b+\zeta(x,b))\right]\Bigg|\\
&\le\left[\frac{1}{2}-\frac{k}{\alpha}\left(B_k\left(\beta+k,\frac{k}{\alpha}\right)-B_k\left(\beta+k,\frac{2k}{\alpha}\right)\right)\right]\frac{\zeta^{\frac{\alpha\beta}{k}+1}(x,a)+\zeta^{\alpha\beta+1}(x,b)}{\alpha^{\frac{\beta}{k}}\zeta(b,a)}|\mathscr{T}'(x)|
\end{aligned}
$$

$$+\left[\frac{1}{2}-\frac{k}{\alpha}B_k\left(\beta+k,\,\frac{2k}{\alpha}\right)\right]\frac{\zeta^{\frac{\alpha\beta}{k}+1}(x,a)|\mathscr{T}'(a)|+\zeta^{\frac{\alpha\beta}{k}+1}(x,b)|\mathscr{T}'(b)|}{\alpha^{\frac{\beta}{k}}\zeta(b,a)}.$$

Proof Using Lemma 1 and property of modulus, we get

$$\Bigg|\frac{\zeta^{\frac{\alpha\beta}{k}}(x,a)\mathscr{T}(a)+\zeta^{\frac{\alpha\beta}{k}}(x,b)\mathscr{T}(b)}{\alpha^{\frac{\beta}{k}}\zeta(b,a)}$$
$$-\frac{\Gamma_k(\beta+k)}{\zeta(b,a)}\left[{}_k^{\beta}\mathscr{H}^{\alpha}_{[a+\zeta(x,a)]^-}\mathscr{T}(a)-{}_k^{\beta}\mathscr{H}^{\alpha}_{b^+}\mathscr{T}(b+\zeta(x,b))\right]\Bigg|$$
$$\leq\frac{\zeta^{\frac{\alpha\beta}{k}+1}(x,a)}{\zeta(b,a)}\int_0^1\left[\frac{1}{\alpha^{\frac{\beta}{k}}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right]|\mathscr{T}'(a+t\zeta(x,a))|\mathrm{d}t$$
$$+\frac{\zeta^{\frac{\alpha\beta}{k}+1}(x,b)}{\zeta(b,a)}\int_0^1\left[\frac{1}{\alpha^{\frac{\beta}{k}}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right]|\mathscr{T}'(b+t\zeta(x,b))|\mathrm{d}t.\quad(3)$$

Using the preinvexity of $|\mathscr{T}'|$, we have

$$\int_0^1\left[\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}-\frac{1}{\alpha^{\frac{\beta}{k}}}\right]|\mathscr{T}'(a+t\zeta(x,a))|\mathrm{d}t$$
$$\leq\frac{1}{\alpha^{\frac{\beta}{k}}}\int_0^1\left[1-(1-(1-t)^{\alpha})^{\frac{\beta}{k}}\right](t|\mathscr{T}'(x)|+(1-t)|\mathscr{T}'(a)|)\mathrm{d}t,\quad(4)$$

and

$$\int_0^1\left[\frac{1}{\alpha^{\frac{\beta}{k}}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right]|\mathscr{T}'(b+t\zeta(x,b))|\mathrm{d}t$$
$$\leq\frac{1}{\alpha^{\frac{\beta}{k}}}\int_0^1\left[1-(1-(1-t)^{\alpha})^{\frac{\beta}{k}}\right](t|\mathscr{T}'(x)|+(1-t)|\mathscr{T}'(b)|)\mathrm{d}t.\quad(5)$$

It can be easily calculated that

$$\int_0^1 t\mathrm{d}t - \int_0^1 t\left[1-(1-(1-t)^{\alpha})^{\frac{\beta}{k}}\right]\mathrm{d}t = \frac{1}{2} - \frac{k}{\alpha}\left(B_k\left(\beta+k, \frac{k}{\alpha}\right) - B_k\left(\beta+k, \frac{2k}{\alpha}\right)\right), \tag{6}$$

and

$$\int_0^1 (1-t)\mathrm{d}t - \int_0^1 (1-t)\left[1-(1-(1-t)^{\alpha})^{\frac{\beta}{k}}\right]\mathrm{d}t = \frac{1}{2} - \frac{k}{\alpha}B_k\left(\beta+k, \frac{2k}{\alpha}\right). \tag{7}$$

Using (4), (5), (6), and (7) in (3), we get the required result.

Theorem 4 *Let* $\mathscr{T} : [a, a+\zeta(b,a)] \to \mathbb{R}$ *be a differentiable function on* $(a, a+\zeta(b,a))$ *with* $\zeta(b,a) > 0$ *and* $\mathscr{T}' \in L[a, a+\zeta(b,a)]$*. Also let* $|\mathscr{T}'|^q$ *be a preinvex function on* $[a, a+\zeta(b,a)]$ *where* $q > 1$, $p^{-1} + q^{-1} = 1$ *and* $\alpha, \beta \in \mathbb{R}^+$*. Then the following inequality for k-fractional conformable integrals holds for* $k > 0$*:*

$$\begin{aligned}
&\left|\frac{\zeta^{\frac{\alpha\beta}{k}}(x,a)\mathscr{T}(a) + \zeta^{\frac{\alpha\beta}{k}}(x,b)\mathscr{T}(b)}{\alpha^{\frac{\beta}{k}}\zeta(b,a)}\right.\\
&\qquad\left.-\frac{\Gamma_k(\beta+k)}{\zeta(b,a)}\left[{}^{\beta}_{k}\mathscr{H}^{\alpha}_{[a+\zeta(x,a)]^-}\mathscr{T}(a) - {}^{\beta}_{k}\mathscr{H}^{\alpha}_{b^+}\mathscr{T}(b+\zeta(x,b))\right]\right|\\
&\le \left[\frac{1}{\alpha^{\frac{\beta p}{k}}} - \frac{k}{\alpha^{\frac{\beta p}{k}+1}}B_k\left(\frac{k}{\alpha}, \beta p + k\right)\right]^{\frac{1}{p}}\\
&\times\left[\frac{\zeta^{\frac{\alpha\beta}{k}+1}(x,a)}{\zeta(b,a)}\left[\frac{|\mathscr{T}'(b)|^q + |\mathscr{T}'(x)|^q}{2}\right]^{\frac{1}{q}} + \frac{\zeta^{\frac{\alpha\beta}{k}+1}(x,b)}{\zeta(b,a)}|\left[\frac{|\mathscr{T}'(b)|^q + |\mathscr{T}'(x)|^q}{2}\right]^{\frac{1}{q}}\right].
\end{aligned}$$

Proof Using Lemma 1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{\zeta^{\frac{\alpha\beta}{k}}(x,a)\mathscr{T}(a) + \zeta^{\frac{\alpha\beta}{k}}(x,b)\mathscr{T}(b)}{\alpha^{\frac{\beta}{k}}\zeta(b,a)}\right.\\
&\qquad\left.-\frac{\Gamma_k(\beta+k)}{\zeta(b,a)}\left[{}^{\beta}_{k}\mathscr{H}^{\alpha}_{[a+\zeta(x,a)]^-}\mathscr{T}(a) - {}^{\beta}_{k}\mathscr{H}^{\alpha}_{b^+}\mathscr{T}(b+\zeta(x,b))\right]\right|\\
&\le \frac{\zeta^{\frac{\alpha\beta}{k}+1}(x,a)}{\zeta(b,a)}\left(\int_0^1\left[\frac{1}{\alpha^{\frac{\beta}{k}}} - \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right]^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 |\mathscr{T}'(a+t\zeta(x,a))|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&+\frac{\zeta^{\frac{\alpha\beta}{k}+1}(x,b)}{\zeta(b,a)}\left(\int_0^1\left[\frac{1}{\alpha^{\frac{\beta}{k}}} - \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right]^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 |\mathscr{T}'(b+t\zeta(x,b))|^q\mathrm{d}t\right)^{\frac{1}{q}}.
\end{aligned}\tag{8}$$

Note that $|a^p - b^p| \le a^p - b^p$ for $a, b > 0$ with $a > b$ and $p > 1$.

Then we can write

$$|1-(1-(1-t)^{\alpha})^{\frac{\beta}{k}}|^p \le 1-|1-(1-t)^{\alpha}|^{\frac{\beta p}{k}}.$$

Therefore,

$$\begin{aligned}
&\int_0^1 \left[\frac{1}{\alpha^{\frac{\beta}{k}}} - \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right]^p \\
&\le \int_0^1 \frac{1}{\alpha^{\frac{\beta p}{k}}}\mathrm{d}t - \int_0^1 \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta p}{k}} \mathrm{d}t \\
&= \frac{1}{\alpha^{\frac{\beta p}{k}}} - \frac{k}{\alpha^{\frac{\beta p}{k}+1}} B_k\left(\frac{k}{\alpha}, \beta p + k\right). \qquad (9)
\end{aligned}$$

Since $|\mathscr{T}'|^q$ is preinvex on $[a, a+\zeta(b,a)]$, we have

$$\int_0^1 |\mathscr{T}'(a+t\zeta(x,a))|^q \mathrm{d}t \le \frac{|\mathscr{T}'(a)|^q + |\mathscr{T}'(x)|^q}{2}, \qquad (10)$$

and

$$\int_0^1 |\mathscr{T}'(b+t\zeta(x,b))|^q \mathrm{d}t \le \frac{|\mathscr{T}'(b)|^q + |\mathscr{T}'(x)|^q}{2}. \qquad (11)$$

Using (9), (10), and (11) in (8) completes the proof.

Theorem 5 *Let $\mathscr{T} : [a, a+\zeta(b,a)] \to \mathbb{R}$ be a differentiable function on $(a, a+\zeta(b,a))$ with $\zeta(b,a) > 0$ and $\mathscr{T}' \in L[a, a+\zeta(b,a)]$. Also let $|\mathscr{T}'|^q$ be a preinvex function on $[a, a+\zeta(b,a)]$, where $q \ge 1$ and $\alpha, \beta \in \mathbb{R}^+$. Then the following inequality for k-fractional conformable integrals holds for $k > 0$:*

$$\begin{aligned}
&\Bigg|\frac{\zeta^{\frac{\alpha\beta}{k}}(x,a)\mathscr{T}(a) + \zeta^{\frac{\alpha\beta}{k}}(x,b)\mathscr{T}(b)}{\alpha^{\frac{\beta}{k}}\zeta(b,a)} \\
&\qquad - \frac{\Gamma_k(\beta+k)}{\zeta(b,a)}\left[{}^{\beta}_{k}\mathscr{H}^{\alpha}_{[a+\zeta(x,a)]^-}\mathscr{T}(a) - {}^{\beta}_{k}\mathscr{H}^{\alpha}_{b^+}\mathscr{T}(b+\zeta(x,b))\right]\Bigg| \\
&\le \frac{\zeta^{\frac{\alpha\beta}{k}+1}(x,a)}{\alpha^{\frac{\beta}{k}+1}\zeta(b,a)}\left[\alpha - kB_k\left(\beta+k, \frac{k}{\alpha}\right)\right]^{1-\frac{1}{q}}
\end{aligned}$$

$$\times \left\{ \left[\frac{\alpha}{2} - k \left(B_k \left(\beta + k, \frac{k}{\alpha} \right) + B_k \left(\beta + k, \frac{2k}{\alpha} \right) \right) \right] |\mathscr{T}'(x)|^q \right.$$

$$\left. + \left[\frac{\alpha}{2} - k B_k \left(\beta + k, \frac{2k}{\alpha} \right) \right] |\mathscr{T}'(a)|^q \right\}^{\frac{1}{q}}$$

$$+ \frac{\zeta^{\frac{\alpha\beta}{k}+1}(x,b)}{\alpha^{\frac{\beta}{k}+1}\zeta(b,a)} \left[\alpha - k B_k \left(\beta + k, \frac{k}{\alpha} \right) \right]^{1-\frac{1}{q}}$$

$$\times \left\{ \left[\frac{\alpha}{2} - k \left(B_k \left(\beta + k, \frac{k}{\alpha} \right) + B_k \left(\beta + k, \frac{2k}{\alpha} \right) \right) \right] |\mathscr{T}'(x)|^q \right.$$

$$\left. + \left[\frac{\alpha}{2} - k B_k \left(\beta + k, \frac{2k}{\alpha} \right) \right] |\mathscr{T}'(b)|^q \right\}^{\frac{1}{q}}.$$

Proof Using Lemma 1 and power mean integral inequality, we have

$$\left| \frac{\zeta^{\frac{\alpha\beta}{k}}(x,a)\mathscr{T}(a) + \zeta^{\frac{\alpha\beta}{k}}(x,b)\mathscr{T}(b)}{\alpha^{\frac{\beta}{k}}\zeta(b,a)} \right.$$

$$\left. - \frac{\Gamma_k(\beta+k)}{\zeta(b,a)} \left[{}_k^{\beta}\mathscr{H}^{\alpha}_{[a+\zeta(x,a)]^-}\mathscr{T}(a) - {}_k^{\beta}\mathscr{H}^{\alpha}_{b^+}\mathscr{T}(b+\zeta(x,b)) \right] \right|$$

$$\leq \frac{\zeta^{\frac{\alpha\beta}{k}+1}(x,a)}{\zeta(b,a)} \left(\int_0^1 \left| \frac{1}{\alpha^{\frac{\beta}{k}}} - \left(\frac{1-(1-t)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right| \mathrm{d}t \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \left| \frac{1}{\alpha^{\frac{\beta}{k}}} - \left(\frac{1-(1-t)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right| |\mathscr{T}'(a+t\zeta(x,a))|^q \mathrm{d}t \right)^{\frac{1}{q}}$$

$$+ \frac{\zeta^{\frac{\alpha\beta}{k}+1}(x,b)}{\zeta(b,a)} \left(\int_0^1 \left| \frac{1}{\alpha^{\frac{\beta}{k}}} - \left(\frac{1-(1-t)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right| \mathrm{d}t \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \left| \frac{1}{\alpha^{\frac{\beta}{k}}} - \left(\frac{1-(1-t)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right| |\mathscr{T}'(b+t\zeta(x,b))|^q \mathrm{d}t \right)^{\frac{1}{q}}. \tag{12}$$

Since $|\mathscr{T}'|^q$ is preinvex on $[a, a+\zeta(b,a)]$, we have

$$\int_0^1 \left| \frac{1}{\alpha^{\frac{\beta}{k}}} - \left(\frac{1-(1-t)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right| |\mathscr{T}'(a+t\zeta(x,a))|^q \mathrm{d}t$$

$$\leq \frac{1}{\alpha^{\frac{\beta}{k}}} \int_0^1 [1-(1-(1-t)^{\alpha})^{\frac{\beta}{k}}](t|\mathscr{T}'(x)|^q + (1-t)|\mathscr{T}'(a)|^q)\mathrm{d}t$$

$$= \frac{1}{\alpha^{\frac{\beta}{k}}} \left[\frac{1}{2} - \frac{k}{\alpha}\left(B_k\left(\beta+k, \frac{k}{\alpha}\right) - B_k\left(\beta+k, \frac{2k}{\alpha}\right)\right)\right] |\mathscr{T}'(x)|^q$$

$$+ \frac{1}{\alpha^{\frac{\beta}{k}}} \left[\frac{1}{2} - \frac{k}{\alpha} B_k\left(\beta+k, \frac{2k}{\alpha}\right)\right] |\mathscr{T}'(a)|^q, \tag{13}$$

similarly

$$\int_0^1 \left|\frac{1}{\alpha^{\frac{\beta}{k}}} - \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right| |\mathscr{T}'(b+t\zeta(x,b))|^q \mathrm{d}t$$

$$\leq \frac{1}{\alpha^{\frac{\beta}{k}}} \int_0^1 [1-(1-(1-t)^{\alpha})^{\frac{\beta}{k}}](t|\mathscr{T}'(x)|^q + (1-t)|\mathscr{T}'(b)|^q)\mathrm{d}t$$

$$= \frac{1}{\alpha^{\frac{\beta}{k}}} \left[\frac{1}{2} - \frac{k}{\alpha}\left(B_k\left(\beta+k, \frac{k}{\alpha}\right) - B_k\left(\beta+k, \frac{2k}{\alpha}\right)\right)\right] |\mathscr{T}'(x)|^q$$

$$+ \frac{1}{\alpha^{\frac{\beta}{k}}} \left[\frac{1}{2} - \frac{k}{\alpha} B_k\left(\beta+k, \frac{2k}{\alpha}\right)\right] |\mathscr{T}'(b)|^q. \tag{14}$$

Also

$$\int_0^1 \left|\frac{1}{\alpha^{\frac{\beta}{k}}} - \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}}\right| \mathrm{d}t = \frac{1}{\alpha^{\frac{\beta}{k}}} \left(\frac{\alpha - kB_k\left(\beta+k, \frac{k}{\alpha}\right)}{\alpha}\right). \tag{15}$$

Using (13), (14), and (15) in (12) completes the proof.

Remark 1 We would like to point out that, if $k = 1$ in the above discussed results, then we have new results for conformable fractional integrals. For $\zeta(m,n) = m - n$, we have new results for k-conformable fractional integrals involving convexity property of the functions. This shows that the results obtained in this paper are quite unifying one. We would like to point out that the main results of this paper can be extended and generalized using the class of h-preinvex functions. Ideas and techniques of this paper may be starting point for further research.

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On the Hyers–Ulam–Rassias Approximately Ternary Cubic Higher Derivations

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Abstract In this paper, we prove the generalized Hyers–Ulam–Rassias stability of ternary cubic higher derivations by using a version of the fixed point theorem.

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1 Introduction

A ternary algebra is a real or complex linear space endowed with a linear mapping, the so-called ternary product $(x, y, z,) \to [xyz]$ of $A \times A \times A$ into A such that

$$[[xyz]tu] = [x[yzt]u] = [xy[ztu]] \text{ for all } x, y, z, t, u \in A.$$

If $(A, .)$ is a usual binary algebra, then an induced ternary multiplication can be defined by $[xyz] = (x.y).z$. Hence, the ternary algebra is a natural generalization of the binary case. If a ternary algebra $(A, [\ ])$ has a unit, i.e., an element $e \in A$ such that $x = [xee] = [eex]$ for all $x \in A$, then A with the binary product $x.y = [xey]$ is a usual algebra.

A normed ternary algebra is a ternary algebra with a norm $\|.\|$ such that

$$\|[xyz]\| \le \|x\|\|y\|\|z\| \text{ for } x, y, z \in A.$$

A Banach ternary algebra is a normed ternary algebra such that the normed linear space with norm $\|.\|$ is complete.

Ternary algebras have been studied during the nineteenth century. Their structures appeared more or less naturally in various domains of mathematical physics and data processing. The discovery of Nambu mechanics and the progress of

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quantum mechanics [20], as well as the work of S. Okubo [21] on the Yang–Baxter equation, provided significant development on ternary algebras (see also [3, 9, 19, 24, 25]). The simplest example of this (non-commutative and non-associative) ternary algebra is given by the following composition rule:

$$[abc]_{ijk} = \sum_{l,m,n=1}^{N} a_{nil} b_{ljm} c_{mkn}, \quad i, j, k = 1, 2, \ldots, N.$$

We say that a functional equation (ξ) is stable if any function g satisfying the equation (ξ) is *approximately* near to a true solution of (ξ). We say that a functional equation is *superstable* if every approximate solution constitutes an exact solution of it.

The stability of functional equations was first introduced by Ulam [26] in 1940. In 1941, Hyers [17] gave a partial solution to $Ulam's$ problem for the case of approximate additive mappings in the context of Banach spaces. In 1950, T. Aoki [5] studied this problem for additive mappings (see also [4, 8, 15] and [16, 23]). In 1978, Th. M. Rassias [23] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p), \quad (\epsilon > 0, p \in [0, 1)).$$

This phenomenon of stability that was introduced by Th. M. Rassias [23] is now known as the Hyers–Ulam–Rassias stability or generalized Hyers–Ulam stability. A further generalization was obtained by Gǎvruta [15], by replacing the Cauchy difference by a control mapping ϕ and also introducing the concept of generalized Hyers–Ulam–Rassias stability in the spirit of Th. M. Rassias' approach (see also [1, 2, 6–8, 10–14, 16–18, 22, 23, 27]).

Chu and Kang [10] introduced the following functional equation:

$$f(x+2y) + f(x-2y) + f(2x) = 2f(x) + 4f(x+y) + f(x-y), \qquad (1.1)$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability for the functional equation (1.1). The function $f(x) = x^3$ satisfies the functional equation (1.1), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function f between real vector spaces X and Y is a solution of (1.1) if and only if there exists a unique function $C : X^3 \to Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, and C is symmetric for each one fixed variable and is additive for two fixed variables. For more detailed definitions of such terminologies, we can refer to [11] and [14].

Throughout this paper, A denotes a ternary algebra and B stands for a Banach ternary algebra.

Definition 1 A mapping $H : A \to B$ is called a ternary cubic homomorphism between ternary algebras A, B if

(1) H is a cubic function,
(2) $H([xyz]) = [H(x)H(y)H(z)]$, for all $x, y, z \in A$.

Definition 2 A mapping $D : A \to A$ is called a ternary cubic derivation on ternary algebra A if

(1) D be a cubic function,
(2) $D([xyz]) = [D(x)y^3z^3] + [x^3D(y)z^3] + [x^3y^3D(z)]$, for all $x, y, z \in A$.

Definition 3 Let $\mathbb{N}$ be the set of natural numbers. For $m \in \mathbb{N}\cup\{0\} = \mathbb{N}_0$, a sequence $H = \{h_0, h_1, \ldots, h_m\}$ (resp., $H = \{h_0, h_1, \ldots, h_n, \ldots\}$) of cubic mappings from A into B is called a ternary cubic higher derivation of rank m (resp., infinite rank) from A into B if

$$h_n[xyz] = \sum_{i+j+k=n} [h_i(x)h_j(y)h_k(z)]$$

holds for each $n \in \{0, 1, \ldots, m\}$ (resp., $n \in \mathbb{N}_0$) and for all $x, y, z \in A$. The ternary cubic higher derivation H on A is called strong if $h_0(x) = x^3$ for all $x \in A$. Of course, a ternary cubic higher derivation of rank 0 from A into B (resp., a strong ternary cubic higher derivation of rank 1 on A) is a ternary cubic homomorphism (resp., a ternary cubic derivation). So a ternary cubic higher derivation is a generalization of both a ternary cubic homomorphism and a ternary cubic derivation.

R. Badora [6] and T. Miura et al. [27] proved the Hyers–Ulam stability and the Isac- and Rassias-type stabilities of derivations. Kyoo-Hong Park and Yong-Soo Jung [22] investigated the stability and superstability of higher ternary derivations via the Cauchy functional equation. Recently, Eshaghi Gordji and Bavand Savadkouhi investigated approximate cubic homomorphisms on Banach algebras. For more detailed definitions of such terminologies, we can refer to [7] and [14].

We apply the following fixed point theorem.

Theorem 1 *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping, that is*

$$d(Jx, Jy) \leq Ld(x, y) \text{ for } x, y \in X \text{ and some } L < 1..$$

Then, for each fixed element $x \in X$, either $d(J^nx, J^{n+1}x) = +\infty$ for all $n \geq 0$ or $d(J^nx, J^{n+1}x) < +\infty$ for all $n \geq n_0$ for some natural n_0. Moreover, if the second alternative holds then:

(i) the sequence (J^nx) is convergent to a fixed point y^* of J;
(ii) y^* is the unique fixed point of J in the set

$$Y := \{y \in X, d(J^{n_0}x, y) < +\infty\}$$

and

$$d(y, y^*) \le \frac{1}{1-L} d(y, Jy) \ \ (x, y \in Y).$$

2 Main Results

In this section, using the fixed point alternative approach, we investigate the generalized Hyers–Ulam–Rassias stability of the functional equation (1.1).

Theorem 2 *Let $\varphi : A^5 \to [0, \infty)$ be a control function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^m x, 2^m y, 2^m t, 2^m u, 2^m z)}{2^{3m}} = 0 \text{ for all } x, y, t, u, z \in A \text{ and}$$
$$\psi(x) = \varphi(x, 0, 0, 0, 0).$$

Let $F = \{f_0, f_1, \ldots, f_n, \ldots\}$ be a sequence of mappings such that

$$\Big\| f_n(x + 2y) + f_n(x - 2y) + f_n(2x) - 2f_n(x) - 4f_n(x + y) - f_n(x - y)$$
$$+ f_n[tuz] - \sum_{i+j+k=n} [f_i(t)f_j(u)f_k(z)]\Big\| \le \varphi(x, y, t, u, z), \tag{2.1}$$

for all $x, y, t, u, z \in A$ and each $n \in \mathbb{N}_0$. Then there exists a unique ternary cubic higher derivation

$$H = \{h_0, h_1, \ldots, h_n, \ldots\}$$

of any rank from A into B such that for each $n \in \mathbb{N}_0$ it holds that

$$\|f_n(x) - h_n(x)\| \le \frac{1}{7}\psi(x), \text{ for all } x \in A.$$

Proof Setting $y = t = u = z = 0$ in (2.1), we obtain

$$\|f_n(2x) - 8f_n(x)\| \le \psi(x). \tag{2.2}$$

Consider the set $X = \{g : g : A \to B\}$ and the generalized metric d on X :

$$d(h, g) = \inf\{M \in (0, \infty) : \|g(x) - h(x)\| \le M\psi(x), \ \ \forall x \in A\},$$

as well as the operator $J : X \to X$ with

$$(Jh)(x) = \frac{1}{8}h(2x) \text{ for all } h \in X \text{ and } x \in A.$$

It follows from (2.2) that

$$\|(Jh)(x) - Jg(x)\| = \left\| \frac{h(2x)}{8} - \frac{g(2x)}{8} \right\| \leq \frac{1}{8}d(h, g) \Rightarrow d(Jh, Jg) \leq \frac{1}{8}d(h, g)$$

for all $h, g \in X$. Thus, J is a strictly contractive mapping with Lipschitz constant $\frac{1}{8}$. On the other hand, by (2.2), we have

$$\|(Jf_n)(x) - f_n(x)\| \leq \frac{1}{8}\psi(x) \Rightarrow d(Jf_n, f_n) \leq \frac{1}{8}.$$

Therefore, it follows from Theorem 1 (i) that there exists a mapping $h_n : A \to B$ such that h_n is a fixed point of J, that is $h_n(2x) = 8h_n(x)$ for all $x \in A$. By Theorem 1 (i) $\lim_{m\to\infty} d(J^m(f_n), f_n) = 0$, we conclude that

$$\lim_{m\to\infty} \frac{f_n(2^m x)}{2^{3m}} = h_n(x) \tag{2.3}$$

for all $x \in A$. The mapping h_n is the unique fixed point of J in the set

$$U_n = \{g \in S : d(f_n, g) < \infty\}.$$

Thus, h_n is the unique fixed point of J such that

$$\| f_n(x) - h_n(x)\| \leq M\psi(x) \text{ for some } M > 0 \text{ and for all } x \in A.$$

Again, by Theorem 1 (ii), we have

$$d(f_n, h_n) \leq \frac{1}{1-\frac{1}{8}}d(f_n, Jf_n) \leq \frac{\frac{1}{8}}{1-\frac{1}{8}} = \frac{1}{7},$$

so

$$\| f_n(x) - h_n(x)\| \leq \frac{1}{7}\psi(x)$$

for all $x \in A$ and each $n \in \mathbb{N}_0$. Replacing in (2.1) the terms x, y by $2^n x, 2^n y$, respectively, as well as setting $t = u = z = 0$, and multiplying both sides of (2.1) by 2^{-3m}, we obtain

$$\|h_n(x + 2y) + h_n(x - 2y) + h_n(2x) - 2h_n(x) - 4h_n(x + y) - h_n(x - y)\| =$$

$$\lim_{m\to\infty}\left\|\frac{f_n(2^m(x+2y))}{2^{3m}}+\frac{f_n(2^m(x-2y))}{2^{3m}}+\frac{f_n(2^{m+1}x)}{2^{3m}}-\frac{4f_n(2^m(x+y))}{2^{3m}}\right.$$
$$\left.-\frac{4f_n(2^m(x-y))}{2^{3m}}-\frac{2f_n(2^m x)}{2^{3m}}\right\|\le\lim_{m\to\infty}\frac{\varphi(2^m x,2^m y,0,0,0)}{2^{3m}}=0$$

for all $x, y \in A$. Thus, h_n is cubic for each $n \in \mathbb{N}_0$. It follows from (2.1) that the function

$$\Omega_n(t,u,z)=f_n[tuz]-\sum_{i+j+k=n}[f_i(t)f_j(u)f_k(z)] \tag{2.4}$$

for each $n \in \mathbb{N}_0$ and all $t, u, z \in A$, is bounded. Hence, we see that

$$\lim_{m\to\infty}\frac{\Omega_n(2^m t,2^m u,2^m z)}{2^{9m}}=0 \tag{2.5}$$

for each $n \in \mathbb{N}_0$ and all $t, u, z \in A$. Using (2.3), (2.4), and (2.5), we get

$$\begin{aligned}h_n[tuz]&=\lim_{m\to\infty}\frac{f_n(2^{3m}[tuz])}{2^{9m}}=\lim_{m\to\infty}\frac{f_n[(2^m t)(2^m u)(2^m z)]}{2^{9m}}\\&=\lim_{m\to\infty}\frac{\sum_{i+j+k=n}[f_i(2^m t)f_j(2^m u)f_j(2^m z)]+\Omega_n(2^m t,2^m u,2^m z)}{2^{9m}}\\&=\lim_{m\to\infty}\sum_{i+j+k=n}\left[\frac{1}{2^{3m}}f_i(2^m t)\frac{1}{2^{3m}}f_j(2^m u)\frac{1}{2^{3m}}f_k(2^m z)\right]\\&\quad+\lim_{m\to\infty}\frac{\Omega_n(2^m t,2^m u,2^m z)}{2^{9m}}=\sum_{i+j+k=n}[h_i(t)h_j(u)h_k(z)]\end{aligned}$$

for all $t, u, z \in A$ and all $n \in \mathbb{N}_0$. This completes the proof of the Theorem. □

As a consequence of Theorem 2, we show the Hyers–Ulam–Rassias stability of ternary cubic higher derivations.

Corollary 1 *Let $0 \le p < 3, \alpha, \beta > 0$ and let $F = \{f_0, f_1, \ldots, f_n, \ldots\}$ be a sequence of mappings from A into B such that*

$$\|f_n(x+2y)+f_n(x-2y)+f_n(2x)-2f_n(x)-4f_n(x+y)-f_n(x-y)$$
$$+f_n[tuz]-\sum_{i+j+k=n}[f_i(t)f_j(u)f_k(z)]\|\le\alpha+\beta(\|x\|^p+\|y\|^p+\|t\|^p+\|u\|^p+\|z\|^p)$$

for all $x, y, t, u, z \in A$ and each $n \in \mathbb{N}_0$. Then there exists a unique ternary cubic higher derivation

$$H = \{h_0, h_1, \ldots, h_n, \ldots\}$$

of any rank from A into B such that

$$\|f_n(x) - h_n(x)\| \leq \frac{\alpha + \beta\|x\|^p}{7}.$$

Proof Set

$$\varphi(x, y, t, u, z) = \alpha + \beta(\|x\|^p + \|y\|^p + \|t\|^p + \|u\|^p + \|z\|^p),$$

in the Theorem 2. □

Similarly to Theorem 2, we can prove the following theorem:

Theorem 3 *Suppose that* $\varphi : A^5 \to [0, \infty)$ *is a control function such that*

$$\lim_{n\to\infty} 2^{3m}\varphi\left(\frac{x}{2^m}, \frac{y}{2^m}, \frac{t}{2^m}, \frac{u}{2^m}, \frac{z}{2^m}\right) = 0$$

for all $x, y, t, u, z \in A$. *Assume that* $F = \{f_0, f_1, \ldots, f_n, \ldots\}$ *is a sequence of mappings such that*

$$\begin{aligned}\|f_n(x+2y) + f_n(x-2y) + f_n(2x) - 2f_n(x) - 4f_n(x+y) - f_n(x-y)\\ + f_n[tuz] - \sum_{i+j+k=n}[f_i(t)f_j(u)f_k(z)]\| \leq \varphi(x, y, t, u, z)\end{aligned} \tag{2.6}$$

for all $x, y, t, u, z \in A$ *and each* $n \in \mathbb{N}_0$. *Suppose that there exists* $0 \leq L < 1$ *such that the mapping* $\gamma(x) = \varphi\left(\frac{x}{2}, 0, 0, 0, 0\right)$ *has the property*

$$8\gamma\left(\frac{x}{2}\right) \leq L\gamma(x)$$

for all $x \in A$. *Then there exists a unique ternary cubic higher derivation* $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ *of any rank from A into B such that for each* $n \in \mathbb{N}_0$,

$$\|f_n(x) - h_n(x)\| \leq \frac{\gamma(x)}{1-L}$$

holds for all $x \in A$.

Proof Setting $y = t = u = z = 0$ in (2.6), we obtain

$$\|f_n(2x) - 8f_n(x)\| \leq \varphi(x, 0, 0, 0, 0).$$

Replacing x by $\frac{x}{2}$ in the above inequality, we get

$$\left\| 8f_n\left(\frac{x}{2}\right) - f_n(x) \right\| \le \varphi\left(\frac{x}{2}, 0, 0, 0, 0\right) = \gamma(x) \tag{2.7}$$

for all $x \in A$ and each $n \in \mathbb{N}_0$. Consider $X = \{g : g : A \to B\}$ and the generalized metric d on X :

$$d(h, g) = \inf\{M \in (0, \infty) : \|g(x) - h(x)\| \le M\gamma(x), \quad \forall x \in A\},$$

as well as the operator $J : X \to X$ with $(Jh)(x) = 8h\left(\frac{x}{2}\right)$ for all $h \in X$. For arbitrary elements $g, h \in X$,, we have

$$d(f, g) < \varepsilon \Rightarrow \|f(x) - h(x)\| \le \varepsilon\gamma(x) \Rightarrow \left\| f\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right) \right\| \le \varepsilon\gamma\left(\frac{x}{2}\right)$$
$$\Rightarrow \|Jf(x) - Jh(x)\| \le 8\varepsilon\gamma\left(\frac{x}{2}\right) \le L\varepsilon\gamma(x) \Rightarrow$$
$$d(Jf, Jh) \le Ld(f, g).$$

Thus, J is a strictly contractive function with the Lipschitz constant L. It follows from (2.7) that

$$d(Jf_n, f_n) \le 1.$$

Moreover, by Theorem 1, there exists a mapping $h_n : A \to B$ such that h_n is a fixed point of J that is $8h_n\left(\frac{x}{2}\right) = h_n(x)$ for all $x \in A$. By Theorem 1, $\lim_{m\to\infty} d(J^m(f_n), f_n) = 0$, we conclude that

$$\lim_{m\to\infty} 2^{3m} f_n(2^{-m}x) = h_n(x) \text{ for all } x \in A.$$

The mapping h_n is the unique fixed point of J in the set

$$U_n = \{g \in S : d(f_n, g) < \infty\}.$$

Hence, h_n is the unique fixed point of J such that

$$\|f_n(x) - h_n(x)\| \le M\gamma(x) \text{ for some } M > 0 \text{ and for all } x \in A.$$

Again, by Theorem 1 (ii), we have

$$d(f_n, h_n) \le \frac{1}{1-L} d(f_n, Jf_n) \le \frac{1}{1-L},$$

that is

$$\|f_n(x) - h_n(x)\| \le \frac{\gamma(x)}{1-L}$$

for all $x \in A$. The rest is similar to the proof of Theorem 2. □

The following corollary is similar to Corollary 1 for the case when $p > 3$.

Corollary 2 *Let $p > 3, \beta > 0$, and $F = \{f_0, f_1, \ldots, f_n, \ldots\}$ be a sequence of mappings from A into B such that*

$$\|f_n(x + 2y) + f_n(x - 2y) + f_n(2x) - 2f_n(x) - 4f_n(x + y) - f_n(x - y)$$

$$+f_n[tuz] - \sum_{i+j+k=n} [f_i(t)f_j(u)f_k(z)]\| \leq \alpha + \beta(\|x\|^p + \|y\|^p + \|t\|^p + \|u\|^p + \|z\|^p)$$

for all $x, y, t, u, z \in A$ and each $n \in \mathbb{N}_0$. Then there exists a unique ternary cubic higher derivation

$$H = \{h_0, h_1, \ldots, h_n, \ldots\}$$

of any rank from A into B such that

$$\|f_n(x) - h_n(x)\| \leq \frac{\beta\|x\|^p}{2^p - 8}$$

for all $x \in A$.

Proof Set

$$\varphi(x, y, t, u, z) = \beta(\|x\|^p + \|y\|^p + \|t\|^p + \|u\|^p + \|z\|^p),$$

and let $L = 2^{3-p}$ in Theorem 3. Then $\gamma(x) = \beta 2^{-p}\|x\|^p$, and there exists a sequence $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ with the required properties. □

Remark An interesting question is to ask whether there exists an approximately ternary cubic higher derivation that is not a ternary cubic higher derivation.

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Hyers–Ulam Stability for Differential Equations and Partial Differential Equations via Gronwall Lemma

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Abstract In this paper, we will study Hyers–Ulam stability for Bernoulli differential equations, Riccati differential equations, and quasi-linear partial differential equations of first order, using Gronwall Lemma, following a method given by Rus.

***MSC*:** 26D10; 34A40; 39B82; 35B20

1 Introduction

In [1–3], Rus has obtained some results regarding Ulam stability of differential and integral equations, using Gronwall inequalities method and weak Picard operators technique. In [4], Rus and Lungu have studied the stability of a partial differential equation of order two of hyperbolic type using the same method. In [5], Craciun and Lungu have studied, using this method, a partial differential equation of order two having a general form. In this paper, we use the same method in order to study the stability of Bernoulli and Riccati equations and also of quasi-linear partial differential equations of first order. We mention that some results regarding Ulam stability of Bernoulli and Riccati differential equations were established by Jung and Rassias [6, 7], using the integrating factor method. The first result proved on the Hyers–Ulam stability of partial differential equations is due to A. Prastaro and Th.M. Rassias [8]. Also Lungu and Popa [9] and Marian and Lungu [10]

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have obtained stability results from some partial differential linear and quasi-linear equations. The Gronwall inequality is used in Quarawani [11] in order to study Hyers–Ulam–Rassias stability for Bernoulli differential equations, and it is also used in [12, 13]. For a broader study of Hyers–Ulam stability for functional equations, the reader is also referred to the following books and papers: [6, 7, 14–26].

In the following, we will use Definitions 2.1, 2.2, 2.3 from [1], p.126 and Remark 2.1, 2.2. from [1], p.127.

2 Main Results

Stability of Bernoulli Differential Equation

Let $(\mathbb{B}, |\cdot|)$ be a (real or complex) Banach space, $a, b \in \mathbb{R}, a < b$, $p, q \in C([a,b], \mathbb{B})$, and $n \in \mathbb{R}\backslash\{0, 1\}$.

We consider the Bernoulli differential equation

$$z'(x) = p(x) z(x) + q(x) z^n(x), x \in [a, b], \tag{2.1}$$

and the inequation

$$\left|y'(x) - p(x) y(x) - q(x) y^n(x)\right| \leq \varepsilon, x \in [a, b]. \tag{2.2}$$

From Remark 2.1 from [1], p.127 follows that $y \in C^1([a,b], \mathbb{B})$ is a solution of the inequation (2.2) if and only if there exists a function $g \in C^1([a,b], \mathbb{B})$ (which depends on y) such that

(i) $|g(x)| \leq \varepsilon, \forall x \in [a, b]$;
(ii) $y'(x) = p(x) y(x) + q(x) y^n(x) + g(x), \forall x \in [a, b]$.

From Remark 2.2 from [1], p.127 follows that if $y \in C^1([a,b], \mathbb{B})$ is a solution of the inequation (2.2), then y is a solution of the following integral inequation

$$\left|y(x) - y(a) - \int_a^x \left[p(t) y(t) + q(t) y^n(t)\right] dt\right| \leq (x - a)\varepsilon, \forall x \in [a, b].$$

Theorem 4 *If*

(i) $a < \infty, b < \infty$;
(ii) $p, q \in C([a,b], \mathbb{B})$;
(iii) *there exists* $L > 0$ *such that*

$$\left|q(x) y^n(x) - q(x) z^n(x)\right| \leq L |y(x) - z(x)|,$$

for all $x \in [a, b]$ *and* $y, z \in C^1([a,b], \mathbb{B})$,

then the equation (2.1) *is Hyers–Ulam stable.*

Proof Let $y \in C^1([a,b], \mathbb{B})$ be a solution of the inequation (2.2) and z the unique solution of the Cauchy problem

$$\begin{cases} z'(x) = p(x)\,z(x) + q(x)\,z^n(x), \ x \in [a,b], \\ z(a) = y(a) \end{cases}. \tag{2.3}$$

We have that

$$z(x) = y(a) + \int_a^x \left[p(t)\,z(t) + q(t)\,z^n(t)\right] dt, x \in [a,b].$$

Let

$$M = \max_{x\in[a,b]} |p(x)|.$$

We consider the difference

$$|y(x) - z(x)| \le \left| y(x) - y(a) - \int_a^x \left[p(t)\,y(t) + q(t)\,y^n(t)\right] dt \right| +$$
$$\left| \int_a^x \left[p(t)\,y(t) + q(t)\,y^n(t) - p(t)\,z(t) - q(t)\,z^n(t)\right] dt \right| \le$$
$$\le \varepsilon(x-a) + \int_a^x \left[|p(t)\,y(t) - p(t)\,z(t)| + \left|\left(q(t)\,y^n(t) - q(t)\,z^n(t)\right)\right|\right] dt \le$$
$$\le \varepsilon(x-a) + \int_a^x \left[|p(t)|\,|y(t) - z(t)| + L\,|y(t) - z(t)|\right] dt =$$
$$= \varepsilon(x-a) + \int_a^x \left[|p(t)| + L\right] |y(t) - z(t)|\, dt.$$

From Gronwall lemma (see [27], p. 6), we have that

$$|y(x) - z(x)| \le \varepsilon(x-a)\,e^{\int_a^x [|p(t)|+L]dt} \le \varepsilon(b-a)\,e^{\int_a^b (M+L)dt} =$$
$$= \varepsilon(b-a)\,e^{(M+L)(b-a)} = c\cdot\varepsilon,$$

where $c = (b-a)\,e^{(M+L)(b-a)}$.

Example 1 We consider the Bernoulli differential equation

$$z' = xz + \frac{x}{1+x^2}\sqrt{z}, \tag{2.4}$$

where $x \in [a,b]$ and $z \ge 1$.We have $p(x) = x$ and $q(x) = \frac{x}{1+x^2}$. Let $D = \{(x,z) \mid x \in [a,b], z \ge 1\}$ and $f(x,z) = \frac{x}{1+x^2}\sqrt{z}$. We have

$$\left|\frac{\partial f}{\partial z}\right| = \left|\frac{x}{1+x^2} \cdot \frac{1}{2\sqrt{z}}\right| \leq \frac{1}{2}\left|\frac{x}{1+x^2}\right| \leq \frac{1}{4}, \forall (x, z) \in D,$$

and hence, the function f satisfies a Lipschitz condition in the variable z, on D, with Lipschitz constant $1/4$. Hence,

$$|f(x, y) - f(x, z)| \leq L|y - z| = \frac{1}{4}|y - z|,$$

that is

$$\left|\frac{x}{1+x^2}\sqrt{y} - \frac{x}{1+x^2}\sqrt{z}\right| \leq \frac{1}{4}|y - z|, x \in [a, b], y \geq 1, z \geq 1.$$

We apply Theorem 4 so the equation (2.4) is Hyers–Ulam stable. Let $y \in C^1([a, b], \mathbb{B})$ be a solution of the inequation

$$\left|z' - xz - \frac{x}{1+x^2}\sqrt{z}\right| \leq \varepsilon, \tag{2.5}$$

and z the unique solution of the Cauchy problem

$$\begin{cases} z' = xz + \frac{x}{1+x^2}\sqrt{z}, \\ z(a) = y(a) \end{cases}. \tag{2.6}$$

We have

$$z(x) = y(a) - \int_a^x \left[tz + \frac{t}{1+t^2}\sqrt{z}\right] dt, x \in [a, b].$$

Let

$$M = \max_{x \in [a,b]} |p(x)| = |b|.$$

We have

$$|y(x) - z(x)| \leq \varepsilon (b - a) e^{\left(|b| + \frac{1}{4}\right)(b-a)}.$$

Stability of Riccati Differential Equation

Let $(\mathbb{B}, |\cdot|)$ be a (real or complex) Banach space, $a, b \in \mathbb{R}, a < b$ and $p, q, r \in C([a, b], \mathbb{B})$.

We consider the Riccati differential equation

$$z'(x) = p(x) z^2(x) + q(x) z(x) + r(x), x \in [a, b], \tag{2.7}$$

and the inequation

$$\left| y'(x) - p(x) y^2(x) - q(x) y(x) - r(x) \right| \le \varepsilon, x \in [a, b]. \tag{2.8}$$

From Remark 2.1 from [1], p.127 follows that $y \in C^1([a, b], \mathbb{B})$ is a solution of the inequation (2.8) if and only if there exists a function $g \in C^1([a, b], \mathbb{B})$ (which depends on y) such that

(i) $|g(x)| \le \varepsilon, \forall x \in [a, b]$;
(ii) $y'(x) = p(x) y^2(x) + q(x) y(x) + r(x) + g(x), \forall x \in [a, b]$.

From Remark 2.2 from [1], p.127 follows that if $y \in C^1([a, b], \mathbb{B})$ is a solution of the inequation (2.8), then y is a solution of the following integral inequation:

$$\left| y(x) - y(a) - \int_a^x \left[p(t) y^2(t) + q(t) y(t) + r(t) \right] dt \right| \le (x - a) \varepsilon, \forall x \in [a, b].$$

Theorem 5 *If*

(i) $a < \infty, b < \infty$;
(ii) $p, q, r \in C([a, b], \mathbb{B})$;
(iii) *there exists* $L > 0$ *such that*

$$\left| p(t) y^2(x) - p(t) z^2(x) \right| \le L |y(x) - z(x)|,$$

for all $x \in [a, b]$ *and* $y, z \in C^1([a, b], \mathbb{B})$,

then the equation (2.7) *is Hyers–Ulam stable.*

Proof Let $y \in C^1([a, b], \mathbb{B})$ be a solution of the inequation (2.8) and z the unique solution of the Cauchy problem

$$\begin{cases} z'(x) = p(x) z^2(x) + q(x) z(x) + r(x), x \in [a, b], \\ z(a) = y(a) \end{cases}. \tag{2.9}$$

We have that

$$z(x) = y(a) + \int_a^x \left[p(t) y^2(t) + q(t) z(t) + r(t) \right] dt, \forall x \in [a, b].$$

Let

$$M = \max_{x \in [a,b]} |q(x)| .$$

We consider the difference

$$|y(x) - z(x)| \le \left| y(x) - y(a) - \int_a^x \left[p(t)\, y^2(t) + q(t)\, y(t) + r(t) \right] dt \right| +$$

$$\left| \int_a^x \left[p(t)\, y^2(t) + q(t)\, y(t) - p(t)\, z^2(t) - q(t)\, z(t) \right] dt \right| \le$$

$$\le \varepsilon (x-a) + \int_a^x \left[\left| p(t)\, y^2(t) - p(t)\, z^2(t) \right| + |(q(t)\, y(t) - q(t)\, z(t))| \right] dt \le$$

$$\le \varepsilon (x-a) + \int_a^x [L\, |y(t) - z(t)| + |q(t)|\, |y(t) - z(t)|]\, dt =$$

$$\le \varepsilon (x-a) + \int_a^x [L + |q(t)|]\, |y(t) - z(t)|\, dt.$$

From Gronwall lemma (see [27], p. 6), we have that

$$|y(x) - z(x)| \le \varepsilon (x-a)\, e^{\int_a^x (L+|q(t)|)dt} \le \varepsilon (b-a)\, e^{\int_a^b (L+M)dt} =$$

$$= \varepsilon (b-a)\, e^{(M+L)(b-a)} = c \cdot \varepsilon,$$

where $c = (b-a)\, e^{(M+L)(b-a)}$.

Hyers–Ulam Stability of Quasi-linear Partial Differential Equation

Hyers–Ulam Stability

Let $(\mathbb{B}, |\cdot|)$ be a (real or complex) Banach space, $a, b \in (0, \infty]$, ε a positive real number, $\varphi \in C([0,a) \times [0,b), \mathbb{R}_+)$ and $p, q, r \in C([0,a) \times [0,b) \times \mathbb{B}, \mathbb{R})$ and $p(x, y, u) \neq 0$.

We consider the following quasi-linear partial differential equation of first order:

$$\frac{\partial u(x,y)}{\partial x} = -\frac{q(x,y,u)}{p(x,y,u)} \frac{\partial u}{\partial y} + \frac{r(x,y,u)}{p(x,y,u)}, \tag{2.10}$$

and the following partial differential inequation:

$$\left| \frac{\partial v(x,y)}{\partial x} + \frac{q(x,y,v)}{p(x,y,v)} \frac{\partial v}{\partial y} - \frac{r(x,y,v)}{p(x,y,v)} \right| \le \varepsilon, \tag{2.11}$$

$$\left|\frac{\partial v(x,y)}{\partial x}+\frac{q(x,y,v)}{p(x,y,v)}\frac{\partial v}{\partial y}-\frac{r(x,y,v)}{p(x,y,v)}\right|\le \varepsilon\cdot\varphi(x,y). \tag{2.12}$$

Remark 1 A function $v \in C([0,a)\times[0,b),\mathbb{B})$ is a solution of the inequation (2.11) if and only if there exists a function $g \in C([0,a)\times[0,b),\mathbb{B})$ such that

(i) $|g(x,y)|\le\varepsilon, \forall (x,y)\in[0,a)\times[0,b)$;
(ii) $\frac{\partial v(x,y)}{\partial x}=-\frac{q(x,y,v)}{p(x,y,v)}v_y(x,y)+\frac{r(x,y,v)}{p(x,y,v)}+g(x,y)$, where $v_y=\frac{\partial v}{\partial y}$.

Remark 2 If $v\in C([0,a)\times[0,b),B)$ is a solution of the inequation (2.11), then v is a solution of the following integral inequation:

$$\left|v(x,y)-v(0,y)-\int_0^x\left[-\frac{q(s,y,v(s,y))}{p(s,y,v(s,y))}v_y(s,y)+\frac{r(s,y,v(s,y))}{p(x,y,v(s,y))}\right]ds\right|\le\varepsilon x,$$

$\forall x\in[0,a), y\in[0,b)$.

Indeed, by Remark 1, we have that

$$\frac{\partial v(x,y)}{\partial x}=-\frac{q(x,y,v(x,y))}{p(x,y,v(x,y))}v_y(x,y)+\frac{r(x,y,v(x,y))}{p(x,y,v(x,y))}+g(x,y),$$

$\forall x\in[0,a), y\in[0,b)$. This implies that

$$v(x,y)=v(0,y)+\int_a^x\left[-\frac{q(s,y,v(s,y))}{p(s,y,v(s,y))}v_y(s,y)+\frac{r(s,y,v(s,y))}{p(x,y,v(s,y))}+g(s,y)\right]ds.$$

From this, it follows that

$$\left|v(x,y)-v(0,y)-\int_0^x\left[-\frac{q(s,y,v(s,y))}{p(s,y,v(s,y))}v_y(s,y)+\frac{r(s,y,v(s,y))}{p(x,y,v(s,y))}\right]ds\right|$$
$$\le\int_0^x|g(s,y)|\,ds\le\varepsilon x.$$

Theorem 6 *We suppose that*

(i) $a<\infty, b<\infty$;
(ii) $p,q,r\in C([0,a]\times[0,b]\times\mathbb{B},\mathbb{B}), p\neq 0$;
(iii) *there exists* $l_1,l_2>0$ *such that*

$$\left|\frac{q(x,y,v_1)}{p(x,y,v_1)}v_{1y}(x,y)-\frac{q(x,y,v_2)}{p(x,y,v_2)}v_{2y}(x,y)\right|\le l_1|v_1-v_2|,$$
$$\left|\frac{r(x,y,v_1)}{p(x,y,v_1)}-\frac{r(x,y,v_2)}{p(x,y,v_2)}\right|\le l_2|v_1-v_2|,$$

$\forall v_1, v_2 \in \mathbb{B}, \forall (x, y) \in [0, a] \times [0, b]$.

Then:

(a) *for* $\psi \in C([0, a], \mathbb{B})$*, the equation* (2.10) *has a unique solution with*

$$u(0, y) = \psi(y), \forall y \in [0, b];$$

(b) *the equation* (2.10) *is Hyers–Ulam stable.*

Proof

(a) This is a known result (see [28]).
(b) Let v be a solution of the inequation (2.11). Denote by u the unique solution of the equation (2.10), which satisfies the condition

$$u(0, y) = v(0, y), \forall y \in [0, b].$$

From Remark 2 and condition (iii), we have that

$$\begin{aligned}
&|v(x, y) - u(x, y)| \le \\
&\le \left| v(x, y) - v(0, y) - \int_0^x \left[-\frac{q(s, y, v(s, y))}{p(s, y, v(s, y))} v_y(s, y) + \frac{r(s, y, v(s, y))}{p(x, y, v(s, y))} \right] ds \right| + \\
&\int_0^x \left| -\frac{q(s, y, v(s, y))}{p(s, y, v(s, y))} v_y(s, y) + \frac{r(s, y, v(s, y))}{p(x, y, v(s, y))} + \frac{q(s, y, u(s, y))}{p(s, y, u(s, y))} u_y(s, y) \right. \\
&\quad \left. - \frac{r(s, y, u(s, y))}{p(x, y, u(s, y))} \right| ds \\
&\quad \le \varepsilon x + \int_0^x [l_1 |v(s, y) - u(s, y)| + l_2 |v(s, y) - u(s, y)|] ds.
\end{aligned}$$

Or,

$$|v(x, y) - u(x, y)| \le \varepsilon x + \int_0^x [l_1 + l_2] |v(s, y) - u(s, y)| ds.$$

From Gronwall lemma (see [27], p. 6), we have

$$|v(x, y) - u(x, y)| \le a e^{a(l_1 + l_2)} \cdot \varepsilon = c \cdot \varepsilon,$$

where $c = a e^{a(l_1 + l_2)}$.

So, the equation (2.10) is Hyers–Ulam stable.

Hyers–Ulam–Rassias Stability of Equation (2.10)

Let us consider the equation (2.10) and the inequation (2.12) in the case $a = \infty$, $b = \infty$.

Theorem 7 *We suppose that*

(i) $p, q, r \in C([0, a) \times [0, b) \times \mathbb{B}, \mathbb{B})$, $p \neq 0$;
(ii) *there exists* $l_1, l_2 \in C^1([0, a) \times [0, b), \mathbb{R}_+)$ *such that*

$$\left|\frac{q(x, y, v_1)}{p(x, y, v_1)} v_{1y}(x, y) - \frac{q(x, y, v_2)}{p(x, y, v_2)} v_{2y}(x, y)\right| \leq l_1(x, y)|v_1 - v_2|;$$
$$\left|\frac{r(x, y, v_1)}{p(x, y, v_1)} - \frac{r(x, y, v_2)}{p(x, y, v_2)}\right| \leq l_2(x, y)|v_1 - v_2|;$$

$\forall v_1, v_2 \in \mathbb{B}, \forall (x, y) \in [0, a) \times [0, b)$;
(iii) $e^{\int_0^\infty [l_1(s,y)+l_2(s,y)]ds}$ *is convergent and there exists a real number* M *such that* $e^{\int_0^\infty [l_1(s,y)+l_2(s,y)]ds} \leq M$, $\forall y \in [0, b)$;
(iv) *there exists* $\lambda_\varphi > 0$ *such that*

$$\int_0^x \varphi(s, y)\, ds \leq \lambda_\varphi \cdot \varphi(x, y), \forall (x, y) \in [0, a) \times [0, b)$$

and φ *increasing.*

Then the equation (2.10) ($a = \infty, b = \infty$) *is Hyers–Ulam–Rassias stable.*

Proof Let v be a solution of the inequation (2.12). Denote by u the unique solution of the problem

$$\begin{cases} \frac{\partial u(x,y)}{\partial x} = -\frac{q(x,y,u)}{p(x,y,u)} u_y(x, y) + \frac{r(x,y,u)}{p(x,y,u)} \\ u(0, y) = v(0, y). \end{cases}$$

We have

$$u(x, y) = v(0, y) + \int_0^x \left[-\frac{q(s, y, u(s, y))}{p(s, y, u(s, y))} u_y(s, y) + \frac{r(s, y, u(s, y))}{p(x, y, u(s, y))}\right] ds$$

and

$$\left|v(x, y) - v(0, y) - \int_0^x \left[-\frac{q(s, y, v(s, y))}{p(s, y, v(s, y))} v_y(s, y) + \frac{r(s, y, v(s, y))}{p(x, y, v(s, y))}\right] ds\right| \leq$$
$$\leq \varepsilon \int_0^x \varphi(s, y)\, ds \leq \varepsilon \lambda_\varphi \cdot \varphi(x, y).$$

Then we have

$$|v(x,y) - u(x,y)| \leq$$

$$\leq \left| v(x,y) - v(0,y) - \int_0^x \left[-\frac{q(s,y,v(s,y))}{p(s,y,v(s,y))} v_y(s,y) + \frac{r(s,y,v(s,y))}{p(x,y,v(s,y))} \right] ds \right| +$$

$$\int_0^x \left| -\frac{q(s,y,v(s,y))}{p(s,y,v(s,y))} v_y(s,y) + \frac{r(s,y,v(s,y))}{p(x,y,v(s,y))} + \frac{q(s,y,u(s,y))}{p(s,y,u(s,y))} u_y(s,y) \right.$$

$$\left. -\frac{r(s,y,u(s,y))}{p(x,y,u(s,y))} \right| ds$$

$$\leq \varepsilon \lambda_\varphi \cdot \varphi(x,y) + \int_0^x \left[l_1(s,y) |v(s,y) - u(s,y)| + l_2(s,y) |v(s,y) - u(s,y)| \right] ds \leq$$

$$\leq \varepsilon \lambda_\varphi \cdot \varphi(x,y) + \int_0^x \left[l_1(s,y) + l_2(s,y) \right] |v(s,y) - u(s,y)| \, ds.$$

From Gronwall lemma (see [27], p. 6), we have that

$$|v(x,y) - u(x,y)| \leq \varepsilon \lambda_\varphi \cdot \varphi(x,y) \, e^{\int_0^x [l_1(s,y) + l_2(s,y)] ds} \leq c_\varphi \cdot \varepsilon \cdot \varphi(x,y),$$

where $c_\varphi = \lambda_\varphi \cdot M$.

So, the equation (2.10) is generalized Hyers–Ulam–Rassias stable.

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On b-Metric Spaces and Brower and Schauder Fixed Point Principles

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Abstract In the paper, we present the basic ideas in b-metric spaces (and b-normed spaces). The main result is the Schauder fixed point principle. For the proof, we use the method presented by Dugundji and Granas in their book [4].

Mathematics Subject Classification (2010): 54D35, 54E50, 54E99, 46S20, 47H10

1 Introduction

We present some basic ideas needed in the paper. We start with the b-metric spaces and b-normed spaces. By $\mathbb{R}$, $\mathbb{R}_+$, N, and N_0, we denote the sets of all real, real nonnegative, natural, and natural with zero, respectively, numbers.

Definition 1 Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

(i) $d(x, y) = 0 \Longleftrightarrow x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leqslant s[d(x, z) + d(z, y)]$,

for all $x, y, z \in X$ and some fixed $s \geqslant 1$, is called a b-metric (ball metric) on X. The pair (X, d) is a b-metric space.
It is clear that for $s = 1$, we get a metric on X.
More information on such spaces the reader may find in [1–3].

Definition 2 Let X be a nonempty linear space. A function $\| \cdot \|$: $X \to \mathbb{R}_+$ such that:

(iv) $\| x \| = 0 \quad \Leftrightarrow \quad x = 0$,
(v) $\| \lambda x \| = |\lambda| \, \| x \|$,

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(vi) $\| x + y \| \leqslant s[\| x \| + \| y \|]$,

for all $x, y \in X$, $\lambda \in \mathbb{R}$ and some fixed $s \geqslant 1$, is called a b-norm on X and $(X, \| \cdot \|)$ is a b-normed linear space.

Definition 3 (see [5]) A mapping $d : X \times X \to \mathbb{R}_+$ we call a strong b-metric if it satisfies (i) and (ii) from the Definition 1 and

(vii) $d(x, y) \leqslant d(x, z) + sd(z, y)$,
for all $x, y, z \in X$ and some $s \geqslant 1$.
Note that a strong b-metric satisfies also the condition (by the symmetry of d)
(viii) $d(x, y) \leqslant sd(x, z) + d(z, y)$,
for all $x, y, z \in X$ and some $s \geqslant 1$.

One can verify that

Remark 1 A strong b-metric satisfies the condition

$$d(x_0, x_n) \leqslant s[d(x_0, x_1) + \cdots + d(x_{n-1}, x_n)], \tag{1}$$

for all $x_0, \cdots, x_n \in X$ and $n \in N$.

Proof We have

$$\begin{aligned} d(x_0, x_n) &\leqslant sd(x_0, x_1) + d(x_1, d_n) \leqslant sd(x_0, x_1) + sd(x_1, x_2) + d(x_2, x_n) \\ \cdots &\leqslant s[d(x_0, x_1) + \cdots + d(x_{n-2}, x_{n-1}] + d(x_{n-1}, x_n) \\ &\leqslant s[d(x_0, x_1) + \cdots + d(x_{n-1}, x_n)]. \end{aligned}$$

We say that d satisfies the s-relaxed triangle inequality if the condition (iii) is fulfilled, and d satisfies the s-relaxed polygonal inequality if the condition (1) holds true. So strong b-metric satisfies the s-relaxed polygonal inequality as well.

Remark 2 One can also consider a strong b-norm and a strong b-normed space, respectively.

Remark 3 If $\| \cdot \|$ is a strong b-norm in a linear space X, then

$$d(x, y) := \| x - y \|, \quad x, y \in X \tag{2}$$

is a strong b-metric in X.

Indeed, for $x, y, z \in X$, one gets

$$\begin{aligned} d(x, y) &= \| x - y \| = \| (x - z) + (z - y) \| \\ &\leqslant \| x - z \| + s \| z - y \| = d(x, z) + sd(z, y). \end{aligned}$$

Remark 4 If $\| \cdot \|$ is a strong b-norm in X, then

$$\| x_1 + x_2 \| \leqslant s \| x_1 \| + \| x_2 \|, \tag{3}$$

$$\| x_1 + \cdots + x_n \| \leqslant \| x_1 \| + s(\| x_2 \| + \cdots + \| x_n \|), \tag{4}$$

$$\| x_1 + \cdots + x_n \| \leqslant s(\| x_1 \| + \cdots + \| x_{n-1} \|) + \| x_n \|, \tag{5}$$

$$\| x_1 + \cdots + x_n \| \leqslant s(\| x_1 \| + \cdots + \| x_n \|), \tag{6}$$

for all $x_1, \cdots, x_n \in X$, and fixed $s \geqslant 1$.
We verify, e.g., (4). One has

$$\begin{aligned} \| x_1 + \cdots + x_n \| &\leqslant \| x_1 + \cdots + x_{n-1} \| + s \| x_n \| \\ &\leqslant \| x_1 + \cdots + x_{n-2} \| + s \| x_{n-1} \| + s \| x_n \| \\ &\cdots \\ &\leqslant \| x_1 \| + s(\| x_2 \| + \cdots + \| x_n \|). \end{aligned}$$

The rest is obvious, so we leave it for the reader.

Remark 5 A strong b-metric is a continuous function. In fact, let $d(x_n, x) \to 0$ as $n \to \infty$ and $d(y_n, y) \to 0$ as $n \to \infty$, $x_n, y_n, x, y \in (X, d, s)$, then one has

$$\begin{aligned} d(x_n, y_n) &\leqslant sd(x_n, x) + d(x, y_n) \\ &\leqslant sd(x_n, x) + d(x, y) + sd(y, y_n) \end{aligned}$$

and hence

$$d(x_n, y_n) - d(x, y) \leqslant s[d(x_n, x) + d(y_n, y)].$$

Similarly,

$$d(x, y) - d(x_n, y_n) \leqslant s[d(x_n, x) + d(y_n, y)],$$

and consequently

$$|d(x_n, y_n) - d(x, y)| \leqslant s[d(x_n, x) + d(y_n, y)], \tag{7}$$

which completes the proof.

Lemma 1 *Let $(X, s, \| \cdot \|), s \geqslant 1$ be a strong b-normed space. Then, $\| \cdot \|$ is a continuous function.*

Proof We have for $x, y \in X$

$$\| x \| = \| y + (x - y) \| \leqslant \| y \| + s \| x - y \|,$$

and hence

$$\| x \| - \| y \| \leqslant s \| x - y \| . \tag{8}$$

Similarly,

$$\| y \| = \| x + (y - x) \| \leqslant \| x \| + s \| x - y \|$$

that is,

$$\| x \| - \| y \| \geqslant -s \| x - y \| . \tag{9}$$

By (8) and (9), one gets

$$\Big| \| x \| - \| y \| \Big| \leqslant s \| x - y \| . \tag{10}$$

Therefore, if $x_n \to x$ as $n \to$, then

$$\Big| \| x_n \| - \| x \| \Big| \leqslant s \| x_n - x \| \to 0, \text{ as } n \to \infty,$$

and consequently

$$\| x_n \| \to \| x \| \text{ as } n \to \infty,$$

i.e., $\| \cdot \|$ is a continuous function.

Definition 4 A mapping $d : X \times X \to \mathbb{R}_+$ satisfies the s-relaxed strong polygonal inequality if

$$d(x_0, x_n) \leqslant d(x_0, x_1) + s[d(x_1, x_2) + \cdots + d(x_{n-1}, x_n)] \tag{11}$$

for some fixed $s \geqslant 1$ and for all $x_0, x_1, \ldots, x_n \in X$ and all $n \in N$.

Lemma 2 *If* $d : X \times X \Rightarrow \mathbb{R}_+$ *satisfies* (11)*, then*

$$d(x_0, x_n) \leqslant s[d(x_0, x_1) + \cdots + d(x_{n-2}, x_{n-1})] + d(x_{n-1}, x_n), \tag{12}$$

for all $x_0, \cdots, x_n \in X$ *and* $n \in N$.

Proof We have, by (11) and (viii),

$$\begin{aligned} d(x_0, x_n) &\leqslant sd(x_0, x_1) + d(x_1, x_n) \\ &\leqslant sd(x_0, x_1) + sd(x_1, x_2) + d(x_2, x_n) \\ \cdots &\leqslant s[d(x_0, x_1) + \cdots + d(x_{n-2}, x_{n-1})] + d(x_{n-1}, x_n) \end{aligned}$$

i.e., we obtain (12).

Remark 6 Clearly, the s-relaxed strong polygonal inequality (11) implies the s-relaxed strong triangle inequality (vii).

Remark 7 The inequality (vii) is equivalent to (11), for all $x = x_0, x_1 = z, x_2, \cdots, x_{n-1}, x_n = y$ and all $n \in N$.

Indeed, it is obvious that (11) implies (vii).
Conversely, one has

$$\begin{aligned} d(x, y) &\leqslant d(x, x_{n-1}) + sd(x_{n-1}, x_n) \\ &\leqslant d(x, x_{n-2}) + sd(x_{n-2}, x_{n-1}) + sd(x_{n-1}, x_n) \\ &\dots \\ &\leqslant d(x, x_1) + s[d(x_1, x_2) + \cdots + d(x_{n-1}, y)], \end{aligned}$$

i.e., (11).

2 Compactness in b-Metric Spaces

Definition 5 Let (X, d, s) be a b-metric space. A set $M \subset X$ is compact, if any $\{x_n\}$ in M contains a subsequence $\{x_{n_k}\}$ that converges (with respect to d) to some $x \in X$. If $x \in M$, then M is called strongly compact.

Theorem 1 *Let $M \subset (X, d, s)$ be strongly compact and $f : M \to \mathbb{R}$ be continuous. Then,*

(a) *f is bounded on M,*
(b) *there exist $x_0, x_1 \in M$ such that*

$$f(x_0) = \inf\{f(x) : x \in M\},$$
$$f(x_1) = \sup\{f(x) : x \in M\}.$$

Proof The proof runs similarly to that one presented in [6]. We show that f is bounded below. For the contrary, assume that

$$\exists_{x_n \in M} f(x_n) < -n. \tag{13}$$

By the compactness of M and continuity of f, there exists a subsequence $\{x_{n_k}\} \subset M$ such that

$$x_{n_k} \to x_0 \in M \text{ and } f(x_{n_k}) \to f(x_0) \in \mathbb{R}.$$

According to (13), we get contradiction.
For (b), let

$$\alpha = inf\{f(x) : x \in M\}.$$

For every $\epsilon_n = \frac{1}{n}$, there exists $x_n \in M$ such that

$$\alpha \leqslant f(x_n) < \alpha + \frac{1}{n}$$

and consequently there exists $\{x_{n_k}\}$, $x_{n_k} \to x_0 \in M$ with

$$\alpha \leqslant f(x_{n_k}) < \alpha + \frac{1}{n_k}$$

and $f(x_{n_k}) \to f(x_0)$ as $k \to \infty$.
Hence,

$$f(x_{n_k}) \to \alpha \text{ and } f(x_{n_k}) \to f(x_0),$$

which means that $\alpha = f(x_0)$, $x_0 \in M$, and the proof is complete.
The verification of other statements is quite similar.

Remark 8 If the assumptions of Theorem 1 are not satisfied, then the result may not be true (see also [6]).

Definition 6 A set $E \subset (X, d, s)$ is called an ϵ-net, $\epsilon > 0$, for a set $M \subset (X, d, s)$, if for every point $x \in M$ there exists a point $u \in E$ such that $d(x, u) < \epsilon$.

Theorem 2 *Let (X, d, s) be a b-metric complete space. Let for every $\epsilon > 0$ there exists finite ϵ-net with paints belonging to $M \subset X$. Then, M is a compact set.*

The proof can be done very similarly to that one given in [6] for a metric space, so the details are left to the reader.

Theorem 3 *Let (X, d, s) be a b-metric space. If $M \subset X$ is compact, then for every $\epsilon > 0$ there exists finite ϵ-net $\{c_1, \ldots, c_n\} \subset M$ for the set M.*

The proof can be done very similarly to the proof presented in [6] for a metric space.

Remark 9 Till now, the existence of completion of b-metric spaces is still an important and open problem.

Theorem 4 *If $M \subset X$ is compact (in b-metric space X), then M is bounded.*

Proof Let

$$T = \{x_1, \dots, x_n\}$$

be 1-net for M. Let $a \in X$. One has for $x \in M, x_i \in T, i = 1, \dots, n,$

$$\begin{aligned} d(x, a) &\leqslant s[d(x, x_i) + d(x_i, a)] \\ &\leqslant s[1 + \max_i d(x_i, a)] \leqslant K < \infty. \end{aligned}$$

Theorem 5 *Every compact b-metric space (X, d, s) is separable.*

Proof Let $\{\epsilon_n\}$ be a sequence of positive, tending to zero, decreasing sequence, and let

$$T_n = \{x_i^n\}, \quad i = 1, 2, \dots, i_n$$

be an ϵ_n-net for X. Let $E = \cup_{i=1}^{\infty} T_n$. Then, T is a countable set. Moreover, for any $x \in X, \epsilon_n < \epsilon$ there exists $x_i^n \in T_n$ for some $i \in N$ such that

$$d(x, x_i^n) < \epsilon_n,$$

i.e., E is dense in X. This is the desired conclusion and finishes the proof.

3 Finite-Dimensional b-Normed Spaces

Let (X, d, s) be an n-dimensional b-normed linear space and $\{e_1, \dots, e_n\}$ be a base of X. Then, we know that any $x \in X$ has a unique representation

$$x = \alpha_1 e_1 + \dots + \alpha_n e_n, \quad \alpha_i \in \mathbb{R}, \quad i = 1, \dots, n. \tag{14}$$

Define

$$||x||_0 := \sum_{i=1}^{n} |\alpha_i|, \tag{15}$$

where x is given by (14).

Theorem 6 *Let $(X, ||\cdot||, s)$ be an n-dimensional b-normed linear space. Then, there exists $\beta > 0$ such that for all $x \in X$*

$$||x|| \leqslant \beta ||x||_0. \tag{16}$$

Proof We have

$$\begin{aligned}
||x|| = ||\sum_{i=1}^{n} \alpha_i e_i|| &\leqslant s|\alpha_1|||e_1|| + \cdots + s^n|\alpha_n|||e_n|| \\
&\leqslant s^n(|\alpha_1|||e_1|| + \ldots + |\alpha_n|||e_n||) \\
&\leqslant s^n \max_i ||e_i|| \sum_{i=1}^{n} |\alpha_1| \\
&\leqslant s^n K \sum_{i=1}^{n} |\alpha_i| = \beta ||x||_0,
\end{aligned}$$

where $\beta = s^n K, \quad K = \max_i ||e_i||$, i.e., (16).

Remark 10 By $(\mathbb{R}^n, ||\cdot||_0)$, we denote the n-dimensional space instead of (X, d, s) with the norm defined by (15).

Theorem 7 *Let* $(X, ||\cdot||, s)$ *be an n-dimensional strong b-normed linear space. Then, there exist* $\alpha > 0$ *and* $\beta > 0$ *such that for all* $x \in X$

$$\alpha ||x||_0 \leqslant ||x|| \leqslant \beta ||x||_0. \tag{17}$$

Proof Let $\{e_1, \ldots, e_n\}$ be a base of X and for $x \in X$,

$$x = \alpha_1 x_1 + \ldots + \alpha_n x_n, \quad \alpha_i \in \mathbb{R}, i = 1, \ldots, n.$$

Let $U := \{x \in X : ||x||_0 = 1\}$. Then, U is bounded: for if $x_1, x_2 \in U, x_k = \alpha_1^k e_1 + \ldots + \alpha_n^k e_n, k = 1, 2$, then

$$\begin{aligned}
||x_1 - x_2|| &= ||\sum_{i=1}^{n} (\alpha_i^1 - \alpha_i^2) e_i|| \\
&\leqslant s^n \sum_{i=1}^{n} |\alpha_i^1 - \alpha_i^2|||e_i|| \leqslant s^n \max_i ||e_i|| (\sum_{i=1}^{n} |\alpha_i^1| + \sum_{i=1}^{n} |\alpha_i^2|) \\
&\leqslant 2s^n K,
\end{aligned}$$

where $K = \max_i ||e_i||, 1 = \sum_{i=1}^{n} |\xi_i^1|$.

By Lemma 1, $f(x) := ||x||$, $f : U \to \mathbb{R}_+$, $U \subset X$, f is continuous.

But $U = \{(\alpha_1, \ldots, \alpha_n)\} \in \mathbb{R}^n : \sum_{i=1}^{n} |\alpha_i| = 1\}$ as a bounded and closed subset of n-dimensional space $\mathbb{R}^n$ is the strongly compact set. Therefore, by Theorem 1, f has infimum α in U different from zero (because if f on U is equal to $f(x_0)$, $x_0 \in U$, so $x_0 \neq 0$).

Therefore,

$$\alpha = \inf_{x \in U} f(x) = \inf_{x \in U} ||x|| = f(\overline{x}) = ||\overline{x}||, \quad \overline{x} \in U, \overline{x} \neq 0.$$

Consequently,

$$||\frac{x}{||x||_0}|| \geqslant \alpha > 0 \text{ for all } x \in X,$$

since

$$||\frac{x}{||x||_0}||_0 = 1 \text{ for all } x \in X.$$

Thus,

$$\alpha ||x||_0 \leqslant ||x||, \quad x \in X. \tag{18}$$

From (16) and (18), one gets

$$\alpha ||x||_0 \leqslant ||x|| \leqslant \beta ||x||_0, \quad x \in X.$$

Remark 11 Moreover, if every b-norm $|| \cdot ||$ is equivalent to $|| \cdot ||_0$, then every b-norms $|| \cdot ||_1$ and $|| \cdot ||_2$ are equivalent too, so we have also (17) with b-norms $|| \cdot ||_1$ and $|| \cdot ||_2$.

More precisely, if

$$\alpha ||x||_0 \leqslant ||x||_1 \leqslant \beta ||x||_0, \quad \alpha > 0, \beta > 0, x \in X,$$

and

$$\alpha_1 ||x||_0 \leqslant ||x||_2 \leqslant \beta_1 ||x||_0, \quad \alpha_1 > 0, \beta_1 > 0, x \in X,$$

then

$$\frac{\alpha_1}{\beta} ||x||_1 \leqslant ||x||_2 \leqslant \frac{\beta_1}{\alpha} ||x||_1, \quad x \in X. \tag{19}$$

This is the desired conclusion.

Lemma 3 *Let* $(X, ||\cdot||, s)$ *be an n-dimensional strong b-normed linear space, and let* $U \subset X$ *be a bounded set. Then,* U *is compact (in* X*).*

Proof Let

$$x = \sum_{i=1}^{n} \alpha_i e_i, \quad \alpha_i \in \mathbb{R}, i = 1, \ldots, n, x \in X,$$

and

$$\overline{x} = \{\alpha_1, \ldots, \alpha_n\}, \quad \overline{x} \in \mathbb{R}^n,$$

Let $f(x) = \overline{x}$, $x \in U$, then $f(U) = \overline{U}$ and by the inequalities

$$\alpha||\overline{x}|| \leqslant ||x|| \leqslant \beta||\overline{x}||, \quad \alpha > 0, \beta > 0,$$

where $||x||$ is the norm of x in X and $||\overline{x}||$ is the norm of corresponding $\overline{x}$ in $\mathbb{R}^n$, we get

1. U bounded in X iff $\overline{U}$ bounded in $\mathbb{R}^n$,
2. a sequence $\{x_n\}$ is convergent in $(X, ||\cdot||, s)$ iff the corresponding sequence $\{\overline{x}_n\}$ is convergent in $\mathbb{R}^n$.

Consequently, the compactness of U bounded in X follows from the compactness of $\overline{U}$ bounded in $\mathbb{R}^n$. This conclude the proof.

Theorem 8 *If the induced space* $(\mathbb{R}^n, ||\cdot||_0)$ *for the strong b-metric n-dimensional linear space* $(X, ||\cdot||, s)$ *is complete, then also* $(X, ||\cdot||, s)$ *is a complete space.*

Proof Let $\{x_m\}$, $x_m = \alpha_1^m e_1 + \ldots + \alpha_n^m e_n$, $\alpha_i^m \in \mathbb{R}$, $i = 1, \ldots, n$, $m \in N$, be a Cauchy sequence of elements from X. Then, also $\{\overline{x}_m\} = \{\alpha_1^m, \ldots, \alpha_n^m\}$ is a Cauchy sequence in $(\mathbb{R}^n, ||\cdot||_0)$: for if $\epsilon > 0$ and

$$||x_m - x_k|| < \epsilon \text{ for } m, k > n_0,$$

then from (17) one has

$$||x_m - x_k||_0 = \sum_{i=1}^{n} |\alpha_i^m - \alpha_i^k| \leqslant \frac{1}{\alpha}||x_m - x_k|| < \frac{\epsilon}{\alpha}.$$

Since $(\mathbb{R}^n, ||\cdot||_0)$ is complete, so for $x_m \to x$ as $m \to \infty$, with respect to $||\cdot||_0$, by (17), one gets

$$||x_m - x|| \leqslant \beta||x_m - x||_0 \to 0 \text{ as } m \to \infty,$$

and $x = \alpha_1 e_1 + \ldots + \alpha_n e_n \in (X, ||\cdot||, s)$, which means that $(X, ||\cdot||, s)$ is a complete space, and the proof of the theorem is finished.

4 Brower Fixed Point Principle in b-Normed Spaces

We know that

Theorem 9 (Brower) *Let U be a nonempty bounded convex closed subset of $\mathbb{R}^n$, and let $T : U \to U$ be a continuous map.*

Then, T has a fixed point $u \in U$.

We prove the following.

Theorem 10 (Brower) *Let $(X_n, ||\cdot||, s)$ be n-dimensional strong b-normed linear space, and let $A \subset X_n$ be a bounded convex closed set. If, moreover, $\varphi : A \to A$ is continuous (in b-norm $||\cdot||$), then there exists $y \in A$ such that $\varphi(y) = y$.*

Proof Let $x \in A$, then $x = \alpha_1 e_1 + \ldots + \alpha_n e_n, \quad \alpha_i \in \mathbb{R}, \quad i = 1, \ldots, n$ and $\{e_1, \ldots, e_n\}$ is a base of X_n;

$$\overline{x} = (\alpha_1, \ldots, \alpha_n) \in \overline{A} \subset \mathbb{R}^n,$$

$$\phi : A \to \overline{A}, \quad \phi(x) = \overline{x}, \quad x \in A.$$

Then, ϕ is a homeomorphism of A onto $\phi(A) = \overline{A}$. In fact, ϕ is one to one. Moreover, ϕ and ϕ^{-1} are continuous. Actually, we verify that for $x, x_0 \in A$,

$$||x - x_0|| \to 0 \text{ implies } ||\phi(x) - \phi(x_0)||_0 \to 0.$$

In fact, one has by (17) Theorem 7

$$||\phi(x) - \phi(x_0)||_0 = ||\overline{x} - \overline{x}_0||_0 \leqslant \frac{1}{\alpha}||x - x_0|| \to 0.$$

so $\phi(x) \to \phi(x_0)$ as $x \to x_0$.

Similarly,

$$\phi^{-1}(\overline{x}) = x, \quad \phi^{-1} : \overline{A} \to A,$$

and if $\overline{x} \to \overline{x}_0$, then $\phi^{-1}(\overline{x}) \to \phi^{-1}(\overline{x}_0)$.

For we have

$$||\phi^{-1}(\overline{x}) - \phi^{-1}(\overline{x}_0)|| = ||x - x_0|| \leqslant \beta||\overline{x} - \overline{x}_0||_0 \to 0,$$

i.e., ϕ^{-1} is continuous in $\overline{A}$.

Now, we verify that $\phi(A) = \overline{A}$ is convex. Let $\overline{x}, \overline{y} \in \phi(A)$,

$$\overline{x} = (\alpha_1, \ldots, \alpha_n), \quad \overline{y} = (\beta_1, \ldots, \beta_n), \quad \alpha_i, \beta_i \in \mathbb{R}, \quad i = 1, \ldots, n.$$

Since A is convex, for $0 \leqslant \lambda \leqslant 1$ one has

$$\lambda\overline{x} + (1-\lambda)\overline{y} = \Big(\lambda\alpha_1 + (1-\lambda)\beta_1, \ldots, \lambda\alpha_n + (1-\lambda)\beta_n\Big) =$$
$$= \phi\Big[\lambda x + (1-\lambda)y\Big] \in \phi(A).$$

Hence,

$$\lambda\overline{x} + (1-\lambda)\overline{y} \in \phi(A), \quad \lambda \in [0, 1],$$

so $\phi(A)$ is convex.

It is easy to show that $\phi(A)$ is bounded: for if $\overline{x}, \overline{y} \in \phi(A)$, then by the boundedness of A and Theorem 6,

$$d(\overline{x}, \overline{y}) = ||\overline{x} - \overline{y}||_0 \leqslant \frac{1}{\alpha}||x - y|| \leqslant \frac{1}{\alpha} \cdot M = K,$$

where M is a constant such that

$$||x - y|| \leqslant M \text{ for } x, y \in A.$$

Therefore, $\phi(A)$ is bounded.

Finally, we show that $\phi(A)$ is closed.

Let $x = \alpha_1 e_1 + \ldots + \alpha_n e_n, \quad \overline{x} = (\alpha_1, \ldots, \alpha_n), \quad \overline{x} \in \phi(A),$

$$x_0 = \gamma_1 e_1 + \ldots + \gamma_n e_n, \quad \overline{x}_0 = (\gamma_1, \ldots, \gamma_n), \quad \alpha_i, \gamma_i \in \mathbb{R}, \quad i = 1, \ldots, n.$$

We have to verify that

$$\Big(||\overline{x} - \overline{x}_0|| \to 0\Big) \quad \Rightarrow \quad \overline{x}_0 \in \phi(A).$$

Really, by the closedness of A,

$$||x - x_0|| \leqslant \beta||\overline{x} - \overline{x}_0||_0 \to 0 \text{ implies } x \to x_0,$$

i.e., $x_0 \in A$, and consequently $\overline{x}_0 \in \phi(A)$.

To finish the proof, define

$$T := \phi\varphi\phi^{-1}, \quad T : \overline{A} \to \overline{A}.$$

By Theorem 9, there exists $x \in \overline{A}$ such that

$$\phi\varphi\phi^{-1}(x) = x,$$

i.e., $\varphi[\phi^{-1}(x)] = \phi^{-1}(x)$.

If $y = \phi^{-1}(x) \in A$, then $\varphi(y) = y$, which ends the proof.

5 Schauder Fixed Point Principle in b-Normed Spaces

Definition 7 (see [4], p. 54) Let $N := \{c_1, \ldots, c_n\}$ be a finite subset of a strong b-normed linear space E, and for any fixed $\epsilon > 0$, let

$$(N, \epsilon) := \bigcup \Big\{ B(c_i, \epsilon) : i = 1, \ldots, n \Big\},$$

and

$$B(c_i, \epsilon) := \Big\{ x \in E : ||x - c_i|| < \epsilon \Big\}, \quad i = 1, \ldots, n.$$

For each $i = 1, \ldots, n$, let $\mu_i : (N, \epsilon) \to \mathbb{R}$ be the map

$$\mu_i(x) := \max \Big[0, \epsilon - ||x - c_i|| \Big].$$

The Schauder–Dugundi–Granas projection (see [4])

$$p_\epsilon : (N, \epsilon) \to conv(N)$$

is given by

$$p_\epsilon(x) := \Big[\sum_{i=1}^{n} \mu_i(x) \Big]^{-1} \sum_{i=1}^{n} \mu_i(x) c_i. \tag{20}$$

Clearly, p_ϵ is well-defined, since each $x \in (N, \epsilon)$ also belongs to some $B(c_i, \epsilon)$, and therefore

$$\sum_{i=1}^{n} \mu_i(x) \neq 0.$$

Also, $p_\epsilon[(N, \epsilon)] \subset conv(N)$ as a convex combination of $c_1, \ldots, c_n$.

Definition 8 (see [4]) Let X and Y be topological spaces. A continuous map $F : X \to X$ is called compact if $F(X)$ is contained in a compact subset of Y.

Note that

Lemma 4 *Let* $E = (E, ||\cdot||, s)$ *be a strong b-normed linear space, and* $c_1, \ldots, c_n \in U \subset E$, *U-convex. Then,*

(ix) $||x - p_\epsilon(x)|| < \epsilon s, \quad x \in (N, \epsilon), \quad N = \{c_1, \ldots, c_n\}$,
(x) $p_\epsilon : (N, \epsilon) \to con(N) \subset U$ *is a continuous compact map.*

Proof We have, by the definition (20) and Remark 1, for $x \in (N, \epsilon)$,

$$
\begin{aligned}
||x - p_\epsilon(x)|| &= \Big[\sum_{i=1}^{n} \mu_i(x)\Big]^{-1} ||\sum_{i=1}^{n} \mu_i(x)[x - c_i]|| \\
&\leqslant \Big[\sum_{i=1}^{n} \mu_i(x)\Big]^{-1} s \sum_{i=1}^{n} \mu_i(x)||x - c_i|| \\
&\leqslant s\epsilon(\sum_{i=1}^{n} \mu_i(x))^{-1}(\sum_{i=1}^{n} \mu_i(x)) = s\epsilon.
\end{aligned}
$$

The continuity of p_ϵ is a consequence of the fact that p_ϵ is a finite sum of continuous functions (see also Lemma 1); compactness follows from Lemma 3.

Remark 12 The values of p_ϵ are in a finite-dimensional b-normed linear space contained in E.

Lemma 5 *Assume that* X *is a topological space and* E *a strong b-normed linear space. Let* U *be a convex subset of* E, *and let* $F : X \to U$ *be a compact map. For every* $\epsilon > 0$, *there exists a finite set*

$$N = \{c_1, \ldots, c_n\} \subset F(x) \subset U$$

and a finite-dimensional map $F_\epsilon : X \to U$
such that:

(xi) $||F_\epsilon(x) - F(x)|| < s\epsilon, \quad x \in X$,
(xii) $F_\epsilon(x) \subset conv(N) \subset U$.

Proof Since $F(X)$ is compact (in E), so by Theorem 3 there exists a finite ϵ-net $\{c_1, \ldots, c_n\} \subset F(X)$. Also, $F(X) \subset (N, \epsilon)$; for if $y \in F(X)$, then $d(y, c_i) < \epsilon$ for some $i \in \{1, \ldots, n\}$, and hence $y \in B(c_i, \epsilon)$, i.e., $y \in (N, \epsilon)$. This shows that $F(X) \subset (N, \epsilon)$.

Now, let $F_\epsilon(x) := p_\epsilon[F(x)], \quad x \in X$. Therefore, if $y = F(x), \quad x \in X$, then

$$||F_\epsilon(x) - F(x)|| = ||p_\epsilon(y) - y|| < s\epsilon$$

because for $y = F(x) \in (N, \epsilon)$, $x \in X$, from Lemma 4,

$$||y - p_\epsilon(y)|| < \epsilon s,$$

and consequently

$$||F_\epsilon(x) - F(x)|| < \epsilon s, \quad x \in X.$$

To verify (xii), let $y \in F_\epsilon(X)$, so $y = p_\epsilon(z), \quad z = F(x) \in (N, \epsilon)$ for some $x \in X$. Let

$$y = p_\epsilon(z) = \sum_{i=1}^{n} \lambda_i c_i, \quad \sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \in \mathbb{R}, \quad i = 1, \ldots, n.$$

Hence, $y \in conv(N) \subset U$. Therefore, since U is convex,

$$F_\epsilon(X) \subset conv(N) \subset U,$$

and we get (xii).

Definition 9 ([6]) Assume that $U \subset E$, and (E, d, s) is a b-metric space. If for a given $\epsilon > 0$, there exists a point $x \in U$ such that $d(x, F(x)) < \epsilon$ for a map $F : U \to E$, then we say that x is an ϵ-fixed point for F.

Note the following:

Theorem 11 *Let (X, b, s) be a strong b-metric space and $A \subset X$ be a closed set. Let $F : A \to X$ be a compact map. Then, F has a fixed point iff for each $\epsilon > 0$ it has an ϵ-fixed point.*

Proof Since a necessary condition is trivial, we verify the sufficient condition only. Let $\epsilon_n = \frac{1}{n}, \quad n \in \mathbb{N}$, and let for each $n \in \mathbb{N}$ there exists $a_n \in A, \quad n \in \mathbb{N}$, ϵ_n-fixed point for F, i.e.,

$$d(a_n, F(a_n)) < \frac{1}{n}, \quad n \in \mathbb{N}. \tag{21}$$

Because $F(X) \subset U \subset X$, where U is compact (in X), then there exists subsequence $\{a_{n_k}\}$, such that $F(a_{n_k}) \to a$ as $k \to \infty, \quad a \in X$. But by (21), for $k \geqslant m_0$ and $\epsilon > 0$

$$\begin{aligned} d(a_{n_k}, a) &\leqslant s[d(a_{n_k}, F(a_{n_k}) + d(F(a_{n_k}, a)] \\ &\leqslant s[\frac{1}{n_k} + \epsilon] < 2s\epsilon \end{aligned}$$

so $a_{n_k} \to a$ as $k \to \infty$, and $a \in A$, since A is closed. Consequently, $F(a_{n_k}) \to a$ and $F(a_{n_k}) \to F(a)$, because F is continuous; consequently, $a = F(a)$, and we get the expected fixed point, which finishes the proof.

The main result of this part is the following:

Theorem 12 (Schauder fixed point principle) *Let $(X, ||\cdot||, s)$ be a strong b-normed linear space, and $U \subset X$ be a nonempty convex closed subset. Let, moreover, $F : U \to U$ be a compact map. Then, there exists $u \in U$ such that $F(u) = u$.*

Proof In view of Theorem 11, we show that for each $\epsilon > 0$, F has an ϵ-fixed point in U. By Lemma 5, for every $\epsilon > 0$, there exists $F_\epsilon : U \to U$ with

(a) $||F_\epsilon(x) - F(x)|| < \epsilon, \quad x \in U,$
(b) $F_\epsilon(U) \subset conv(N) \subset U.$

But $F_\epsilon : conv(N) \to conv(N)$.
Indeed, $conv(N) \subset U$ and

$$F_\epsilon[conv(N)] \subset F_\epsilon(U) \subset conv(N).$$

Also, by Theorem 10 and Lemma 5, there exists $x_\epsilon \in U$ such that $F_\epsilon(x_\epsilon) = x_\epsilon$. Finally, by Theorem 11, there exists $u \in U$ with $F(u) = u$, which concludes the proof.

Remark 13 In [4], Theorem 3.2, it is stated that C is not necessarily closed, but in Theorem 3.1 (which is used in the proof), the set A is closed. Something is not quite clear.

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Deterministic Prediction Theory

Nicholas J. Daras

Abstract We give a general method for predicting spatio-temporal regions with "strange" systemic occurrences. To do so, we consider systemic indices and their measurements into the under consideration fixed spatio-temporal region. Given a set of preselected future points, the magnitude of the (Euclidean or not) distance between the surface of these systemic indices and a parametrized surface that interpolates or passes very close to the points of systemic measurements and given preselected vector values may be viewed as a measure for assessing the appearance of peculiar systemic incidents over the region under consideration; so, depending on these preselected points, we provide a general algorithmic framework for predicting spatio-temporal regions into which crucial systemic events are expected.

AMS Subject Classification: 60G25, 65D05, 65D07, 65D10, 93A10, 93A30, 93B 15, 93E14, 93E24

1 Introduction

In many modern scientific studies, quantifying assumptions, data and variables can contribute to the accurate description of the phenomena through appropriate mathematical models [1–4, 6–10, 13–15, 17]. The first purpose of the paper is to provide a general method to predict time intervals of appearance of peculiar systemic incidents during a given period. To do so, we consider *systemic indices* and their *measurements* over a fixed under consideration domain in the space-time. The magnitude of the (Euclidean or not) distance between the surface of systemic indices and a parametrized surface that interpolates or passes very close to systemic measurements and preselected vector values can be considered as a measure for assessing occurrence of peculiar systemic incidents over the region

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under consideration; so, depending on these preselected points, we provide a general algorithmic framework predicting spatio-temporal subregions into which appearance of peculiar systemic incidents is expected.

Of course, two basic and reasonable questions arise immediately and may also constitute central subjects of discussion. The first one relates to the *subjectivity of systemic choices and personal priorities*, since it is very doubtful whether a set of systemic indices could be considered as exhaustive, in the sense that it could guarantee the ultimate reliability of the corresponding prediction. We treat this question in a forthcoming paper [5]. Here, in order to simplify the formulation of the model, we will assume constantly that *there is a complete objectivity in all systemic options and personal priorities*, i.e., all systemic analysts have agreed for the finalized selection of all systemic indices. The second question concerns the *reliability of systemic measurements and how much it could affect the validity of prediction*. Again, for simplification reasons, we will assume continuously that all *systemic measurements are carried out with sufficient reliability* to such an extent as to preclude any discrepancy in the estimates of the predictions.

The chapter is structured as follows. Section 2 introduces basic aspects and methodology for a qualitative systemic analysis and provides basic systemic definitions, such as the systemic index, the regularity interval, the two precarity intervals, the two danger intervals and the predictable system.

Subsequently, the next section focuses on the algebraic approach that gives possibility of introducing new concepts, such as the concept of systemic indices over a system, the concept of systemic fibre at a point of the space-time and the concept of systemic affinity between two systems.

Section 4 deals with geometric formalities permitting us to examine the structure of universalities of systemic indices, that is of parametrized surfaces passing only from given places of systemic indices over the spatio-temporal region under consideration. In the same section, we overview the meaning of systemic measurements (at discrete moments and locations) and then discuss the deviations and the smooth parametrized surface of such a systemic measurement from a given universality of systemic indices.

Based on this background, in Section 5, we consider the magnitude of the (Euclidean or not) distance between the smooth parametrized surface of such a systemic measurement from a given universality of systemic indices and another surface that interpolates or passes close to the measurement points and some future balance points. The measurement points are taken at predefined locations that system administrators chose having put the requirement that, at optioned next spatio-temporal moments, there will be no deviation from the regularity universality. This approach allows predicting time moments and locations at which peculiar systemic incidents are expected to happen: *if at some spatio-temporal point, the distance between the two surfaces exceeds a given critical tolerance value, then it means that at this point peculiar systemic incidents are expected*. As it is clear, this prediction may be well described in two remarkable cases of main interest: the limit case where the location remains constant and the general case where the measurements are conducted at discrete time moments and over different locations.

So, in Section 5, we first provide a general algorithmic framework for determining time intervals and locations into which peculiar systemic incidents are expected and, next, we limit ourselves to considering consequent computational algorithms only for the case where the measurements are carried out at discrete points in time and the position remains always fixed. These ideas are specified through alternative and independent directions using interpolation methods and least square techniques.

Finally, in Section 6, we apply these approximations and give indicative numerical examples to determine time intervals into which peculiar systemic incidents are expected.

2 Systemic Indices

Having regard to what has been mentioned in the previous section, suppose S is a given *system* (or *complex* [7]) of which we want to predict behaviour. To this end, we accept that *the system is identified by its own* $(\ell + 1)$ *system characteristics* (see, for instance, http://www.tezu.ernet.in/dba/new/faculty/heera/SAD.pdf, https://www.kenyaplex.com/questions/22895-outline-the-characteristics-of-a-system.aspx, https://managingresearchlibrary.org/glossary/system-characteristics and http://www.ddegjust.ac.in/studymaterial/pgdca/ms-04.pdf), which we can fully know one by one and depend on the time and their location.

We need to quantify the behaviour of each characteristic j.

Definition 1 A **systemic index** of S is a numerical function $g_S^{(j)} = g_S^{(j)}(t, x, y, z)$, which represents the states of the characteristic j at any date $t \in \mathbb{R}$ and location $(x, y, z) \in \mathbb{R}^3$ depending on its intrinsic physical features.

To simplify, *any systemic index of* S *is supposed to be a piecewise continuous function at* (t, x, y, z). Furthermore, we will assume that, *for any system characteristic* j, $(j = 1, 2, \ldots, \ell + 1)$ *and at any date* $t \in \mathbb{R}$ *and location* $(x, y, z) \in \mathbb{R}^3$ into S, *we are given*

1. a ***regularity interval*** $\left[\tilde{r}_S^{(j)}, \tilde{r}_S^{(j)}\right] \subset \mathbb{R}$ into which there is ***no*** *change* in the behaviour of the characteristic j within the system, affecting both the other systemic indices and the power and influence of others systemic characteristics acting in the complex.
2. the ***under-weighted precarity interval*** $\left[\tilde{p}_S^{(j)}, \tilde{r}_S^{(j)}\right] \subset \mathbb{R}$ and the ***over-weighted interval*** $\left[r_S^{(j)}, p_S^{(j)}\right] \subset \mathbb{R}$ into which there is only a *slight change* in the behaviour of the characteristic j within the system, affecting both the other systemic indices and the power and influence of others systemic characteristics acting in the complex.
3. the ***under-weighted danger interval*** $\left[\tilde{d}_S^{(j)}, \tilde{p}_S^{(j)}\right] \subset \mathbb{R}$ and the ***over-weighted danger interval*** $\left[p_S^{(j)}, d_S^{(j)}\right] \subset \mathbb{R}$ into which into which there is a *major change* in the behaviour of the characteristic j within the system, affecting both the other

systemic indices and the power and influence of others systemic characteristics acting in the complex.

In the exterior of $\left[\tilde{d}_S^{(j)}, d_S^{(j)}\right]$, there is a *catastrophic change* in the behaviour of the characteristic j within the system, affecting both the other systemic indices and the power and influence of others systemic characteristics acting in the complex.

The vectors $\left(\tilde{r}_S^{(j_1)}, \dots, \tilde{r}_S^{(j_k)}\right) \in \mathbb{R}^k$ and $\left(r_S^{(j_1)}, \dots, r_S^{(j_k)}\right) \in \mathbb{R}^k$ are, respectively, the ***lowest*** and ***highest thresholds of regularity*** over the system characteristics $j_1, j_2, \dots, j_k$ at date $t \in \mathbb{R}$ and location $(x, y, z) \in \mathbb{R}^3$. Especially, for $(j_1, j_2, \dots, j_k) = (1, 2, \dots, \ell + 1)$, we prefer to use the notation

$$\tilde{r} = \left(\tilde{r}_S^{(1)}, \tilde{r}_S^{(2)}, \dots, \tilde{r}_S^{(\ell+1)}\right) \text{ and } r = \left(r_S^{(1)}, r_S^{(2)}, \dots, r_S^{(\ell+1)}\right).$$

The correspondence that associates each element of the space-time $\mathbb{R}^4$ with the corresponding regularity interval of the characteristic j is the ***regularity state mapping*** of the system characteristic j over the space-time. Any point in its image is a ***regularity state*** or ***regularity point***, and any set in its graph is a ***regularity zone*** for j. If, for instance, we have fixed the location (x, y, z) and we let the time t to vary from a moment T_0 to another moment T_3, then a regularity zone for a characteristic j may have a form like that of the graph in Figure 1.

Definition 2 Given any (t, x, y, z), the closed interval

$$[\tilde{r}_S^{(j)}(t, x, y, z), r_S^{(j)}(t, x, y, z)]$$

is the **regularity tolerance** of the system characteristic j at date $t \in \mathbb{R}$ and location $(x, y, z) \in \mathbb{R}^3$ into S.

Remark 1 It is not excluded the limit situation $\tilde{r}_S^{(j)} = r_S^{(j)} = R_S^{(j)}$. In such a case, the systemic index $R_S^{(j)}$ is the unique ***regularity value*** of the system characteristic j at date $t \in \mathbb{R}$ and location $(x, y, z) \in \mathbb{R}^3$ into S.

Similarly, the vectors $\left(\tilde{p}_S^{(j_1)}, \dots, \tilde{p}_S^{(j_k)}\right) \in \mathbb{R}^k$ and $\left(p_S^{(j_1)}, \dots, p_S^{(j_k)}\right) \in \mathbb{R}^k$ are, respectively, the ***lowest threshold of under-weighted precarity*** and the ***highest threshold of over-weighted precarity*** over the system characteristics $j_1, \dots, j_k$ at date $t \in \mathbb{R}$ and location $(x, y, z) \in \mathbb{R}^3$. Especially, for $(j_1, j_2, \dots, j_k) = (1, 2, \dots, \ell + 1)$, we prefer to use the notation

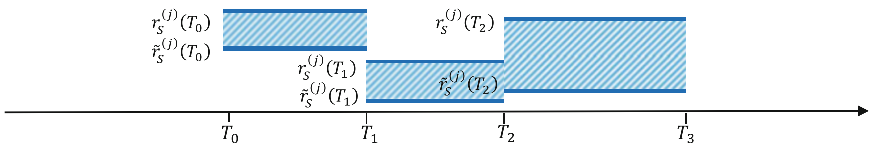

Fig. 1 Regularity zone for a characteristic j, if we have fixed the location (x, y, z) and we let the time t to vary from a moment T_0 to another moment T_3

$$\tilde{p} = \left(\tilde{p}_S^{(1)}, \tilde{p}_S^{(2)}, \ldots, \tilde{p}_S^{(\ell+1)}\right) \text{ and } p = \left(p_S^{(1)}, p_S^{(2)}, \ldots, p_S^{(\ell+1)}\right).$$

Notice that it is not excluded the case of coincidence

$$\tilde{p}_S^{(j)} = \tilde{r}_S^{(j)} \text{ nor } p_S^{(j)} = r_S^{(j)}.$$

The mappings that assign each element of the space-time to the corresponding precarity intervals of the characteristic j are the ***precarity state mappings*** of the system characteristic j over the space-time. A point in the image of a precarity state mapping is a ***precarity state*** or ***precarity point***, while a set in its graph is said to be a ***precarity zone*** for j. The ***tolerance of the under-weighted precarity*** and the ***tolerance of the over-weighted precarity*** at time t and location (x, y, z) into S are defined to be the differences

$$\tilde{\delta}_{\text{critical}}^{(j)} := |\tilde{p}_S^{(j)} - \tilde{r}_S^{(j)}| \text{ and } \delta_{\text{critical}}^{(j)} := |p_S^{(j)} - r_S^{(j)}|, \text{ respectively.}$$

Finally, the vectors $\left(\tilde{d}_S^{(j_1)}, \ldots, \tilde{d}_S^{(j_k)}\right) \in \mathbb{R}^k$ and $\left(d_S^{(j_1)}, \ldots, d_S^{(j_k)}\right) \in \mathbb{R}^k$ are, respectively, the ***lowest threshold of under-weighted danger*** and the ***highest threshold of over-weighted danger*** over the systemic characteristics $j_1, \ldots, j_k$ at date $t \in \mathbb{R}$ and location $(x, y, z) \in \mathbb{R}^3$. Especially, for $(j_1, j_2, \ldots, j_k) = (1, 2, \ldots, \ell + 1)$, we prefer to use the notation

$$\tilde{d} = \left(\tilde{d}_S^{(1)}, \tilde{d}_S^{(2)}, \ldots, \tilde{d}_S^{(\ell+1)}\right) \text{ and } p = \left(d_S^{(1)}, d_S^{(2)}, \ldots, d_S^{(\ell+1)}\right).$$

Notice again that it is not excluded the case of coincidence

$$\tilde{d}_S^{(j)} = \tilde{p}_S^{(j)} \text{ nor } d_S^{(j)} = p_S^{(j)}.$$

The mappings that assign each element of the space-time to the corresponding danger intervals of the characteristic j are the ***danger state mappings*** of the system characteristic j over the space-time. A point in the image of a danger state mapping is a ***danger state*** or ***danger point***, and a set in its graph is said to be a ***danger zone*** for j. The ***extents of under-weighted danger*** and ***over-weighted danger*** at time t and location (x, y, z) into S are defined to be the differences

$$\tilde{\epsilon}_{\text{critical}}^{(j)} := |\tilde{d}_S^{(j)} - \tilde{r}_S^{(j)}| \text{ and } \epsilon_{\text{critical}}^{(j)} := |d_S^{(j)} - r_S^{(j)}|, \text{ respectively.}$$

Any point that does not belong to a closed interval of the form

$$\left[\tilde{d}_S^{(j)}(t, x, y, z) - d_S^{(j)}(t, x, y, z)\right]$$

for some (t, x, y, z) is a ***disaster point***.

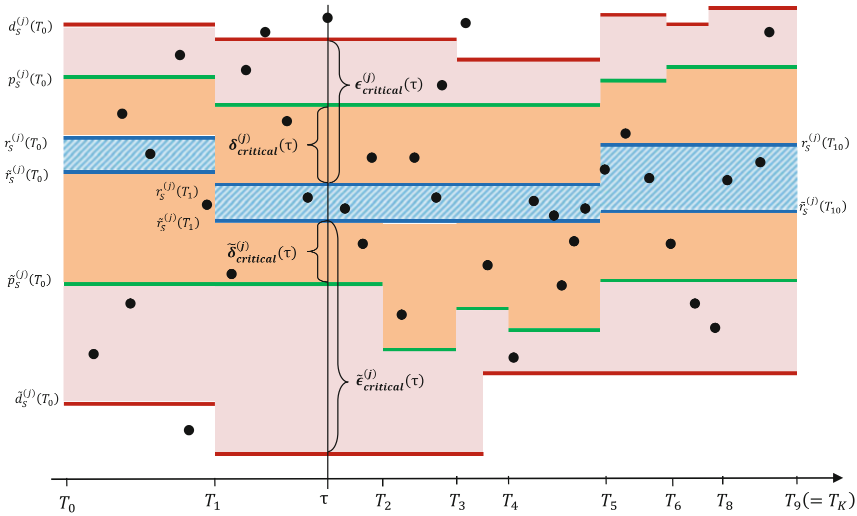

Fig. 2 A schematic representation of the above concepts for a fixed location $(x, y, z) \in \mathbb{R}^3$

Figure 2 provides a schematic representation of the above concepts for a fixed location $(x, y, z) \in \mathbb{R}^3$.

Definition 3 A system S endowed with the above defined tolerances of regularity, precarity and danger is a **predictable system**.

Remark 2 The case of coincidence $\tilde{r}_S^{(j)} = r_S^{(j)} = R_S^{(j)}$ does not allow the consideration of the two orientations, the first of which is introduced in the direction drawn from a lowest to a highest threshold, while the second one is introduced in the direction drawn from a highest to a lowest threshold. Instead, in this coincidence, there is only one direction. This is the direction in which, simply, one of the three successive situations may be happen: the **precarious situation**, the **danger situation** and the **disaster situation**.

Remark 3 The concepts of regularity zone, precarity zone and danger zone could be considered as analogues of the concepts that can be understood by saying low-risk zone, medium-risk zone and high-risk zone, respectively.

For obvious reasons of simplifying the technical handling of our reasoning, we make the following assumption

Assumption 1 *The system analysts, who study the given system, have agreed for a finalized, unique and discrete selection of all systemic indices governing the system behaviour.*

On the other hand, we are concerned about current numerical values $g_j^{(S)}$ (t, x, y, z) of the relevant selected system characteristics at a given discrete set of time moments and locations. However, systemic measurements performed by a single person or computational block or body on the same item and under the same conditions may contain errors due to various causes, such as rounding of measurements, erroneous information, limited databases, etc. In order to avoid any confounding effect, we will assume continuously the ideal situation.

Assumption 2 *All systemic measurements are carried out with sufficient reliability to such an extent as to preclude any discrepancy in the estimates of the predictions.*

3 Basic Algebraic Considerations

The Space of Systemic Indices Over a System

It is assumed that *there are a* ***finite number*** **of** $\ell + 1$ *distinguishable systemic indices of the system* S, *say* $g_S^{(1)}, g_S^{(2)}, \ldots, g_S^{(\ell+1)}$ *for any date* t *and any location* (x, y, z).

Definition 4

i. If every unit vector

$$e^{(j)} = \underbrace{(0, \ldots 0, 1, 0, \ldots, 0)}_{j}$$

of the vector space $\mathbb{R}^{\ell+1}$ is identified with one unit of the vector space $\mathbb{R}^{\ell+1}$ of the system S at date t and location (x, y, z) $(j = 1, 2, \ldots, \ell + 1)$, then the linear space $\mathbb{G}_{(t,x,y,z)}(S) := \left\{g = g_S = \lambda_1 g_S^{(1)} + \ldots + \lambda_{\ell+1} g_S^{(\ell+1)} : \lambda_1, \lambda_2, \ldots, \lambda_{\ell+1} \in \mathbb{R}\right\} \equiv \mathbb{R}^{\ell+1}$ with the usual Euclidean distance in $\mathbb{R}^{\ell+1}$ is the **space of the instantaneous local systemic indices** in S at date t and location (x, y, z).

ii. The linear space

$$\mathbb{G}(S) = \left\{(t, (x, y, z), g_S) : t \in \mathbb{R}, (x, y, z) \in \mathbb{R}^3, g_S \in \mathbb{G}_{(t,x,y,z)}(S)\right\} \equiv \mathbb{R}^{\ell+5},$$

endowed with the usual Euclidean topology in $\mathbb{R}^{\ell+5}$, is the **space of the systemic indices** over the system S. The elements g_S of $\mathbb{G}(S)$ are the **systemic indices** of the system S.

It is clear that $\mathbb{G}(S)$ can be endowed with a continuous projection $\pi_S : \mathbb{G}(S) \to \mathfrak{B} \subset \mathbb{R}^4$, such that *for each point* $(t, x, y, z) \in \mathfrak{B}$, *the space* $\mathbb{G}_{(t,x,y,z)}(S)$ *coincides with the* ***systemic fibre*** $\pi_S^{-1}(t, x, y, z)$ *of* $\mathbb{G}(S)$ *at the point* (t, x, y, z). Since the space of systemic indices $\mathbb{G}(S)$ is separable and connected, *the cardinality of the each systemic fibre* $\pi_S^{-1}(t, x, y, z) \equiv \mathbb{G}_{(t,x,y,z)}(S)$ *does not exceed the infinite cardinality of any basis of open sets in* $\mathfrak{B}$.

The systemic index space $\mathbb{G}(S)$ is a trivial bundle of discrete fibres $\mathbb{G}_{(t,x,y,z)}(S) = \mathbb{R}^{\ell+1}$, and therefore $\mathbb{G}(S)$, *endowed with the continuous projection* π_S, *is* (also) *a systemic covering space of* $\mathfrak{B}$.

On the other hand, it is important to see that the inverse image $\pi_S^{-1}(K)$ of any compact set K in $\mathfrak{B}$ is also compact in $\mathbb{G}(S)$. Thus, *the systemic index space* $\mathbb{G}(S)$ is a ***quasi-compact space*** in the following sense: For any $(t, x, y, z) \in \mathfrak{B}$ and any family $(V_i)_{i\in I}$ of open subsets of $\mathbb{G}(S)$ such that $\cup_{i\in I} V_i \supset \pi_S^{-1}(t, x, y, z)$, there exists a finite part J of I and an open neighbourhood $\mathcal{V}$ of (t, x, y, z) such that $\cup_{i\in I} V_i \supset \pi_S^{-1}(\mathcal{V})$. In particular, we have the following.

Proposition 1 *The systemic index space* $\mathbb{G}(S)$ *is a proper space over* $\mathfrak{B}$.

Affinities Between Systems

Let S and T be two systems. Let us consider the corresponding systemic index spaces $\mathbb{G}(S)$ and $\mathbb{G}(T)$, with projections π_S and π_T, respectively.

Definition 5 A continuous mapping $\chi : \mathbb{G}(S) \to \mathbb{G}(T)$ is said to be a **systemic affinity** between the systems S and T if the following diagram commutes:

$$\begin{array}{ccc} \mathbb{G}(S) & \xrightarrow{\chi} & \mathbb{G}(S) \\ \pi_S \searrow & & \swarrow \pi_T \\ & \mathbb{B} & \end{array} .$$

Evidently, if χ is a systemic affinity between the systems S and T, then for any $(t, x, y, z) \in \mathfrak{B}$, χ induces a mapping

$$\chi_{(t,x,y,z)} : \mathbb{G}_{(t,x,y,z)}(S) \to \mathbb{G}_{(t,x,y,z)}(T)$$

of the momentary local systemic index space of the system S at date t and location (x, y, z) into the momentary local systemic index space of the system T at date t and location (x, y, z).

It is easy to verify the following.

Proposition 2 *Any systemic affinity* $\chi : \mathbb{G}(S) \to \mathbb{G}(T)$ *between the systems* S *and* T *is **onto** the systemic index space* $\mathbb{G}(T)$. *If, moreover, there exists a point* $(t, x, y, z) \in \mathfrak{B}$ *such that the induced mapping* $\chi_{(t,x,y,z)} : \mathbb{G}_{(t,x,y,z)}(S) \to \mathbb{G}_{(t,x,y,z)}(T)$ *is one-to-one, then the systemic affinity between the systems* S *and* T *is an isomorphism.*

Given any two systemic affinities χ *and* ψ *between the systems* S and T, *the set of all* $\mathcal{D} \in \mathbb{G}(T)$ *such that* $\chi(\mathcal{D}) = \psi(\mathcal{D})$ *is open and closed in* $\mathbb{G}(T)$. In particular, since the systemic index space $\mathbb{G}(T)\left(\mathbb{R}^{\ell+5}\right)$ is connected, we infer the following result.

Proposition 3 *Whenever χ and ψ are two systemic affinities between the systems S and T,*

i. *if there exists a systemic index $g \in \mathbb{G}(T)$ such that $\chi(g) = \psi(g)$, the systemic affinities χ and ψ coincide*
ii. *if there exists a $(t, x, y, z) \in \mathfrak{B}$ such that $\chi_{(t,x,y,z)} = \psi_{(t,x,y,z)}$, the systemic affinities χ and ψ coincide.*

The category which has elements the systemic index spaces and morphisms the systemic affinities between two systems is called the ***category of systemic systems***. It will be denoted by $\mathfrak{B} - Top$. The ***sum*** of $\mathbb{G}(S)$ and $\mathbb{G}(T)$ into the category $\mathfrak{B} - Top$ of systemic systems is the disjoint union $\mathbb{G}(S) \sqcup \mathbb{G}(T)$ endowed with the projection inducing π_S onto $\mathbb{G}(S)$ and π_T onto $\mathbb{G}(T)$. It holds $(\mathbb{G}(S) \sqcup \mathbb{G}(T))_{(t,x,y,z)} = \mathbb{G}_{(t,x,y,z)}(S) \sqcup \mathbb{G}_{(t,x,y,z)}(T)$.

The Fibre Product of Two Systemic Index Spaces

Let S and T be two systems, with corresponding systemic index spaces $\mathbb{G}(S)$ and $\mathbb{G}(T)$ and projections π_S and π_T, respectively. The ***fibre product*** $\mathbb{G}(S) \times_{\mathfrak{B}} \mathbb{G}(T)$ of $\mathbb{G}(S)$ and $\mathbb{G}(T)$ over the systems S and T is the subspace of the topological space $\mathbb{G}(S) \times \mathbb{G}(T)$ consisting in all pairs (D_S, D_T) satisfying $\pi_S(D_S) = \pi_T(D_T)$. The fibre product $\mathbb{G}(S) \times_{\mathfrak{B}} \mathbb{G}(T)$ endowed with the mapping $(D_S, D_T) \to \pi_S(D_S)$ is the **product of the systemic index spaces** $\mathbb{G}(S)$ **and** $\mathbb{G}(T)$ **into the category of systems**. It is clear that

$$\big(\mathbb{G}(S) \times_{\mathfrak{B}} \mathbb{G}(T)\big)(t, x, y, z) = \mathbb{G}_{(t,x,y,z)}(S) \times \mathbb{G}_{(t,x,y,z)}(T),$$

whenever $(t, x, y, z) \in \mathfrak{B}$. Letting now $h : \mathfrak{B} \equiv \mathbb{R} \times \mathbb{R}^3 \to \mathfrak{B} \equiv \mathbb{R} \times \mathbb{R}^3$ be a continuous mapping, the topological space $\mathbb{G}^*(S) := h(\mathfrak{B}) \times_{\mathfrak{B}} \mathbb{G}(S)$ endowed with the first projection $\mathbb{G}^*(S) \to h(\mathfrak{B})$ is a *space* over the topological space $h(\mathfrak{B})$, which is called the ***space over*** $h(\mathfrak{B})$ ***obtained from*** $\mathbb{G}(S)$ by base change from $\mathfrak{B}$ to $h(\mathfrak{B})$. The fibre of $\mathbb{G}^*(S)$ at a point b' of $h(\mathfrak{B})$ is identified with the fibre of $\mathbb{G}(S)$ at $h(b')$.

4 Geometric Foundations

Let S be any predictable system/complex with corresponding systemic index space $\mathbb{G}(S)$. A tool that would allow us a thorough study of the measurements carried out in the weighted systemic space is to attach systemic vector field measurements on all points of the space of systemic indices. For now, we will always assume that *the values obtained from the measurements are reliable and accurate and will compare them with respect to the given and fixed values of the systemic indices*.

Universalities of Systemic Indices

Let U be a non-empty open subset of $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ representing a ***spatio-temporal historical phase***.

Definition 6

i. The mapping

$$\mathfrak{D} : U \to \mathbb{G}(S) : (t, x, y, z) \mapsto \left(t, x, y, z; g_S^{(1)}, \ldots, g_S^{(\ell+1)}\right)$$

is called a **universality of systemic indices** for the system S over the spatio-temporal historical phase U, or simply **system universality**.

ii. If the mapping $\mathfrak{D}$ is smooth and regular, i.e., its differential $\mathfrak{D}_{(t,x,y,z)}$ is non-singular (:has rank 4) for each $(t, x, y, z) \in \mathbb{R} \times \mathbb{R}^3$, then $\mathfrak{D}$ is a parametrized surface of dimension 4 in the systemic index space $\mathbb{G}(S)$. In such a case, we say that the image of the system universality $S_{\mathfrak{D}} = \mathfrak{D}(U)$ or simply $\mathfrak{D}$ is the **parametrized surface of the systemic indices** for the system S over U.

Smooth Parametrized Surfaces of Systemic Indices

We will first assume that the universality $\mathfrak{D} : U \to \mathbb{G}(S) \equiv \mathbb{R}^{\ell+5}$ is smooth and regular. The ***differential*** of $\mathfrak{D}$ is the smooth map $d\mathfrak{D} : U \times \mathbb{R}^4 \to \mathbb{G}(S) \times \mathbb{G}(S)$ defined as follows. A point $v \in U \times \mathbb{R}^4$ is a vector $v = ((t, x, y, z), u)$ at a point $(t, x, y, z) \in U$. Let $\alpha : I \to U$ be any parametrized curve in U with $\alpha(t_0) = v$. Then, $d\mathfrak{D}(v)$ is the vector at $\mathfrak{D}(t, x, y, z)$ ($d\mathfrak{D}(v) \in \mathbb{R}^{\ell+5}_{\mathfrak{D}(t,x,y,z)} \subset \mathbb{G}(S) \times \mathbb{G}(S)$) defined by $d\mathfrak{D}(v) := \mathfrak{D} \circ \alpha(t_0)$. Note that the value of $d\mathfrak{D}(v)$ does not depend on the choice of parametrized curve α, because

$$\mathfrak{D} \circ \alpha(t_0) =$$

$$\left(\mathfrak{D} \circ \alpha(t_0), \left(\mathfrak{D}_S^{(1)} \circ \alpha\right)'(t_0), \ldots, \left(\mathfrak{D}_S^{(l+1)} \circ \alpha\right)'(t_0)\right) =$$

$$\left(\mathfrak{D}(t, x, y, z), \nabla\mathfrak{D}_S^{(1)}(\alpha(t_0)) \cdot \alpha(t_0), \ldots \ \ldots, \nabla\mathfrak{D}_S^{(l+1)}(\alpha(t_0)) \cdot \alpha(t_0)\right) =$$

$$\left(\mathfrak{D}(t, x, y, z), \nabla\mathfrak{D}_S^{(1)}(t, x, y, z) \cdot v, \ldots \ \ldots, \nabla\mathfrak{D}_S^{(l+1)}(t, x, y, z) \cdot v\right),$$

so

$$d\mathfrak{D}(v) = \left(D(t, x, y, z), \nabla_v\mathfrak{D}_S^{(1)}, \ldots, \nabla_v\mathfrak{D}_S^{(l+1)}\right).$$

It follows immediately from the above formula that the restriction $d\mathfrak{D}_{(t,x,y,z)}$ of $d\mathfrak{D}$ to $\mathbb{R}^4_{(t,x,y,z)}$ (: the vectors at (t, x, y, z)) is a linear map

$$d\mathfrak{D}_{(t,x,y,z)} : \mathbb{R}^4_{(t,x,y,z)} \to \mathbb{R}^{\ell+5}_{\mathfrak{D}(t,x,y,z)}.$$

Its matrix relative to the standard bases for $\mathbb{R}^4_{(t,x,y,z)}$ and $\mathbb{R}^{\ell+5}_{\mathfrak{D}(t,x,y,z)}$ is just the Jacobian matrix of $\mathfrak{D}$ at (t, x, y, z). The regularity condition on $\mathfrak{D}$ guarantees the following:

Proposition 4 *For any $(t, x, y, z) \in U$, the image $d\mathfrak{D}_{(t,x,y,z)}\left(\mathbb{R}^4_{(t,x,y,z)}\right)$ of $d\mathfrak{D}_{(t,x,y,z)}$ is a four-dimensional subspace of $\mathbb{R}^{l+5}_{\mathfrak{D}(t,x,y,z)}$ tangent to the parametrized hypersurface $\mathfrak{D}$ of dimension 4 in the systemic index space $\mathbb{G}(S)$ corresponding to the point $(t, x, y, z) \in U$.*

Notice that the parametrized surface $\mathfrak{D}$ of dimension 4 in the systemic index space $\mathbb{G}(S)$ does not need to be one-to-one, and that $\mathfrak{D}(t, x, y, z) = \mathfrak{D}\left(t', x', y', z'\right)$ for $(t, x, y, z) \neq \left(t', x', y', z'\right)$ **does not** necessarily imply that the image of $d\mathfrak{D}_{(t,x,y,z)}$ is equal to the image of $d\mathfrak{D}_{\left(t',x',y',z'\right)}$. In other words, the following general inequality applies:

$$d\mathfrak{D}_{(t,x,y,z)}\left(\mathbb{R}^4_{(t,x,y,z)}\right) \neq d\mathfrak{D}_{\left(t',x',y',z'\right)}\left(\mathbb{R}^4_{\left(t',x',y',z'\right)}\right).$$

A ***systemic vector field*** *along the parametrized surface* $S_{\mathfrak{D}}$ of the systemic indices for the system S over U is a map $\mathfrak{f}$ that assigns to each point $p = (t, x, y, z) \in U$ a vector $\mathfrak{f}(p) \in \mathbb{R}^{\ell+5}_{\mathfrak{D}(t,x,y,z)}$. A comprehensive study of the systemic vector fields along parametrized surfaces requires some additional concepts.

Definition 7 Let

$$\mathfrak{f} : U \to \mathbb{R}^{\ell+5} : p = (t, x, y, z) \longmapsto \mathfrak{f}(p) = (\mathfrak{D}(p); \mathfrak{f}_1, \ldots, \mathfrak{f}_{\ell+5}) \in \mathbb{R}^{\ell+5}_{\mathfrak{D}(t,x,y,z)}$$

be a systemic vector field along the parametrized surface $S_{\mathfrak{D}}$.

i. We say that $\mathfrak{f}$ is **smooth** if each coordinate $\mathfrak{f}_j : U \to \mathbb{R}$ is smooth.

ii. We say that $\mathfrak{f}$ is **tangent to the parametrized surface** $S_{\mathfrak{D}}$ of the systemic indices for the system S over U if $\mathfrak{f}$ is of the form

$$\mathfrak{f}(p) = d\mathfrak{D}_{(t,x,y,z)}(\mathfrak{y}(p))$$

for some vector field $\mathfrak{y}$ on U.

iii. We say that $\mathfrak{f}$ is **normal to the parametrized surface** $S_{\mathfrak{D}}$ of the systemic indices for the system S over U if

$$\mathfrak{f}(p) \perp d\mathfrak{D}_{(t,x,y,z)}\left(\mathbb{R}^4_{(t,x,y,z)}\right) \text{ for all } (t,x,y,z) \in U.$$

Let us now give a generalization of the concept of the *velocity field* in the case of a systemic vector field along the parametrized surface describing the universality of the systemic indices for the system S over U. Let

$$\mathfrak{E}^{(1)}, \mathfrak{E}^{(2)}, \mathfrak{E}^{(3)} \text{ and } \mathfrak{E}^{(4)}$$

denote the tangent vector fields along the parametrized surface $S_{\mathfrak{D}}$ defined by

$$\mathfrak{E}^{(i)}(t,x,y,z) = d\mathfrak{D}_{(t,x,y,z)}\left((t,x,y,z); 0,\ldots,0,1,0\ldots,0\right),$$

where the 1 is in the $(i+1)$th spot (i spots after the $(t,x,y,z) \in U$).

Proposition 5 *The components of $\mathfrak{E}^{(i)}$ are just the entries in the ith column of the Jacobian matrix for $\mathfrak{D}$ at $(t,x,y,z) \in U$:*

$$\begin{aligned}
\mathfrak{E}^{(1)}(t,x,y,z) &= \left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{D}}{\partial t}(t,x,y,z)\right) \\
&= \left(\mathfrak{D}; \frac{\partial t}{\partial t}, \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}, \frac{\partial \mathfrak{D}_1}{\partial t}, \ldots, \frac{\partial \mathfrak{D}_{\ell+1}}{\partial t}\right)(t,x,y,z), \\
\mathfrak{E}^{(2)}(t,x,y,z) &= \left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{D}}{\partial x}(t,x,y,z)\right) \\
&= \left(\mathfrak{D}; \frac{\partial t}{\partial x}, \frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}, \frac{\partial \mathfrak{D}_1}{\partial x}, \ldots, \frac{\partial \mathfrak{D}_{\ell+1}}{\partial x}\right)(t,x,y,z), \\
\mathfrak{E}^{(3)}(t,x,y,z) &= \left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{D}}{\partial y}(t,x,y,z)\right) \\
&= \left(\mathfrak{D}; \frac{\partial t}{\partial y}, \frac{\partial x}{\partial y}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial y}, \frac{\partial \mathfrak{D}_1}{\partial y}, \ldots, \frac{\partial \mathfrak{D}_{\ell+1}}{\partial y}\right)(t,x,y,z), \\
\mathfrak{E}^{(4)}(t,x,y,z) &= \left(\mathfrak{D}(t,x,y,z); \frac{\partial \mathfrak{D}}{\partial z}(t,x,y,z)\right) \\
&= \left(\mathfrak{D}; \frac{\partial t}{\partial z}, \frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z}, \frac{\partial \mathfrak{D}_1}{\partial z}, \ldots, \frac{\partial \mathfrak{D}_{\ell+1}}{\partial z}\right)(t,x,y,z),
\end{aligned}$$

where

$$\mathfrak{D}(t,x,y,z) = \left(t,x,y,z; \mathfrak{D}_S^{(1)}(t,x,y,z), \ldots, \mathfrak{D}_S^{(\ell+1)}(t,x,y,z)\right).$$

Note that $\mathfrak{E}^{(i)}(t,x,y,z)$ is simply the ***velocity at*** $(t,x,y,z) \in U$ of the coordinate curve

$$u_i \longmapsto \mathfrak{D}\left(u_1, u_2, u_3, u_4\right)$$

(all u_j held constant except u_i) passing through $\mathfrak{D}\left(t, x, y, z\right)$. Here, $u_1 = t$, $u_2 = x$, $u_3 = y$ and $u_4 = z$. Since $d\mathfrak{D}_{(t,x,y,z)}$ is non-singular, we infer the following proposition.

Proposition 6

i. *The tangent vector fields $\mathfrak{E}^{(1)}$, $\mathfrak{E}^{(2)}$, $\mathfrak{E}^{(3)}$ and $\mathfrak{E}^{(4)}$ are linearly independent at each point $(t, x, y, z) \in U$.*
ii. *For each point $(t, x, y, z) \in U$, the tangent vector fields $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$, $\mathcal{E}^{(3)}$ and $\mathcal{E}^{(4)}$ form a basis for the tangent space defined by $Image\left[d\mathfrak{D}_{(t,x,y,z)}\right]$.*

Definition 8 For any smooth systemic vector field $\mathfrak{f} : U \to \mathbb{R}^{\ell+5}$ along the parametrized surface $S_{\mathfrak{D}}$ of the systemic indices for the system S, the derivative

$$\nabla_u \mathfrak{f} \in \mathbb{R}^{\ell+5}_{\mathfrak{D}(t,x,y,z)}$$

of $\mathfrak{f}$ with respect to $u \in \mathbb{R}^4_{(t,x,y,z)}$ $(t, x, y, z) \in U$ is defined by

$$\nabla_u \mathfrak{f} = \left(\mathfrak{D}\left(t, x, y, z\right), \left.\frac{d}{d\tau}\right|_{\tau_0} \left(\mathfrak{f} \circ \alpha\right)\right) = \left(\mathfrak{D}\left(t, x, y, z\right), \nabla_u \mathfrak{f}_1, \ldots, \nabla_u \mathfrak{f}_{\ell+5}\right),$$

where α is any parametrized curve in U with $\widehat{\alpha}\left(\tau_0\right) = u$.

Notice that, when

$$\begin{aligned} u \in \Big\{ & e_1 := (t, x, y, z; 1, 0, 0, 0)\, e_2 = (t, x, y, z; 0, 1, 0, 0)\,, \\ & e_3 = (t, x, y, z; 0, 0, 1, 0)\,, e_4 = (t, x, y, z; 0, 0, 0, 1) \Big\}, \end{aligned}$$

we have

$$\begin{aligned}
\nabla_{e_1} \mathfrak{f} &= \left(\mathfrak{D}\left(t, x, y, z\right); \frac{\partial \mathfrak{f}}{\partial t}\left(t, x, y, z\right)\right) \\
&= \left(\mathfrak{D}\left(t, x, y, z\right); \frac{\partial \mathfrak{f}_1}{\partial t}\left(t, x, y, z\right), \ldots, \frac{\partial \mathfrak{f}_{\ell+5}}{\partial t}\left(t, x, y, z\right)\right), \\
\nabla_{e_2} \mathfrak{f} &= \left(\mathfrak{D}\left(t, x, y, z\right); \frac{\partial \mathfrak{f}}{\partial x}\left(t, x, y, z\right)\right) \\
&= \left(\mathfrak{D}\left(t, x, y, z\right); \frac{\partial \mathfrak{f}_1}{\partial x}\left(t, x, y, z\right), \ldots, \frac{\partial \mathfrak{f}_{\ell+5}}{\partial x}\left(t, x, y, z\right)\right), \\
\nabla_{e_3} \mathfrak{f} &= \left(\mathfrak{D}\left(t, x, y, z\right); \frac{\partial \mathfrak{f}}{\partial y}\left(t, x, y, z\right)\right)
\end{aligned}$$

$$= \left(\mathfrak{D}\,(t,x,y,z)\,;\, \frac{\partial \mathfrak{f}_1}{\partial y}\,(t,x,y,z)\,, \ldots, \frac{\partial \mathfrak{f}_{\ell+5}}{\partial y}\,(t,x,y,z) \right),$$

$$\nabla_{e_4}\mathfrak{f} = \left(\mathfrak{D}\,(t,x,y,z)\,;\, \frac{\partial \mathfrak{f}}{\partial z}\,(t,x,y,z) \right)$$

$$= \left(\mathfrak{D}\,(t,x,y,z)\,;\, \frac{\partial \mathfrak{f}_1}{\partial z}\,(t,x,y,z)\,, \ldots, \frac{\partial \mathfrak{f}_{\ell+5}}{\partial z}\,(t,x,y,z) \right).$$

Discontinuous Universalities of Systemic Indices

It is quite reasonable to assume that all the components $g_S^{(1)}, \ldots, g_S^{(\ell+1)}$ of a universality of systemic indices for a predictable system S remain constant over long or short periods and for large or small areas. In other words, we can assume that the spatio-temporal historical phase U is partitioned into different (closed) regions, each associated with a different constant expression of the systemic indices:

There are

- *a finite partition* $\left\{\widehat{U_i} : U_i \textit{ is a nonvoid open subset of } U \textit{ and } i = 1, 2, \ldots, I\right\}$ *of* U, *such that*

$$U_i \bigcap U_{i'} = \emptyset \textit{ whenever } i \neq i', \textit{ and}$$

- *a finite set of constant vectors* $c^{(i)} = \left(c_1^{(i)}, \ldots, c_{l+1}^{(i)}\right)$ in $\mathbb{R}^{l+1}$, $i = 1, 2, \ldots, I$, *such that*

$$\left(g_1^{(S)}, \ldots, g_{\ell+1}^{(S)}\right) = \left(c_1^{(i)}, \ldots, c_{\ell+1}^{(i)}\right),$$

for any $(t, x, y, z) \in U_i$.

The intersection $\Upsilon_{i,i'} := \overline{U}_i \bigcap \overline{U}_{i'}$ between the closure (: the set plus its boundary) of the sets U_i and $U_{i'}$ is either an $\mathbb{R}^3$-dimensional manifold included in the boundaries ∂U_i and $\partial U_{i'}$ or the empty set. A set $\Upsilon_{i,i'}$ is termed to be a ***systemic border*** or ***systemic discontinuous boundary***.

Systemic Measurements

Systemic Measurement Deviations

Let U be any non-empty subset of $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ representing a spatio-temporal historical phase. Suppose $\mathfrak{M}_{\mathcal{F}}$ is a ***systemic measurement of size*** $K + 1$ in U.

This means that it has been selected a certain process $\mathcal{F}$ by which each actual value $g_S^{(j)} = g_S^{(j)}(t, x, y, z)$ is assigned to $K + 1$ numbers

$$\mathcal{F}\big(g_j^{(S)}\big)\,(t_1, x_1, y_1, z_1)\big), \ldots, \mathcal{F}\big(g_j^{(S)}\big)\,(t_{K+1}, x_{K+1}, y_{K+1}, z_{K+1})\big),$$

whenever $(t_\nu, x_\nu, y_\nu, z_\nu)$ is in a given discrete set $\mathcal{E}_{K+1} = \big\{(t_\nu, x_\nu, y_\nu, z_\nu) \in U, \nu = 0, 1, 2, \ldots, K\big\}$ of cardinality $K + 1$. Letting

$$\mathcal{F}\,(t_\nu, x_\nu, y_\nu, z_\nu) := \Bigg(\underbrace{\mathcal{F}\left(g_S^{(1)}\right)(t_\nu, x_\nu, y_\nu, z_\nu)}_{\mathcal{F}_\nu^{(1)}}, \ldots, \underbrace{\mathcal{F}\left(g_S^{(\ell+1)}\right)(t_\nu, x_\nu, y_\nu, z_\nu)}_{\mathcal{F}_\nu^{(\ell+1)}}\Bigg),$$

the systemic measurement $\mathfrak{M}_{\mathcal{F}}$ can be understood as a mapping, which is expressed in the following form:

$$\mathfrak{M}_{\mathcal{F}} : \mathcal{E}_{K+1} \to \mathbb{G}(S) : (t_\nu, x_\nu, y_\nu, z_\nu) \mapsto \big(t_\nu, x_\nu, y_\nu, z_\nu, \mathcal{F}_\nu^{(1)}, \ldots, \mathcal{F}_\nu^{(\ell+1)}\big).$$

Definition 9 Assume that the space $\mathbb{G}(S)$ is endowed with a (Euclidean or not) metric $dist$, the choice of which may depend on the formulation or nature of the problem under consideration.

i. The function

$$\mathcal{W}_* : \mathcal{E}_{K+1} \to \mathbb{R} : (t_\nu, x_\nu, y_\nu, z_\nu) \mapsto \mathcal{W}_*\,(t_\nu, x_\nu, y_\nu, z_\nu) := \\ \operatorname{dist}\left(\tilde{r}_S\,(t_\nu, x_\nu, y_\nu, z_\nu)\,, \mathcal{F}\,(t_\nu, x_\nu, y_\nu, z_\nu)\right)$$

is the **systemic measurement deviation from the lowest threshold of regularity** at the points of $\mathcal{E}_{K+1}$.

ii. The function

$$\mathcal{W}^* : \mathcal{E}_{K+1} \to \mathbb{R} : (t_\nu, x_\nu, y_\nu, z_\nu) \mapsto \mathcal{W}^*\,(t_\nu, x_\nu, y_\nu, z_\nu) := \\ \operatorname{dist}\left(r_S\,(t_\nu, x_\nu, y_\nu, z_\nu)\,, \mathcal{F}\,(t_\nu, x_\nu, y_\nu, z_\nu)\right)$$

is the **systemic measurement deviation from the highest threshold of regularity** at the points of $\mathcal{E}_{K+1}$.

iii. In the case of coincidence $\tilde{r}_S^{(j)} = r_S^{(j)} = R_S^{(j)}$ $(\forall j = 1, 2, \ldots, \ell + 1)$, the function

$$\mathcal{W} : \mathcal{E}_{K+1} \to \mathbb{R} : (t_k, x_k, y_k, z_k) \mapsto \mathcal{W}\,(t_k, x_k, y_k, z_k) := \\ \operatorname{dist}\left(R_S\,(t_k, x_k, y_k, z_k)\,, \mathcal{F}\,(t_k, x_k, y_k, z_k)\right)$$

is the **systemic measurement deviation from the regularity value** of S at the points of $\mathcal{E}_{K+1}$.

Since U is a *separable* topological space, it is possible to choose a sequence

$$\cdots \subsetneq \mathcal{E}_K \subsetneq \mathcal{E}_{K+1} \subsetneq \mathcal{E}_{K+2} \subsetneq \cdots.$$

of *finite* sets of points of U, such that

- their union $\mathcal{E} = \bigcup_{K=1}^{\infty} \mathcal{E}_{K+1}$ is dense in U and
- $\mathcal{E}_{K+1}$ contains only one element more than $\mathcal{E}_K$, say $(t_{K+1}, x_{K+1}, y_{K+1}, z_{K+1})$.

Hence, for any $(t, x, y, z) \in U$, there exists a well-defined sequence

$$(t_{K+1}, x_{K+1}, y_{K+1}, z_{K+1}) \in \mathcal{E}_{K+1} (K = 1, 2, \ldots)$$

such that

$$(t, x, y, z) = \lim_{K+1 \to \infty} (t_{K+1}, x_{K+1}, y_{K+1}, z_{K+1}) .$$

Defining

$$\underline{\mathcal{F}}^{(j)}(t, x, y, z) := liminf_{K+1 \to \infty} \mathcal{F}^{(j)}(t_{k+1}, x_{k+1}, y_{k+1}, z_{k+1}) \; (j=1, 2, \ldots, \ell + 1)$$

and

$$\overline{\mathcal{F}}^{(j)}(t, x, y, z) := limsup_{K+1 \to \infty} \mathcal{F}^{(j)}(t_{k+1}, x_{k+1}, y_{k+1}, z_{k+1}) \; (j=1, 2, \ldots, \ell+1),$$

it is clear that $\underline{\mathcal{F}}^{(j)}$ and $\overline{\mathcal{F}}^{(j)}$ can be viewed as two processes by means of which the actual value $g_j^{(S)}(t, x, y, z)$ corresponds to two real numbers $\underline{\mathcal{F}}^{(j)}(t, x, y, z)$ and $\overline{\mathcal{F}}^{(j)}(t, x, y, z)$, respectively, whenever $(t, x, y, z) \in U$. We are reasonably directed to the next definition.

Definition 10

i. The mappings

$$\underline{\mathcal{F}} : U \to \mathbb{G}(S) : (t, x, y, z) \mapsto \left(t, x, y, z; \underline{\mathcal{F}}^{(1)}(t, x, y, z), \ldots, \underline{\mathcal{F}}^{(\ell+1)}(t, x, y, z)\right)$$

and

$$\overline{\mathcal{F}} : U \to \mathbb{G}(S) : (t, x, y, z) \mapsto \left(t, x, y, z; \overline{\mathcal{F}}^{(1)}(t, x, y, z), \ldots, \overline{\mathcal{F}}^{(\ell+1)}(t, x, y, z)\right)$$

are called, respectively, the **lower** and the **upper sections of the systemic measurement** $\mathfrak{M}_{\mathcal{F}}$ for the predictable system S over U.

ii. If the set U is open in $\mathbb{R}^4$ and the two mappings $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are smooth and regular in U, i.e., their differentials $d\underline{\mathcal{F}}_{(t,x,y,z)}$ and $d\overline{\mathcal{F}}_{(t,x,y,z)}$ are non-singular (: they have rank 4) for each $(t, x, y, z) \in U$, then

- $\underline{\mathcal{F}} = \overline{\mathcal{F}} = \tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}(t,x,y,z) = \left(\tilde{\mathcal{F}}^{(1)}(t,x,y,z), \ldots, \tilde{\mathcal{F}}^{(\ell+1)}(t,x,y,z)\right)$ is called a **smooth and regular extension of the systemic measurement** $\mathfrak{M}_{\mathcal{F}}$ for the predictable system S over U and
- the image $\mathcal{S}_{\mathcal{F}} := \tilde{\mathcal{F}}(U)$ is a **parametrized surface of the systemic measurement** $\mathfrak{M}_{\mathcal{F}}$ for the predictable system S over U.

Analogously, by defining

$$\underline{\mathcal{W}}_*(t,x,y,z) := liminf_{K+1\to\infty}\mathcal{W}_*\left(t_{k+1}, x_{k+1}, y_{k+1}, z_{k+1}\right),$$

$$\overline{\mathcal{W}}_*(t,x,y,z) := limsup_{K+1\to\infty}\mathcal{W}_*\left(t_{k+1}, x_{k+1}, y_{k+1}, z_{k+1}\right),$$

and

$$\underline{\mathcal{W}}^*(t,x,y,z) := liminf_{K+1\to\infty}\mathcal{W}^*\left(t_{k+1}, x_{k+1}, y_{k+1}, z_{k+1}\right),$$

$$\overline{\mathcal{W}}^*(t,x,y,z) := limsup_{K+1\to\infty}\mathcal{W}^*\left(t_{k+1}, x_{k+1}, y_{k+1}, z_{k+1}\right),$$

it is obvious that $\underline{\mathcal{W}}_*(t,x,y,z)$, $\overline{\mathcal{W}}_*(t,x,y,z)$, $\underline{\mathcal{W}}^*(t,x,y,z)$ and $\overline{\mathcal{W}}^*(t,x,y,z)$ are four functions representing distances between, on the one hand, $\underline{\mathcal{F}}(t,x,y,z)$ and $\overline{\mathcal{F}}(t,x,y,z)$ and, on the other hand, the lowest and highest thresholds of regularity, respectively, at every point $(t,x,y,z) \in U$. More precisely, we are led reasonably to the next definition.

Definition 11

i. The function

$$\underline{\mathcal{W}}_* : U \to \mathbb{R} : (t,x,y,z) \mapsto \underline{\mathcal{W}}_*(t,x,y,z) := \mathrm{dist}\left(\tilde{r}_S(t,x,y,z), \overline{\mathcal{F}}(t,x,y,z)\right)$$

is the **upper deviation of the systemic measurement at the points of** U **from the lowest threshold of regularity** over the predictable system S.

ii. The function

$$\underline{\mathcal{W}}^* : U \to \mathbb{R} : (t,x,y,z) \mapsto \underline{\mathcal{W}}^*(t,x,y,z) := \mathrm{dist}\left(\tilde{r}_S(t,x,y,z), \underline{\mathcal{F}}(t,x,y,z)\right)$$

is the **lower deviation of the systemic measurement at the points of** U **from the lowest threshold of regularity** over the predictable system S.

iii. The function

$$\overline{\mathcal{W}}^* : U \to \mathbb{R} : (t,x,y,z) \mapsto \overline{\mathcal{W}}^*(t,x,y,z) := \mathrm{dist}\left(r_S(t,x,y,z), \overline{\mathcal{F}}(t,x,y,z)\right)$$

is the **upper deviation of the systemic measurement at the points of** U **from the highest threshold of regularity** over the predictable system S.

iv. The function

$$\overline{\mathcal{W}}_* : U \to \mathbb{R} : (t, x, y, z) \mapsto \overline{\mathcal{W}}_*(t, x, y, z) := \mathrm{dist}\left(r_S(t, x, y, z), \underline{\mathcal{F}}(t, x, y, z)\right)$$

is the **lower deviation of the systemic measurement at the points of U from the highest threshold of regularity** over the predictable system S.

If, in particular, $\tilde{r}_S = r_S =: R_S$, then $\underline{\mathcal{W}}^* \equiv \overline{\mathcal{W}}^* =: \mathcal{W}^*$ and $\underline{\mathcal{W}}_* \equiv \overline{\mathcal{W}}_* =: \mathcal{W}_*$. In such a case, it is straightforward to see that the function $\mathcal{W}^*(t, x, y, z)$ equals the distance $\mathrm{dist}\left(\overline{\mathcal{F}}(t, x, y, z), \mathfrak{D}(t, x, y, z)\right)$ between the ***upper section*** $\overline{\mathcal{F}}(t, x, y, z)$ of the systemic measurement $\mathfrak{M}_{\mathcal{F}}$ and the universality $\mathfrak{D}(t, x, y, z)$ of the systemic indices for the predictable system S in U. Similarly, the function $\mathcal{W}_*(t, x, y, z)$ equals the distance $\mathrm{dist}\left(\underline{\mathcal{F}}(t, x, y, z), \mathfrak{D}(t, x, y, z)\right)$ between the ***lower section*** $\underline{\mathcal{F}}(t, x, y, z)$ of the systemic measurement $\mathfrak{M}_{\mathcal{F}}$ and the universality $\mathfrak{D}(t, x, y, z)$ of the systemic indices for the predictable system S in U.

Thus, we are led reasonably to the next definition.

Definition 12

i. The function

$$\mathcal{W}^* : U \to \mathbb{R} : (t, x, y, z) \mapsto \mathcal{W}^*(t, x, y, z) := \mathrm{dist}\left(\mathfrak{D}(t, x, y, z), \overline{\mathcal{F}}(t, x, y, z)\right)$$

is the **upper deviation of the systemic measurement at the points of U from the lowest threshold of regularity** over the predictable system S.

ii. The function

$$\mathcal{W}_* : U \to \mathbb{R} : (t, x, y, z) \mapsto \mathcal{W}_*(t, x, y, z) := \mathrm{dist}\left(\mathfrak{D}(t, x, y, z), \underline{\mathcal{F}}(t, x, y, z)\right)$$

is the **lower deviation of the systemic measurement at the points of U from the lowest threshold of regularity** over the predictable system S.

Smooth Parametrized Surfaces of Systemic Measurement

We can now make some useful general observations.

If U is a non-empty open subset of $\mathbb{R}^4$ and if the map $\tilde{\mathcal{F}} : U \to \mathbb{R}^{l+5}$ is smooth and regular, its ***differential*** is the smooth map

$$d\tilde{\mathcal{F}} : U \times \mathbb{R}^4 \to \mathbb{R}^{l+5} \times \mathbb{R}^{l+5}$$

defined as follows. A point $v \in U \times \mathbb{R}^4$ is a vector $v = ((t, x, y, z), u)$ at a point $(t, x, y, z) \in U$. Let $\alpha : I \to U$ be any parametrized curve in U with $\alpha(t_0) = v$. Then, $d\tilde{\mathcal{F}}(v)$ is the vector at

$$\tilde{\mathcal{F}}(t, x, y, z) \left(d\hat{F}(v) \in R^{l+5}_{D(t,x,y,z)} \subset \mathbb{R}^{l+5} \times \mathbb{R}^{l+5}\right)$$

defined by

$$d\tilde{\mathcal{F}}\,(v) = \tilde{\mathcal{F}} \circ \alpha\,(t_0)\,.$$

Note that the value of $d\tilde{\mathcal{F}}\,(v)$ does not depend on the choice of parametrized curve α, because

$$\tilde{\mathcal{F}} \circ \alpha\,(t_0) = \left(\tilde{\mathcal{F}} \circ \alpha\,(t_0)\,, \left(\tilde{\mathcal{F}}_S^{(1)} \circ \alpha\right)^{'}(t_0)\,, \ldots, \left(\tilde{\mathcal{F}}_S^{(l+1)} \circ \alpha\right)^{'}(t_0)\right) =$$

$$\left(\tilde{\mathcal{F}}\,(t,x,y,z)\,, \nabla\tilde{\mathcal{F}}_S^{(1)}\,(\alpha\,(t_0)) \cdot \dot{\alpha}\,(t_0)\,, \ldots \;\; \ldots, \nabla\tilde{\mathcal{F}}_S^{(l+1)}\,(\alpha\,(t_0)) \cdot \dot{\alpha}\,(t_0)\right) =$$

$$\left(\tilde{\mathcal{F}}\,(t,x,y,z)\,, \nabla\tilde{\mathcal{F}}_S^{(1)}\,(t,x,y,z) \cdot v, \ldots \;\; \ldots, \nabla\tilde{\mathcal{F}}_S^{(l+1)}\,(t,x,y,z) \cdot v\right),$$

so

$$d\tilde{\mathcal{F}}\,(v) = \left(\tilde{\mathcal{F}}\,(t,x,y,z)\,, \nabla_v\tilde{\mathcal{F}}_S^{(1)}, \ldots, \nabla_v\tilde{\mathcal{F}}_S^{(l+1)}\right).$$

It follows immediately from the above formula that the restriction $d\tilde{\mathcal{F}}_{(t,x,y,z)}$ of $d\tilde{\mathcal{F}}$ to $\mathbb{R}^4_{(t,x,y,z)}$ (: the vectors at (t,x,y,z)) is a linear map $d\tilde{\mathcal{F}}_{(t,x,y,z)} : \mathbb{R}^4_{(t,x,y,z)} \to \mathbb{R}^{l+5}_{\tilde{F}(t,x,y,z)}$. Its matrix relative to the standard bases for $\mathbb{R}^4_{(t,x,y,z)}$ and $\mathbb{R}^{l+1}_{D(t,x,y,z)}$ is just the Jacobian matrix of $\tilde{\mathcal{F}}$ at (t,x,y,z).

The regularity condition on $\tilde{\mathcal{F}}$ guarantees the following:

Proposition 7

i. *The image $d\tilde{\mathcal{F}}_{(t,x,y,z)}\left(\mathbb{R}^4_{(t,x,y,z)}\right)$ of $d\tilde{\mathcal{F}}_{(t,x,y,z)}$ is a four-dimensional subspace of $\mathbb{R}^{l+5}_{\tilde{F}(t,x,y,z)}$ for each $(t,x,y,z) \in U$.*

ii. *Furthermore, the image $d\tilde{\mathcal{F}}_{(t,x,y,z)}\left(\mathbb{R}^4_{(t,x,y,z)}\right)$ of $d\tilde{\mathcal{F}}_{(t,x,y,z)}$ is the tangent space to the four-dimensional parametrized surface $S_{\tilde{\mathcal{F}}} = \tilde{\mathcal{F}}(U)$ in the systemic index space $\mathbb{G}\,(S)$ corresponding to the point $(t,x,y,z) \in U$.*

Note that a parametrized surface $\tilde{\mathcal{F}}$ in the systemic index space $\mathbb{G}\,(S)$ does not need to be one-to-one, and that $\tilde{\mathcal{F}}\,(t,x,y,z) = \tilde{\mathcal{F}}\left(t^{'},x^{'},y^{'},z^{'}\right)$ for $(t,x,y,z) \neq \left(t^{'},x^{'},y^{'},z^{'}\right)$ **does not** necessarily imply that the image $d\tilde{F}_{(t,x,y,z)}\left(\mathbb{R}^4_{(t,x,y,z)}\right)$ of $d\hat{F}_{(t,x,y,z)}$ is equal to the image $d\tilde{\mathcal{F}}_{(t',x',y',z')}\left(\mathbb{R}^4_{(t',x',y',z')}\right)$ of $d\tilde{\mathcal{F}}_{(t',x',y',z')}$, i.e.,

$$\text{Image}\left[d\tilde{\mathcal{F}}_{(t,x,y,z)}\right] \neq \text{Image}\left[d\tilde{\mathcal{F}}_{\left(t',x',y',z'\right)}\right].$$

Definition 13 A **systemic vector field along a parametrized surface** $S_{\tilde{\mathcal{F}}}$ **of systemic measurement** $\mathfrak{M}_{\mathcal{F}}$ **for the system** S **over** U is a map $\mathfrak{F} = \left(\tilde{\mathcal{F}}; \mathfrak{F}_1, \ldots, \mathfrak{F}_{\ell+5}\right)$ that assigns to each point $p = (t, x, y, z) \in U$ a vector $\mathfrak{F}(p) \in \mathbb{R}^{\ell+5}_{\tilde{\mathcal{F}}(t,x,y,z)}$.

The study of the systemic vector fields along a parametrized surface $S_{\tilde{\mathcal{F}}}$ requires consideration of some additional concepts.

Definition 14 Let $\mathfrak{F} : U \to \mathbb{R}^{\ell+5} : p = (t, x, y, z) \longmapsto \mathfrak{F}(p) = \left(\tilde{\mathcal{F}}(p); \mathfrak{F}_1, \ldots, \mathfrak{F}_{\ell+5}\right) \in R^{\ell+5}_{\hat{F}(t,x,y,z)}$ be a systemic vector field along a parametrized surface $S_{\tilde{\mathcal{F}}}$.

i. We say that $\mathfrak{F} = \left(\tilde{\mathcal{F}}; \mathfrak{F}_1, \ldots, \mathfrak{F}_{\ell+5}\right)$ is **smooth** if each coordinate $\mathfrak{F}_j : U \to \mathbb{R}$ is smooth ($j = 1, 2, \ldots, l + 5$).

ii. We say that $\mathfrak{F} = \left(\tilde{\mathcal{F}}; \mathfrak{F}_1, \ldots, \mathfrak{F}_{\ell+5}\right)$ is **tangent to the parametrized surface** $S_{\tilde{\mathcal{F}}}$ **of the systemic indices for the system** S **over** U if $\mathfrak{F}$ is of the form $\mathfrak{F}(p) = d\tilde{\mathcal{F}}_{(t,x,y,z)}(\mathfrak{y}(p))$ for some vector field $\mathfrak{y}$ on U.

iii. We say that $\mathfrak{F} = \left(\tilde{\mathcal{F}}; \mathfrak{F}_1, \ldots, \mathfrak{F}_{\ell+5}\right)$ is **normal to the parametrized surface** $S_{\tilde{\mathcal{F}}}$ **of the systemic measurement** $\mathfrak{M}_{\mathcal{F}}$ **for the system** S **over** U if

$$\mathfrak{F}(p) \perp Image\left[d\tilde{\mathcal{F}}_{(t,x,y,z)}\right] \text{ for all } (t, x, y, z) \in U.$$

Let us now give a generalization of the concept of the ***velocity field*** in the case of a systemic vector field along a parametrized surface $S_{\tilde{\mathcal{F}}}$ of a systemic measurement $\mathfrak{M}_{\mathcal{F}}$ for the system S over U. Let $\mathcal{G}^{(1)}$,$\mathcal{G}^{(2)}$, $\mathcal{G}^{(3)}$ and $\mathcal{G}^{(4)}$ denote the tangent vector fields along the parametrized surface $S_{\tilde{\mathcal{F}}}$ defined by

$$\mathcal{G}^{(i)}(t, x, y, z) = d\tilde{\mathcal{F}}_{(t,x,y,z)}((t, x, y, z); 0, \ldots, 0, 1, 0 \ldots, 0),$$

where the 1 is in the $(i + 1)$th spot (i spots after the $(t, x, y, z) \in U$).

Proposition 8 *The components of* $\mathcal{G}^{(i)}$ *are just the entries in the* i*th column of the Jacobian matrix for* $\tilde{\mathcal{F}}$ *at* $(t, x, y, z) \in U$.

Note that $\mathcal{G}^{(i)}(t, x, y, z)$ is simply the ***velocity*** *at* $(t, x, y, z) \in U$ of the coordinate curve $u_i \longmapsto \tilde{\mathcal{F}}(u_1, u_2, u_3, u_4)$ (all u_j held constant except u_i) passing through $\tilde{\mathcal{F}}(t, x, y, z)$. Here, $u_1 = t$, $u_2 = x$, $u_3 = y$ and $u_4 = z$. Furthermore, since $d\tilde{\mathcal{F}}_{(t,x,y,z)}$ is non-singular, we infer the following proposition.

Proposition 9

i. *The tangent vector fields* $\mathcal{G}^{(1)}$, $\mathcal{G}^{(2)}$, $\mathcal{G}^{(3)}$ *and* $\mathcal{G}^{(4)}$ *are linearly independent at each point* $(t, x, y, z) \in U$.

ii. *For each point* $(t, x, y, z) \in U$*, the tangent vector fields* $\mathcal{G}^{(1)}$, $\mathcal{G}^{(2)}$, $\mathcal{G}^{(3)}$ *and* $\mathcal{G}^{(4)}$ *form a basis for the tangent* $Image\left[d\tilde{\mathcal{F}}_{(t,x,y,z)}\right]$.

Definition 15 For any smooth systemic vector field $\mathfrak{F} : U \to \mathbb{R}^{\ell+5}$ (U open in $\mathbb{R}^4{=}\mathbb{R}\times\mathbb{R}^3$) along the parametrized surface $S_{\tilde{\mathcal{F}}}$ of the systemic measurement $\mathfrak{M}_{\mathcal{F}}$ for the system S, **the derivative** $\nabla_u\mathfrak{F} \in \mathbb{R}^{\ell+5}_{\tilde{\mathcal{F}}(t,x,y,z)}$ of $\mathfrak{F}$ with respect to $u \in \mathbb{R}^4_{(t,x,y,z)}$ $(t, x, y, z) \in U$ is defined by

$$\nabla_u\mathfrak{F} = \left(\tilde{\mathcal{F}}(t, x, y, z), \left.\frac{d}{d\tau}\right|_{\tau_0} (\mathfrak{F} \circ \alpha)\right) = \left(\tilde{\mathcal{F}}(t, x, y, z), \nabla_u\mathfrak{F}_1, \ldots, \nabla_u\mathfrak{F}_{\ell+5}\right),$$

where

- $\mathfrak{F} = (\mathfrak{F}_1, \ldots, \mathfrak{F}_{\ell+5})$ is the vector part of $\mathfrak{F}$ $\left(:\mathfrak{F}(q) = \left(\tilde{\mathcal{F}}(q); \mathfrak{F}_1(q), \ldots, \mathfrak{F}_{\ell+5}(q)\right)\right.$ for $q \in U$) and
- *α is any parametrized curve in U with $\dot{\alpha}(\tau_0) = u$.*

Note that, when

$$\begin{aligned} u \in \{e_1 = ((t, x, y, z); 1, 0, 0, 0),\ e_2 = ((t, x, y, z); 0, 1, 0, 0), \\ e_3 = ((t, x, y, z); 0, 0, 1, 0),\ e_4 = ((t, x, y, z); 0, 0, 0, 1)\}, \end{aligned}$$

we have

$$\nabla_{e_1}\mathfrak{F} = \left(\tilde{\mathcal{F}}(t, x, y, z); \frac{\partial\mathfrak{F}}{\partial t}(t, x, y, z)\right),$$

$$\nabla_{e_2}\mathfrak{F} = \left(\tilde{\mathcal{F}}(t, x, y, z); \frac{\partial\mathfrak{F}}{\partial x}(t, x, y, z)\right),$$

$$\nabla_{e_3}\mathfrak{F} = \left(\tilde{\mathcal{F}}(t, x, y, z); \frac{\partial\mathfrak{F}}{\partial y}(t, x, y, z)\right),$$

$$\nabla_{e_4}\mathfrak{F} = \left(\tilde{\mathcal{F}}(t, x, y, z); \frac{\partial\mathfrak{F}}{\partial z}(t, x, y, z)\right).$$

5 Distance Between the Universality of Systemic Indices and a Parametrized Surface Passing Through the Points of a Systemic Measurement

We will now use measurement results to predict dates and locations where there will be future systemic incidents. To this end, it would suffice to construct the lower and upper sections $\underline{F}$ and $\overline{F}$ of a systemic measurement $\mathfrak{M}_{\mathcal{F}}$ in a predictable system S and then identify the four systematic deviations to investigate whether some of them are greater or less than corresponding tolerances given in advance.

For any systemic characteristic j and any date $t \in \mathbb{R}$, let us consider the midpoint $\mu_S^{(j)}$ of the regularity tolerance $[\tilde{r}_S^{(j)}(t,x,y,z), r_S^{(j)}(t,x,y,z)]$. The point $\mu = \left(\mu_S^{(1)}, \ldots, \mu_S^{(\ell+1)}\right)$ is the ***focus of regularity*** in S at the time t and location (x,y,z). The hyperplane that perpendicularly intersects the regularity tolerance on this focus μ divides the space-time into two parts, the ***bottom focal half-space*** $\mathcal{P}_1$ and ***upper focal half-space*** $\mathcal{P}_2$, in such a way that

- if $Z = (Z_1, Z_2, \ldots, Z_{\ell+5}) \in \mathcal{P}_1$, then

$$\operatorname{dist}\left(\tilde{r}_S(t,x,y,z) - Z\right) < \operatorname{dist}\left(r_S(t,x,y,z) - Z\right),$$

  and
- if $Z = (Z_1, Z_2, \ldots, Z_{\ell+5}) \in \mathcal{P}_2$, then

$$\operatorname{dist}\left(\tilde{r}_S(t,x,y,z) - Z\right) > \operatorname{dist}\left(r_S(t,x,y,z) - Z\right).$$

It is clear that only four situations may occur: either

$$\overline{\mathcal{F}}(t,x,y,z) \in \mathcal{P}_1 \text{ and } \underline{\mathcal{F}}(t,x,y,z) \in \mathcal{P}_2 \text{ or}$$
$$\overline{\mathcal{F}}(t,x,y,z) \in \mathcal{P}_1 \text{ and } \underline{\mathcal{F}}(t,x,y,z) \in \mathcal{P}_1 \text{ or}$$
$$\overline{\mathcal{F}}(t,x,y,z) \in \mathcal{P}_2 \text{ and } \underline{\mathcal{F}}(t,x,y,z) \in \mathcal{P}_1 \text{ or}$$
$$\overline{\mathcal{F}}(t,x,y,z) \in \mathcal{P}_2 \text{ and } \underline{\mathcal{F}}(t,x,y,z) \in \mathcal{P}_1.$$

In the first and third of these situations, we will say that the measurement in (t,x,y,z) has a ***bifurcated ending*** and (t,x,y,z) is a ***point with bifurcated measurement trend***. In the second of the previous situations, we will say that the measurement in (t,x,y,z) is ***downward*** and (t,x,y,z) is a ***point of downtrend measurement***, while in the fourth situation, we will say that the measurement in (t,x,y,z) is **upward** and (t,x,y,z) is a ***point of uptrend measurement***.

Definition 16 Let

$$\tilde{\delta}_{\text{critical}}(t,x,y,z) = \left(\tilde{\delta}^{(1)}_{\text{critical}}(t,x,y,z), \ldots, \tilde{\delta}^{(\ell+1)}_{\text{critical}}(t,x,y,z)\right),$$

$$\tilde{\epsilon}_{\text{critical}}(t,x,y,z) = \left(\tilde{\epsilon}^{(1)}_{\text{critical}}(t,x,y,z), \ldots, \tilde{\epsilon}^{(\ell+1)}_{\text{critical}}(t,x,y,z)\right),$$

$$\delta_{\text{critical}}(t,x,y,z) = \left(\delta^{(1)}_{\text{critical}}(t,x,y,z), \ldots, \delta^{(\ell+1)}_{\text{critical}}(t,x,y,z)\right) \text{ and}$$

$$\epsilon_{\text{critical}}(t,x,y,z) = \left(\epsilon^{(1)}_{\text{critical}}(t,x,y,z), \ldots, \epsilon^{(\ell+1)}_{\text{critical}}(t,x,y,z)\right)$$

be critical mappings that represent distances from the lowest and highest thresholds outside of which the structure of regularity ceases to exist.

i. Suppose $(\tau, x, y, z) \in U$ is a point of downtrend measurement.

a. If

$$0 < \max\left\{\underline{\mathcal{W}}_*(\tau, x, y, z), \underline{\mathcal{W}}^*(\tau, x, y, z)\right\} < ||\tilde{\delta}_{\text{critical}}(\tau, x, y, z)||,$$

the point (τ, x, y, z) is a **precarity point, due to low performance or sub-sufficiency**.

b. If

$$||\tilde{\delta}_{\text{critical}}(\tau, x, y, z)|| \leq \min\left\{\underline{\mathcal{W}}_*(\tau, x, y, z), \underline{\mathcal{W}}^*(\tau, x, y, z)\right\},$$
$$\leq \max\left\{\underline{\mathcal{W}}_*(\tau, x, y, z), \underline{\mathcal{W}}^*(\tau, x, y, z)\right\} < ||\tilde{\epsilon}_{\text{critical}}(\tau, x, y, z)||,$$

the point (τ, x, y, z) is a **dangerous point, due to low performance or sub-sufficiency**.

c. If

$$||\tilde{\epsilon}_{\text{critical}}(\tau, x, y, z)|| \leq \min\left\{\underline{\mathcal{W}}_*(\tau, x, y, z), \underline{\mathcal{W}}^*(\tau, x, y, z)\right\},$$

the point (τ, x, y, z) is a **collapse point, due to low performance or sub-sufficiency**.

ii. Suppose $(\tau, x, y, z) \in U$ is a point of uptrend measurement.

a. If

$$0 < \max\left\{\overline{\mathcal{W}}_*(\tau, x, y, z), \overline{\mathcal{W}}^*(\tau, x, y, z)\right\} < ||\delta_{\text{critical}}(\tau, x, y, z)||,$$

the point (τ, x, y, z) is a **precarity point, due to high performance or ultra-sufficiency**.

b. If

$$||\delta_{\text{critical}}(\tau, x, y, z)|| \leq \min\left\{\overline{\mathcal{W}}_*(\tau, x, y, z), \overline{\mathcal{W}}^*(\tau, x, y, z)\right\},$$
$$\leq \max\left\{\overline{\mathcal{W}}_*(\tau, x, y, z), \overline{\mathcal{W}}^*(\tau, x, y, z)\right\} < ||\epsilon_{\text{critical}}(\tau, x, y, z)||,$$

the point (τ, x, y, z) is a **dangerous point, due to high performance or ultra-sufficiency**.

c. If

$$||\epsilon_{\text{critical}}(\tau, x, y, z)|| \leq \min\left\{\overline{\mathcal{W}}_*(\tau, x, y, z), \overline{\mathcal{W}}^*(\tau, x, y, z)\right\},$$

the point (τ, x, y, z) is a **collapse point, due to high performance or ultra-sufficiency**.

In practice, it seems often difficult to identify the lower and the upper sections $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ of a systemic measurement $\mathfrak{M}_{\mathcal{F}}$ for the predictable system S over an open set $U \subset \mathbb{R}^4$. Therefore, in this section, it is intuitively preferable to be searched for parametrized surfaces $H(t, x, y, z)$ passing very close to the systemic measurement points, in order to determine deviations between these surfaces and the universality of systemic indices. To this end, we give the following two definitions.

Definition 17 If $H : \mathbb{R}^4 \to \mathbb{G}(S) : (t, x, y, z) \mapsto H(t, x, y, z)$ is a parametrized surface in the space $\mathbb{G}(S)$ of the systemic indices over the predictable system S, then the functions

$$\mathcal{V}_* : U \to \mathbb{R} : (t, x, y, z) \mapsto \mathcal{V}_*(t, x, y, z) := dist\left(\tilde{r}_S(t, x, y, z), H(t, x, y, z)\right) \text{ and}$$

$$\mathcal{V}^* : U \to \mathbb{R} : (t, x, y, z) \mapsto \mathcal{V}^*(t, x, y, z) := dist\left(r_S(t, x, y, z), H(t, x, y, z)\right)$$

are, respectively, the **deviations of the parametrized surface H from the lowest and highest thresholds of regularity at the points of** U over the system S. In the case of coincidence $r_S = \tilde{r}_S = R_S$, the common function

$$\mathcal{V} : U \to \mathbb{R} : (t, x, y, z) \mapsto \mathcal{V}(t, x, y, z) := dist\left(R_S(t, x, y, z), H(t, x, y, z)\right)$$

is called the **deviation of the parametrized surface H from the regularity value of** S at the points of U.

Definition 18 As in Definition 16, let us consider the critical mappings

$$\tilde{\delta}_{\text{critical}}(t, x, y, z) = \left(\tilde{\delta}^{(1)}_{\text{critical}}(t, x, y, z), \ldots, \tilde{\delta}^{(\ell+1)}_{\text{critical}}(t, x, y, z)\right),$$
$$\tilde{\epsilon}_{\text{critical}}(t, x, y, z) = \left(\tilde{\epsilon}^{(1)}_{\text{critical}}(t, x, y, z), \ldots, \tilde{\epsilon}^{(\ell+1)}_{\text{critical}}(t, x, y, z)\right),$$
$$\delta_{\text{critical}}(t, x, y, z) = \left(\delta^{(1)}_{\text{critical}}(t, x, y, z), \ldots, \delta^{(\ell+1)}_{\text{critical}}(t, x, y, z)\right) \text{ and}$$
$$\epsilon_{\text{critical}}(t, x, y, z) = \left(\epsilon^{(1)}_{\text{critical}}(t, x, y, z), \ldots, \epsilon^{(\ell+1)}_{\text{critical}}(t, x, y, z)\right).$$

Let also $H : \mathbb{R}^4 \to \mathbb{G}(S) : (t, x, y, z) \mapsto H(t, x, y, z)$ be a parametrized surface in the space $\mathbb{G}(S)$of the systemic indices over the predictable system S.

i. Suppose $(\tau, x, y, z) \in U$ is a point such that $H(\tau, x, y, z)$ in the **bottom focal half-space** $\mathcal{P}_1$.

a. If

$$0 < \mathcal{V}_*(\tau, x, y, z) < ||\tilde{\delta}_{\text{critical}}(\tau, x, y, z)||,$$

we say that the point (τ, x, y, z) is a **potential point to display precarious incident, because of low performance or sub-sufficiency**.

b. If

$$||\tilde{\delta}_{\text{critical}}(\tau, x, y, z)|| \leq \mathcal{V}_*(\tau, x, y, z) < ||\tilde{\epsilon}_{\text{critical}}(\tau, x, y, z)||,$$

we say that the point (τ, x, y, z) is a **potential point to display dangerous incident, because of low performance or sub-sufficiency**.

c. If

$$||\tilde{\epsilon}_{\text{critical}}(\tau, x, y, z)|| \leq \mathcal{V}^*(\tau, x, y, z),$$

we say that the point (τ, x, y, z) is a **potential point to display disastrous incident, because of low performance or sub-sufficiency**.

ii. Suppose $(\tau, x, y, z) \in U$ is a point such that $H(\tau, x, y, z)$ in the **upper focal half-space** $\mathcal{P}_2$.

a. If

$$0 < \mathcal{V}_*(\tau, x, y, z) < ||\delta_{\text{critical}}(\tau, x, y, z)||,$$

we say that the point (τ, x, y, z) is a **potential point to display precarious incident, because of high performance and ultra-sufficiency**.

b. If

$$||\delta_{\text{critical}}(\tau, x, y, z)|| \leq \mathcal{V}^*(\tau, x, y, z) < ||\epsilon_{\text{critical}}(\tau, x, y, z)||,$$

we say that the point (τ, x, y, z) is a **potential point to display dangerous incident, because of high performance or ultra-sufficiency**.

c. If

$$||\epsilon_{\text{critical}}(\tau, x, y, z)|| \leq \mathcal{V}^*(\tau, x, y, z),$$

we say that the point (τ, x, y, z) is a **potential point to display disastrous incident, because of high performance or ultra-sufficiency**.

Having now defined the necessary theoretical background, we are able to look for numerical or approximate constructions of parametrized surfaces H_M passing through $M + 1$ systemic measurement results at the points of a given finite subset of U, in order to determine deviations between these surfaces and the universality of systemic indices at each point (t, x, y, z) of $U \subset\subset \mathbb{R}^4$.

Suppose $\mathcal{E}_{M+1} = \left\{ (t_\nu, x_\nu, y_\nu, z_\nu) \in U : \nu = 0, 1, \ldots, M \right\}$ is a given finite set of $M + 1$ different points.

Let also $0 \leq k < M$. Assume that, for any $\nu = 0, 1, 2, \ldots, k$, we know the corresponding measurement points. Specifically, this means that for any $j = 1, 2, \ldots, \ell + 1$, we know the measured values

$$f_j\,(t_\nu, x_\nu, y_\nu, z_\nu)$$

of the jth systemic index $g_S^{(j)}$ according to a systemic measurement $\mathfrak{M}_{\mathcal{F}}$ at the $k+1$ discrete points t_ν, $\nu = 0, 1, \ldots, k$.

Below, we will formulate a general approximate method to identify all those time intervals into the region $(t_k, t_{k+1}) \times (x_k, x_{k+1}) \times (y_k, y_{k+1}) \times (z_k, z_{k+1})$, during which peculiar incidents in the system may occur.

General Algorithmic Framework to Determine Times and Locations of Peculiar Systemic Incidents

1. **For each** $j = 1, 2, \ldots, \ell + 1$, construct a well manageable numerical function $H_M^{(j)}(t, x, y, z)$, which passes very close to the $M + 1$ measured values

$$f_j(t_\nu, x_\nu, y_\nu, z_\nu)\ (\nu = 0, 1, \ldots, k).$$

2. **Construct** the *parametrized surface*

$$H_M : \mathbb{R}^4 \to \mathbb{G}(S) : (t, x, y, z) \mapsto H_M(t, x, y, z) :=$$
$$\left(t, x, y, z, H_M^{(1)}(t, x, y, z), \ldots \ \ldots, H_M^{(\ell+1)}(t, x, y, z)\right).$$

3. **Choose** four *critical tolerance functions*

$$\tilde{\delta}_{\text{critical}}(t, x, y, z) = \left(\tilde{\delta}^{(1)}_{\text{critical}}(t, x, y, z), \ldots, \tilde{\delta}^{(\ell+1)}_{\text{critical}}(t, x, y, z)\right),$$
$$\tilde{\epsilon}_{\text{critical}}(t, x, y, z) = \left(\tilde{\epsilon}^{(1)}_{\text{critical}}(t, x, y, z), \ldots, \tilde{\epsilon}^{(\ell+1)}_{\text{critical}}(t, x, y, z)\right),$$
$$\delta_{\text{critical}}(t, x, y, z) = \left(\delta^{(1)}_{\text{critical}}(t, x, y, z), \ldots, \delta^{(\ell+1)}_{\text{critical}}(t, x, y, z)\right) \text{ and}$$
$$\epsilon_{\text{critical}}(t, x, y, z) = \left(\epsilon^{(1)}_{\text{critical}}(t, x, y, z), \ldots, \epsilon^{(\ell+1)}_{\text{critical}}(t, x, y, z)\right),$$

which represent distances from the lowest and highest thresholds outside of which the regularity is repealed banded.

4. **If** $\tilde{r} = r$, then

   i. Find the set $\mathbb{P}$ of all points $(\tau, \chi, \psi, \zeta)$ satisfying

$$t_k < \tau < t_{k+1},$$
$$x_k < \chi < x_{k+1},$$
$$y_k < \psi < y_{k+1},$$
$$z_k < \zeta < z_{k+1}.$$

   ii. Solve in $\mathbb{P}$ the inequality

$$0 < \mathcal{V}(\tau, \chi, \psi, \zeta) < ||\delta_{\text{critical}}(\tau, \chi, \psi, \zeta)||;$$

any point $(\tau, \chi, \psi, \zeta)$ in $\mathbb{P}$ satisfying this inequality is a **potential point to display precarious incident**.

iii. Solve in $\mathbb{P}$ the inequalities

$$||\delta_{\text{critical}}(\tau, \chi, \psi, \zeta)|| \leq \mathcal{V}(\tau, \chi, \psi, \zeta) < ||\epsilon_{\text{critical}}(\tau, \chi, \psi, \zeta)||;$$

any point $(\tau, \chi, \psi, \zeta)$ in $\mathbb{P}$ satisfying this inequality is a **potential point to display dangerous incident**.

iv. Solve in $\mathbb{P}$ the inequalities

$$||\epsilon_{\text{critical}}(\tau, \chi, \psi, \zeta)|| \leq \mathcal{V}(\tau, \chi, \psi, \zeta);$$

any point $(\tau, \chi, \psi, \zeta)$ in $\mathbb{P}$ satisfying this inequality is a **potential point to display disastrous incident**.

5. Else

i. Find the set $\mathbb{P}_1$ of all points $(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta}) \in \mathbb{R}^4$ satisfying

$$\text{dist}\left(\tilde{r}_S(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta}) - H_M(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta})\right) < \text{dist}\left(r_S(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta}) - H_M(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta})\right),$$

$$t_k < \tilde{\tau} < t_{k+1},$$

$$x_k < \tilde{\chi} < x_{k+1},$$

$$y_k < \tilde{\psi} < y_{k+1},$$

$$z_k < \tilde{\zeta} < z_{k+1}.$$

ii. Solve in $\mathbb{P}_1$ the inequality

$$0 < \mathcal{V}_*(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta}) < ||\delta_{\text{critical}}(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta})||;$$

any point $(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta})$ in $\mathbb{P}_1$ satisfying this inequality is a **potential point to display precarious incident because of low performance or sub-sufficiency**.

iii. Solve in $\mathbb{P}_1$ the inequalities

$$||\delta_{\text{critical}}(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta})|| \leq \mathcal{V}_*(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta}) < ||\epsilon_{\text{critical}}(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta})||;$$

any point $(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta})$ in $\mathbb{P}_1$ satisfying this inequality is a **potential point to display dangerous incident because of low performance or sub-sufficiency**.

iv. Solve in $\mathbb{P}_1$ the inequalities

$$||\epsilon_{\text{critical}}(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta})|| \leq \mathcal{V}_*(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta});$$

any point $(\tilde{\tau}, \tilde{\chi}, \tilde{\psi}, \tilde{\zeta})$ in $\mathbb{P}_1$ satisfying this inequality is a **potential point to display disastrous incident because of low performance or sub-sufficiency**.

v. Find the set $\mathbb{P}_2$ of all points $(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta}) \in \mathbb{R}^4$ satisfying

$$\mathrm{dist}\Big(\tilde{r}_S(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta}) - H_M(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta})\Big) > \mathrm{dist}\Big(r_S(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta}) - H_M(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta})\Big),$$

$$t_k < \hat{\tau} < t_{k+1},$$
$$x_k < \hat{\chi} < x_{k+1},$$
$$y_k < \hat{\psi} < y_{k+1},$$
$$z_k < \hat{\zeta} < z_{k+1}.$$

vi. Solve in $\mathbb{P}_2$ the inequality

$$0 < \mathcal{V}^*(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta}) < ||\delta_{\text{critical}}(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta})||;$$

any point $(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta})$ in $\mathbb{P}_2$ satisfying this inequality is a **potential point to display precarious incident because of high performance or ultra-sufficiency**.

vii. Solve in $\mathbb{P}_2$ the inequalities

$$||\delta_{\text{critical}}(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta})|| \leq \mathcal{V}^*(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta}) < ||\epsilon_{\text{critical}}(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta})||;$$

any point $(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta})$ in $\mathbb{P}_2$ satisfying this inequality is a **potential point to display dangerous incident because of high performance or ultra-sufficiency**.

viii. Solve in $\mathbb{P}_2$ the inequalities

$$||\epsilon_{\text{critical}}(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta})|| \leq \mathcal{V}^*(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta});$$

any point $(\hat{\tau}, \hat{\chi}, \hat{\psi}, \hat{\zeta})$ in $\mathbb{P}_2$ satisfying this inequality is a **potential point to display disastrous incident because of high performance or ultra-sufficiency**.

In order to simplify the computational complexity of our approach, we will assume that ***the systemic study is carried out in a fixed location***, say

$$x = x_0 = const,\ y = y_0 = const \text{ and } z = z_0 = const.$$

The general case cited in the algorithmic framework above will be considered in a forthcoming paper [6].

Here, to find well manageable numerical functions

$$H_M^{(j)}\,(t, x_0, y_0, z_0)\;(j = 1, 2, \ldots, \ell + 1)$$

passing very close to the $M + 1$ values $f_j\,(t_\nu, x_0, y_0, z_0)$ $(\nu = 0, 1, \ldots, M + 1)$, we will use *interpolation techniques* and *least square polynomial approximation*.

Obviously, these methods are not the only ones that could be used to determine such well manageable numerical functions. However, for the main purpose of this chapter, it is sufficiently indicative to consider only these methods, since for a multitude of cases, they can be applied and give satisfactory prediction results.

The Linear Splines Interpolation Method

Suppose

$$\mathcal{E}_{K+1} = \{t_\nu \in [T_0, T_\nu] : \nu = 0, 1, \ldots, K\}$$

is a given finite set of $M+1$ different time moments in a fixed time interval $[T_0, T_\nu]$, such that $(t_\nu, x_0, y_0, t_0) \in U$ and $t_\nu < t_{\nu'}$, whenever $\nu, \nu' \in 0, 1, \ldots, K$ satisfy $\nu < \nu'$.

Let also $k < K$. Assume that, for any $\nu = 0, 1, \ldots, k$, we know the corresponding measurement points. Specifically, this means that for any such ν and any $j = 0, 1, \ldots, \ell + 1$, we know the measured values

$$f_j\,(t_\nu) := f_j\,(t_\nu, x_0, y_0, z_0)$$

of the jth systemic index $g_S^{(j)}$ accordingly to a systemic measurement $\mathfrak{M}_{\mathcal{F}}$ at the $k + 1$ discrete points t_ν, $\nu = 0, 1, \ldots, k$.

Furthermore, assume that ***for any*** $\nu = k + 1, k + 2, \ldots, M$, ***the point*** (t_ν, x_0, y_0, t_0) ***is a regularity state***.

Below, we will formulate a general approximate method to identify all those time intervals in the region (t_k, t_{k+1}), during which peculiar incidents in the system may occur (Figure 3).

The advantage of application of a linear splines interpolation method [11] consists in its low computational complexity, not only for computing the linear splines but also for computing the roots and the intervals in which the approximate tolerance deviations are negative or positive. In case of few interpolating points, this method will give inaccurate results. But if there are enough interpolating points, the method is efficient, so it is proposed in case that there are enough interpolating points. Of course, in the general theoretical case, the effectiveness of the method may be directly dependent on the number of the linear spline zeros that are within the period of measurements. However, usually in practice, this is not a problem,

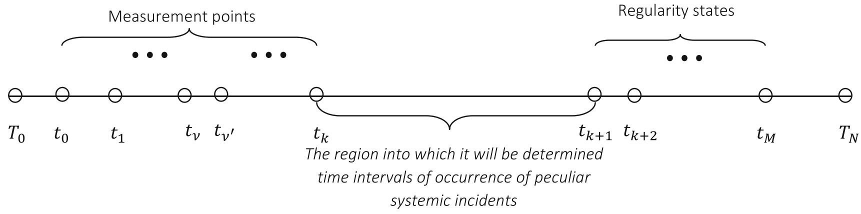

Fig. 3 Time intervals

since all measurement values are situated a bit far from regularity points that, in most cases, take positive values.

Now, our *general algorithmic framework* specializes as follows.

Algorithm 1:
Deterministic Prediction Using Linear Splines

Input: - the points $(t_\nu, f_j(t_\nu))$;
- the k measurement points;
- the $M-k$ regularity points.

Output: - the zeroes of the functions

$$\begin{aligned}
\mathsf{F}_{\text{LinearSpline}}(\tau) &= ||\delta_{\text{critical}}(\tau)|| - \mathcal{V}(\tau),\\
\mathsf{F}'_{\text{LinearSpline}}(\tau) &= ||\epsilon_{\text{critical}}(\tau)|| - \mathcal{V}(\tau),\\
\tilde{\mathsf{G}}_{\text{LinearSpline}}(\tilde{\tau}) &= ||\tilde{\delta}_{\text{critical}}(\tilde{\tau})|| - \mathcal{V}_*(\tilde{\tau}),\\
\tilde{\mathsf{G}}'_{\text{LinearSpline}}(\tilde{\tau}) &= ||\tilde{\epsilon}_{\text{critical}}(\tilde{\tau})|| - \mathcal{V}_*(\tilde{\tau}),\\
\mathsf{G}_{\text{LinearSpline}}(\tau) &= ||\delta_{\text{critical}}(\tau)|| - \mathcal{V}^*(\tau),\\
\mathsf{G}'_{\text{LinearSpline}}(\tau) &= ||\epsilon_{\text{critical}}(\tau)|| - \mathcal{V}^*(\tau)
\end{aligned}$$

in a given interval (t_k, t_{k+1});
- the intervals into which the following inequalities are satisfied:

$$\begin{aligned}
\mathsf{F}_{\text{LinearSpline}}(\tau) &< 0,\\
\mathsf{F}'_{\text{LinearSpline}}(\tau) &< 0,\\
\tilde{\mathsf{G}}_{\text{LinearSpline}}(\tilde{\tau}) &< 0,\\
\tilde{\mathsf{G}}'_{\text{LinearSpline}}(\tilde{\tau}) &< 0,\\
\mathsf{G}_{\text{LinearSpline}}(\tau) &< 0,\\
\mathsf{G}'_{\text{LinearSpline}}(\tau) &< 0.
\end{aligned}$$

1. **For each** $j = 1, 2, \ldots, \ell + 1$, **compute** *the Linear Spline*

$$\mathfrak{S}_M^{(j)}(t) = \begin{cases} \sigma_1(t) = f_j(t_0)\,\frac{t-t_1}{t_0-t_1} + f_j(t_1)\,\frac{t-t_0}{t_1-t_0}, & t \in [t_0, t_1]\\ \sigma_2(t) = f_j(t_1)\,\frac{t-t_2}{t_1-t_2} + f_j(t_2)\,\frac{t-t_1}{t_1-t_1}, & t \in [t_1, t_2]\\ \vdots & \\ \sigma_M(t) = f_j(t_{M-1})\,\frac{t-t_M}{t_{M-1}-t_M} + f_j(t_M)\,\frac{t-t_{M-1}}{t_M-t_{M-1}}, & t \in [t_{M-1}, t_M] \end{cases}$$

in the given interval based on the $M+1$ values $(t_\nu, f_j(t_\nu))$.

2. **Construct** the *curve*

$$H_M(t) \equiv \mathfrak{S}_M\,(t, x_0, y_0, z_0)$$

with

$$\mathfrak{S}_M : \mathbb{R}^4 \to \mathbb{G}\,(S) : t \mapsto \mathfrak{S}_M\,(t, x_0, y_0, z_0) :=$$
$$\left(t, x_0, y_0, z_0, \mathfrak{S}_M^{(1)}\,(t, x_0, y_0, z_0)\,, \ldots, \mathfrak{S}_M^{(\ell+1)}\,(t, x_0, y_0, z_0)\right).$$

3. **Choose** four *critical tolerance functions*

$$\tilde{\delta}_{\text{critical}}(t) = \left(\tilde{\delta}^{(1)}_{\text{critical}}(t), \ldots, \tilde{\delta}^{(\ell+1)}_{\text{critical}}(t)\right),$$
$$\tilde{\epsilon}_{\text{critical}}(t) = \left(\tilde{\epsilon}^{(1)}_{\text{critical}}(t), \ldots, \tilde{\epsilon}^{(\ell+1)}_{\text{critical}}(t)\right),$$
$$\delta_{\text{critical}}(t) = \left(\delta^{(1)}_{\text{critical}}(t), \ldots, \delta^{(\ell+1)}_{\text{critical}}(t)\right) \text{ and}$$
$$\epsilon_{\text{critical}}(t) = \left(\epsilon^{(1)}_{\text{critical}}(t), \ldots, \epsilon^{(\ell+1)}_{\text{critical}}(t)\right)$$

representing distances from the lowest and highest thresholds outside of which the regularity is repealed banded.

4. If $\tilde{r}_S = r_S$, then

   i. **Compute** *the zeroes of the deviations*

   $$\mathsf{F}_{\text{LinearSpline}}(\tau) \text{ and } \mathsf{F}'_{\text{LinearSpline}}(\tau)$$

   in the given interval (t_k, t_{k+1}).

   ii. **Determine** *the intervals into which the tolerance deviation* $\mathsf{F}_{\text{LinearSpline}}(\tau)$ is positive using the computed zeroes; *any point* τ in (t_k, t_{k+1}) *satisfying this inequality is a* ***potential point to display precarious incident***.

   iii. **Determine** *the intervals into which the tolerance deviations*

   $$\mathsf{F}_{\text{LinearSpline}}(\tau) \text{ and } \mathsf{F}'_{\text{LinearSpline}}(\tau)$$

   are negative and positive, respectively; *any point* τ *in* (t_k, t_{k+1}) *satisfying these inequalities is a* ***potential point to display a dangerous incident***.

   iv. **Determine** *the intervals into which* $\mathsf{F}'_{\text{LinearSpline}}(\tau) < 0$; *any point* τ *in* (t_k, t_{k+1}) *satisfying this inequality is a* ***potential point to display a dangerous incident***.

5. Else

i. **Determine** *the sets* $\tilde{\mathbb{P}}$ *and* $\mathbb{P}$ *of all points* $\tilde{\tau}, \tau \in (t_k, t_{k+1})$ satisfying

$$\text{dist}\,(\tilde{r}_S(\tilde{\tau}), H_M(\tilde{\tau})) < \text{dist}\,(r_S(\tilde{\tau}), H_M(\tilde{\tau})) \text{ and}$$
$$\text{dist}\,(\tilde{r}_S(\tau), H_M(\tau)) > \text{dist}\,(r_S(\tau), H_M(\tau))\,.$$

ii. **Compute** *the zeroes of the tolerance deviations*

$$\tilde{\mathsf{G}}_{\text{LinearSpline}}(\tilde{\tau}),\, \tilde{\mathsf{G}}'_{\text{LinearSpline}}(\tilde{\tau}),$$
$$\mathsf{G}_{\text{LinearSpline}}(\tau) \textit{ and } \mathsf{G}'_{\text{LinearSpline}}(\tau).$$

iii. **Determine**

a. *the intervals* $I \subset \tilde{\mathbb{P}}$ *into which* $\tilde{\mathsf{G}}_{\text{LinearSpline}}(\tilde{\tau}) > 0$; *any point* $\tilde{\tau} \in \tilde{\mathbb{P}}$ *satisfying this inequality is a* ***potential point to display precarious incident, because of low performance or sub-sufficiency***;

b. *the intervals* $I \subset \tilde{\mathbb{P}}$ *into which* $\mathsf{G}_{\text{LinearSpline}}(\tau) > 0$; *any point* $\tau \in \mathbb{P}$ *satisfying this inequality is a* ***potential point to display precarious incident, because of high performance or ultra-sufficiency***.

iv. **Determine**

a. *in* $\tilde{\mathbb{P}}$ *the intervals in which* $\tilde{\mathsf{G}}_{\text{LinearSpline}}(\tilde{\tau}) < 0$ and $\tilde{\mathsf{G}}'_{\text{LinearSpline}}(\tilde{\tau}) > 0$; *any point* $\tilde{\tau} \in (t_k, t_{k+1})$ *satisfying these inequalities is a* ***potential point to display dangerous incident, because of low performance or sub-sufficiency***;

b. *in* $\mathbb{P}$ *the intervals in which* $\mathsf{G}_{\text{LinearSpline}}(\tau) < 0$ and $\mathsf{G}'_{\text{LinearSpline}}(\tau) > 0$; *any point* $\tau \in (t_k, t_{k+1})$ *satisfying this inequality is a* ***potential point to display dangerous incident, because of high performance or ultra-sufficiency***;

v. **Determine**

a. the intervals $I \subset \tilde{\mathbb{P}}$ into which $\tilde{\mathsf{G}}'_{\text{LinearSpline}}(\tilde{\tau}) < 0$; *any point* $\tilde{\tau} \in (t_k, t_{k+1})$ *satisfying this inequality is a* ***potential point to display disastrous incident, because of low performance or sub-sufficiency***;

b. the intervals $I \subset \tilde{\mathbb{P}}$ into which $\mathsf{G}'_{\text{LinearSpline}}(\tau) < 0$; *any point* $\tau \in (t_k, t_{k+1})$ *satisfying this inequality is a* ***potential point to display disastrous incident, because of high performance or ultra sufficiency***.

The Lagrange Interpolation Method

The advantage of Lagrange interpolation method [12, 16, 18] is its unified expression into the whole interval of interest. But, on the other hand, its computational complexity is greater than that of linear splines approximation and, in case of many interpolating points, the resulting polynomial will be of large degree, which may cause problems due to cancellation of significant digits during floating point operations, with subsequent increment in the computational complexity for the computation of its roots. Thus, this method is recommended in the case of a few interpolating points. Note that, the effectiveness of the polynomial interpolation method seems to be dependent on the number of polynomial zeros located into the period of measurements. However, as before, this is not a real problem, since, usually in practice, the measurement values are all taken to be positive. Using Lagrange interpolation, our general algorithmic framework becomes as follows.

Algorithm 2:
Deterministic Prediction Using Lagrange Interpolation

Input: - the interpolation points $(t_\nu, f_j(t_\nu)$;
- the k measurement points;
- the $M - k$ regularity points.

Output: - the zeroes of the functions

$$\begin{aligned}
\mathsf{F}_{\text{Interpolation}}(\tau) &= \|\delta_{\text{critical}}(\tau)\| - \mathcal{V}(\tau),\\
\mathsf{F}'_{\text{Interpolation}}(\tau) &= \|\epsilon_{\text{critical}}(\tau)\| - \mathcal{V}(\tau),\\
\tilde{\mathsf{G}}_{\text{Interpolation}}(\tilde{\tau}) &= \|\tilde{\delta}_{\text{critical}}(\tilde{\tau})\| - \mathcal{V}_*(\tilde{\tau}),\\
\tilde{\mathsf{G}}'_{\text{Interpolation}}(\tilde{\tau}) &= \|\tilde{\epsilon}_{\text{critical}}(\tilde{\tau})\| - \mathcal{V}_*(\tilde{\tau}),\\
\mathsf{G}_{\text{Interpolation}}(\tau) &= \|\delta_{\text{critical}}(\tau)\| - \mathcal{V}^*(\tau),\\
\mathsf{G}'_{\text{Interpolation}}(\tau) &= \|\epsilon_{\text{critical}}(\tau)\| - \mathcal{V}^*(\tau)
\end{aligned}$$

in a given interval (t_k, t_{k+1});
- the intervals into which the following inequalities are satisfied:

$$\begin{aligned}
\mathsf{F}_{\text{Interpolation}}(\tau) &< 0,\\
\mathsf{F}'_{\text{Interpolation}}(\tau) &< 0,\\
\tilde{\mathsf{G}}_{\text{Interpolation}}(\tilde{\tau}) &< 0,\\
\tilde{\mathsf{G}}'_{\text{Interpolation}}(\tilde{\tau}) &< 0,\\
\mathsf{G}_{\text{Interpolation}}(\tau) &< 0,\\
\mathsf{G}'_{\text{Interpolation}}(\tau) &< 0.
\end{aligned}$$

1. **For each** $j = 1, 2, \ldots, \ell + 1$, **compute** *the unique Lagrange polynomial of degree at most M*

$$\mathcal{L}_M^{(j)}(t) = \sum_{\nu=1}^{M} f_j(t_\nu) \prod_{\nu'=0(\nu'\neq\nu)}^{M} \frac{t - t_{\nu'}}{t_\nu - t_{\nu'}}$$

interpolating the $M + 1$ given values $(t_\nu, f_j(t_\nu))$.

2. **Construct** the *curve*

$$H_M(t) \equiv \mathcal{L}_M(t)$$

with

$$\mathcal{L}_M : \mathbb{R} \to \mathbb{R}^{\ell+1} : t \mapsto \mathcal{L}_M(t) := \left(t, \mathcal{L}_M^{(1)}(t), \ldots, \mathcal{L}_M^{(\ell+1)}(t)\right).$$

3. **Choose** four *critical tolerance functions*

$$\tilde{\delta}_{\text{critical}}(t) = \left(\tilde{\delta}^{(1)}_{\text{critical}}(t), \ldots, \tilde{\delta}^{(\ell+1)}_{\text{critical}}(t)\right),$$

$$\tilde{\epsilon}_{\text{critical}}(t) = \left(\tilde{\epsilon}^{(1)}_{\text{critical}}(t), \ldots, \tilde{\epsilon}^{(\ell+1)}_{\text{critical}}(t)\right),$$

$$\delta_{\text{critical}}(t) = \left(\delta^{(1)}_{\text{critical}}(t), \ldots, \delta^{(\ell+1)}_{\text{critical}}(t)\right) \text{ and}$$

$$\epsilon_{\text{critical}}(t) = \left(\epsilon^{(1)}_{\text{critical}}(t), \ldots, \epsilon^{(\ell+1)}_{\text{critical}}(t)\right)$$

representing distances from the lowest and highest thresholds outside of which the regularity is repealed banded.

4. If $\tilde{r}_S = r_S$, then

 i. **Compute** *the tolerance deviations*

$$\mathsf{F}_{\text{Interpolation}}(\tau) \text{ and } \mathsf{F}'_{\text{Interpolation}}(\tau)$$

 in the given interval (t_k, t_{k+1}).

 ii. **Determine** *the intervals* (α_i, β_i), $(\alpha'_i, \beta'_i) \subset (t_k, t_{k+1})$ into which the tolerance deviations $\mathsf{F}_{\text{Interpolation}}(\tau)$ and $\mathsf{F}'_{\text{Interpolation}}(\tau)$ are changing sign, respectively;

 For every interval (α_i, β_i) and (α'_i, β'_i),

 Apply Bisection method for approaching zeroes of $\mathsf{F}_{\text{Interpolation}}(\tau)$ in (α_i, β_i) and zeroes of $\mathsf{F}'_{\text{Interpolation}}(\tau)$ in (α'_i, β'_i);

 Apply Newton's method for computing zeroes of $\mathsf{F}_{\text{Interpolation}}(\tau)$ in (α_i, β_i) and zeroes of $\mathsf{F}'_{\text{Interpolation}}(\tau)$ in (α'_i, β'_i);

 iii. **Determine** *the intervals into which the tolerance deviation* $\mathsf{F}_{\text{Interpolation}}(\tau)$ is positive using the computed zeroes; *any point* τ in (t_k, t_{k+1}) *satisfying this inequality is a* ***potential point to display precarious incident***.

 iv. **Determine** *the intervals into which the tolerance deviations*

$$\mathsf{F}_{\text{Interpolation}}(\tau) \text{ and } \mathsf{F}'_{\text{Interpolation}}(\tau)$$

 are negative and positive, respectively; *any point* τ *in* (t_k, t_{k+1}) *satisfying these inequalities is a* ***potential point to display a dangerous incident*** .

 v. **Determine** *the intervals into which* $\mathsf{F}'_{\text{Interpolation}}(\tau) < 0$; *any point* τ *in* (t_k, t_{k+1}) *satisfying this inequality is a* ***potential point to display a dangerous incident***.

5. Else

 i. **Determine** *the sets* $\tilde{\mathbb{P}}$ *and* $\mathbb{P}$ *of all points* $\tilde{\tau}$. $\tau \in (t_k, t_{k+1})$ satisfying

$$\text{dist}\left(\tilde{r}(\tilde{\tau}), H_M(\tilde{\tau})\right) < \text{dist}\left(r(\tilde{\tau}), H_M(\tilde{\tau})\right) \text{ and}$$

$$\text{dist}\left(\tilde{r}(\tau), H_M(\tau)\right) > \text{dist}\left(r(\tau), H_M(\tau)\right).$$

ii. **Compute** *the tolerance deviations*

$$\tilde{\mathsf{G}}_{\text{Interpolation}}(\tilde{\tau}),\ \tilde{\mathsf{G}}'_{\text{Interpolation}}(\tilde{\tau}),$$
$$\mathsf{G}_{\text{Interpolation}}(\tau) \textit{ and } \mathsf{G}'_{\text{Interpolation}}(\tau).$$

iii. **Determine** *the intervals* $(\tilde{c}_i, \tilde{d}_i)$, $(\tilde{c}'_i, \tilde{d}'_i)$, (c_i, d_i), $(c'_i, d'_i) \subset (t_k, t_{k+1})$ *into which the tolerance deviations* $\tilde{\mathsf{G}}_{\text{Interpolation}}(\tilde{\tau})$, $\tilde{\mathsf{G}}'_{\text{Interpolation}}(\tilde{\tau})$, $\mathsf{G}_{\text{Interpolation}}(\tau)$ *and* $\mathsf{G}'_{\text{Interpolation}}(\tau)$ are changing sign, respectively;

For every interval $(\tilde{c}_i, \tilde{d}_i)$, $(\tilde{c}'_i, \tilde{d}'_i)$, (c_i, d_i), $(c'_i, d'_i) \subset (t_k, t_{k+1})$,

Apply Bisection method for approaching zeroes of $\tilde{\mathsf{G}}_{\text{Interpolation}}(\tilde{\tau})$ in $(\tilde{c}_i, \tilde{d}_i)$, zeroes of $\tilde{\mathsf{G}}'_{\text{Interpolation}}(\tilde{\tau})$ in $(\tilde{c}'_i, \tilde{d}'_i)$, zeroes of $\mathsf{G}_{\text{Interpolation}}(\tau)$ in (c_i, d_i) and zeroes of $\mathsf{G}'_{\text{Interpolation}}(\tau)$ in (c'_i, d'_i);

Apply Newton's method for computing zeroes of $\tilde{\mathsf{G}}_{\text{Interpolation}}(\tilde{\tau})$ in $(\tilde{c}_i, \tilde{d}_i)$, zeroes of $\tilde{\mathsf{G}}'_{\text{Interpolation}}(\tilde{\tau})$ in $(\tilde{c}'_i, \tilde{d}'_i)$, zeroes of $\mathsf{G}_{\text{Interpolation}}(\tau)$ in (c_i, d_i) and zeroes of $\mathsf{G}'_{\text{Interpolation}}(\tau)$ in (c'_i, d'_i).

iv. **Determine**

a. the intervals $I \subset \tilde{\mathbb{P}}$ into which $\tilde{\mathsf{G}}_{\text{Interpolation}}(\tilde{\tau}) > 0$; *any point* $\tilde{\tau} \in \tilde{\mathbb{P}}$ *satisfying this inequality is a* ***potential point to display precarious incident, because of low performance or sub-sufficiency***;

b. the intervals $I \subset \tilde{\mathbb{P}}$ into which $\mathsf{G}_{\text{Interpolation}}(\tau) > 0$; *any point* $\tau \in \mathbb{P}$ *satisfying this inequality is a* ***potential point to display precarious incident, because of high performance or ultra-sufficiency***.

v. **Determine**

a. in $\tilde{\mathbb{P}}$ the intervals in which $\tilde{\mathsf{G}}_{\text{Interpolation}}(\tilde{\tau}) < 0$ and $\tilde{\mathsf{G}}'_{\text{Interpolation}}(\tilde{\tau}) > 0$; *any point* $\tilde{\tau} \in (t_k, t_{k+1})$ *satisfying these inequalities is a* ***potential point to display dangerous incident, because of low performance or sub-sufficiency***;

b. in $\mathbb{P}$ the intervals in which $\mathsf{G}_{\text{Interpolation}}(\tau) < 0$ and $\mathsf{G}'_{\text{Interpolation}}(\tau) > 0$; *any point* $\tau \in (t_k, t_{k+1})$ *satisfying this inequality is a* ***potential point to display dangerous incident, because of high performance or ultra-sufficiency***;

vi. **Determine**

a. the intervals $I \subset \tilde{\mathbb{P}}$ into which $\tilde{\mathsf{G}}'_{\text{Interpolation}}(\tilde{\tau}) < 0$; *any point* $\tilde{\tau} \in (t_k, t_{k+1})$ *satisfying this inequality is a* ***potential point to display disastrous incident, because of low performance or sub-sufficiency***;

b. the intervals $I \subset \tilde{\mathbb{P}}$ into which $\mathsf{G}'_{\text{Interpolation}}(\tau) < 0$; *any point* $\tau \in (t_k, t_{k+1})$ *satisfying this inequality is a* ***potential point to display disastrous incident, because of high performance or ultra-sufficiency***.

The Least Squares Polynomial Approximation Method

The third method that we use is the least squares polynomial approximation [12, 16]. The advantages of the method are the united formula for the whole interval, and the degree of the computed polynomial is selected from the user, thus it can be small. It can be used even if the number of the known points is large. The disadvantage is that the computed polynomial is not interpolating all (or any of) the given points.

The method consists in the following thinking. Given $(M+1)$ points

$$(t_i, f_j(t_i, x_0, y_0, z_0))$$

and a degree m, $m < M+1$, we will find an optimal polynomial $p_M^{(j)}(t)$ of degree m that minimizes the 2-norm of the distance of $p_M^{(j)}(t)$ from the given $(M+1)$ points. In order to evaluate the minimization of the 2-norm, we use the QR factorization and the fact that $\|Q\|_2 = 1$, since Q is an orthogonal matrix.

Having regard to the above considerations, in Algorithm 3 below, we will use the following remarks.

1. For every point $(t_i, f_j(t_i, x_0, y_0, z_0))$, we put

$$f_j(t_i, x_0, y_0, z_0) = a_m^{(j)} t_i^m + a_{m-1}^{(j)} t_i^{m-1} + \cdots + a_1^{(j)} t_i + a_0^{(j)}. \quad (1)$$

2. Let A be an $\lambda \times m$ matrix with $\lambda > m$, b a vector of length λ, and suppose that we want to minimize the 2-norm of $At - b$. The QR factorization of A has the following form:

$$R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix},$$

where R_1 is an $m \times m$ upper triangular matrix. Thus, $A = QR$, and for minimizing the 2-norm of $At - b$, it holds

$$\|At - b\|_2 = \|QRt - b\|_2 = \left\|Q^T\right\|_2 \|QRt - b\|_2 = \left\|Q^T QRt - Q^T b\right\|_2$$

$$\left\|Rt - Q^T b\right\|_2 = \left\|\begin{bmatrix} R_1 \\ 0 \end{bmatrix} t - \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right\|_2 = \left\|\begin{bmatrix} R_1 t - c_1 \\ -c_2, \end{bmatrix}\right\|_2$$

where

$$Q^T b = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \text{ with } length\,(c_1) = m \text{ and } length\,(c_2) = \lambda - m.$$

Since c_2 is constant, the norm is minimized when $R_1 t - c_1 = 0$; thus, in order to minimize the initial norm, we have to solve the liner system

$$R_1 t = c_1 \tag{2}$$

The algorithm evaluating the previous technique is the following.

Algorithm 3:
Deterministic Prediction Using Least Squares

Input: - the interpolation points $(t_\nu, f_j(t_\nu))$;
- the k measurement points;
- the $M-k$ regularity points.

Output: - the zeroes of the functions

$$\begin{aligned}
\mathsf{F}_{\text{LeastSquares}}(\tau) &= ||\delta_{\text{critical}}(\tau)|| - \mathcal{V}(\tau),\\
\mathsf{F}'_{\text{LeastSquares}}(\tau) &= ||\epsilon_{\text{critical}}(\tau)|| - \mathcal{V}(\tau),\\
\tilde{\mathsf{G}}_{\text{LeastSquares}}(\tilde{\tau}) &= ||\tilde{\delta}_{\text{critical}}(\tilde{\tau})|| - \mathcal{V}_*(\tilde{\tau}),\\
\tilde{\mathsf{G}}'_{\text{LeastSquares}}(\tilde{\tau}) &= ||\tilde{\epsilon}_{\text{critical}}(\tilde{\tau})|| - \mathcal{V}_*(\tilde{\tau}),\\
\mathsf{G}_{\text{LeastSquares}}(\tau) &= ||\delta_{\text{critical}}(\tau)|| - \mathcal{V}^*(\tau),\\
\mathsf{G}'_{\text{LeastSquares}}(\tau) &= ||\epsilon_{\text{critical}}(\tau)|| - \mathcal{V}^*(\tau)
\end{aligned}$$

in a given interval (t_k, t_{k+1});

- the intervals into which the following inequalities are satisfied:

$$\begin{aligned}
\mathsf{F}_{\text{LeastSquares}}(\tau) &< 0,\\
\mathsf{F}'_{\text{LeastSquares}}(\tau) &< 0,\\
\tilde{\mathsf{G}}_{\text{LeastSquares}}(\tilde{\tau}) &< 0,\\
\tilde{\mathsf{G}}'_{\text{LeastSquares}}(\tilde{\tau}) &< 0,\\
\mathsf{G}_{\text{LeastSquares}}(\tau) &< 0,\\
\mathsf{G}'_{\text{LeastSquares}}(\tau) &< 0.
\end{aligned}$$

1. **For each** $j = 1, 2, \ldots, \ell + 1$,
   form *the linear system*

$$T * a^{(j)} = f_j(T)$$

$$\left(\Leftrightarrow \underbrace{\begin{pmatrix} t_M^m \; t_M^{m-1} & \cdots & t_M \; 1 \\ \vdots \; \vdots & \cdots & \vdots \; \vdots \\ t_0^m \; t_0^{m-1} & \cdots & t_0 \; 1 \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} a_m^{(j)} \\ \vdots \\ a_0^{(j)} \end{pmatrix}}_{a^{(j)}} = \underbrace{\begin{pmatrix} f_j(t_M, x_0, y_0, z_0) \\ \vdots \\ f_j(t_0, x_0, y_0, z_0) \end{pmatrix}}_{f_j(T)} \right)$$

   resulting from equation (1).
   Apply the **QR factorization** to T**:** $[\mathbf{Q}, \mathbf{R}] = \mathbf{q}r(\mathbf{T})$
   Compute the coefficients $a_i^{(j)}$, $i = m, m-1, \ldots, 0$ of the polynomial $p_M^{(j)}(t)$ by **solving** the linear system

$$R_1 a^{(j)} = c_1$$

resulting from (2) applying the **LU factorization** with partial pivoting.

2. **Construct** the *curve*

$$H_M(t) \equiv P_M\,(t)$$

with

$$P_M : \mathbb{R} \to \mathbb{R}^{\ell+1} : t \mapsto P_M\,(t) := \left(t,\, P_M^{(1)}\,(t)\,, \ldots,\, P_M^{(\ell+1)}\,(t)\right).$$

3. **Choose** four *critical tolerance functions*

$$\tilde{\delta}_{\text{critical}}(t) = \left(\tilde{\delta}^{(1)}_{\text{critical}}(t), \ldots, \tilde{\delta}^{(\ell+1)}_{\text{critical}}(t)\right),$$

$$\tilde{\epsilon}_{\text{critical}}(t) = \left(\tilde{\epsilon}^{(1)}_{\text{critical}}(t), \ldots, \tilde{\epsilon}^{(\ell+1)}_{\text{critical}}(t)\right),$$

$$\delta_{\text{critical}}(t) = \left(\delta^{(1)}_{\text{critical}}(t), \ldots, \delta^{(\ell+1)}_{\text{critical}}(t)\right) \text{ and}$$

$$\epsilon_{\text{critical}}(t) = \left(\epsilon^{(1)}_{\text{critical}}(t), \ldots, \epsilon^{(\ell+1)}_{\text{critical}}(t)\right)$$

representing distances from the lowest and highest thresholds outside of which the structure of regularity ceases to exist.

4. If $\tilde{r}_S = r_S$, then

   i. **Compute** *the tolerance deviations*

$$\mathsf{F}_{\text{LeastSquares}}(\tau) \text{ and } \mathsf{F}'_{\text{LeastSquares}}(\tau)$$

   in the given interval (t_k, t_{k+1}).

   ii. **Determine** *the intervals* (α_i, β_i), $(\alpha'_i, \beta'_i) \subset (t_k, t_{k+1})$ into which the tolerance deviations $\mathsf{F}_{\text{LeastSquares}}(\tau)$ and $\mathsf{F}'_{\text{LeastSquares}}(\tau)$ are changing sign, respectively;

   For every interval (α_i, β_i) and (α'_i, β'_i),

   Apply Bisection method for approaching zeroes of $\mathsf{F}_{\text{LeastSquares}}(\tau)$ in (α_i, β_i) and zeroes of $\mathsf{F}'_{\text{LeastSquares}}(\tau)$ in (α'_i, β'_i);

   Apply Newton's method for computing zeroes of $\mathsf{F}_{\text{LeastSquares}}(\tau)$ in (α_i, β_i) and zeroes of $\mathsf{F}'_{\text{LeastSquares}}(\tau)$ in (α'_i, β'_i);

   iii. **Determine** *the intervals into which the tolerance deviation* $\mathsf{F}_{\text{LeastSquares}}(\tau)$ is positive using the computed zeroes; *any point* τ in (t_k, t_{k+1}) *satisfying this inequality is a* ***potential point to display precarious incident***.

   iv. **Determine** *the intervals into which the tolerance deviations*

$$\mathsf{F}_{\text{LeastSquares}}(\tau) \text{ and } \mathsf{F}'_{\text{LeastSquares}}(\tau)$$

are negative and positive, respectively; *any point* τ *in* (t_k, t_{k+1}) *satisfying these inequalities is a* ***potential point to display a dangerous incident***.

v. **Determine** *the intervals into which* $\mathsf{F}'_{\text{LeastSquares}}(\tau) < 0$; *any point* τ *in* (t_k, t_{k+1}) *satisfying this inequality is a* ***potential point to display a dangerous incident***.

5. Else

i. **Determine** *the sets* $\tilde{\mathbb{P}}$ *and* $\mathbb{P}$ *of all points* $\tilde{\tau}$. $\tau \in (t_k, t_{k+1})$ satisfying

$$\operatorname{dist}(\tilde{r}(\tilde{\tau}), P_M(\tilde{\tau})) < \operatorname{dist}(r(\tilde{\tau}), P_M(\tilde{\tau})) \text{ and}$$
$$\operatorname{dist}(\tilde{r}(\tau), P_M(\tau)) > \operatorname{dist}(r(\tau), P_M(\tau)).$$

ii. **Compute** *the tolerance deviations*

$$\tilde{\mathsf{G}}_{\text{LeastSquares}}(\tilde{\tau}), \tilde{\mathsf{G}}'_{\text{LeastSquares}}(\tilde{\tau}),$$
$$\mathsf{G}_{\text{LeastSquares}}(\tau) \textit{ and } \mathsf{G}'_{\text{LeastSquares}}(\tau).$$

iii. **Determine** *the intervals* $(\tilde{c}_i, \tilde{d}_i)$, $(\tilde{c}'_i, \tilde{d}'_i)$, (c_i, d_i), $(c'_i, d'_i) \subset (t_k, t_{k+1})$ *into which the four respective tolerance deviations*

$$\tilde{\mathsf{G}}_{\text{LeastSquares}}(\tilde{\tau}), \tilde{\mathsf{G}}'_{\text{LeastSquares}}(\tilde{\tau}),$$
$$\mathsf{G}_{\text{LeastSquares}}(\tau), \mathsf{G}'_{\text{LeastSquares}}(\tau)$$

are changing sign;

For every interval $(\tilde{c}_i, \tilde{d}_i)$, $(\tilde{c}'_i, \tilde{d}'_i)$, (c_i, d_i), $(c'_i, d'_i) \subset (t_k, t_{k+1})$,

Apply Bisection method for approaching zeroes of $\tilde{\mathsf{G}}_{\text{LeastSquares}}(\tilde{\tau})$ in $(\tilde{c}_i, \tilde{d}_i)$, zeroes of $\tilde{\mathsf{G}}'_{\text{LeastSquares}}(\tilde{\tau})$ in $(\tilde{c}'_i, \tilde{d}'_i)$, zeroes of $\mathsf{G}_{\text{LeastSquares}}(\tau)$ in (c_i, d_i) and zeroes of $\mathsf{G}'_{\text{LeastSquares}}(\tau)$ in (c'_i, d'_i);

Apply Newton's method for computing zeroes of $\tilde{\mathsf{G}}_{\text{LeastSquares}}(\tilde{\tau})$ in $(\tilde{c}_i, \tilde{d}_i)$, zeroes of $\tilde{\mathsf{G}}'_{\text{LeastSquares}}(\tilde{\tau})$ in $(\tilde{c}'_i, \tilde{d}'_i)$, zeroes of $\mathsf{G}_{\text{LeastSquares}}(\tau)$ in (c_i, d_i) and zeroes of $\mathsf{G}'_{\text{LeastSquares}}(\tau)$ in (c'_i, d'_i).

iv. **Determine**

a. the intervals $I \subset \tilde{\mathbb{P}}$ into which $\tilde{\mathsf{G}}_{\text{LeastSquares}}(\tilde{\tau}) > 0$; *any point* $\tilde{\tau} \in \tilde{\mathbb{P}}$ *satisfying this inequality is a* ***potential point to display precarious incident, because of low performance or sub-sufficiency***;

b. the intervals $I \subset \tilde{\mathbb{P}}$ into which $\mathsf{G}_{\text{LeastSquares}}(\tau) > 0$; *any point* $\tau \in \mathbb{P}$ *satisfying this inequality is a* ***potential point to display precarious incident, because of high performance or ultra-sufficiency***.

v. **Determine**

a. in $\tilde{\mathbb{P}}$ the intervals in which $\tilde{\mathsf{G}}_{\text{LeastSquares}}(\tilde{\tau}) < 0$ and $\tilde{\mathsf{G}}'_{\text{LeastSquares}}(\tilde{\tau}) > 0$; *any point* $\tilde{\tau} \in (t_k, t_{k+1})$ *satisfying these inequalities is a* ***potential point to display dangerous incident, because of low performance or sub-sufficiency***;

b. in $\mathbb{P}$ the intervals in which $\mathsf{G}_{\text{LeastSquares}}(\tau) < 0$ and $\mathsf{G}'_{\text{Interpolation}}(\tau) > 0$; *any point* $\tau \in (t_k, t_{k+1})$ *satisfying this inequality is a* ***potential point to display dangerous incident, because of high performance or ultra-sufficiency***;

vi. **Determine**

a. the intervals $I \subset \tilde{\mathbb{P}}$ into which $\tilde{\mathsf{G}}'_{\text{LeastSquares}}(\tilde{\tau}) < 0$; *any point* $\tilde{\tau} \in (t_k, t_{k+1})$ *satisfying this inequality is a* ***potential point to display disastrous incident, because of low performance or sub-sufficiency***;

b. the intervals $I \subset \tilde{\mathbb{P}}$ into which $\mathsf{G}'_{\text{LeastSquares}}(\tau) < 0$; *any point* $\tau \in (t_k, t_{k+1})$ *satisfying this inequality is a* ***potential point to display disastrous incident, because of high performance or ultra-sufficiency***.

6 Numerical Results

In this section, we present some analytical numerical results evaluating the algorithms of Section 5, and we comment the behaviour of the algorithms.

Numerical Examples

Below, we present numerical examples, implementing our algorithms.
Let

$$H_M(t, x_0, y_0, z_0) = (H_M^{(1)}(t, x_0, y_0, z_0), H_M^{(2)}(t, x_0, y_0, z_0), H_M^{(3)}(t, x_0, y_0, z_0)),$$

where $H_M^{(1)}$ is a function that passes through the points

$$(t_1^{(1)}, w_1^{(1)}) = (0.227995363116753, 0.005887473341432)$$

$$(t_2^{(1)}, w_2^{(1)}) = (0.666935900301706, -0.002451487989435)$$

$$(t_3^{(1)}, w_3^{(1)}) = (1.088758378092684, -0.026536596177317)$$

$$(t_4^{(1)}, w_4^{(1)}) = (1.222118728436067, -0.083738292365262)$$

$$(t_5^{(1)}, w_5^{(1)}) = (1.919487917032162, 0.000000000000002),$$

$H_M^{(2)}$ a function that passes through the points

$$(t_1^{(2)},\ w_1^{(2)}) = (1.084431059927606, 0.001824976373948)$$

$$(t_2^{(2)},\ w_2^{(2)}) = (1.217471981917524, -0.008796025804148)$$

$$(t_3^{(2)},\ w_3^{(2)}) = (1.303998173014984, -0.005794329946252)$$

$$(t_4^{(2)},\ w_4^{(2)}) = (1.306656948627908, -0.005645884524371)$$

$$(t_5^{(2)},\ w_5^{(2)}) = (1.918582850410889, 0.000000000000004)$$

and $H_M^{(3)}$ a function that passes through the points

$$(t_1^{(3)},\ w_1^{(3)}) = (0.477049937449979, 0.003546169430236)$$

$$(t_2^{(3)},\ w_2^{(3)}) = (0.831405751301321, 0.037059412329365)$$

$$(t_3^{(3)},\ w_3^{(3)}) = (0.969342971854446, 0.076154493784762)$$

$$(t_4^{(3)},\ w_4^{(3)}) = (1.136154854287057, 0.100711757050127)$$

$$(t_5^{(3)},\ w_5^{(3)}) = (1.858527246374456, 0.000000000000001).$$

Let also $\mathfrak{D} = \left(t, x_0, y_0, z_0; g_S^{(1)}, g_S^{(2)}, g_S^{(3)}\right)$ be a system universality defined by

$$g_S^{(1)}(t, x_0, y_0, z_0) = \begin{cases} 0.3, & t \in [0, 1] \\ 1, & t \in (1, 2] \end{cases},$$

$$g_S^{(2)}(t, x_0, y_0, z_0) = \begin{cases} 5, & t \in [0, 0.5] \\ 4.8, & t \in (0.5, 1.5] \\ 4.5, & t \in (1.5, 2] \end{cases}$$

and

$$g_S^{(3)}(t, x_0, y_0, z_0) = 1, \quad t \in [0, 2]$$

Let us finally take

$$\epsilon_{\text{critical}} = 0.4.$$

Application of the Lagrange Interpolation Method

Applying the Lagrange interpolation method to the first five pairs of points, we construct the following polynomial of degree 4:

$$H_M^{(1)}(t, x_0, y_0, z_0) = t^4 - 3.834982837445468t^3 + 4.754550511049691t^2$$
$$-2.230278241661984t + 0.309978747075857.$$

Similarly, applying again the Lagrange interpolation method to the other two sets of pairs of points, we construct the following polynomials:

$$H_M^{(2)}(t, x_0, y_0, z_0) = t^4 - 6.192973860723464t^3 + 14.172259131310970t^2$$
$$-14.183279563983136t + 5.230995486528229$$

and

$$H_M^{(3)}(t, x_0, y_0, z_0) = t^4 - 4.674114367931685 - t^3 7.506905149871919t^2$$
$$-4.781612611847812t + 1.031873971027329.$$

<u>*In the interval* $[0, 0.5]$</u>:

$$g_S^{(1)}(t, x_0, y_0, z_0) = 0.3,\ g_S^{(2)}(t, x_0, y_0, z_0) = 5 \text{ and } g_S^{(3)}(t, x_0, y_0, z_0) = 1.$$

Thus,

$$\mathcal{D}(t^*) := \text{dist}\left(\mathfrak{D}\left(t^*, x_0, y_0, z_0\right), H_M\left(t^*, x_0, y_0, z_0\right)\right) - \epsilon_{\text{critical}} =$$

$$\Bigg(\left[H_M^1(t^*, x_0, y_0, z_0) - g_S^{(1)}(t^*, x_0, y_0, z_0)\right]^2$$
$$+\left[H_M^2(t^*, x_0, y_0, z_0) - g_S^{(2)}(t^*, x_0, y_0, z_0)\right]^2$$
$$+\left[H_M^3(t^*, x_0, y_0, z_0) - g_S^{(3}(t^*, x_0, y_0, z_0)\right]^2\Bigg)^{\frac{1}{2}}$$
$$- 0.4$$

$$= 10^2(0.030000000000000(t^*)^8 - 0.294041421322012(t^*)^7 + 1.277747933120760(t^*)^6$$
$$- 3.245707071889221(t^*)^5 + 5.178371259645941(t^*)^4 - 4.982519922531551(t^*)^3$$
$$+ 2.361242738501194(t^*)^2 - 0.069018758555689(t^*) + 0.00014474440218668).$$

Applying a few steps of Bisection and Newton's methods, we compute the only two real zeroes (t_1^*) and (t_2^*) of this polynomial:

$$t_1^* = 0.002273099702700 \text{ and } t_2^* = 0.028800172170581.$$

These two zeroes belong to [0, 0.5], and thus we may investigate in which subinterval, the distance $\text{dist}\left(\mathfrak{D}\left(t_i^*, x_0, y_0, z_0\right), H_M\left(t_i^*, x_0, y_0, z_0\right)\right)$ exceeds $\epsilon_{\text{critical}}$, by determining the sign of $\mathcal{D}(t_i^*) := \text{dist}\left(\mathfrak{D}\left(t_i^*, x_0, y_0, z_0\right), H_M\left(t_i^*, x_0, y_0, z_0\right)\right) - \epsilon_{\text{critical}}$. To do so, we observe that the computation of the middle of an interval $[a, b]$ is numerically more stable using the formula $a + (b - a)/2$ instead of $(a + b)/2$, so we have

$$\mathcal{D}\left(\frac{0 + t_1^*}{2}\right) = 0.006934396087571 > 0: \text{ exceeds}$$

$$\mathcal{D}\left(t_1^* + \frac{t_2^* - t_1^*}{2}\right) = 0.037415583826047 < 0: \text{ does not exceed}$$

$$\mathcal{D}\left(t_2^* + \frac{0.5 - t_2^*}{2}\right) = 7.639373481094819 > 0: \text{ exceeds.}$$

Thus, the subintervals of [0, 0.5] in which $\text{dist}\left(\mathfrak{D}\left(t^*, x_0, y_0, z_0\right), H_M\left(t^*, x_0, y_0, z_0\right)\right)$ exceeds $\epsilon_{\text{critical}}$ are

$$[0, t_1^*] = [0, 0.002273099702700] \text{ and } [t_2^*, 0.5] = [0.028800172170581, 0.5].$$

These results can be verified in the following graphs (Figure 4):
Similarly, we may proceed in the other intervals:

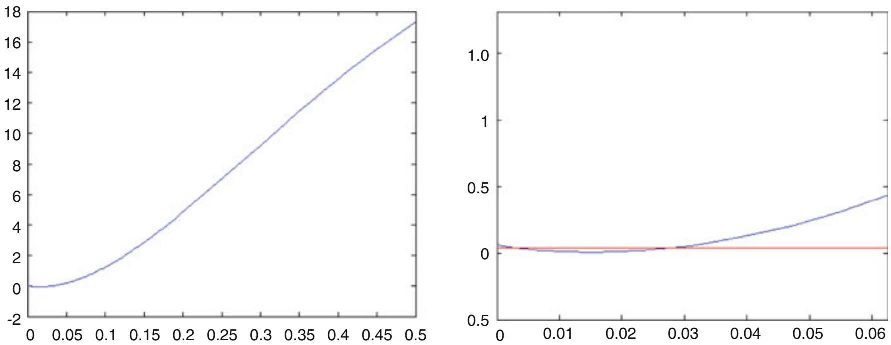

Fig. 4 Left. $\mathcal{D}(t_i^*) = \text{dist}\left(\mathfrak{D}\left(t_i^*, x_0, y_0, z_0\right), H_M\left(t_i^*, x_0, y_0, z_0\right)\right) - \epsilon_{\text{critical}}$ Right. Blue: $\mathcal{D}(t_i^*) = \text{dist}\left(\mathfrak{D}\left(t_i^*, x_0, y_0, z_0\right), H_M\left(t_i^*, x_0, y_0, z_0\right)\right)$, Red: $\epsilon_{\text{critical}}$

<u>*In the interval* [0.5, 1]:</u>

$$g_S^{(1)}(t, x_0, y_0, z_0) = 0.3,\ g_S^{(2)}(t, x_0, y_0, z_0) = 4.8 \text{ and } g_S^{(3)}(t, x_0, y_0, z_0) = 1.$$

Thus,

$$\mathcal{D}(t^*) := \text{dist}\left(\mathfrak{D}\left(t^*, x_0, y_0, z_0\right), H_M\left(t^*, x_0, y_0, z_0\right)\right) - \epsilon_{\text{critical}} =$$

$$\begin{aligned} = 10^2(&0.030000000000000(t^*)^8 - 0.294041421322012(t^*)^7 + 1.277747933120760(t^*)^6 \\ &- 3.245707071889221(t^*)^5 + 5.182371259645940(t^*)^4 - 5.007291817974445(t^*)^3 \\ &+ 2.417931775026438(t^*)^2 - 0.125751876811622(t^*) + 0.00146872634829960). \end{aligned}$$

This polynomial has no real roots in [0.5, 1]. Thus, $\mathcal{D}$ does not change sign in [0.5, 1]. We check the sign of $\mathcal{D}$ in [0.5, 1]:

$$dist\left(0.5+\frac{1-0.5}{2}, x_0, y_0, z_0\right) = 21.546863660167102 > 0.$$

Thus, any point in the interval [0.5, 1] satisfies the inequality. This result is also verified from Figure 5.

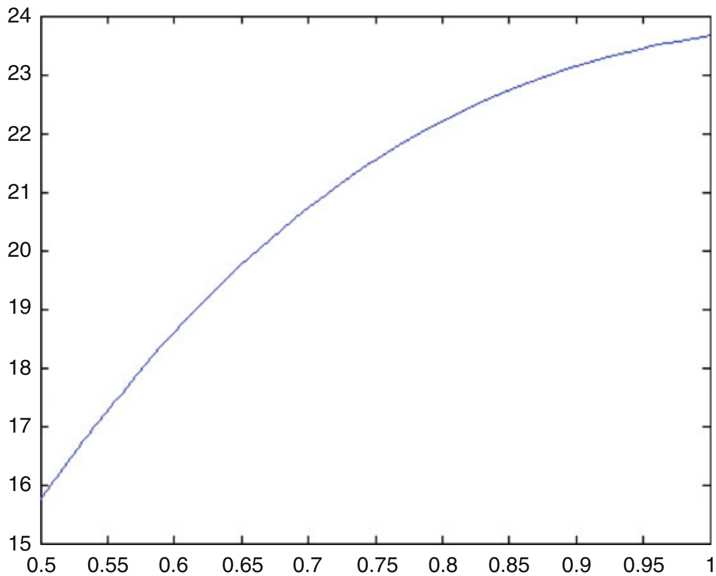

Fig. 5 $\mathcal{D}(t_i^*) = \text{dist}\left(\mathfrak{D}\left(t_i^*, x_0, y_0, z_0\right), H_M\left(t_i^*, x_0, y_0, z_0\right)\right) - \epsilon_{\text{critical}} = 0$, in [0.5, 1]

<u>*In the interval* [1, 1.5]</u>:

$$g_S^{(1)}(t, x_0, y_0, z_0) = 1, g_S^{(2)}(t, x_0, y_0, z_0) = 4.8 and g_S^{(3)}(t, x_0, y_0, z_0) = 1.$$

Thus,

$$\mathcal{D}(t^*) := \text{dist}\left(\mathfrak{D}\left(t^*, x_0, y_0, z_0\right), H_M\left(t^*, x_0, y_0, z_0\right)\right) - \epsilon_{critical} =$$

$$= 10^2(0.030000000000000(t^*)^8 - 0.294041421322012(t^*)^7 + 1.277747933120760(t^*)^6 - 3.245707071889221(t^*)^5 + 5.168371259645941(t^*)^4 - 4.953602058250208(t^*)^3 + 2.351368067871742(t^*)^2 - 0.094527981428354(t^*) + 0.006629023889238).$$

This polynomial has no real roots neither in interval [1, 1.5] nor in $\mathbb{R}$. Thus, $\mathcal{D}$ does not change sign in [1, 1.5]. We check the sign of $\mathcal{D}$ in [0.5, 1]:

$$dist\left(1 + \frac{1.5 - 1}{2}, x_0, y_0, z_0\right) = 25.099633861696560 > 0.$$

Thus, any point of the interval [1, 1.5] satisfies the inequality. This result is also verified from Figure 6.

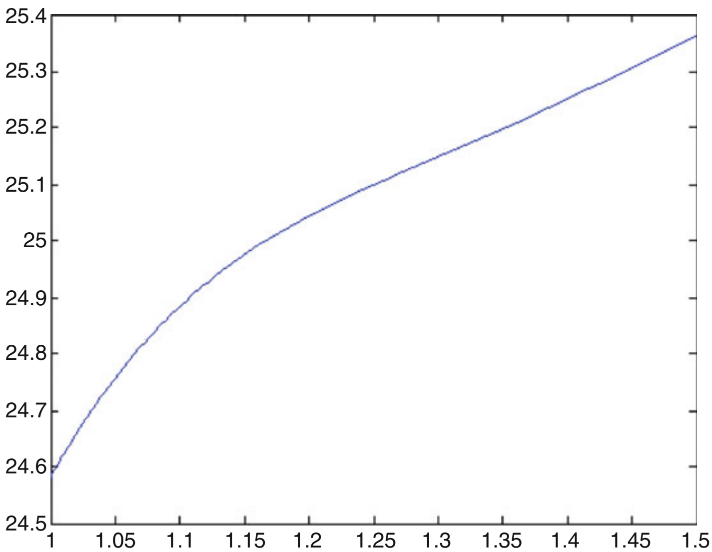

Fig. 6 $\mathcal{D}(t_i^*) = \text{dist}\left(\mathfrak{D}\left(t_i^*, x_0, y_0, z_0\right), H_M\left(t_i^*, x_0, y_0, z_0\right)\right) - \epsilon_{\text{critical}} = 0$, in [1, 1.5]

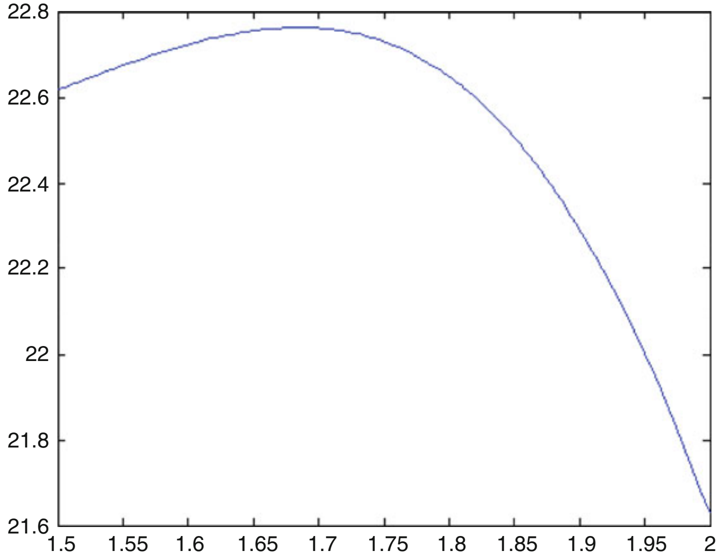

Fig. 7 $\mathcal{D}(t_i^*) = \text{dist}\left(\mathfrak{D}\left(t_i^*, x_0, y_0, z_0\right), H_M\left(t_i^*, x_0, y_0, z_0\right)\right) - \epsilon_{\text{critical}} = 0$, in $[1.5, 2]$

<u>*In the interval* $[1.5, 2]$</u> :

$$g_S^{(1)}(t, x_0, y_0, z_0) = 1,\ g_S^{(2)}(t, x_0, y_0, z_0) = 4.8 \text{ and } g_S^{(3)}(t, x_0, y_0, z_0) = 1.$$

Thus,

$$\mathcal{D}(t^*) := \text{dist}\left(\mathfrak{D}\left(t^*, x_0, y_0, z_0\right), H_M\left(t^*, x_0, y_0, z_0\right)\right) - \epsilon_{\text{critical}} =$$

$$\begin{aligned} = 10^2(&0.030000000000000(t^*)^8 - 0.294041421322012(t^*)^7 + 1.277747933120760(t^*)^6 \\ &- 3.245707071889221(t^*)^5 + 5.168371259645941(t^*)^4 - 4.990759901414549(t^*)^3 \\ &+ 2.436401622659608(t^*)^2 - 0.179627658812253(t^*) + 0.00971499680840698). \end{aligned}$$

This polynomial has no real roots neither in interval $[1.5, 2]$ nor in $\mathbb{R}$ Thus, $\mathcal{D}$ does not change sign in $[1.5, 2]$. We check the sign of the previous function in $[1.5, 2]$:

$$dist\left(1.5 + \frac{2 - 1.5}{2}, x_0, y_0, z_0\right) = 22.689534856364734 > 0.$$

Thus, any point of the interval $[1.5, 2]$ satisfies the inequality. This result is also verified from Figure 7.

A special case: *Measurement in the Chebyshev Points of* [0, 2]
Assume that the measurements have been done in the Chebyshev points, properly transformed for the interval [0, 2]:

$$1.951056516295154,\ 1.587785252292473,\ 1.000000000000000,$$

$$0.412214747707527 \text{ and } 0.048943483704846.$$

Thus,
$H_M^{(1)}$ is a function that passes through the points

$$(t_1^{(1)},\ w_1^{(1)}) = (1.951056516295154, 0.065558230683740),$$

$$(t_2^{(1)},\ w_2^{(1)}) = (1.587785252292473, -0.240026117055866),$$

$$(t_3^{(1)},\ w_3^{(1)}) = (1.000000000000000, -0.000731820981905),$$

$$(t_4^{(1)},\ w_4^{(1)}) = (0.412214747707527, -0.041221034220753),$$

$$(t_5^{(1)},\ w_5^{(1)}) = (0.048943483704846, 0.211766633448476),$$

$H_M^{(2)}$ a function that passes through the points

$$(t_1^{(2)},\ w_1^{(2)}) = (1.951056516295154, 0.002603154846258),$$

$$(t_2^{(2)},\ w_2^{(2)}) = (1.587785252292473, 0.006004566492993),$$

$$(t_3^{(2)},\ w_3^{(2)}) = (1.000000000000000, 0.027001193132598),$$

$$(t_4^{(2)},\ w_4^{(2)}) = (1.306656948627908, 1.387695745483932),$$

$$(t_5^{(2)},\ w_5^{(2)}) = (1.918582850410889, 4.570045178558653),$$

and $H_M^{(3)}$ a function that passes through the points

$$(t_1^{(3)},\ w_1^{(3)}) = (1.951056516295154, 0.054654055471936),$$

$$(t_2^{(3)},\ w_2^{(3)}) = (1.587785252292473, 0.010790868704285),$$

$$(t_3^{(3)},\ w_3^{(3)}) = (1.000000000000000, 0.083052141119750),$$

$$(t_4^{(3)},\ w_4^{(3)}) = (0.412214747707527, 0.037883303951495),$$

$$(t_5^{(3)},\ w_5^{(3)}) = (0.048943483704846, 0.815285451543443).$$

Evaluating the previous procedure (Lagrange interpolation) to the above Chebyshev points and solving the inequality of the relevant algorithm, we may conclude that the inequality is satisfied in the following intervals:

$$[0, 0.002273099702700], [0.028800172170581, 0.5], [0.5, 1], [1, 1.5] \text{ and } [1.5, 2].$$

The norm-2 of the difference of the results obtained from the algorithms is

$$1.314636333821229 \cdot 10^{-12}.$$

Application of the Least Squares Method

In case of many measurements, the use of polynomial interpolation will result to a polynomial of high degree, which means that it cannot be handled efficiently due to floating point errors and its increased computational complexity. The use of least squares concluding to a polynomial of manageable degree is more appropriate.

Supposing that we have 100 measurements (Lagrange interpolation would lead to a polynomial of degree 99!), we apply Algorithm 3, evaluating the least squares technique to derive a polynomial of degree 4. The intervals in which the resulting inequality is positive are

$$[0, 0.002273099702699], [0.028800172170581, 0.5], [0.5, 1], [1, 1.5] \text{ and } [1.5, 2].$$

Application of the Linear Splines Method

We used 500 sets of measured points in each interval (:[0,0.5], [0.5,1], [1,1.5], [1.5,2]) for every component of H.

Approximating H_1, H_2 and H_3 using linear splines and computing the intervals where the difference is positive, we conclude to the following result:

$$[0, 0.002257794228617], [0.029015423603700, 0.5], [0.5, 1], [1, 1.5] \text{ and } [1.5, 2]$$

The 2-norm error with the other methods is

$$2.15 \cdot 10^{-4}.$$

Using 50 instead of 500 measurements in each interval, for every component of H, the error is similar:

$$2.31 \cdot 10^{-4},$$

but the decrease of the computational time is too significant.

An interesting task is the case that there are some measurement errors in the initial points.

In this case, the most appropriate method is the least squares one, which minimizes the 2-norm of the system $At = b$, analysed in Subsection 5. Suppose that there are errors in measured y_is of order of $O(10^{-3})$. Evaluating Algorithm 3 to 100 points, for the functions defined in our example, we get the following intervals:

$$[0, 0.001903961570183], [0.028255098676371, 0.5], [0.5, 1], [1, 1.5] \text{ and } [1.5, 2].$$

The absolute error in the first interval is $3.691381325159999 \cdot 10^{-4}$ and in the second is $5.450734942099994 \cdot 10^{-4}$ for measurement errors of order of 10^{-3} in each measured point.

Applying Lagrange's interpolation, the corresponding algorithm fails to compute the roots of the final distance, and thus the sign of the final computed polynomial does not change and it is positive for the whole initial interval. Thus, the inequality holds for every t in $[0, 2]$, which is not correct.

If the measured points are the Chebyshev ones, then the result is quite close to the real ones:

$$[0, 0.001447041826483], [0.027872115109573, 0.5], [0.5, 1], [1, 1.5] \text{ and } [1.5, 2].$$

For measurement errors of order of 10^{-3} in each measured point, the absolute errors are $8.260578762160000 \cdot 10^{-4}$ in the first interval and $9.280570610080002 \cdot 10^{-4}$ in the second interval.

Comparison of Algorithms

Comparing the algorithms, we may conclude the following results:

Method	Number of measured points	Efficiency	Proposal	Computational complexity
Linear splines	Many	High	Proposed	Low
Linear splines	Few	Low	Not proposed	Low
Langrange	Many	Low	Not proposed	Too high
Lagrange	Few	Good	Proposed	Low
Chebyshev	Many	High	Not proposed	Too high
Chebyshev	Few	Good	Proposed	Low
Least squares	Many	High	Proposed	Low[a]
Least squares	Few	Good	Proposed	Low[a]

[a]Low complexity for polynomials of low degree

According to this table, we infer that Lagrange and Chebyshev's interpolation methods are not proposed in case of many measured points, since the polynomial that is computed is of high degree, causing instability issues due to floating point operations and resulting to high computation complexity. The second one can be evaluated in case that there is the opportunity the measurements to be carried out at Chebyshev points. In case of few measured points, the behaviour of both methods is good, and thus they are proposed.

Least squares method is proposed for the case that there are many measured points or there are measurement errors. The method is efficient, and the computational complexity is low for polynomials of low degree.

Finally, linear splines method is proposed in case of many measured points. The computational complexity is low and the computation of the roots of the inequality of the relevant algorithm is stable, since the splines are polynomials of degree 1. Also, linear splines require only the continuity of the function in the interpolating points. In case of few measured points, the method is not proposed.

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Accurate Approximations of the Weighted Exponential Beta Function

Silvestru Sever Dragomir and Farzad Khosrowshahi

Abstract In this chapter, we provide several error bounds in approximating the *Weighted Exponential Beta* function

$$F(\alpha, \beta; \gamma) := \int_0^1 \exp\left[\gamma x^\alpha (1-x)^\beta\right] dx,$$

where α, β and γ are positive numbers, with some simple quadrature rules of Beta-Taylor, Ostrowski and Trapezoid type.

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1 Introduction

Both contractor and subcontractors' failure to meet the liabilities to the suppliers and financial institutions can force an otherwise successful organization into liquidity, which is the ultimate cause of insolvency (Davis 1999, [1]). The mishaps tend to cause damaging impact during both depressed and buoyant economic situations. These are manifested in cash flow failures by overtrading in boom periods and income constraints of recessed periods. Construction project expenditure patterns tend to display growth behaviour and cumulatively take the familiar 'S' curve. The expenditure pattern of construction projects is typically represented by exponential curves where the rate of growth is proportional to the state of the growth, and

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each value represents a constant percentage of the neighbouring value. While the traditional methods require extensive knowledge about the project and project plan, the mathematical approaches are somewhat alienating to the user, as the logic of the forecast is embedded within the data underpinning the model. The traditional models typically generate a forecast, which is then depicted graphically. An alternative approach to forecasting project expenditure has been proposed by Khosrowshahi [12]. The method takes a reverse approach to the traditional methods. Instead of forecasting the expenditure values, the method reconstructs the likely shape of the expenditure pattern and then converts the shape into figure. The shape of the periodic project expenditure profile embodies characteristics associated with the physical properties of the project. These shape criteria consist of the following general and specific characteristics:

General Characteristics. These characteristics apply to all projects.

- Negation of negative values
- Periodic values are discrete and form a pattern
- The baseline periodic pattern is a two-phased monotonic curve monotonically increasing towards a peak and monotonically decreasing towards the end.
- The commencing and the terminating final values are both zero.

  Specific Characteristics. These are characteristics that define the specifics of each project.
- The position of the peak point on the time and the cost axes.
- The intensity of expenditure from the start to the peak point.
- The distortion of the underlying pattern causing acceleration or retardation resulting in the generation of additional peaks and troughs.

Therefore, the shape of the expenditure pattern is defined in terms of these variables. The role of the mathematical model is to generate a pattern converted by transforming the shape variables into a graphical pattern.

Extensive analysis of project expenditure patterns has revealed that the main features of the shape of the project periodic expenditure pattern are defined in terms of a number of variables represented by the following expression (see [11]):

$$Y_C := \exp\left[bx^a(1-x)^d\right] - 1,$$

where

$$x_p := R = \frac{a}{a+d} \text{ and } y_p := Q = \exp\left[bR^a(1-R)^d\right] - 1,$$

where

- Q and R, represent the positions of the project expenditure peak on both the cost and time axes.
- a and b are parameterized in terms of x_p and y_p as follows:

$$a = \frac{x_p d}{1 - x_p}, \; b = \frac{\ln\left(1 + y_p\right)}{x_p^a \left(1 - x_p\right)^d}.$$

– Parameter d is calculated through numerical method that is derived to rapidly converge towards a solution within desired error tolerance.

A relationship is established between the properties of the project and the physical shape of the project expenditure pattern. These are then related and reflected on the mathematical expression through its parameters.

Motivated by the above considerations, in this paper we introduce the three-parameter family of functions

$$f_{\alpha,\beta,\gamma}(x) := \exp\left[\gamma x^{\alpha}(1-x)^{\beta}\right], \; x \in [0,1], \; \alpha, \beta, \gamma > 0$$

and the "*Weighted Exponential Beta*" function defined by the integral

$$F(\alpha,\beta;\gamma) := \int_0^1 f_{\alpha,\beta,\gamma}(x)\,dx = \int_0^1 \exp\left[\gamma x^{\alpha}(1-x)^{\beta}\right]dx, \; \alpha, \; \beta, \; \gamma > 0.$$

In the following, by making use of Theory of Inequalities, we provide several error bounds in accurately approximating the *Weighted Exponential Beta* function with some simple quadrature rules of Beta-Taylor, Ostrowski and Trapezoid type.

2 Basic Facts on the Generating Function $f_{\alpha,\beta,\gamma}$

In Mathematics, the *Beta function*, also called the *Euler integral of the first kind*, is a special function defined by

$$B(\alpha,\beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx, \; \alpha > 0, \; \beta > 0. \tag{2.1}$$

The utility of the Beta function is often overshadowed by that of the *Gamma function*, partly perhaps because it can be evaluated in terms of the Gamma function. However, since it occurs so frequently in practice, a special designation for it is widely accepted.

We consider the three-parameter generating function $f_{\alpha,\beta,\gamma} : [0,1] \to [0,\infty)$,

$$f_{\alpha,\beta,\gamma}(x) := \exp\left[\gamma x^{\alpha}(1-x)^{\beta}\right],$$

where α, β and γ are positive constants. This family can be extended for negative numbers α and β by eliminating either ends of the closed interval $[0,1]$. However, we do not consider this case here.

Define the simpler two-parameter family that generates the Beta function, $g_{\alpha,\beta} : [0, 1] \to [0, \infty)$,

$$g_{\alpha,\beta}(x) = x^{\alpha}(1-x)^{\beta}, \tag{2.2}$$

where α and β are positive constants.

We start with the simple fact incorporated in the following:

Proposition 1 *Let* $\alpha, \beta, \gamma > 0$. *The function* $f_{\alpha,\beta,\gamma}$ *is increasing on* $\left[0, \frac{\alpha}{\alpha+\beta}\right]$, *decreasing on* $\left[\frac{\alpha}{\alpha+\beta}, 1\right]$, *and*

$$\max_{x\in[0,1]} f_{\alpha,\beta,\gamma}(x) = f_{\alpha,\beta,\gamma}\left(\frac{\alpha}{\alpha+\beta}\right) = \exp\left[\gamma\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right]. \tag{2.3}$$

Proof We have

$$f_{\alpha,\beta,\gamma}(x) = \exp\left[\gamma g_{\alpha,\beta}(x)\right]$$

and

$$f'_{\alpha,\beta,\gamma}(x) = \gamma g'_{\alpha,\beta}(x)\exp\left[\gamma g_{\alpha,\beta}(x)\right],\ x \in [0,1], \tag{2.4}$$

showing that the sign of $f'_{\alpha,\beta,\gamma}$ on $[0, 1]$ is the same with the one of $g'_{\alpha,\beta}$.

Furthermore, we have

$$\begin{aligned} g'_{\alpha,\beta}(x) &= \alpha x^{\alpha-1}(1-x)^{\beta} - \beta x^{\alpha}(1-x)^{\beta-1} = x^{\alpha-1}(1-x)^{\beta-1}[\alpha(1-x) - \beta x] \\ &= x^{\alpha-1}(1-x)^{\beta-1}[\alpha - (\alpha+\beta)x],\ x \in (0,1). \end{aligned}$$

This shows that $g'_{\alpha,\beta}(x) > 0$ for $x \in \left(0, \frac{\alpha}{\alpha+\beta}\right)$ and $g'_{\alpha,\beta}(x) < 0$ for $\left(\frac{\alpha}{\alpha+\beta}, 1\right)$, which proves the statement. □

We need the following lemma that is of interest in itself, see also [5]:

Lemma 1 *Let* $\alpha, \beta, \gamma > 0$.

(i) *If* $0 < \alpha + \beta \le 1$,, *then* $g_{\alpha,\beta}$ *is strictly concave on* $[0, 1]$.
Define

$$x_{1,\alpha,\beta} := \frac{\alpha(\alpha+\beta-1) - \sqrt{\alpha\beta(\alpha+\beta-1)}}{(\alpha+\beta)(\alpha+\beta-1)} < \frac{\alpha}{\alpha+\beta}$$

and

$$x_{2,\alpha,\beta} := \frac{\alpha(\alpha+\beta-1)+\sqrt{\alpha\beta(\alpha+\beta-1)}}{(\alpha+\beta)(\alpha+\beta-1)} > \frac{\alpha}{\alpha+\beta}.$$

(ii) *If* $\alpha, \beta \in (0,1)$ *with* $\alpha+\beta > 1$, *then* $g_{\alpha,\beta}$ *is strictly concave on* $[0,1]$.
(iii) *If* $\alpha > 1$ *and* $\beta \in (0,1)$ *then* $g_{\alpha,\beta}$ *is strictly convex on* $\left(0, x_{1,\alpha,\beta}\right)$ *and strictly concave on* $\left(x_{1,\alpha,\beta}, 1\right)$.
(iv) *If* $\alpha \in (0,1)$ *and* $\beta > 1$, *then* $g_{\alpha,\beta}$ *is strictly concave on* $\left(0, x_{2,\alpha,\beta}\right)$ *and strictly convex on* $\left(x_{2,\alpha,\beta}, 1\right)$.
(v) *If* $\alpha, \beta > 1$, *then* $g_{\alpha,\beta}$ *is strictly concave on* $\left(x_{1,\alpha,\beta}, x_{2,\alpha,\beta}\right)$ *and strictly convex on* $\left(0, x_{1,\alpha,\beta}\right) \cup \left(x_{2,\alpha,\beta}, 1\right)$.

Proof If we take the second derivative of $g_{\alpha,\beta}$ on $(0,1)$, then we get

$$\begin{aligned} g''_{\alpha,\beta}(x) &= \alpha(\alpha-1)x^{\alpha-2}(1-x)^{\beta} - \alpha\beta x^{\alpha-1}(1-x)^{\beta-1} \\ &\quad - \alpha\beta x^{\alpha-1}(1-x)^{\beta-1} + \beta(\beta-1)x^{\alpha}(1-x)^{\beta-2} \\ &= \alpha(\alpha-1)x^{\alpha-2}(1-x)^{\beta} - 2\alpha\beta x^{\alpha-1}(1-x)^{\beta-1} + \beta(\beta-1)x^{\alpha}(1-x)^{\beta-2} \\ &= x^{\alpha-2}(1-x)^{\beta-2}\left[\alpha(\alpha-1)(1-x)^2 - 2\alpha\beta x(1-x) + \beta(\beta-1)x^2\right] \end{aligned}$$

for all $\alpha, \beta > 0$ and $x \in (0,1)$.

Now, consider the two-parameter family of parabolas

$$h_{\alpha,\beta}(x) := \alpha(\alpha-1)(1-x)^2 - 2\alpha\beta x(1-x) + \beta(\beta-1)x^2,\ x \in \mathbb{R}.$$

We have

$$\begin{aligned} h_{\alpha,\beta}(x) &= \alpha(\alpha-1)\left(x^2-2x+1\right) - 2\alpha\beta\left(x-x^2\right) + \beta(\beta-1)x^2 \\ &= [\alpha(\alpha-1) + 2\alpha\beta + \beta(\beta-1)]x^2 - 2(\alpha(\alpha-1)+\alpha\beta)x + \alpha(\alpha-1) \\ &= \left[\alpha^2 + 2\alpha\beta + \beta^2 - (\alpha+\beta)\right]x^2 - 2\alpha(\alpha+\beta-1)x + \alpha(\alpha-1) \\ &= \left[(\alpha+\beta)^2 - (\alpha+\beta)\right]x^2 - 2\alpha(\alpha+\beta-1)x + \alpha(\alpha-1) \\ &= (\alpha+\beta)(\alpha+\beta-1)x^2 - 2\alpha(\alpha+\beta-1)x + \alpha(\alpha-1) \end{aligned}$$

for $x \in \mathbb{R}$.

The discriminant of this family of parabolas is

$$\begin{aligned} \Delta_{\alpha,\beta} &:= 4\alpha^2(\alpha+\beta-1)^2 - 4(\alpha+\beta)(\alpha+\beta-1)\alpha(\alpha-1) \\ &= 4\alpha(\alpha+\beta-1)[\alpha(\alpha+\beta-1) - (\alpha+\beta)(\alpha-1)] \\ &= 4\alpha(\alpha+\beta-1)\left(\alpha^2 + \alpha\beta - \alpha - \alpha^2 - \alpha\beta + \alpha + \beta\right) \end{aligned}$$

$$= 4\alpha\beta\,(\alpha + \beta - 1)$$

for $\alpha,\ \beta > 0$.

Now, if $0 < \alpha+\beta < 1$, then $\Delta_{\alpha,\beta} < 0$, which shows that the parabola $h_{\alpha,\beta}(x) < 0$ for all $x \in \mathbb{R}$, implying that $g''_{\alpha,\beta}(x) < 0$ for $x \in (0, 1)$, namely $g_{\alpha,\beta}$ is strictly concave on $[0, 1]$.

If $\alpha + \beta = 1$, then $h_{\alpha,\beta}(x) = \alpha(\alpha - 1) < 0$, namely $g_{\alpha,\beta}$ is strictly concave on $[0, 1]$.

If $\alpha + \beta > 1$ with $\alpha,\ \beta > 0$ then $\Delta_{\alpha,\beta} > 0$ and the parabola $h_{\alpha,\beta}(\cdot)$ has two distinct interceptions with the axis ox, namely

$$x_{1,\alpha,\beta} = \frac{\alpha(\alpha + \beta - 1) - \sqrt{\alpha\beta(\alpha + \beta - 1)}}{(\alpha + \beta)(\alpha + \beta - 1)}$$

and

$$x_{2,\alpha,\beta} = \frac{\alpha(\alpha + \beta - 1) + \sqrt{\alpha\beta(\alpha + \beta - 1)}}{(\alpha + \beta)(\alpha + \beta - 1)}.$$

The x coordinate for the vertex is

$$x_{V,\alpha,\beta} = \frac{\alpha}{\alpha + \beta} \in (0, 1)$$

for all $\alpha,\ \beta > 0$.

We also have $h_{\alpha,\beta}(0) = \alpha(\alpha - 1)$ and $h_{\alpha,\beta}(1) = \beta(\beta - 1)$.

Now, if $\alpha,\ \beta \in (0, 1)$ with $\alpha + \beta > 1$, then $x_{1,\alpha,\beta} < 0$ and $x_{2,\alpha,\beta} > 1$ showing that $h_{\alpha,\beta}(x) < 0$, namely $g_{\alpha,\beta}$ is strictly concave on $[0, 1]$.

If $\alpha > 1$ and $\beta \in (0, 1)$, then $\alpha + \beta > 1$, $x_{1,\alpha,\beta} \in \left(0, \frac{\alpha}{\alpha+\beta}\right)$, $x_{2,\alpha,\beta} > 1$, which shows that $h_{\alpha,\beta}(x) > 0$ for $x \in \left(0, x_{1,\alpha,\beta}\right)$ and $h_{\alpha,\beta}(x) < 0$ for $x \in \left(x_{1,\alpha,\beta}, 1\right)$ showing that $g_{\alpha,\beta}$ is strictly convex on $\left(0, x_{1,\alpha,\beta}\right)$ and strictly concave on $\left(x_{1,\alpha,\beta}, 1\right)$.

If $\alpha \in (0, 1)$ and $\beta > 1$, then $\alpha + \beta > 1$, $x_{1,\alpha,\beta} < 0$, $x_{2,\alpha,\beta} \in \left(\frac{\alpha}{\alpha+\beta}, 1\right)$, which shows that $h_{\alpha,\beta}(x) < 0$ for $x \in \left(0, x_{2,\alpha,\beta}\right)$ and $h_{\alpha,\beta}(x) > 0$ for $x \in \left(x_{2,\alpha,\beta}, 1\right)$ showing that $g_{\alpha,\beta}$ is strictly concave on $\left(0, x_{2,\alpha,\beta}\right)$ and strictly convex on $\left(x_{2,\alpha,\beta}, 1\right)$.

If $\alpha,\ \beta > 1$, then $x_{1,\alpha,\beta} \in \left(0, \frac{\alpha}{\alpha+\beta}\right)$ and $x_{2,\alpha,\beta} \in \left(\frac{\alpha}{\alpha+\beta}, 1\right)$, which shows that $h_{\alpha,\beta}(x) < 0$ for $\left(x_{1,\alpha,\beta}, x_{2,\alpha,\beta}\right)$ and $h_{\alpha,\beta}(x) > 0$ for $x \in \left(0, x_{1,\alpha,\beta}\right) \cup \left(x_{2,\alpha,\beta}, 1\right)$ showing that $g_{\alpha,\beta}$ is strictly concave on $\left(x_{1,\alpha,\beta}, x_{2,\alpha,\beta}\right)$ and strictly convex on $\left(0, x_{1,\alpha,\beta}\right) \cup \left(x_{2,\alpha,\beta}, 1\right)$. □

We can state the following fact concerning the logarithmic convexity of $f_{\alpha,\beta,\gamma}$.

Proposition 2 *Let $\alpha,\ \beta,\ \gamma > 0$. Define $x_{1,\alpha,\beta}$ and $x_{2,\alpha,\beta}$ as in Lemma 1.*

(1) *If* $\alpha, \beta \in (0,1)$, *then* $f_{\alpha,\beta,\gamma}(x)$ *is strictly log-concave on* $[0,1]$.
(2) *If* $\alpha > 1$ *and* $\beta \in (0,1)$, *then* $f_{\alpha,\beta,\gamma}(x)$ *is strictly log-convex on* $\left(0, x_{1,\alpha,\beta}\right)$ *and strictly log-concave on* $\left(x_{1,\alpha,\beta}, 1\right)$.
(3) *If* $\alpha \in (0,1)$ *and* $\beta > 1$, *then* $f_{\alpha,\beta,\gamma}(x)$ *is strictly log-concave on* $\left(0, x_{2,\alpha,\beta}\right)$ *and strictly log-convex on* $\left(x_{2,\alpha,\beta}, 1\right)$.
(4) *If* $\alpha, \beta > 1$, *then* $f_{\alpha,\beta,\gamma}(x)$ *is strictly log-concave on* $\left(x_{1,\alpha,\beta}, x_{2,\alpha,\beta}\right)$ *and strictly log-convex on* $\left(0, x_{1,\alpha,\beta}\right) \cup \left(x_{2,\alpha,\beta}, 1\right)$.

The proof is obvious by Lemma 1 observing that $\ln\left[f_{\alpha,\beta,\gamma}(x)\right] = \gamma g_{\alpha,\beta}(x) = \gamma x^{\alpha}(1-x)^{\beta}$, $x \in [0,1]$ and $\alpha, \beta, \gamma > 0$.

3 Taylor's Type Expansion for $f_{\alpha,\beta,\gamma}$

We have the following representation result:

Theorem 1 *Let* $\alpha, \beta, \gamma > 0$, *then for all* $x \in [0,1]$ *and natural number* $n \geq 1$, *we have*

$$f_{\alpha,\beta,\gamma}(x) = 1 + \sum_{k=1}^{n} \frac{1}{k!}\gamma^{k} x^{\alpha k}(1-x)^{\beta k} + R_n(\alpha,\beta,\gamma,x), \tag{3.1}$$

where

$$\begin{aligned} &R_n(\alpha,\beta,\gamma,x) \\ &= \frac{1}{n!}\gamma^{n+1}x^{\alpha(n+1)}(1-x)^{\beta(n+1)} \int_0^1 \exp\left[\gamma s x^{\alpha}(1-x)^{\beta}\right](1-s)^n\,ds. \end{aligned} \tag{3.2}$$

Proof Let $I \subset \mathbb{R}$ be a closed interval, $c \in I$, and let n be a positive integer. If $f : I \longrightarrow \mathbb{C}$ is such that the n-derivative $f^{(n)}$ is absolutely continuous on I, then for each $y \in I$

$$f(y) = T_n(f;c,y) + R_n(f;c,y), \tag{3.3}$$

where $T_n(f;c,y)$ is *Taylor's polynomial*, i.e.,

$$T_n(f;c,y) := \sum_{k=0}^{n} \frac{(y-c)^k}{k!} f^{(k)}(c). \tag{3.4}$$

Note that $f^{(0)} := f$ and $0! := 1$ and the remainder is given by

$$R_n(f;c,y) := \frac{1}{n!}\int_c^y (y-t)^n f^{(n+1)}(t)\,dt. \tag{3.5}$$

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t = (1-s)c + sd$, $s \in [0,1]$ that

$$\int_c^d h(t)\,dt = (d-c)\int_0^1 h((1-s)c + sd)\,ds.$$

Therefore,

$$\begin{aligned}
&\int_c^y f^{(n+1)}(t)(y-t)^n\,dt \\
&= (y-c)\int_0^1 f^{(n+1)}((1-s)c+sy)(x-(1-s)c-sy)^n\,ds \\
&= (y-c)^{n+1}\int_0^1 f^{(n+1)}((1-s)c+sy)(1-s)^n\,ds,
\end{aligned}$$

and from (3.5), we get the representation

$$\begin{aligned}
f(y) = {}& \sum_{k=0}^{n}\frac{(y-c)^k}{k!}f^{(k)}(c) \\
&+ \frac{1}{n!}(y-c)^{n+1}\int_0^1 f^{(n+1)}((1-s)c+sy)(1-s)^n\,ds
\end{aligned} \tag{3.6}$$

for all $y, c \in I$.

Now, if we write the equality (3.6) for the exponential function $f(y) = e^y$, $y \in \mathbb{R}$, and the point $c = 0$, we get

$$\exp y - 1 = \sum_{k=1}^{n}\frac{y^k}{k!} + \frac{1}{n!}y^{n+1}\int_0^1 \exp(sy)(1-s)^n\,ds \tag{3.7}$$

for any real number $y \in \mathbb{R}$.

If we take in (3.7) $y = \gamma g_{\alpha,\beta}(x)$, $x \in [0,1]$, we get

$$\begin{aligned}
\exp\left[\gamma g_{\alpha,\beta}(x)\right] - 1 = {}& \sum_{k=1}^{n}\gamma^k\frac{\left[g_{\alpha,\beta}(x)\right]^k}{k!} \\
&+ \frac{1}{n!}\gamma^{n+1}\left[g_{\alpha,\beta}(x)\right]^{n+1}\int_0^1 \exp\left[s\gamma g_{\alpha,\beta}(x)\right](1-s)^n\,ds,
\end{aligned}$$

which produces the desired result (3.1). □

We have some simple upper and lower bounds as follows:

Corollary 1 *Let* $\alpha, \beta, \gamma > 0$, *then for all* $x \in [0, 1]$ *and natural number* $n \geq 1$, *we have*

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{1}{k!}\gamma^k x^{\alpha k} (1-x)^{\beta k} \\
&\leq f_{\alpha,\beta,\gamma}(x) - 1 \\
&\leq \sum_{k=1}^{n} \frac{1}{k!}\gamma^k x^{\alpha k} (1-x)^{\beta k} + \frac{e^{\gamma}}{(n+1)!}\gamma^{n+1} x^{\alpha(n+1)} (1-x)^{\beta(n+1)}.
\end{aligned} \tag{3.8}$$

Proof The inequalities in (3.8) follow by (3.1) observing that

$$\begin{aligned}
0 &\leq \frac{1}{n!}\gamma^{n+1}\left[g_{\alpha,\beta}(x)\right]^{n+1} \int_0^1 \exp\left[\gamma s g_{\alpha,\beta}(x)\right](1-s)^n \, ds \\
&\leq \frac{1}{n!}\gamma^{n+1}\left[g_{\alpha,\beta}(x)\right]^{n+1} \max_{s\in[0,1]} \exp\left[s\gamma g_{\alpha,\beta}(x)\right] \int_0^1 (1-s)^n \, ds \\
&\leq \frac{e^{\gamma}}{(n+1)!}\gamma^{n+1}\left[g_{\alpha,\beta}(x)\right]^{n+1}
\end{aligned}$$

for all $x \in [0, 1]$. ◻

Corollary 2 *Let* $\alpha, \beta, \gamma > 0$, *then we have function series expansion*

$$f_{\alpha,\beta,\gamma}(x) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\gamma^k x^{\alpha k} (1-x)^{\beta k}$$

uniformly on the interval $[0, 1]$.

Proof Let $\alpha, \beta, \gamma > 0$. By (3.1), we have

$$\begin{aligned}
&\left| f_{\alpha,\beta,\gamma}(x) - 1 - \sum_{k=1}^{n} \gamma^k \frac{\left[g_{\alpha,\beta}(x)\right]^k}{k!} \right| \\
&= \left| \frac{1}{n!}\gamma^{n+1}\left[g_{\alpha,\beta}(x)\right]^{n+1} \int_0^1 \exp\left[s\gamma g_{\alpha,\beta}(x)\right](1-s)^n \, ds \right| \\
&\leq \frac{1}{n!}\gamma^{n+1}\left[g_{\alpha,\beta}(x)\right]^{n+1} \int_0^1 \left|\exp\left[s\gamma g_{\alpha,\beta}(x)\right](1-s)^n\right| ds \\
&\leq \frac{e^{\gamma}}{n!}\gamma^{n+1} \int_0^1 (1-s)^n \, ds = e^{\gamma}\frac{\gamma^{n+1}}{(n+1)!}
\end{aligned}$$

for $x \in [0, 1]$ and $n \geq 1$.

Consider the positive sequence

$$a_n := \frac{\gamma^{n+1}}{(n+1)!}, \; n \geq 1.$$

Then,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{\gamma^{n+2}}{(n+2)!}}{\frac{\gamma^{n+1}}{(n+1)!}} = \lim_{n\to\infty} \frac{\gamma}{(n+2)} = 0.$$

By using the ratio test for sequences, we conclude that $\lim_{n\to\infty} a_n = 0$, which proves the statement. □

Now, we can introduce the three variable function $F : (0,\infty) \times (0,\infty) \times (0,\infty) \to (0,\infty)$, which we can call the *weighted Exponential Beta function,* defined by the integral

$$F(\alpha,\beta;\gamma) := \int_0^1 \exp\left[\gamma x^\alpha (1-x)^\beta\right] dx > 1.$$

Then, we have the following representation result in terms of the Beta function:

Theorem 2 *For any natural number $n \geq 1$ and any α, β, $\gamma > 0$, we have the Beta-Taylor representation*

$$F(\alpha,\beta;\gamma) = 1 + \sum_{k=1}^{n} \frac{1}{k!}\gamma^k B(\alpha k+1, \beta k+1) + R_n(\alpha,\beta;\gamma), \tag{3.9}$$

where the remainder $R_n(\alpha,\beta;\gamma)$ is given by

$$R_n(\alpha,\beta;\gamma) := \frac{1}{n!}\gamma^{n+1} \int_0^1 \left(\int_0^1 \left\{x^{\alpha(n+1)} (1-x)^{\beta(n+1)} \exp\left[s\gamma x^\alpha (1-x)^\beta\right]\right\} dx\right) (1-s)^n \, ds. \tag{3.10}$$

Proof If we integrate the identity (3.1), we get

$$\begin{aligned} F(\alpha,\beta;\gamma) &= \int_0^1 f_{\alpha,\beta,\gamma}(x)\,dx \qquad (3.11) \\ &= 1 + \sum_{k=1}^{n} \frac{1}{k!}\gamma^k \int_0^1 x^{\alpha k}(1-x)^{\beta k} dx + \int_0^1 R_n(\alpha,\beta,\gamma,x)\,dx \end{aligned}$$

$$= 1 + \sum_{k=1}^{n} \frac{1}{k!} \gamma^k B\left(\alpha k + 1, \beta k + 1\right) + \int_0^1 R_n\left(\alpha, \beta, \gamma, x\right) dx.$$

Also

$$\begin{aligned}
&\int_0^1 R_n\left(\alpha, \beta, \gamma, x\right) dx \\
&= \frac{1}{n!}\gamma^{n+1} \int_0^1 x^{\alpha(n+1)} \left(1-x\right)^{\beta(n+1)} \left(\int_0^1 \exp\left[\gamma s x^{\alpha} \left(1-x\right)^{\beta}\right] \left(1-s\right)^n ds\right) dx \\
&= \frac{1}{n!}\gamma^{n+1} \int_0^1 \left(\int_0^1 x^{\alpha(n+1)} \left(1-x\right)^{\beta(n+1)} \exp\left[\gamma s x^{\alpha} \left(1-x\right)^{\beta}\right] dx\right) \left(1-s\right)^n ds,
\end{aligned}$$

where for the last equality, we used Fubini's theorem. □

Corollary 3 *We have the following Beta-Taylor series expansion*

$$F\left(\alpha, \beta; \gamma\right) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \gamma^k B\left(\alpha k + 1, \beta k + 1\right) \tag{3.12}$$

uniformly over $\alpha,\ \beta,\ \gamma > 0$.

Proof Observe that

$$\begin{aligned}
0 \le R_n\left(\alpha, \beta; \gamma\right) &\le \frac{1}{n!}\gamma^{n+1} \int_0^1 \exp\left(s\gamma\right) \left(1-s\right)^n ds \\
&= \frac{1}{n!}\gamma^{n+1} e^{\gamma} \int_0^1 \left(1-s\right)^n ds = e^{\gamma} \frac{\gamma^{n+1}}{(n+1)!} \to 0
\end{aligned}$$

for $n \to \infty$. This proves the claim. □

4 Error Bounds Via Ostrowski Type Inequalities

The following lemma provides an error estimate in approximating the integral mean by a value of the function in the case when the derivative is bounded. It was obtained in 1938 by Ostrowski, see [14].

Lemma 2 *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b), *whose derivative is bounded on* (a, b) *and let* $\left\|f'\right\|_{\infty,(a,b)} := \sup_{t\in(a,b)} \left|f'(t)\right| < \infty$. *Then,*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_{\infty,(a,b)} \tag{4.1}$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

For a recent survey on this inequality, see [2] and the references therein.

We need the following lemma:

Lemma 3 *For $\alpha, \beta > 1$, we have*

$$\max_{x\in[0,1]} \left| g'_{\alpha,\beta}(x) \right| \leq \max\{\alpha, \beta\} \left(\frac{\alpha-1}{\alpha+\beta-2} \right)^{\alpha-1} \left(\frac{\beta-1}{\alpha+\beta-2} \right)^{\beta-1}. \tag{4.2}$$

Proof From the definition of $g_{\alpha,\beta}(x)$, we have

$$g'_{\alpha,\beta}(x) = g_{\alpha-1,\beta-1}(x) [\alpha - (\alpha+\beta) x], \ x \in [0, 1], \tag{4.3}$$

which implies that for $\alpha, \beta > 1$, we have

$$\begin{aligned} \max_{x\in[0,1]} \left| g'_{\alpha,\beta}(x) \right| &\leq \max_{x\in[0,1]} g_{\alpha-1,\beta-1}(x) \max_{x\in[0,1]} |\alpha - (\alpha+\beta) x| \\ &= \max\{\alpha, \beta\} \max_{x\in[0,1]} g_{\alpha-1,\beta-1}(x). \end{aligned} \tag{4.4}$$

From (4.3), we get

$$g'_{\alpha-1,\beta-1}(x) = g_{\alpha-2,\beta-2}(x) [\alpha - 1 - (\alpha+\beta-2) x], x \in (0, 1).$$

This shows that $g'_{\alpha-1,\beta-1}(x) > 0$ for $x \in \left(0, \frac{\alpha-1}{\alpha+\beta-2}\right)$ and $g'_{\alpha-1,\beta-1}(x) < 0$ for $\left(\frac{\alpha-1}{\alpha+\beta-2}, \infty\right)$, which gives that

$$\begin{aligned} \max_{x\in[0,1]} g_{\alpha-1,\beta-1}(x) &= g_{\alpha-1,\beta-1}\left(\frac{\alpha-1}{\alpha+\beta-2} \right) \\ &= \left(\frac{\alpha-1}{\alpha+\beta-2} \right)^{\alpha-1} \left(\frac{\beta-1}{\alpha+\beta-2} \right)^{\beta-1}. \end{aligned} \tag{4.5}$$

By (4.4) and (4.5), we get the desired inequality (4.2). □

We have the following result via Ostrowski's inequality:

Theorem 3 *For $\alpha, \beta > 1$ and $\gamma > 0$, we have*

$$\left|F(\alpha,\beta;\gamma) - f_{\alpha,\beta,\gamma}(x)\right| \leq \left[\frac{1}{4} + \left(x - \frac{1}{2}\right)^2\right] \times \gamma \max\{\alpha,\beta\} \left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1} \left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1} \times \exp\left[\gamma \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha} \left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right] \tag{4.6}$$

for all $x \in [0, 1]$.

In particular,

$$\left|F(\alpha,\beta;\gamma) - \exp\left(\frac{\gamma}{2^{\alpha+\beta}}\right)\right| \leq \frac{1}{4}\gamma \max\{\alpha,\beta\} \left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1} \left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1} \times \exp\left[\gamma \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha} \left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right]. \tag{4.7}$$

Proof If we write Ostrowski's inequality for the function $f_{\alpha,\beta,\gamma}$ on the interval $[0, 1]$, then we have

$$\left|f_{\alpha,\beta,\gamma}(x) - \int_0^1 f_{\alpha,\beta,\gamma}(t)\,dt\right| \leq \left[\frac{1}{4} + \left(x - \frac{1}{2}\right)^2\right] \left\|f'_{\alpha,\beta,\gamma}\right\|_{\infty,[0,1]} \tag{4.8}$$

for all $x \in [0, 1]$.

From (2.4), we have

$$f'_{\alpha,\beta,\gamma}(x) = \gamma g'_{\alpha,\beta}(x) \exp\left[\gamma g_{\alpha,\beta}(x)\right] = \gamma g'_{\alpha,\beta}(x)\, f_{\alpha,\beta,\gamma}(x),\ \ x \in [0, 1],$$

which shows that

$$\max_{x\in[0,1]} \left|f'_{\alpha,\beta,\gamma}(x)\right| \leq \gamma \max_{x\in[0,1]} \left|g'_{\alpha,\beta}(x)\right| \max_{x\in[0,1]} f_{\alpha,\beta,\gamma}(x) \leq \gamma \max\{\alpha,\beta\} \left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1} \left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1} \times \exp\left[\gamma \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha} \left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right],$$

where for the last inequality, we used Lemmas 2 and 3.

By employing (4.8), we obtain the desired result (4.6). □

In 1997, Dragomir and Wang proved the following Ostrowski type inequality [6], see also [2, p. 26]:

Lemma 4 *Let* $f : [a, b] \to \mathbb{R}$ *be an absolutely continuous function on* $[a, b]$. *Then, we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right] \left\| f' \right\|_{[a,b],1}, \tag{4.9}$$

for all $x \in [a, b]$, *where* $\|\cdot\|_1$ *is the Lebesgue norm on* $L_1[a, b]$, *i.e., we recall it*

$$\|g\|_{[a,b],1} := \int_a^b |g(t)|\,dt.$$

The constant $\frac{1}{2}$ *is best possible.*

Note the fact that $\frac{1}{2}$ is the best constant for differentiable functions was proved in [15].

Theorem 4 *For* $\alpha, \beta > 1$, $\gamma > 0$, *we have*

$$\begin{aligned} \left| \frac{f_{\alpha,\beta,\gamma}(x)}{F(\alpha, \beta; \gamma)} - 1 \right| &\le \left[\frac{1}{2} + \left| x - \frac{1}{2} \right| \right] \\ &\quad \times \gamma \max\{\alpha, \beta\} \left(\frac{\alpha - 1}{\alpha + \beta - 2} \right)^{\alpha - 1} \left(\frac{\beta - 1}{\alpha + \beta - 2} \right)^{\beta - 1} \end{aligned} \tag{4.10}$$

for all $x \in [0, 1]$ *and, in particular,*

$$\left| \frac{\exp\left(\frac{\gamma}{2^{\alpha+\beta}} \right)}{F(\alpha, \beta; \gamma)} - 1 \right| \le \frac{1}{2}\gamma \max\{\alpha, \beta\} \left(\frac{\alpha - 1}{\alpha + \beta - 2} \right)^{\alpha - 1} \left(\frac{\beta - 1}{\alpha + \beta - 2} \right)^{\beta - 1}. \tag{4.11}$$

For $\alpha, \beta, \gamma > 0$, *we also have*

$$\begin{aligned} &\left| F(\alpha, \beta; \gamma) - f_{\alpha,\beta,\gamma}(x) \right| \\ &\le \left[\frac{1}{2} + \left| x - \frac{1}{2} \right| \right] \\ &\times \gamma \max\{\alpha, \beta\} B(\alpha, \beta) \exp\left[\gamma \left(\frac{\alpha}{\alpha + \beta} \right)^{\alpha} \left(\frac{\beta}{\alpha + \beta} \right)^{\beta} \right] \end{aligned} \tag{4.12}$$

for all $x \in [0, 1]$ *and, in particular,*

$$\left| F(\alpha, \beta; \gamma) - \exp\left(\frac{\gamma}{2^{\alpha+\beta}} \right) \right| \tag{4.13}$$

$$\leq \frac{1}{2}\gamma \max\{\alpha, \beta\} B(\alpha, \beta) \exp\left[\gamma \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha} \left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right],$$

where $B(\cdot, \cdot)$ is Euler's Beta function.

Proof If we write the inequality (4.9) for $f_{\alpha,\beta,\gamma}$ on the interval $[0, 1]$, then we have

$$\left|f_{\alpha,\beta,\gamma}(x) - F(\alpha, \beta; \gamma)\right| \leq \left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right] \left\|f'_{\alpha,\beta,\gamma}\right\|_{[0,1],1}, \tag{4.14}$$

for all $x \in [0, 1]$.

Now, observe that

$$\begin{aligned} &\left\|f'_{\alpha,\beta,\gamma}\right\|_{[0,1],1} \\ &= \int_0^1 \left|f'_{\alpha,\beta,\gamma}(t)\right| dt = \gamma \int_0^1 \left|g'_{\alpha,\beta}(t)\right| \exp\left[\gamma g_{\alpha,\beta}(t)\right] dt \\ &= \gamma \int_0^1 \left|g'_{\alpha,\beta}(t)\right| f_{\alpha,\beta,\gamma}(t)\, dt \\ &= \gamma \int_0^1 g_{\alpha-1,\beta-1}(t) \left|\alpha - (\alpha+\beta)t\right| f_{\alpha,\beta,\gamma}(t)\, dt \\ &\leq \gamma \max_{t\in[0,1]} \left|\alpha - (\alpha+\beta)t\right| \int_0^1 g_{\alpha-1,\beta-1}(t) f_{\alpha,\beta,\gamma}(t)\, dt \\ &= \gamma \max\{\alpha, \beta\} \int_0^1 g_{\alpha-1,\beta-1}(t) f_{\alpha,\beta,\gamma}(t)\, dt. \end{aligned} \tag{4.15}$$

Since

$$\begin{aligned} &\int_0^1 g_{\alpha-1,\beta-1}(t) f_{\alpha,\beta,\gamma}(t)\, dt \\ &\leq \max_{t\in[0,1]} g_{\alpha-1,\beta-1}(t) \int_0^1 f_{\alpha,\beta,\gamma}(t)\, dt \\ &= \left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1} \left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1} F(\alpha, \beta; \gamma), \end{aligned} \tag{4.16}$$

and hence by (4.14)–(4.16), we get

$$\left|f_{\alpha,\beta,\gamma}(x) - F(\alpha, \beta; \gamma)\right| \leq \left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right]$$

$$\times \gamma \left(\frac{\alpha - 1}{\alpha + \beta - 2} \right)^{\alpha - 1} \left(\frac{\beta - 1}{\alpha + \beta - 2} \right)^{\beta - 1} F(\alpha, \beta; \gamma)$$

that is equivalent to (4.10).

We also have

$$\begin{aligned} &\int_0^1 g_{\alpha-1,\beta-1}(t) f_{\alpha,\beta,\gamma}(t)\,dt \qquad (4.17)\\ &\le \max_{t\in[0,1]} f_{\alpha,\beta,\gamma}(t) \int_0^1 g_{\alpha-1,\beta-1}(t)\,dt \\ &= \exp\left[\gamma \left(\frac{\alpha}{\alpha+\beta} \right)^{\alpha} \left(\frac{\beta}{\alpha+\beta} \right)^{\beta} \right] B(\alpha,\beta), \end{aligned}$$

and hence by (4.14), (4.15) and (4.17), we get (4.12). □

In 1998, Dragomir and Wang proved the following Ostrowski type inequality for p-norms of the derivative [7].

Lemma 5 *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f' \in L_p[a,b]$, then we have the inequality*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \qquad (4.18)\\ &\le \frac{1}{(q+1)^{1/q}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{b-x}{b-a} \right)^{q+1} \right]^{1/q} (b-a)^{1/q} \left\| f' \right\|_{[a,b],p}, \end{aligned}$$

for all $x \in [a,b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p} := \left(\int_a^b |g(t)|^p\,dt \right)^{1/p}.$$

Using this tool, we can prove the following result as well:

Theorem 5 *For $\alpha, \beta > 1$, $\gamma > 0$, we have*

$$\begin{aligned} \left| F(\alpha,\beta;\gamma) - f_{\alpha,\beta,\gamma}(x) \right| &\le \frac{1}{(q+1)^{1/q}} \left[x^{q+1} + (1-x)^{q+1} \right]^{1/q} \qquad (4.19)\\ &\times \gamma \max\{\alpha,\beta\} \exp\left[\gamma \left(\frac{\alpha}{\alpha+\beta} \right)^{\alpha} \left(\frac{\beta}{\alpha+\beta} \right)^{\beta} \right] \\ &\times \left[B(p(\alpha-1)+1, p(\beta-1)+1) \right]^{1/p} \end{aligned}$$

for all $x \in [0,1]$, *where* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

In particular,

$$\left|F(\alpha,\beta;\gamma) - \exp\left(\frac{\gamma}{2^{\alpha+\beta}}\right)\right| \tag{4.20}$$
$$\leq \frac{1}{2(q+1)^{1/q}}\gamma \max\{\alpha,\beta\} \exp\left[\gamma\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right]$$
$$\times \left[B(p(\alpha-1)+1, p(\beta-1)+1)\right]^{1/p},$$

where $B(\cdot,\cdot)$ *is Euler's Beta function.*

Proof If we write the inequality (4.18) for the function for $f_{\alpha,\beta}$ on the interval $[0,1]$, then we have

$$\left|f_{\alpha,\beta,\gamma}(x) - F(\alpha,\beta;\gamma)\right| \tag{4.21}$$
$$\leq \frac{1}{(q+1)^{1/q}}\left[x^{q+1} + (1-x)^{q+1}\right]^{1/q}\left\|f'_{\alpha,\beta,\gamma}\right\|_{[0,1],p},$$

for all $x \in [a,b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Observe that

$$\left\|f'_{\alpha,\beta,\gamma}\right\|^{p}_{[0,1],p} \tag{4.22}$$
$$= \int_0^1 \left|f'_{\alpha,\beta,\gamma}(t)\right|^p dt = \gamma^p \int_0^1 \left|g'_{\alpha,\beta}(t)\right|^p \left(\exp\left[\gamma g_{\alpha,\beta}(t)\right]\right)^p dt$$
$$= \gamma^p \int_0^1 \left|g'_{\alpha,\beta}(t)\right|^p f^p_{\alpha,\beta,\gamma}(t)\,dt$$
$$= \gamma^p \int_0^1 g^p_{\alpha-1,\beta-1}(t)\left|\alpha - (\alpha+\beta)t\right|^p f^p_{\alpha,\beta,\gamma}(t)\,dt$$
$$\leq \gamma^p \max\left\{\alpha^p, \beta^p\right\}\int_0^1 x^{p(\alpha-1)}(1-x)^{p(\beta-1)} f^p_{\alpha,\beta,\gamma}(t)\,dt.$$

Since, by (2.3), we have

$$\max_{t\in[0,1]} f^p_{\alpha,\beta,\gamma}(t) = f^p_{\alpha,\beta,\gamma}\left(\frac{\alpha}{\alpha+\beta}\right) = \exp\left[p\gamma\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right]$$

and hence by (4.22), we get

$$
\begin{aligned}
&\left\| f'_{\alpha,\beta,\gamma} \right\|_{[0,1],p}^{p} \\
&\leq \gamma^{p} \max\left\{\alpha^{p}, \beta^{p}\right\} \exp\left[p\gamma \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha} \left(\frac{\beta}{\alpha+\beta}\right)^{\beta} \right] \\
&\times \int_0^1 x^{p(\alpha-1)} (1-x)^{p(\beta-1)}\, dt \\
&= \gamma^{p} \max\left\{\alpha^{p}, \beta^{p}\right\} \exp\left[p\gamma \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha} \left(\frac{\beta}{\alpha+\beta}\right)^{\beta} \right] \\
&\times B\left(p(\alpha-1)+1, p(\beta-1)+1\right),
\end{aligned}
$$

namely

$$
\begin{aligned}
\left\| f'_{\alpha,\beta,\gamma} \right\|_{[0,1],p} &\leq \gamma \max\{\alpha, \beta\} \exp\left[\gamma \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha} \left(\frac{\beta}{\alpha+\beta}\right)^{\beta} \right] \\
&\times \left[B\left(p(\alpha-1)+1, p(\beta-1)+1\right)\right]^{1/p}.
\end{aligned}
$$

Therefore, by (4.21), we get the desired result (4.19). □

5 Quadrature Rules of Ostrowski and Trapezoid Type

Let

$$I_k : a = x_0 < x_1 < \ldots < x_{k-1} < x_k = b$$

be a *division* of the interval $[a, b]$, α_i $(i = 0, \ldots, k+1)$ be '$k+2$' points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, k)$ and $\alpha_{k+1} = b$. Define

$$h_i := x_{i+1} - x_i \ \ (i = 0, \ldots, k-1) \ \text{ and } \nu(h) := \max\{h_i \mid i = 0, \ldots, k-1\}.$$

Consider the equality

$$\int_a^b f(t)\, dt = \Omega_k(f, I_k, \alpha_{k+1}) + R_k(f, I_k, \alpha_{k+1}), \tag{5.1}$$

where

$$\Omega_k(f, I_k, \alpha) := \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \tag{5.2}$$

is the *Ostrowski quadrature rule* associated with the division I_k and the "$k+2$" points $\alpha_{k+1} := (\alpha_0, \alpha_1, \ldots, \alpha_k, \alpha_{k+1})$, while $R_k(f, I_k, \alpha_{k+1})$ is the error in approximating the integral $\int_a^b f(t)\,dt$ by the quadrature $\Omega_k(f, I_k, \alpha)$.

If we chose in (5.5)

$$\alpha_0 = a,\ \alpha_1 = \frac{a+x_1}{2},\ \alpha_2 = \frac{x_1+x_2}{2}, \ldots,$$
$$\alpha_{k-1} = \frac{x_{k-2}+x_{k-1}}{2},\ \alpha_k = \frac{x_{k-1}+x_k}{2},\ \alpha_{k+1} = b,$$

then we get after some arrangements that

$$\Omega_k(f, I_k, \alpha) = \frac{1}{2}\left[(x_1-a)f(a) + \sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i) + (b-x_{k-1})f(b)\right]$$
$$=: T_k(f, I_k),$$

where $T_k(f, I_k)$ is called the *Trapezoid quadrature rule* associated with the function f and the division I_k.

In this situation, we have

$$\int_a^b f(t)\,dt = T_k(f, I_k) + R_k(f, I_k), \tag{5.3}$$

where $R_k(f, I_k)$ is the error in approximating the integral by the trapezoid rule $T_k(f, I_k)$.

Let

$$I_k:\ x_i := a + (b-a)\frac{i}{k},\ i = 0, \ldots, k$$

be the equidistant partitioning of $[a, b]$. We can then consider the *equidistant Trapezoid rule* given by

$$T_k(f) := \frac{1}{k}\frac{f(a)+f(b)}{2}(b-a) + \frac{b-a}{k}\sum_{i=1}^{k-1} f\left(a+(b-a)\frac{i}{k}\right)$$

for $k \geq 2$.

Furthermore, we can approximate the integral as

$$\int_a^b f(t)\,dt = T_k(f) + R_k(f), \tag{5.4}$$

where $R_k(f)$ is the error in this equidistant approximation.

Assume that f is absolutely continuous on $[a, b]$.

If f' is essentially bounded on $[a,b]$, namely, $f' \in L_\infty [a,b]$, then we have the error bounds [8, p. 19]

$$\begin{aligned} &|R_k(f, I_k, \alpha_{k+1})| \qquad (5.5) \\ &\le \left[\frac{1}{4} \sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1} \left(\alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right)^2 \right] \|f'\|_{\infty,[a,b]} \\ &\le \frac{1}{2} \sum_{i=0}^{k-1} h_i^2 \|f'\|_{\infty,[a,b]} \le \frac{1}{2}(b-a) \|f'\|_{\infty,[a,b]} \nu(h). \end{aligned}$$

The trapezoid rule error $R_k(f, I_k)$ satisfies the better bounds

$$|R_k(f, I_k)| \le \frac{1}{4} \left(\sum_{i=0}^{k-1} h_i^2 \right) \|f'\|_{\infty,[a,b]} \le \frac{1}{4}(b-a) \|f'\|_{\infty,[a,b]} \nu(h),$$

and the equidistant error $R_k(f)$ satisfies the inequality

$$|R_k(f)| \le \frac{1}{4k}(b-a)^2 \|f'\|_{\infty,[a,b]}.$$

In terms of 1-norm, we have the error bounds [3], see also [8, p. 51],

$$\begin{aligned} &|R_k(f, I_k, \alpha_{k+1})| \qquad (5.6) \\ &\le \left[\frac{1}{2} \nu(h) + \max_{i=1,\dots,n} \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \right] \|f'\|_{1,[a,b]} \le \|f'\|_{1,[a,b]} \nu(h). \end{aligned}$$

In particular, we have

$$|R_k(f, I_k)| \le \frac{1}{2} \nu(h) \|f'\|_{1,[a,b]}$$

and

$$|R_k(f)| \le \frac{1}{2k}(b-a) \|f'\|_{1,[a,b]}.$$

If $f' \in L_p[a,b]$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then [4], see also [8, p. 35],

$$\begin{aligned} &|R_k(f, I_k, \alpha_{k+1})| \qquad (5.7) \\ &\le \frac{1}{(q+1)^{1/q}} \left[\sum_{i=0}^{k-1} (\alpha_{i+1} - x_i)^{q+1} + (x_{i+1} - \alpha_{i+1})^{q+1} \right]^{1/q} \|f'\|_{p,[a,b]} \end{aligned}$$

$$\leq \frac{1}{(q+1)^{1/q}} \left\| f' \right\|_{p,[a,b]} \left(\sum_{i=0}^{k-1} h_i^{q+1} \right)^{1/q}$$
$$\leq \frac{1}{(q+1)^{1/q}} (b-a)^{1/q} \left\| f' \right\|_{p,[a,b]} \nu(h).$$

Moreover, we have

$$|R_k(f, I_k)| \leq \frac{1}{2(q+1)^{1/q}} \left\| f' \right\|_{p,[a,b]} \left(\sum_{i=0}^{k-1} h_i^{q+1} \right)^{1/q}$$
$$\leq \frac{1}{2(q+1)^{1/q}} (b-a)^{1/q} \left\| f' \right\|_{p,[a,b]} \nu(h)$$

and

$$|R_k(f)| \leq \frac{1}{2k(q+1)^{1/q}} (b-a)^{1+1/q} \left\| f' \right\|_{p,[a,b]}.$$

Let

$$I_k : 0 = x_0 < x_1 < \ldots < x_{k-1} < x_k = 1$$

be a division of the interval $[0, 1]$ and $\alpha_0 = 0$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, k)$ and $\alpha_{k+1} = 1$. We define the following Ostrowski type quadrature rule for the exponential Beta function by

$$\Omega_k\left(f_{\alpha,\beta,\gamma}, I_k, \alpha\right) := \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) \exp\left[\gamma x_i^{\alpha} (1 - x_i)^{\beta}\right]$$

and the Trapezoid rule by

$$T_k\left(f_{\alpha,\beta,\gamma}, I_k\right) := \frac{1}{2}\left[x_1 + \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) \exp\left[\gamma x_i^{\alpha} (1 - x_i)^{\beta}\right] + 1 - x_{k-1}\right].$$

Consider also the *equidistant Trapezoid rule* given by

$$T_k\left(f_{\alpha,\beta,\gamma}\right) := \frac{1}{k} + \frac{1}{k} \sum_{i=1}^{k-1} \exp\left[\gamma \left(\frac{i}{k}\right)^{\alpha} \left(1 - \frac{i}{k}\right)^{\beta}\right]$$

for $k \geq 2$.

Theorem 6 *Let I_k, α be as defined above. Then,*

$$F(\alpha,\beta;\gamma)=\Omega_k\left(f_{\alpha,\beta,\gamma},I_k,\alpha\right)+R_k\left(f_{\alpha,\beta,\gamma},I_k,\alpha_{k+1}\right),$$

where the remainder $R_k\left(f_{\alpha,\beta},I_k,\alpha_{k+1}\right)$ *satisfies the bounds*

$$\begin{aligned}
&\left|R_k\left(f_{\alpha,\beta,\gamma},I_k,\alpha_{k+1}\right)\right| \\
&\le\left[\frac{1}{4}\sum_{i=0}^{k-1}h_i^2+\sum_{i=0}^{k-1}\left(\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right)^2\right]\left\|f'_{\alpha,\beta,\gamma}\right\|_{\infty,[0,1]} \\
&\le\left[\frac{1}{4}\sum_{i=0}^{k-1}h_i^2+\sum_{i=0}^{k-1}\left(\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right)^2\right] \\
&\times\gamma\max\{\alpha,\beta\}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1} \\
&\times\exp\left[\gamma\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right],\ \alpha,\ \beta>1,
\end{aligned}\tag{5.8}$$

$$\begin{aligned}
&\left|R_k\left(f_{\alpha,\beta,\gamma},I_k,\alpha_{k+1}\right)\right| \\
&\le\left[\frac{1}{2}\nu(h)+\max_{i=1,\dots,n}\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\left\|f'_{\alpha,\beta,\gamma}\right\|_{1,[0,1]} \\
&\le\left[\frac{1}{2}\nu(h)+\max_{i=1,\dots,n}\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right] \\
&\times\gamma\max\{\alpha,\beta\}B(\alpha,\beta)\exp\left[\gamma\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right],\ \alpha,\ \beta>0
\end{aligned}\tag{5.9}$$

and

$$\begin{aligned}
&\left|R_k\left(f_{\alpha,\beta,\gamma},I_k,\alpha_{k+1}\right)\right| \\
&\le\frac{1}{(q+1)^{1/q}}\left[\sum_{i=0}^{k-1}(\alpha_{i+1}-x_i)^{q+1}+(x_{i+1}-\alpha_{i+1})^{q+1}\right]^{1/q}\left\|f'_{\alpha,\beta,\gamma}\right\|_{p,[0,1]} \\
&\le\frac{1}{(q+1)^{1/q}}\left[\sum_{i=0}^{k-1}(\alpha_{i+1}-x_i)^{q+1}+(x_{i+1}-\alpha_{i+1})^{q+1}\right]^{1/q} \\
&\times\gamma\max\{\alpha,\beta\}\exp\left[\gamma\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right] \\
&\times\left[B\left(p(\alpha-1)+1,p(\beta-1)+1\right)\right]^{1/p},\ \alpha,\ \beta>1,
\end{aligned}\tag{5.10}$$

where $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

The proof follows from the inequalities (5.5), (5.6) and (5.7), and the fact that from the previous section, we have the following upper bounds for the norms of $f'_{\alpha,\beta,\gamma}$

$$\left\| f'_{\alpha,\beta,\gamma} \right\|_{\infty,[0,1]} \leq \gamma \max\{\alpha,\beta\} \left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1} \left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1} \times \exp\left[\gamma \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha} \left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right],\ \alpha,\beta>1,$$

$$\left\| f'_{\alpha,\beta,\gamma} \right\|_{1,[0,1]} \leq \gamma \max\{\alpha,\beta\} \exp\left[\gamma \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha} \left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right] B(\alpha,\beta),\ \alpha,\beta>0$$

and

$$\left\| f'_{\alpha,\beta,\gamma} \right\|_{[0,1],p} \leq \gamma \max\{\alpha,\beta\} \exp\left[\gamma \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha} \left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right] \times \left[B\left(p(\alpha-1)+1,\, p(\beta-1)+1\right)\right]^{1/p},\ \alpha,\beta>1.$$

Corollary 4 *Let I_k be as defined above. Then,*

$$F(\alpha,\beta;\gamma) = T_k\left(f_{\alpha,\beta,\gamma}, I_k\right) + R_k\left(f_{\alpha,\beta,\gamma}, I_k\right),$$

where the remainder $R_k\left(f_{\alpha,\beta}, I_k\right)$ satisfies the bounds

$$\left|R_k\left(f_{\alpha,\beta,\gamma}, I_k\right)\right| \leq \frac{1}{4}\sum_{i=0}^{k-1} h_i^2 \left\| f'_{\alpha,\beta,\gamma} \right\|_{\infty,[0,1]} \leq \left[\frac{1}{4}\sum_{i=0}^{k-1} h_i^2\right] \times \gamma \max\{\alpha,\beta\} \left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1} \left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1} \times \exp\left[\gamma \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha} \left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right],\ \alpha,\ \beta>1, \tag{5.11}$$

$$\left|R_k\left(f_{\alpha,\beta,\gamma}, I_k\right)\right| \leq \left[\frac{1}{2}\nu(h)\right] \left\| f'_{\alpha,\beta,\gamma} \right\|_{1,[0,1]} \leq \left[\frac{1}{2}\nu(h)\right] \times \gamma \max\{\alpha,\beta\} B(\alpha,\beta) \exp\left[\gamma \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha} \left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right],\ \alpha,\ \beta>0 \tag{5.12}$$

and

$$\begin{aligned}\left|R_k\left(f_{\alpha,\beta,\gamma}, I_k\right)\right| &\leq \frac{1}{2(q+1)^{1/q}}\left(\sum_{i=0}^{k-1} h_i^{q+1}\right)^{1/q}\left\|f'_{\alpha,\beta,\gamma}\right\|_{p,[0,1]} \\ &\leq \frac{1}{2(q+1)^{1/q}}\left(\sum_{i=0}^{k-1} h_i^{q+1}\right)^{1/q} \\ &\quad\times \gamma \max\{\alpha,\beta\}\exp\left[\gamma\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right] \\ &\quad\times \left[B\left(p(\alpha-1)+1, p(\beta-1)+1\right)\right]^{1/p}, \ \alpha,\ \beta > 1.\end{aligned} \tag{5.13}$$

Remark 1 Finally, we mention the following simple trapezoid quadrature rule

$$F(\alpha,\beta;\gamma) = T_k\left(f_{\alpha,\beta,\gamma}\right) + R_k\left(f_{\alpha,\beta,\gamma}\right),$$

where the remainder $R_k\left(f_{\alpha,\beta,\gamma}\right)$ satisfies the bounds

$$\begin{aligned}\left|R_k\left(f_{\alpha,\beta,\gamma}\right)\right| &\leq \frac{1}{4k}\left\|f'_{\alpha,\beta,\gamma}\right\|_{\infty,[0,1]} \\ &\leq \frac{1}{4k}\gamma\max\{\alpha,\beta\}\left(\frac{\alpha-1}{\alpha+\beta-2}\right)^{\alpha-1}\left(\frac{\beta-1}{\alpha+\beta-2}\right)^{\beta-1} \\ &\quad\times \exp\left[\gamma\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right], \ \alpha,\ \beta > 1,\end{aligned} \tag{5.14}$$

$$\begin{aligned}&\left|R_k\left(f_{\alpha,\beta,\gamma}\right)\right| \\ &\leq \frac{1}{2k}\left\|f'_{\alpha,\beta,\gamma}\right\|_{1,[0,1]} \\ &\leq \frac{1}{2k}\gamma\max\{\alpha,\beta\}\exp\left[\gamma\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right]B(\alpha,\beta), \ \alpha,\ \beta > 0\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}\left|R_k\left(f_{\alpha,\beta,\gamma}\right)\right| &\leq \frac{1}{2k(q+1)^{1/q}}\left\|f'_{\alpha,\beta,\gamma}\right\|_{p,[0,1]} \\ &\leq \frac{1}{2k(q+1)^{1/q}}\gamma\max\{\alpha,\beta\}\exp\left[\gamma\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\right]\end{aligned} \tag{5.16}$$

$$\times\left[B\left(p\left(\alpha-1\right)+1,p\left(\beta-1\right)+1\right)\right]^{1/p},\ \alpha,\ \beta>1.$$

The bounds above show that $R_k\left(f_{\alpha,\beta,\gamma}\right)\to 0$ when $k\to\infty$, and therefore $F\left(\alpha,\beta;\gamma\right)=\lim_{k\to\infty}T_k\left(f_{\alpha,\beta,\gamma}\right)$ for $\alpha,\ \beta>1$ and $\gamma>0$.

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On the Multiplicity of the Zeros of Polynomials with Constrained Coefficients

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Abstract We survey a few recent results focusing on the multiplicity of the zero at 1 of polynomials with constrained coefficients. Some closely related problems and results are also discussed.

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1 On the Multiplicity of the Zero at 1 of Polynomials with Constrained Coefficients

In [17] and [18], we examined a number of problems concerning polynomials with coefficients restricted in various ways. We were particularly interested in how small such polynomials can be on [0, 1]. For example, we proved that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$e^{-c_1\sqrt{n}} \le \min_{0 \not\equiv Q \in \mathcal{F}_n} \left\{ \max_{x \in [0,1]} |Q(x)| \right\} \le e^{-c_2\sqrt{n}}$$

for every $n \ge 2$, where $\mathcal{F}_n$ denotes the set of all polynomials of degree at most n with coefficients from $\{-1, 0, 1\}$.

Littlewood considered minimization problems of this variety on the unit disk. His most famous, now solved, conjecture was that the L_1 norm of an element $f \in \mathcal{F}_n$ on the unit circle grows at least as fast as $c \log N$, where N is the number of non-zero coefficients in f and $c > 0$ is an absolute constant.

When the coefficients are required to be integers, the questions have a Diophantine nature and have been studied from a variety of points of view.

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One key to the analysis is the study of the related problem of giving an upper bound for the multiplicity of the zero these restricted polynomials can have at 1. In [17] and [18], we answer this latter question precisely for the class of polynomials of the form

$$Q(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad |a_j| \le 1\,, \quad a_j \in \mathbb{C}\,, \quad j = 1, 2, \ldots, n\,,$$

with fixed $|a_0| \neq 0$.

Various forms of these questions have attracted considerable study, though rarely have precise answers been possible to give. Indeed, the classical, much studied, and presumably very difficult problem of Prouhet, Tarry, and Escott rephrases as a question of this variety. (Precisely: what is the maximal vanishing at 1 of a polynomial with integer coefficients with l_1 norm $2n$? It is conjectured to be n.)

For $n \in \mathbb{N}$, $L > 0$, and $p \ge 1$, let $\kappa_p(n, L)$ be the largest possible value of k for which there is a polynomial $Q \not\equiv 0$ of the form

$$Q(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad a_j \in \mathbb{C}\,, \qquad |a_0| \ge L\left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p}\,,$$

such that $(x-1)^k$ divides $Q(x)$.

For $n \in \mathbb{N}$ and $L > 0$, let $\kappa_\infty(n, L)$ be the largest possible value of k for which there is a polynomial $Q \not\equiv 0$ of the form

$$Q(x) = \sum_{j=0}^{n} a_j x^j\,, \qquad a_j \in \mathbb{C}\,, \qquad |a_0| \ge L \max_{1 \le j \le n} |a_j|\,,$$

such that $(x-1)^k$ divides $Q(x)$.

In [17], we proved that there is an absolute constant $c_3 > 0$ such that

$$\min\left\{\frac{1}{6}\sqrt{n(1-\log L)} - 1\,, n\right\} \le \kappa_\infty(n, L) \le \min\left\{c_3\sqrt{n(1-\log L)}\,, n\right\}$$

for every $n \in \mathbb{N}$ and $L \in (0, 1]$. However, we were far from being able to establish the right result in the case of $L \ge 1$. In [18], we proved the right order of magnitude of $\kappa_\infty(n, L)$ and $\kappa_2(n, L)$ in the case of $L \ge 1$. Our results in [17] and [18] sharpen and generalize the results of Schur [62], Amoroso [1], Bombieri and Vaaler [6], and Hua [49] who gave versions of this result for polynomials with integer coefficients. Our results in [17] have turned out to be related to a number of recent and old publications from a rather wide range of research areas. See [1–16, 18–67], for example. More results on the zeros of polynomials with Littlewood type coefficient constraints may be found in [37]. Markov and Bernstein type inequalities under Erdős type coefficient constraints are surveyed in [36].

For $n \in \mathbb{N}$, $L > 0$, and $q \geq 1$, let $\mu_q(n, L)$ be the smallest value of k for which there is a polynomial of degree k with complex coefficients such that

$$|Q(0)| > \frac{1}{L}\left(\sum_{j=1}^{n} |Q(j)|^q\right)^{1/q}.$$

For $n \in \mathbb{N}$ and $L > 0$, let $\mu_\infty(n, L)$ be the smallest value of k for which there is a polynomial of degree k with complex coefficients such that

$$|Q(0)| > \frac{1}{L} \max_{1 \leq j \leq n} |Q(j)|.$$

It is a simple consequence of Hölder's inequality (see Lemma 3.6 in [42]) that

$$\kappa_p(n, L) \leq \mu_q(n, L)$$

whenever $n \in \mathbb{N}$, $L > 0$, $1 \leq p, q \leq \infty$, and $1/p + 1/q = 1$.

In [42], we have found the size of $\kappa_p(n, L)$ and $\mu_q(n, L)$ for all $n \in \mathbb{N}$, $L > 0$, and $1 \leq p, q \leq \infty$. The result about $\mu_\infty(n, L)$ is due to Coppersmith and Rivlin, [27], but our proof presented in [42] is completely different and much shorter even in that special case. Another short proof of the Coppersmith–Rivlin inequality is presented in [41].

Our results in [17] may be viewed as finding the size of $\kappa_\infty(n, L)$ and $\mu_1(n, L)$ for all $n \in \mathbb{N}$ and $L \in (0, 1]$.

Our results in [18] may be viewed as finding the size of $\kappa_\infty(n, L)$, $\mu_1(n, L)$, $\kappa_2(n, L)$, and $\mu_2(n, L)$ for all $n \in \mathbb{N}$ and $L > 0$.

Our main results in [42] are stated below.

Theorem 1 *Let $p \in (1, \infty]$ and $q \in [1, \infty)$ satisfy $1/p + 1/q = 1$. There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\sqrt{n}(c_1 L)^{-q/2} - 1 \leq \kappa_p(n, L) \leq \mu_q(n, L) \leq \sqrt{n}(c_2 L)^{-q/2} + 2$$

for every $n \in \mathbb{N}$ and $L > 1/2$, and

$$c_3 \min\left\{\sqrt{n(-\log L)}, n\right\} \leq \kappa_p(n, L) \leq \mu_q(n, L) \leq c_4 \min\left\{\sqrt{n(-\log L)}, n\right\} + 4$$

for every $n \in \mathbb{N}$ and $L \in (0, 1/2]$. Here, $c_1 := 1/53$, $c_2 := 40$, $c_3 := 2/7$, and $c_4 := 13$ are the appropriate choices.

Theorem 2 *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1\sqrt{n(1-L)} - 1 \leq \kappa_1(n, L) \leq \mu_\infty(n, L) \leq c_2\sqrt{n(1-L)} + 1$$

for every $n \in \mathbb{N}$ and $L \in (1/2, 1]$, and

$$c_3 \min\left\{\sqrt{n(-\log L)}, n\right\} \le \kappa_1(n,L) \le \mu_\infty(n,L) \le c_4 \min\left\{\sqrt{n(-\log L)}, n\right\} + 4$$

for every $n \in \mathbb{N}$ *and* $L \in (0, 1/2]$. *Note that* $\kappa_1(n,L) = \mu_\infty(n,L) = 0$ *for every* $n \in \mathbb{N}$ *and* $L > 1$. *Here,* $c_1 := 1/5$, $c_2 := 1$, $c_3 := 2/7$, *and* $c_4 := 13$ *are the appropriate choices.*

Note that in [39], extending a result of Totik and Varjú in [66], we showed that if the modulus of a monic polynomial P of degree at most n, with complex coefficients, on the unit circle of the complex plane is at most $1 + o(1)$ uniformly, then the multiplicity of each zero of P outside the open unit disk is $o(n^{1/2})$. Equivalently, if a polynomial P of degree at most n, with complex coefficients and constant term 1, has modulus at most $1 + o(1)$ uniformly on the unit circle, then the multiplicity of each zero of P in the closed unit disk is $o(n^{1/2})$. These observations were obtained in [39] as a consequence of our "one-sided" improvement of an old Erdős–Turán Theorem in [43]. Namely in [39], we proved that if the zeros of

$$P(z) := \sum_{j=0}^{n} a_j z^j\,, \qquad a_j \in \mathbb{C}\,, \quad a_0 a_n \neq 0$$

are denoted by

$$z_j = r_j \exp(i\varphi_j)\,, \qquad r_j > 0\,, \quad \varphi_j \in [0, 2\pi)\,, \quad j = 1, 2, \cdots, n\,,$$

then for every $0 \le \alpha < \beta \le 2\pi$, we have

$$\sum_{j \in I_1(\alpha,\beta)} 1 - \frac{\beta - \alpha}{2\pi} n \le 16\sqrt{n \log R_1}\,,$$

and

$$\sum_{j \in I_2(\alpha,\beta)} 1 - \frac{\beta - \alpha}{2\pi} n \le 16\sqrt{n \log R_2}\,,$$

where

$$R_1 := |a_n|^{-1} \|P\|\,, \qquad R_2 := |a_0|^{-1} \|P\|\,,$$

and

$$I_1(\alpha,\beta) := \{j : \alpha \le \varphi_j \le \beta,\ r_j \ge 1\}\,, \qquad I_2(\alpha,\beta) := \{j : \alpha \le \varphi_j \le \beta,\ r_j \le 1\}\,.$$

Here, $\|P\|$ denotes the maximum modulus of P on the closed unit disk of the complex plane. For better constants in the Erdős–Turán Theorem in [43], see the recent paper [64] by Soundararajan, who also offers a very elegant new approach to prove the Erdős–Turán Theorem in [43].

2 Remarks and Problems

A question we have not considered in [42] is if there are examples of n, L, and p for which the values of $\kappa_p(n, L)$ are significantly smaller if the coefficients are required to be rational (perhaps together with other restrictions). The same question may be raised about $\mu_q(n, L)$. As the conditions on the coefficients of the polynomials in Theorems 1 and 2 are homogeneous, assuming rational coefficients and integer coefficients lead to the same results. Four special classes of interest are

$$\mathcal{F}_n := \left\{ Q : Q(z) = \sum_{j=0}^{n} a_j z^j \,, \quad a_j \in \{-1, 0, 1\} \right\} ,$$

$$\mathcal{N}_n := \left\{ Q : Q(z) = \sum_{j=0}^{n} a_j z^j \,, \quad a_j \in \{0, 1\} \right\} ,$$

$$\mathcal{L}_n := \left\{ Q : Q(z) = \sum_{j=0}^{n} a_j z^j \,, \quad a_j \in \{-1, 1\} \right\} ,$$

and

$$\mathcal{K}_n := \left\{ Q : Q(z) = \sum_{j=0}^{n} a_j z^j \,, \quad a_j \in \mathbb{C}, \quad |a_j| = 1 \right\} .$$

Elements of $\mathcal{F}_n$ are often called Borwein polynomials of degree at most n. Elements of $\mathcal{N}_n$ are often called Newman polynomials of degree at most n. Elements of $\mathcal{L}_n$ are often called Littlewood polynomials of degree n. Elements of $\mathcal{K}_n$ are often called unimodular or Kahane polynomials of degree n. In [17], we proved the following result.

Theorem 3 *Let $p \le n$ be a prime. Suppose $Q \in \mathcal{F}_n$ and Q has exactly k zeros at* 1 *and exactly m zeros at a primitive pth root of unity. Then*

$$p(m+1) \ge k \, \frac{\log p}{\log(n+1)} \,.$$

The proof of Theorem 3 is so simple that we reproduce it here.

Proof of Theorem 3 Let

$$\xi_j := \exp\left(\frac{2\pi i j}{p}\right), \qquad j = 1, 2, \ldots, p-1\,.$$

Let $Q \in \mathcal{F}_n$ be of the form

$$Q(x) = (x-1)^k R(x)\,,$$

where R is a polynomial of degree at most $n-k$ with integer coefficients. Then, for every integer $m \le k$, we have

$$Q^{(m)}(x) = (x-1)^{k-m} S(x)\,,$$

where S is a polynomial of degree at most $n-k$ with integer coefficients. Hence,

$$K := \prod_{j=1}^{p-1} Q^{(m)}(\xi_j) = \prod_{j=1}^{p-1} (\xi_j - 1)^{k-m} \prod_{j=1}^{p-1} S(\xi_j) =: p^{k-m} N\,,$$

where both K and N are integers by the fundamental theorem of symmetric polynomials. Further,

$$|K| \le \prod_{j=1}^{p-1} (n+1)n^m \le (n+1)^{(p-1)(m+1)}\,.$$

Hence, $K \neq 0$ implies

$$p^{k-m} \le (n+1)^{(p-1)(m+1)}\,,$$

that is,

$$k - m \le \frac{(p-1)(m+1)\log(n+1)}{\log p}\,,$$

and the result follows. □

The following three problems arise naturally, and they have been already raised in [10], for example.

Problem 1 How many zeros can a polynomial $0 \not\equiv Q \in \mathcal{F}_n$ have at 1?

Problem 2 How many zeros can a polynomial $Q \in \mathcal{L}_n$ have at 1?

Problem 3 How many zeros can a polynomial $Q \in \mathcal{K}_n$ have at 1?

The case when $p = \infty$ and $L = 1$ in our Theorem 1 gives that every $0 \not\equiv Q \in \mathcal{F}_n$, every $Q \in \mathcal{L}_n$, and every $Q \in \mathcal{K}_n$ can have at most $cn^{1/2}$ zeros at 1 with an absolute constant $c > 0$, but one may expect better results by utilizing the additional pieces of information on their coefficients.

It was observed in [17] that for every integer $n \geq 2$ there is a $Q \in \mathcal{F}_n$ having at least $c(n/\log n)^{1/2}$ zeros at 1 with an absolute constant $c > 0$. This is a simple pigeon hole argument. However, as far as we know, closing the gap between $cn^{1/2}$ and $c(n/\log n)^{1/2}$ in Problem 1 is an open and most likely very difficult problem.

It is proved in [11] that every polynomial P of the from

$$P(z) = \sum_{j=0}^{n} a_j z^j \,, \qquad |a_0| = 1, \;\; |a_j| \leq 1 \,, \;\; a_j \in \mathbb{C}$$

has at most $c\sqrt{n}$ zeros inside any polygon with vertices on the unit circle ∂D, where c depends only on the polygon. One of the main results of [19] gives explicit estimates for the number and location of zeros of polynomials with bounded coefficients. Namely, if

$$\delta_n := 33\pi \frac{\log n}{\sqrt{n}} \leq 1 \,,$$

then every polynomial P of the from

$$P(z) = \sum_{j=0}^{n} a_j z^j \,, \qquad |a_0| = |a_n| = 1, \;\; |a_j| \leq 1 \,, \;\; a_j \in \mathbb{C}$$

has at least $8\sqrt{n}\log n$ zeros in any disk with center on the unit circle and radius δ_n. More on Littlewood polynomials may be found in [7, 37], for example.

As far as Problem 2.3 is concerned, Boyd [23] showed that for $n \geq 3$ every $Q \in \mathcal{L}_n$ has at most

$$\frac{c(\log n)^2}{\log \log n} \tag{2.1}$$

zeros at 1, and this is the best known upper bound even today. Boyd's proof is very clever and up to an application of the Prime Number Theorem is completely elementary. It is reasonable to conjecture that for $n \geq 2$ every $Q \in \mathcal{L}_n$ has at most $c \log n$ zeros at 1. It is easy to see that for every integer $n \geq 2$ there are $Q_n \in \mathcal{L}_n$ with at least $c \log n$ zeros at 1 with an absolute constant $c > 0$. Indeed, the polynomials P_k defined by

$$P_k(z) = \prod_{j=0}^{k} (z^{2^j} - 1) \,, \qquad k = 1, 2, \ldots$$

has degree $2^{k+1} - 1$ and a zero of multiplicity $k + 1$ at 1. By using Boyd's elegant method, it is easy to prove also that if M_k denotes the largest possible multiplicity that a zero of a $P \in \mathcal{L}_k$ can have at 1 and (C_k) is an arbitrary sequence of positive integers tending to ∞, then

$$\lim_{n\to\infty} \frac{1}{n} \left|k \in \{1, 2, \dots, n\} : M_k \geq C_k\right| = 0\,.$$

This was proposed as a problem in the Monthly [40] in 2009, and a few people have solved it.

As far as Problem 3 is concerned, one may suspect that for $n \geq 2$ every $Q \in \mathcal{K}_n$ has at most $c \log n$ zeros at 1. However, just to see if Boyd's bound (2.1) holds for every $Q \in \mathcal{K}_n$ seems quite challenging and beyond reach at the moment.

Problem 4 How many zeros can a polynomial $P \in \mathcal{F}_n$ have at α if $|\alpha| \neq 1$ and $\alpha \neq 0$? Can it have as many as we want?

Problem 5 How many zeros can a polynomial $P \in \mathcal{L}_n$ have at α if $|\alpha| \neq 1$ and $\alpha \neq 0$? Can it have as many as we want?

The Mahler measure

$$M_0(P) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|P(e^{it})|\,dt\right)$$

is defined for bounded measurable functions P defined on the unit circle. It is well known that

$$M_0(P) := \lim_{q\to 0+} M_q(P)\,,$$

where, for $q > 0$,

$$M_q(P) := \left(\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{it})\right|^q\,dt\right)^{1/q}\,.$$

It is a simple consequence of the Jensen formula that

$$M_0(P) = |c| \prod_{k=1}^{n} \max\{1, |z_k|\}$$

for every polynomial of the form

$$P(z) = c\prod_{k=1}^{n}(z - z_k)\,, \qquad c, z_k \in \mathbb{C}\,.$$

Lehmer's conjecture is a problem in number theory raised by Derrick Henry Lehmer. The conjecture asserts that there is an absolute constant $\mu > 1$ such that for every polynomial P with integer coefficients satisfying $P(0) \neq 0$ we have either $M_0(P) = 1$ (that is, P is monic and has all its zeros on the unit circle) or $M_0(P) \geq \mu$.

The smallest known Mahler measure greater than 1 is taken for the "Lehmer's polynomial"

$$P(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1,$$

for which

$$M_0(P) = 1.176280818\ldots .$$

It is widely believed that this example represents the true minimal value: that is,

$$\mu = 1.176280818\ldots$$

in Lehmer's conjecture.

In 1973, Pathiaux [57] proved that if Q is an irreducible polynomial with integer coefficients and $M_0(Q) < 2$, then there exists a polynomial $P \in \mathcal{F}_n$ such that Q divides P. A remark at the end of this paper notes that the proof may be modified to establish the same result for reducible polynomials. Mignotte [52] found a simpler proof of this statement for irreducible polynomials Q with integer coefficients and derived an upper bound on the degree of P in terms of the degree of Q and $M_0(Q)$. His proof may also be extended to the reducible case. These results were generalized and strengthened by Bombieri and Vaaler in [6], as an application of their improved formulation of Siegel's lemma.

Similarly, it is a simple counting argument to show that if $k \geq 2$ is an integer, the monic polynomial Q has only integer coefficients, and $M_0(Q) < k$, then there is a polynomial P with integer coefficients in $[-k+1, k-1]$ such that Q divides P. See the hint to E.8 on page 23 of [7].

The result of Pathiaux [57] leads us to the following observations.

Remark 1 If

$$Q(z) := (z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1)^4\,,$$

then $M_0(Q) = (1.176280818\ldots)^4 < 2$, hence there is a polynomial $P \in \mathcal{F}_n$ such that Q divides P.

Remark 2 If Lehmer's conjecture is false, then the answer to Problem 4 is yes. Indeed, if Lehmer's conjecture is false, then for every $1 < \mu < 2$ there is a monic polynomial Q such that $1 < M_0(Q) \leq \mu$, so if $k := \lfloor \log 2/\log \mu \rfloor - 1$, then Q^k divides a polynomial $P \in \mathcal{F}_n$.

Remark 3 It remains open whether or not a polynomial $P \in \mathcal{F}_n$ with $P(0) = 1$ can have a zero α of multiplicity at least 5 outside the unit circle.

To find examples of Newman polynomials with constant term 1 and with at least one repeated zero outside the unit circle had been asked by Odlyzko and Poonen [56]. This question was later answered by Mossinghoff [54] who found examples of several such polynomials with repeated zeros outside the unit circle.

To find examples of Littlewood polynomials with at least one repeated zero outside the unit circle is also a very interesting problem. It is easy to see that such Littlewood polynomials must have odd degree. Drungilas, Jankauskas, and Šiurys [29] have found a Littlewood polynomial P of degree 195 such that $(x^3 - x + 1)^2$ divides P. See more in [29, 30, 34, 48].

We close this section by a version of an old and hard unsolved problem known as the already mentioned Tarry–Escott Problem.

Problem 6 Let $N \in \mathbb{N}$ be fixed. Let $a(N)$ be the smallest value of m for which there is a polynomial $P \in \cup_{n=1}^{\infty} \mathcal{F}_n$ with exactly m non-zero terms in it and with a zero at 1 with multiplicity at least N. Prove or disprove that $a(N) = 2N$.

To prove that $a(N) \geq 2N$ is simple. The fact that $a(N) \leq 2N$ is known for $N = 1, 2, \ldots, 12$, but the problem is open for every $N \geq 13$. In 1999, S. Chen found the first ideal solution with $N \geq 12$:

$$\begin{aligned} &0^k + 11^k + 24^k + 65^k + 90^k + 129^k + 173^k + 212^k + 237^k + 278^k + 291^k + 302^k \\ =&3^k + 5^k + 30^k + 57^k + 104^k + 116^k + 186^k + 198^k + 245^k + 272^k + 297^k + 299^k , \end{aligned}$$

valid for all $k = 1, 2, \ldots, 11$.

The best known upper bound for $a(N)$ in general seems to be $a(N) \leq cN^2 \log N$ with an absolute constant $c > 0$. See [21]. Even improving this (like dropping the factor $\log N$) would be a significant achievement. Note that for every integer $n \geq 2$ there is a polynomial $Q \in \mathcal{F}_n$ having at least $c(n/\log n)^{1/2}$ zeros at 1 with an absolute constant $c > 0$. This was observed in [17] based on a simple counting argument. The inequality $a(N) \leq cN^2 \log N$ with an absolute constant $c > 0$ follows simply from this.

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Generalized Barycentric Coordinates and Sharp Strongly Negative Definite Multidimensional Numerical Integration

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Abstract This paper is devoted to study and construct a family of multidimensional numerical integration formulas (cubature formulas), which approximate all *strongly* convex functions from above. We call them strongly negative definite cubature formulas (or for brevity snd-formulas). We attempt to quantify their sharp approximation errors when using continuously differentiable functions with Lipschitz continuous gradients. We show that the error estimates based on such cubature formulas are always controlled by the Lipschitz constants of the gradients and the error associated with using the quadratic function. Moreover, assuming the integrand is itself strongly convex, we establish sharp upper as well as lower refined bounds for their error estimates. Based on the concepts of barycentric coordinates with respect to an arbitrary polytope P, we provide a necessary and sufficient condition for the existence of a class of snd-formulas on P. It consists of checking that such coordinates exist on P. Then, the Delaunay triangulation is used as a convenient partition of the integration domain for constructing the best piecewise snd-formulas in L^1 metric. Finally, we present numerical examples illustrating the proposed method.

1 Introduction, Motivation, and Terminology

This paper constitutes the progression of previous works [4, 5, 7, 8], which focused on the study of some classes of multidimensional numerical integration in the context of the classical notion of convexity. Here, our objective is to extend the

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results given there for strongly convex functions. To describe our problem of integration from a numerical standpoint more precisely, let $\Omega \subset \mathbb{R}^d$ be a non-empty compact convex set and $f : \Omega \to \mathbb{R}$ be a given function. We sometimes know beforehand that the function f satisfies various known structural and regularity properties. For example, it may be known that f has some additional kind of convexity; therefore, we would wish to use this information in order to get most appropriate methods for numerical integration of f. In this paper, to get a better approximation of the integral of our function, we try to approximate it using cubature formulas, which approximate the integral of all *strongly* convex functions from above. The strongly convex functions are widely applied in economic theory (see [23]) and are also central to optimization theory (see [14]). Indeed, in the framework of function minimization, this convexity notion has important and well-known implications. As we will see, the key advantage of using cubature formulas of such kind is that their associated approximation errors can always be controlled by the error associated with using the quadratic function. Hence, if we want a more accurate approximation of the integral of our function, we need to find a better approximation of the integral of the quadratic function.

To appreciate the problem more clearly, let us start by describing briefly a specific one-dimensional example, since its simplicity helps us better understand all the necessary steps through very simple explicit computations. Assume that μ is a fixed nonnegative real number. In one-dimensional numerical integration, say on an interval $[a, b]$, a simple way of approximating the integral of a given real μ-strongly convex function $f : [a, b] \to \mathbb{R}$ is first to choose a partition $P := \{x_0, x_1, \dots, x_n\}$ of the interval $[a, b]$, such that $a = x_0 < x_1 < \dots < x_n = b$, and then to apply the classical local trapezoidal quadrature rule $T_i(f) = \frac{f(x_{i-1})+f(x_i)}{2}$ on each subinterval $I_i := [x_{i-1}, x_i]$, $i = 1, \dots, n$, and to sum up the results. Among its many important properties, this rule satisfies the well-known Hermite–Hadamard inequality, which ensures an upper estimate for the exact value of the integral of any convex function:

$$\frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} f(t)dt \le T_i(f), \ (i = 1, \dots, n), \tag{1}$$

where the sign of equality being achieved if f is an affine function. Recall that the local trapezoidal rule $T_i(f)$ could be obtained by integrating the barycentric approximation operator:

$$B_i[f](x) := \lambda_{i-1}(x) f(x_{i-1}) + \lambda_i(x) f(x_i), \ (x \in I_i),$$

where $\lambda_{i-1}(x)$ and $\lambda_i(x)$ are the barycentric coordinates of x with respect to I_i, which are defined as

$$\lambda_{i-1}(x) := \frac{x - x_i}{x_{i-1} - x_i}, \ \lambda_i(x) := \frac{x - x_{i-1}}{x_i - x_{i-1}}, \ (x \in I_i).$$

Observe that B_i is a first-order barycentric polynomial interpolating f at two points, x_{i-1} and x_i and that the weights λ_{i-1} and λ_i can expressed as

$$\lambda_{i-1}(x) = \frac{1}{\text{length}(I_i)}\begin{vmatrix}1 & x\\ 1 & x_i\end{vmatrix},\ \lambda_i(x) = \frac{1}{\text{length}(I_i)}\begin{vmatrix}1 & x_{i-1}\\ 1 & x\end{vmatrix}.$$

Rearranging terms, it is clear that these weights are nonnegative on I_i, and moreover they satisfy

$$\lambda_{i-1}(x) + \lambda_i(x) = 1,\ x = \lambda_{i-1}(x)x_{i-1} + \lambda_i(x)x_i, (x \in I_i). \tag{2}$$

The trapezoidal rule is the simplest, most well-known, and widely used quadrature rule. The reason for this popularity lies in the large number of useful theoretical and computational properties of this rule. It actually served as basic ingredients for constructing more accurate and adaptive formulas. For this reason, this rule together with its fundamental inequality (1) has been an effective starting point for several subsequent investigations, see [2, 6]. Furthermore, in the local error analysis of the rule $T_i(f)$,

$$ET_i(f) := T_i(f) - \frac{1}{x_i - x_{i-1}}\int_{x_{i-1}}^{x_i} f(t)dt,$$

estimate of (1) is a very useful tool. Indeed, let $(.)^2$ denote the square function $t \to t^2$, and assume that the first derivative of f is a Lipschitz function with a Lipschitz constant $L(f')$ in $[a, b]$ (or $f \in C^{1,1}[a, b]$), then Hermite–Hadamard inequality implies the following upper local estimation:

$$|ET_i(f)| \leq \frac{ET_i\left((.)^2\right)}{2}L(f') \tag{3}$$

$$= \frac{T_i\left(\left(. - \frac{x_{i-1}+x_i}{2}\right)^2\right)}{3}L(f') \tag{4}$$

$$= \frac{(x_i - x_{i-1})^2}{12}L(f'), \tag{5}$$

where equality is attained for all quadratic functions. The literature contains a number of variations of these estimations, some statements employing the largest absolute value of the second derivative over the interval $[a, b]$. In addition, if f is μ-strongly convex, then the following lower local estimation also holds for all $i = 1, \ldots, n$,

$$ET_i(f) \geq \frac{ET_i\left((.)^2\right)}{2}\mu \tag{6}$$

$$= \frac{T_i\left(\left(\cdot - \frac{x_{i-1}+x_i}{2}\right)^2\right)}{3}\mu \tag{7}$$

$$= \frac{(x_i - x_{i-1})^2}{12}\mu. \tag{8}$$

We did not find any reference to such result. However, the abovementioned estimates can be derived as an immediate consequence of our multivariate general results, see Remark 3. Estimates (3) and (6) say that for the trapezoidal rule, we can always control its approximation error by the Lipschitz constants of the first derivative, the parameter (of the strong convexity), and the error associated with using the quadratic function. It should also be noted that equalities in (3) and (6) are satisfied for all μ-strongly convex functions of the form

$$f(x) = a(x) + \frac{\mu}{2}x^2, \tag{9}$$

where $a(\cdot)$ is any affine function. Therefore, in this sense, the error estimates (3) and (6) are sharp for the class of μ-strongly convex functions having Lipschitz continuous first derivatives. This provides the starting point of the present study. Indeed, the present contributions of this paper are twofold: first, we would like to consider the general multivariate variable case. More precisely, this paper deals with the problem of approximation of the integral of multivariate functions by snd-formulas, that is, those which approximate from above all strongly convex functions with Lipschitz continuous gradients. Geometrically, if a function f belongs to such class, then its gradient ∇f cannot change too quickly and it cannot change too slowly either. Functions satisfying these conditions are widely used in the optimization literature, we refer to Nesterov's book [14].

Hence, the questions that arise, as a natural consequence of the estimates (3) and (6), are the following:

- Can we extend the one-dimensional approach to construct a natural multivariate version of the trapezoidal quadrature rule in any polytope?
- Can the approximation errors for such cubature formulas satisfy similar lower and upper bounds in the multidimensional case?

We will answer these questions positively by defining and studying a class of snd-formulas on an arbitrary polytope to approximate the integral of a function by piecewise cubature formulas. Our extensions are derived in a natural way by using the generalized barycentric coordinates, which turn out to be appropriate to the more general multivariate setting. In particular, we will show how the Delaunay triangulation can be used as a convenient partition of the integration domain for constructing the best piecewise snd-formulas in L^1 metric.

This paper is organized as follows: In the next section, we briefly recall key notions and notations. Then, we introduce the notion of strong convexity and establish two general characterization results (see Lemmas 1 and 2). These general results provide two equivalent conditions for a linear functional to be negative in the set of convex functions. We then use them to establish a first characterization of the approximation error of our class of cubature formulas. In order to provide a second characterization result, Section 3 defines the notion of generalized barycentric coordinates on polytopes and gives an existence result of them in any polytope. Here, we provide a necessary and sufficient condition for the existence of the snd-formulas. It consists of checking the existence of a set of these coordinates. Section 4 uses the generalized barycentric coordinates to construct a multivariate version of the classical trapezoidal rule in arbitrary higher-dimensional polytopes. As a result, we get explicit lower and upper bounds for the approximation error when using continuously differentiable functions with Lipschitz continuous gradients. Indeed, analogously to the one-dimensional estimates (3) and (6), we offer sharp error estimates that only depend on the parameter of the strong convexity, the Lipschitz constants of the gradients, and the error associated with using the quadratic function. In Section 5, using the Delaunay triangulation as a partition of a polytope, we present an explicit construction of our sharp cubature schemes. Finally, Section 6 will provide a numerical example to illustrate the efficiency of this approach.

2 General Setting

Our main results in this section first concern two characterization results of any negative linear functional in the set of convex functions, which hold in a general framework and will be repeatedly applied in the sequel. We will start in this section with some of the basic properties of strong convex functions. But first, we need to introduce some notations, which follow closely those of [3]. Let Ω be a subset of $\mathbb{R}^d$. As usual, we mean by Ω° the interior of Ω. We say that Ω is *measurable* if it has a finite Lebesgue measure, which we denote by $|\Omega|$. For measurable Ω, the class $L^1(\Omega)$ comprises all Lebesgue integrable functions $f : \Omega \to \mathbb{R}$. A property holds *almost everywhere* (abbreviated by a.e.) on Ω if it holds on Ω except for a set of measure zero. Furthermore, we denote by $C(\Omega)$ the class of all real-valued continuous functions on Ω and by $C^k(\Omega)$, where $k \in \mathbb{N}$, the subclass of all functions that are k times continuously differentiable. It is convenient to agree that $C^0(\Omega) = C(\Omega)$. We denote by $\|.\|$ the Euclidean norm in $\mathbb{R}^d$ and $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ the standard inner product of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$. By $C^{1,1}(\Omega)$, we denote the subclass of all functions f, which are continuously differentiable on Ω with Lipschitz continuous gradients, i.e., there exists $L(\nabla f)$ such that

$$\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\| \leq L(\nabla f)\, \|\boldsymbol{x} - \boldsymbol{y}\|\,, \ (\boldsymbol{x}, \boldsymbol{y} \in \Omega).$$

We now present the notion of strong convexity, which generalizes the classical definition of convexity.

Definition 1 A function f is called strongly convex with parameter $\mu > 0$ if $\operatorname{dom} f$ is convex and the strong Jensen inequality holds: for any $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{dom} f$ and $t \in [0, 1]$,

$$f(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \leq tf(\boldsymbol{x}) + (1-t)f(\boldsymbol{y}) - \frac{\mu}{2}t(1-t)\|\boldsymbol{x} - \boldsymbol{y}\|^2. \tag{10}$$

A simple calculation reveals that this definition is equivalent to the convexity of $g := f - \frac{\mu}{2}\|.\|^2$. See [11, Prop 1.1.2] for a direct proof of this result, what was derived using the identity

$$(1-t)\|\boldsymbol{x}\|^2 + t\|\boldsymbol{y}\|^2 - \|(1-t)\boldsymbol{x} + t\boldsymbol{y}\|^2 = t(1-t)\|\boldsymbol{x} - \boldsymbol{y}\|^2.$$

Obviously, every strongly convex function is convex. Observe also that, for instance, affine functions are not strongly convex and if $\mu = 0$, we can get the classical definition of convexity.

Remark 1 For any positive real number μ, the following functions are μ-strongly convex functions:

1. $\frac{\rho}{2}\|.\|^2$, $(\mu \leq \rho)$.
2. Addition of a convex function to a strongly convex function gives a strongly convex function with the same modulus of strong convexity. Therefore, adding a convex function to $\frac{\mu}{2}\|.\|^2$ does not affect μ-strong convexity.

Now, we state a first characterization result of linear functionals, which are negative in the set of convex functions. It is shown that in order to prove such property for the given functional E, it suffices to check that E is negative in a subset of strongly convex functions with a given fixed strong convexity parameter. Recall the obvious inclusion, the set of strongly convex functions is contained in the set of convex functions.

Lemma 1 *Let $\Omega \subset \mathbb{R}^d$ be a compact convex set. Let μ be an arbitrary, fixed real number, and let E be a linear functional defined on $C(\Omega)$. Then, the following conditions are equivalent:*

(i) *For every convex function $f \in C(\Omega)$, we have*

$$E(f) \leq 0.$$

(ii) *For every μ-strongly convex function $f \in C(\Omega)$, we have*

$$E(f) \leq 0.$$

Proof (i) implies (ii) is the trivial part of the proof. Indeed, assume that (i) holds. Let f be μ-strongly convex function. Set $g := f - \frac{\mu}{2}\|.\|^2$. By definition, g is therefore convex. Hence, applying property (i), it follows, by linearity of E

$$E(f) \leq \frac{\mu}{2} E(\|.\|^2).$$

Since $\|.\|^2$ is convex, then again by (i) we have $E(\|.\|^2) \leq 0$. This shows that (ii) holds.
Now, assume that (ii) holds. Let ε be a positive real number, and let f be a convex function. Define the function g by

$$g := f + \frac{\varepsilon}{2}\|.\|^2.$$

Noting that

$$\frac{\mu}{\varepsilon} f = \frac{\mu}{\varepsilon} g - \frac{\mu}{2}\|.\|^2$$

and since $\frac{\mu}{\varepsilon} f$ is convex, then by the definition of strong convexity $\frac{\mu}{\varepsilon} g$ is μ-strongly convex. Hence, by (ii), we can conclude that

$$E\left(\frac{\mu}{\varepsilon} g\right) \leq 0.$$

Thus, it follows that

$$E(g) \leq 0,$$

or equivalently, by virtue of the linearity of E,

$$E(f) \leq -\frac{\varepsilon}{2} E(\|.\|^2).$$

In view of the fact that this inequality holds for all $\varepsilon > 0$, then by letting $\varepsilon \downarrow 0$, it follows that

$$E(f) \leq 0.$$

Hence, the desired statement (i) is valid and thus means that these two statements are equivalent. □

If, in addition, the functions belong to $C^{1,1}(\Omega)$, then our second characterization result is given in the following:

Lemma 2 *Let $\Omega \subset \mathbb{R}^d$ be a compact convex set. Let $E : C^k(\Omega) \to \mathbb{R}$, where $k \in \{0, 1\}$, be a linear functional, and let μ be a positive real number. Then, the two following statements are equivalent:*

(i) *For every μ-strongly convex function $g \in C^{1,1}(\Omega)$, we have*

$$E[g] \leq 0. \tag{11}$$

(ii) *For every* $f \in C^{1,1}(\Omega)$ *with* $L(\nabla f)$*-Lipschitz gradient, we have*

$$|E[f]| \leq -E\left[\|.\|^2\right].\frac{L(\nabla f)}{2}. \tag{12}$$

Equality is attained for all functions of the form

$$f(\boldsymbol{x}) := a(\boldsymbol{x}) + c\|.\|^2, \tag{13}$$

where $c \in \mathbb{R}$, *and* $a(\cdot)$ *is any affine function.*

Proof First, we prove (i) implies (ii). Let f be any function from $C^{1,1}(\Omega)$ with $L(\nabla f)$-Lipschitz gradient. Define the following two functions:

$$g_{\pm} := \|.\|^2 \frac{L(\nabla f)}{2} \pm f.$$

Then, according to [3, Proposition 2.2], we know that both of these functions belong to $C^{1,1}(\Omega)$ and are also convex. Hence, by (i) and Lemma 1, it follows that the functions g_- and g_+ satisfy

$$E[g_{\pm}] \leq 0.$$

Then, by linearity of E and a simple manipulation, we find that

$$E\left[\|.\|^2\right]\frac{L(\nabla f)}{2} \leq E[f] \leq -E\left[\|.\|^2\right]\frac{L(\nabla f)}{2}.$$

This is equivalent to (12) and shows that property (ii) also holds.

Now, let us assume that (ii) holds. Then, we deduce that

$$E\left[\|.\|^2\right] \leq 0, \tag{14}$$

Let $g \in C^{1,1}(\Omega)$ be any μ-strongly convex function and set

$$f := \frac{L(\nabla g)}{2}\|.\|^2 - g.$$

Then, according to [3, Proposition 2.2], we have

$$f \in C^{1,1}(\Omega) \text{ and } L(\nabla f) \leq L(\nabla g). \tag{15}$$

Since

$$g = \frac{L(\nabla g)}{2}\|.\|^2 - f,$$

it can be written as follows:

$$g = \left(\|.\|^2 \frac{L(\nabla f)}{2} - f \right) + \|.\|^2 \left(\frac{L(\nabla g)}{2} - \frac{L(\nabla f)}{2} \right),$$

we therefore obtain

$$E[g] = E\left[\|.\|^2 \frac{L(\nabla f)}{2} - f \right] + E\left[\|.\|^2 \right] \left(\frac{L(\nabla g)}{2} - \frac{L(\nabla f)}{2} \right).$$

Finally, by combining (ii), (14), and (15), we can conclude that (i) is valid. For the statement on the occurrence of equality, it is enough to note that a linear functional E satisfying (11) for all convex functions must vanish for affine functions. □

We now define our new general class of cubature formulas, which we formulate as follows:

Definition 2 Let $\Omega \subset \mathbb{R}^d$ be a compact set, and let μ be a positive real number. For n points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \Omega$, called nodes, and associated positive numbers $A_1, \ldots, A_n$, we say that

$$\left\{ (A_i, \boldsymbol{x}_i) : i = 1, \ldots, n \right\} \tag{16}$$

defines the μ-strongly negative definite cubature formula

$$\int_\Omega f(\boldsymbol{x}) d\boldsymbol{x} = \sum_{i=1}^{n} A_i f(\boldsymbol{x}_i) + E[f], \tag{17}$$

if the approximation error E satisfies

$$E[f] \le 0, \tag{18}$$

for all μ-strongly convex functions $f \in C(\Omega)$.

We say that (17) is a μ *snd-formula* for short. We also call (16) a μ *snd-system*, which is said to be of *length* n if the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are distinct. Let us mention that any μ snd-cubature formula approximates the exact value of the integral of a μ-strongly convex function from above. This means that the approximation error for such cubature formulas is negative on the set of μ-strongly convex functions.

Remark 2 Note that a μ snd-cubature formula as specified in Definition 2 is always of order two. In fact, by Lemma 2 and inequality (12), the functional E vanishes for affine functions, and so the order is at least two. However, if the order were greater than two, then (12) would imply that $E[f] = 0$ for all $f \in C^{1,1}(\Omega)$. Recall that, in the univariate case, a quadrature rule is snd-formula if and only if its second Peano kernel is greater than zero or less than zero; see [9, Chap.II.4] or [10, Chap. 4.3].

In the theory of inequalities, inequality (18), with E defined by (17) and valid for all μ-strongly convex functions, has also been called upper Hermite–Hadamard inequality.

We now present a characterization of our class of cubature formulas in terms of their associated error functionals. Indeed, we show that for functions in $C^{1,1}(\Omega)$, the error estimates based on such cubature formulas are always controlled by the Lipschitz constants of the gradients, the strong convexity parameter, and the error associated with using the quadratic function. This result is a direct consequence of Lemmas 1 and 2.

Theorem 1 *Let $\Omega \subset \mathbb{R}^d$ be a compact convex set. A cubature formula* (17) *is μ-strongly snd-formula if and only if for all μ-strongly convex functions $f \in C^{1,1}(\Omega)$, its error functional satisfies*

$$-\frac{\mu}{2}E\left[\|.\|^2\right] \leq E\left[f\right] \leq -E\left[\|.\|^2\right].\frac{L(\nabla f)}{2}. \tag{19}$$

In (19)*, equality is attained for all functions of the form*

$$f(\boldsymbol{x}) := a(\boldsymbol{x}) + \frac{\mu}{2}\|.\|^2,$$

where $a(\cdot)$ is any affine function.

In order to describe the second constructive method, we introduce the following notion.

3 Generalized Barycentric Coordinates on Polytopes

In this section, we start by giving a brief overview of the basic elements of barycentric coordinates in d dimensions, see, e.g., [12, pp. 132–135] for more details. Let us quickly recall how these so-called coordinates are defined. Fix an integer $n \geq 1$, and let $W := \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n\}$ be a finite subset of distinct but otherwise arbitrary points in $\mathbb{R}^d$. The following linear combination,

$$\boldsymbol{b} = \sum_{i=0}^{n} \alpha_i \boldsymbol{x}_i \tag{20}$$

is called a convex combination if the coefficients α_i are all nonnegative. All convex combinations of points of the set W define the convex hull of the set W. The resulting set is a convex set $conv(W)$, i.e., the smaller convex set containing W. Following the terminology of [22], a *convex* polytope Ω, or simply a polytope, we mean a set that is the convex hull of a non-empty finite set of points $W \subset \mathbb{R}^d$.

From now on, let $\Omega \subset \mathbb{R}^d$ be a (convex) polytope generated from a finite subset of points in $\mathbb{R}^d$, $W := \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n\}$, i.e., $\Omega = conv(W)$.
A vector $\boldsymbol{x} \in \mathbb{R}^d$ is an extreme point of Ω if $\boldsymbol{x} \in \Omega$ and $\boldsymbol{x}$ cannot be expressed as a convex combination of two vectors of Ω, both of which are different from $\boldsymbol{x}$. The set of extreme points of the polytope Ω shall be denoted by $Vert(\Omega)$. It is well known that the convex hull of a finite set W is compact, and its set of extreme points is non-empty and included in W. That is, $Vert(\Omega) \neq \emptyset$ and $Vert(\Omega) \subset W$. In what follows, we assume that the number of vertices of Ω is greater than 2.
Introduced by Möbius in 1827 as mass points to define a coordinate-free geometry [20], barycentric coordinates over polytopes are a very common tool in many computations and have many useful applications, ranging from Gouraud and Phong shading, rendering of quadrilaterals, image warping, mesh deformation, and finite element applications, see, e.g., [15, 21]. Given a polytope $\Omega = conv(\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n\})$, we wish to construct one coordinate function $\lambda_i(\boldsymbol{x})$ per point $\boldsymbol{x}_i$ for all $\boldsymbol{x} \in \Omega$. These functions are called barycentric coordinates with respect to $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n\}$ (or Ω) if they satisfy three properties. First, the coordinate functions are nonnegative on Ω,

$$\lambda_i(\boldsymbol{x}) \geq 0 \tag{21}$$

for all $\boldsymbol{x} \in \Omega$. Second, the functions form a partition of unity, which means that the equation

$$\sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) = 1 \tag{22}$$

is obtained for all $\boldsymbol{x} \in \Omega$. Finally, the functions act as coordinates in that, given a value of $\boldsymbol{x}$, weighting each point $\boldsymbol{x}_i$ by $\lambda_i(\boldsymbol{x})$ returns back $\boldsymbol{x}$, i.e.,

$$\boldsymbol{x} = \sum_{i=0}^{n} \lambda_i(\boldsymbol{x})\boldsymbol{x}_i. \tag{23}$$

This last property is also sometimes referred to as linear precision since the coordinate functions can reproduce linear functions. For most potential applications, it is also preferable that these coordinate functions are as smooth as possible. Constructing the barycentric coordinates of a point $\boldsymbol{x}$ with respect to some given points in a polytope Ω is often not a trivial task. For simplices, barycentric coordinates are a very common tool in many computations. Basically, they are defined as follows: let $\boldsymbol{X}_d = \{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d\}$ be any linearly independent set of $d+1$ points in $\mathbb{R}^d$, and the simplex T with the set of vertices $\boldsymbol{X}_d$ is the convex hull of $\boldsymbol{X}_d$ (e.g., a triangle in 2D or a tetrahedron in 3D). Let $A_i(\boldsymbol{x})$ be the signed volume (or area) of the subsimplex of T created with the vertex $\boldsymbol{v}_i$ replaced by $\boldsymbol{x}$. Then, the barycentric coordinate functions $\{\lambda_0, \ldots, \lambda_d\}$ of the simplex T with respect to its vertices are uniquely defined by

$$\lambda_i(\boldsymbol{x}) = \frac{A_i(\boldsymbol{x})}{vol(T)}, \tag{24}$$

where $vol(T)$ will mean the volume measure of T. It is easily seen that each point $\boldsymbol{x}$ of T has a (unique) representation and that $\boldsymbol{x} = \sum_{i=0}^{d} \lambda_i(\boldsymbol{x})\boldsymbol{v}_i$ and the barycentric coordinates $\{\lambda_0, \dots, \lambda_d\}$ are nonnegative affine functions on T. The uniqueness of this representation allows the weights $\lambda_i(\boldsymbol{x})$ to be interpreted as an alternative set of coordinates for point $\boldsymbol{x}$, the so-called barycentric coordinates. Note that a d-simplex is a special polytope given as the convex hull of $d+1$ vertices in d dimensions, each pair of which is joined by an edge. For $n > d$, which is the case of interest in this paper, the linear constraints form an under-determined system.
Barycentric coordinates also exist for more general types of polytopes and will be a crucial ingredient in what follows. Indeed, we have, see [13, Theorem 2]:

Theorem 2 *Let $W = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_n\}$ be a set of finite points of $\mathbb{R}^d$, and let the polytope $\Omega = conv(W)$. Then, there exist nonnegative real-valued continuous functions $\lambda_0, \lambda_1, \dots, \lambda_n$ defined on Ω such that*

$$\boldsymbol{x} = \sum_{i=0}^{n} \lambda_i(\boldsymbol{x})\boldsymbol{x}_i \quad \textit{and} \quad \sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) = 1, , \tag{25}$$

for each $\boldsymbol{x} \in \Omega$.

Thus, from now on, it proves useful to work with barycentric coordinates. Therefore, unless otherwise indicated, it is assumed that $\lambda_i(\boldsymbol{x}), i = 0, \dots, n$, are the barycentric coordinates of $\boldsymbol{x}$ with respect to a set of finite fixed points $\{\boldsymbol{x}_0, \dots, \boldsymbol{x}_n\}$ of the polytope

$$\Omega = \mathrm{conv}(\{\boldsymbol{x}_0, \dots, \boldsymbol{x}_n\}).$$

We shall not always trouble to repeat this at each stage. Furthermore, they do not need to be the vertices of Ω, of course, the polytope Ω may be generated by another different set of points $\left\{\boldsymbol{y}_0, \dots, \boldsymbol{y}_k\right\}$ on Ω.

Note also that Equation (25) can be rewritten in the following general way:

$$\sum_{i=0}^{n} \lambda_i(\boldsymbol{x})\,(\boldsymbol{x} - \boldsymbol{x}_i) = 0, \tag{26}$$

which obviously implies

$$\sum_{i=0}^{n} \int_{\Omega} \lambda_i(\boldsymbol{x})\,(\boldsymbol{x} - \boldsymbol{x}_i)\ d\boldsymbol{x} = 0. \tag{27}$$

Characterization of snd-Cubature Formulas in Terms of the Existence of a Set of Barycentric Coordinates

From now on, let $\Omega \subset \mathbb{R}^d$ be a compact convex polytope of positive measure, and let $\boldsymbol{X} := \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ be a finite subset that includes the vertices of Ω. Thus, the convex hull of $\boldsymbol{X}$ must be equal to Ω. Now, we provide a necessary and sufficient condition for the existence of the snd-formulas. It consists of checking the existence of a set of barycentric coordinates.

Theorem 3 *A set $\mathfrak{a} = \{(A_i, \boldsymbol{x}_i) : i = 1, \ldots, n\}$ defines a μ snd-cubature formula on Ω if and only if there exists a set of barycentric coordinates $\{\lambda_1, \ldots, \lambda_n\}$ on Ω such that*

$$\boldsymbol{x} = \sum_{i=1}^{n} \lambda_i(\boldsymbol{x})\boldsymbol{x}_i \quad (a.e.\ on\ \Omega), \tag{28}$$

and

$$A_i = \int_{\Omega} \lambda_i(\boldsymbol{x}) d\boldsymbol{x} \quad (i = 1, \ldots, n). \tag{29}$$

Proof Let $\{(A_i, \boldsymbol{x}_i) : i = 1, \ldots, n\}$ define a μ snd-cubature formula on Ω. Then, according to the definition, the error functional E satisfies, for any μ-strongly convex function f,

$$E[f] \leq 0. \tag{30}$$

We deduce then by Lemma 1 that, for every convex function $g \in C(\Omega)$, we have

$$E[g] \leq 0. \tag{31}$$

This means that the estimate

$$\int_{\Omega} g(\boldsymbol{x}) d\boldsymbol{x} \leq \sum_{i=1}^{n} A_i g(\boldsymbol{x}_i)$$

holds for every convex function $g \in C(\Omega)$. Hence, by [7, Theorem 2.1a, p.97], there exists a set of barycentric coordinates $\{\lambda_1, \ldots, \lambda_n\}$ on Ω, which satisfies the required conditions (28) and (29).

Conversely, assume that there exists a set of barycentric coordinates $\{\lambda_1, \ldots, \lambda_n\}$ on Ω, such that conditions (28) and (29) hold. Let f be convex on Ω. Then, since f is convex, by Jensen's inequality, it follows from (28) that

$$f(\boldsymbol{x}) \leq \sum_{i=1}^{n} \lambda_i(\boldsymbol{x}) f(\boldsymbol{x}_i).$$

Integrating both sides over Ω and using (29), we obtain the inequality

$$E[f] := \int_{\Omega} f(\boldsymbol{x}) d\boldsymbol{x} - \sum_{i=1}^{n} A_i f(\boldsymbol{x}_i) \leq 0.$$

Since the above inequality holds for every convex function, then according to Lemma 1, we also have, for every μ-strongly convex function,

$$E[f] \leq 0. \tag{32}$$

This shows that $\{(A_i, \boldsymbol{x}_i) : i = 1, \ldots, n\}$ defines a μ snd-cubature formula on Ω.

□

4 Integral Approximation Using Barycentric Coordinates

Many of useful properties of the classical trapezoidal quadrature rule (1) on the interval $[a, b]$ can be carried over directly to the d-dimensional hypercube $\prod_{i=1}^{d}[a_i, b_i]$ by using tensor products of d copies of this latter. Non-tensorial constructions of the trapezoidal curbature formula are rare in the case of an arbitrary polytope. In general, leaving the tensor product setting causes a lot of difficulties in theoretical as well as in computational aspects. From the theoretical point of view, it gets harder to find a suitable set of barycentric coordinates needed for their constructions as we did for the one-dimensional case. An example of a non-tensorial construction on surplices with the derivation of an efficient computational scheme for the trapezoidal cubature formulas can be found in [6]. Using generalized barycentric coordinates, this section shows how the simple univariate trapezoidal rule (1) can be extended to arbitrary higher-dimensional polytopes. To this end, let $X_m = \{\boldsymbol{x}_i\}_{i=0}^{m}$ be a given finite set of pairwise distinct points in $\Omega \subset \mathbb{R}^d$, with $\Omega = \text{conv}(X_m)$ denoting the convex hull of the point set X_m. We are interested in approximating the integral of an unknown function $f : \Omega \to \mathbb{R}$ from given function values $f(\boldsymbol{y}_0), \ldots, f(\boldsymbol{y}_n)$, where $Y_n := \{\boldsymbol{y}_i\}_{i=0}^{n} \subset \Omega$. In order to obtain a simple and stable global approximation of the integral of f on Ω, we may consider a μ snd-cubature formula of the following form:

$$I_n[f] := \sum_{i=0}^{n} A_i f(\boldsymbol{y}_i). \tag{33}$$

Theorem 3 tells us that there exists a set of barycentric coordinates $\{\lambda_1, \ldots, \lambda_n\}$ on Ω such that

$$\boldsymbol{x} = \sum_{i=0}^{n} \lambda_i(\boldsymbol{x})\boldsymbol{y}_i \quad (a.e.\ on\ \Omega), \tag{34}$$

and

$$A_i = \int_{\Omega} \lambda_i(\boldsymbol{x})d\boldsymbol{x} \quad (i = 0, \ldots, n). \tag{35}$$

For any function $f \in C^{1,1}(\Omega)$, the functional

$$E_n[f] := E_n[f, \boldsymbol{\lambda}] = I_n[f] - \int_{\Omega} f(\boldsymbol{x})\, d\boldsymbol{x} \tag{36}$$

will be reserved exclusively to denote the incurred approximation error between the integral of f and its approximation $I_n[f]$.

We now give a simple expression of the error $E_n[\|.\|^2]$ in terms of the barycentric coordinates $\{\lambda_0, \ldots, \lambda_n\}$.

Lemma 3 *The error $E_n[\|.\|^2]$ when approximating the integral of the quadratic function $\|.\|^2$ by $I_n[\|.\|^2]$ can be expressed as*

$$E_n[\|.\|^2](\boldsymbol{x}) = \sum_{i=0}^{n} \int_{\Omega} \lambda_i(\boldsymbol{x}) \left\|\boldsymbol{x} - \boldsymbol{y}_i\right\|^2\, d\boldsymbol{x}. \tag{37}$$

Proof For $f(\boldsymbol{x}) = \|\boldsymbol{x}\|^2$, we find by a simple calculation that

$$f(\boldsymbol{x}) + \left\langle \nabla f(\boldsymbol{x}), \boldsymbol{y}_i - \boldsymbol{x} \right\rangle = \left\|\boldsymbol{y}_i\right\|^2 - \left\|\boldsymbol{x} - \boldsymbol{y}_i\right\|^2.$$

Hence, multiplying on each side by λ_i, summing up with respect to i from 0 to n, using the linear precision property of barycentric coordinates, and rearranging, we get the desired result and complete the proof of the lemma. □

The following lemma shows that if the cubature formula I_n approximates every strongly convex function from above, then it generates a sharp lower bound for the error of any strongly convex function.

Lemma 4 *Let μ be a positive real number. If the barycentric coordinate approximation functional I_n approximates every μ-strongly convex function from above, then for every μ-strongly convex function f, it holds*

$$\frac{\mu}{2} \sum_{i=0}^{n} \int_{\Omega} \lambda_i(\boldsymbol{x}) \left\|\boldsymbol{x} - \boldsymbol{y}_i\right\|^2\, d\boldsymbol{x} \leq I_n[f] - \int_{\Omega} f(\boldsymbol{x})\, d\boldsymbol{x}. \tag{38}$$

Equality in (38) *is attained for all functions of the form*

$$f(\boldsymbol{x}) = a(\boldsymbol{x}) + \frac{\mu}{2}\|\boldsymbol{x}\|^2, \tag{39}$$

where $a(\cdot)$ *is any affine function.*

Proof Let us fix f as a μ-strongly convex function. By the Jensen convexity for μ-strongly convex functions, see [11], we get

$$f(\boldsymbol{y}_i) \geq f(\boldsymbol{x}) + \left\langle \nabla f(\boldsymbol{x}), \boldsymbol{y}_i - \boldsymbol{x} \right\rangle + \frac{\mu}{2}\left\|\boldsymbol{x} - \boldsymbol{y}_i\right\|^2.$$

Hence, multiplying on each side by λ_i, summing up with respect to i from 0 to n, and integrating each term, we get the desired result and complete the proof of the lemma. The case of equality is easily verified. □

The following lemma gives an upper bound for the absolute value of the error of any function possessing Lipschitz continuous gradient:

Lemma 5 *The following error estimate holds for every function* $f \in C^{1,1}(\Omega)$:

$$\left| I_n[f] - \int_\Omega f(\boldsymbol{x})\, d\boldsymbol{x} \right| \leq \frac{L(\nabla f)}{2} \sum_{i=0}^{n} \int_\Omega \lambda_i(\boldsymbol{x}) \left\|\boldsymbol{x} - \boldsymbol{y}_i\right\|^2. \tag{40}$$

Equality in (40) *is attained for all functions of the form*

$$f(\boldsymbol{x}) = a(\boldsymbol{x}) + \frac{\mu}{2}\|\boldsymbol{x}\|^2, \tag{41}$$

where $a(\cdot)$ *is any affine function.*

Proof This lemma is an immediate consequence of Theorem 1 and Lemma 3. The case of equality is easily verified. □

Now, everything is set for giving an upper bound and a lower bound for the approximation error estimate $E_n[f] = I_n[f] - \int_\Omega f(\boldsymbol{x})\, d\boldsymbol{x}$ of any μ-strongly convex function f, having Lipschitz continuous gradient.

Theorem 4 *Let* μ *be a positive real number. Then, for every* μ*-strongly convex function* $f \in C^{1,1}(\Omega)$ *and any* $\boldsymbol{x} \in \Omega$*, it holds*

$$\frac{\mu}{2} \sum_{i=0}^{n} \int_\Omega \lambda_i(\boldsymbol{x}) \left\|\boldsymbol{x} - \boldsymbol{y}_i\right\|^2 \leq I_n[f] - \int_\Omega f(\boldsymbol{x})\, d\boldsymbol{x} \leq \frac{L(\nabla f)}{2} \sum_{i=0}^{n} \int_\Omega \lambda_i(\boldsymbol{x}) \left\|\boldsymbol{x} - \boldsymbol{y}_i\right\|^2. \tag{42}$$

Equality in (42) *is attained for all functions of the form*

$$f(\boldsymbol{x}) = a(\boldsymbol{x}) + \frac{\mu}{2}\|\boldsymbol{x}\|^2, \tag{43}$$

where $a(\cdot)$ is any affine function.

Proof This is an immediate consequence of Lemmas 3, 4, and 5 and Theorem 1. The case of equality is easily verified. □

Remark 3 In the univariate case, a simple inspection of the error estimates (42) reveals that (42) is nicely reduced to the simple form given in (3) and (6).

5 Practical Construction of snd-Cubature Formulas

We now turn to a practical construction of snd-cubature formulas. To this end, let us first consider the case, where Ω is a non-degenerate simplex in $\mathbb{R}^d$ with $\boldsymbol{x}_i, i = 1, \ldots, d+1$, being the set of its vertices. Then, each $\boldsymbol{x} \in \Omega$ has a unique representation as a convex combination

$$\boldsymbol{x} = \sum_{i=1}^{d+1} \lambda_i(\boldsymbol{x})\boldsymbol{x}_i, \tag{44}$$

where λ_i is the restriction to Ω of the affine function that attains the value 1 at $\boldsymbol{x}_i$ and is zero at all the other vertices of Ω. The value $\lambda_i(\boldsymbol{x})$ is the barycentric coordinate of $\boldsymbol{x}$ with respect to $\boldsymbol{x}_i$. Then, if f is convex, by Jensen's inequality it follows from (44) that

$$f(\boldsymbol{x}) \leq \sum_{i=1}^{n} \lambda_i(\boldsymbol{x}) f(\boldsymbol{x}_i).$$

Integrating both sides over Ω and using the fact that $\int_\Omega \lambda_i(\boldsymbol{x})d\boldsymbol{x} = \frac{|\Omega|}{d+1}, i = 1, \ldots, d+1$, we deduce that

$$\int_\Omega f(\boldsymbol{x})d\boldsymbol{x} \leq Q^{\mathrm{TraR}}(f), \tag{45}$$

$$Q^{\mathrm{TraR}}(f) := \sum_{i=1}^{d+1} \frac{|\Omega|}{d+1} f(\boldsymbol{x}_i). \tag{46}$$

Consequently, by Lemma 1, the set of barycentric coordinates $\lambda_1, \ldots, \lambda_{d+1}$ produces the snd-system

$$\left\{ \left(\frac{|\Omega|}{d+1}, \boldsymbol{x}_i \right) : i = 1, \ldots, d+1 \right\}.$$

It is the only snd-system on Ω, which has no other nodes than the vertices.

Now, let $\boldsymbol{X} = \left\{\boldsymbol{x}_i \in \mathbb{R}^d, i = 1, \ldots, n\right\}$ be an arbitrary set of points of $\mathbb{R}^d$. The previous approach can be generalized when $\Omega = conv\,(\boldsymbol{X})$ is an arbitrary polytope in $\mathbb{R}^d$. A triangulation $\mathscr{T}$ of Ω with respect to $\boldsymbol{X}$ is a decomposition of Ω into d-dimensional simplices such that $\boldsymbol{X}$ is the set of all their vertices, and the intersection of any two simplices consists of a common lower-dimensional simplex or is empty. Triangulations of compact convex polytopes exist.[1] Indeed, given any finite set $\boldsymbol{X}$ of points that do not all lie on a hyperplane, Chen and Xu [1, p. 301] describe a lifting-and-projection procedure that results in a triangulation of the convex hull of $\boldsymbol{X}$ with respect to $\boldsymbol{X}$. For an explicit statement on the existence of triangulations with a proof based on an algorithmic method, see [16, Theorem 3, part a].
Now, let $\mathbf{S}_1, \ldots, \mathbf{S}_l$ be the simplices of $\mathscr{T}$, and let N_i be the set of all integers j such that $\boldsymbol{x}_i$ is a vertex of $\mathbf{S}_j$. If $\boldsymbol{x} \in \mathbf{S}_j$ and $j \in N_i$, then we denote by $\lambda_{ij}(\boldsymbol{x})$ the barycentric coordinate of $\boldsymbol{x}$ with respect to $\boldsymbol{x}_i$ for the simplex $\mathbf{S}_j$. It is easily verified that if $\boldsymbol{x} \in \mathbf{S}_j \bigcap \mathbf{S}_k$, then $\lambda_{ij}(\boldsymbol{x}) = \lambda_{ik}(\boldsymbol{x})$ if $j, k \in N_i$ and $\lambda_{ij}(\boldsymbol{x}) = 0$ if $j \in N_i, k \notin N_i$. Therefore, setting

$$\phi_i(\boldsymbol{x}) := \begin{cases} \lambda_{ij}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \mathbf{S}_j \quad \text{and} \quad j \in N_i \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, n$, we obtain well-defined barycentric coordinates $\phi_1, \ldots, \varphi_n$. This obviously produces the snd-formula

$$\int_\Omega f(\boldsymbol{x})d\boldsymbol{x} = Q^{\mathrm{tra}}(f) + E[f], \tag{47}$$

where

$$Q^{\mathrm{tra}}(f) = \sum_{i=1}^{n} \left(\sum_{j \in N_i} \frac{|\mathbf{S}_j|}{d+1} \right) f(\boldsymbol{x}_i). \tag{48}$$

Let $T(\Omega)$ be any triangulation of the point set X_n. Then, $\boldsymbol{\lambda}^{T(\Omega)} := \left\{\lambda_i^{T(\Omega)}\right\}_{i=0}^{n}$ denotes the set of barycentric coordinates associated with each $\boldsymbol{x}_i$ of X_n. Now, we list the basic properties of $\boldsymbol{\lambda}^{T(\Omega)}$, which are particularly relevant to us:

(1) They are well defined, piecewise linear, and nonnegative real-valued continuous functions.
(2) The function $\lambda_i^{T(\Omega)}$ satisfies the delta property, which equals 1 at $\boldsymbol{x}_i$ and 0 at all other points in $\boldsymbol{X}_n \setminus \{\boldsymbol{x}_i\}$, that is, $\lambda_i^{T(\Omega)}(\boldsymbol{x}_j) = \delta_{ij}$ (δ is the Kronecker delta).

[1] It seems that in dimension $d = 3$, the existence was already known to mathematicians like Euler and Dirichlet.

We denote by

$$E_n^{T(\Omega)}[f](\boldsymbol{x}) := \sum_{i=0}^{n} \lambda_i^{T(\Omega)}(\boldsymbol{x}) f(\boldsymbol{x}_i) - f(\boldsymbol{x}). \tag{49}$$

As regards the error estimates (49), it was shown that Delaunay triangulation is the triangulation that minimizes the L^p norm of the approximation error $E_n^{T(\Omega)}[\|.\|^2]$ among all triangulations, see [3, Theorem 4.10]. This optimality condition also characterizes Delaunay triangulation.

6 Numerical Experiments in 3*D*

In this section, we provide some numerical tests, which we perform in order to validate our theoretical predictions. We have considered the following function of three variables as test function:

$$g(x, y, z) = \exp(ax + by + cz),$$

and the domain of integration is the pyramid Pyr given in the Cartesian coordinate system (x, y, z) by the inequalities:

$$Pyr = \{(x, y; z) \in \mathbb{R}^3 : 0.3z < x < 1 - 0.3z, 0.3z < y < 1 - 0.3z, 0 < z < 1\}. \tag{50}$$

The algorithm for computing the approximate values of the integral is as follows:

1. Pyramid should be decomposed into tetrahedra, see figure 1a.
2. Each of tetrahedra should be mapped onto the reference one, see figure 1b.
3. For integration of function g over the reference tetrahedron, the method $Q^{\mathrm{TraR}}(g)$ should be applied. Where $Q^{\mathrm{TraR}}(g)$ is defined by formula (46).
4. The results are the sums of approximate values of integrals over all tetrahedra in the decomposition of the pyramid.

Let us give more details about these steps.

For decomposition of the domain Pyr, the DistMesh package was used that is a simple triangular mesh generator in MATLAB based on Delaunay triangulation. A detailed description of the program is provided in [18, 19] or http://persson.berkeley.edu/distmesh. Specifically, we used the code of the Problem #3 from the web page available at the address:

https://people.sc.fsu.edu/~jburkardt/m_src/distmesh_3d/distmesh_3d.html.

For computing the errors of our methods, we need to compute the exact value of integral of function $g(x, y, z)$ over the pyramid Pyr, assuming that Pyr is given by its H-representation (50) or, alternatively, by its corresponding V-representation. We should mention that some useful methods for computing such integrals are discussed

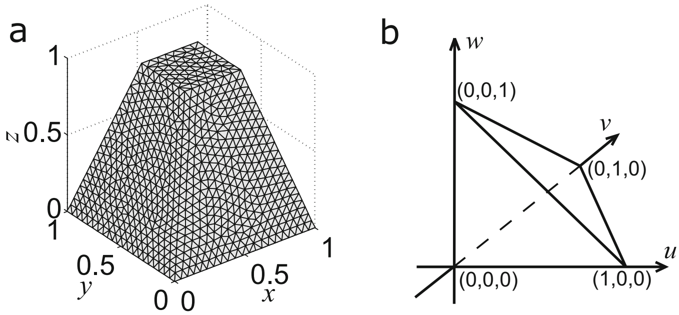

Fig. 1 Domain of the pyramid and its decomposition into tetrahedra generated by DistMesh (**a**). The characteristic linear size of tetrahedra is $1/21$. Reference tetrahedron (**b**)

in [17, Section 2]. The exact value of this integral is

$$I_{pyr}(g)=K\left(Aa^3+Ba^2b+Ca^2c+Db^2a+Eb^2c+Fb^3+Gc^3+Hc^2a+Ic^2b+Jabc\right),$$

where $K = \dfrac{10}{ab(3a-3b-10c)(3a+3b-10c)(3a-3b+10c)(3a+3b+10c)}$,

$$\begin{aligned}
A &= 27(e^b - e^a + e^{a+b} + \alpha + \beta - \gamma - \theta - 1),\\
B &= 27(e^b - e^a - e^{a+b} - \alpha + \beta - \gamma + \theta + 1),\\
C &= 90(e^{a+b} - e^a - e^b - \alpha + \beta + \gamma - \theta + 1),\\
D &= 27(e^a - e^b - e^{a+b} - \alpha - \beta + \gamma + \theta + 1),\\
E &= 90(-e^a - e^b + e^{a+b} - \alpha + \beta + \gamma - \theta + 1),\\
F &= 27(e^a - e^b + e^{a+b} + \alpha - \beta + \gamma - \theta - 1),\\
G &= 1000(e^a + e^b - e^{a+b} + \alpha - \beta - \gamma + \theta - 1),\\
H &= 300(e^a - e^b - e^{a+b} - \alpha - \beta + \gamma + \theta + 1),\\
I &= 300(e^b - e^a - e^{a+b} - \alpha + \beta - \gamma + \theta + 1),\\
J &= 180(-e^a - e^b - e^{a+b} + \alpha + \beta + \gamma + \theta - 1),\\
\alpha &= e^{0.3a+0.3b+c},\ \beta = e^{0.7a+0.3b+c},\ \gamma = e^{0.3a+0.7b+c},\ \theta = e^{0.7a+0.7b+c}.
\end{aligned}$$

After applying the above algorithm, we got the asymptotics of the relative errors of our formulas for the case of function g with $a = 1$, $b = 2$, and $c = 3$. The expression of the relative error for Q_3 cubature formula is as follows:

$$E_N^{\mathrm{Tra}}(g) = \frac{Q^{\mathrm{tra}}(g) - I_{pyr}(g)}{I_{pyr}(g)},$$

where $Q^{\mathrm{tra}}(g)$ is defined by formula (48). In Table 1, the values of the relative errors of integration are given for the case of test with $a = 1$, $b = 2$, and $c = 3$.

Table 2 shows the orders of convergence obtained for the test with $a = 1$, $b = 2$, and $c = 3$. The orders are close to 2.

Table 1 Errors obtained while integrating g with $a = 1$, $b = 2$, and $c = 3$ over pyramid Pyr

N	4	8	16	32	64	128
$E_N^{\mathrm{Tra}}(g)$	3.441E-01	6.520E-02	1.478E-02	3.420E-03	8.312E-04	2.074E-04

Table 2 Orders of convergence obtained while integrating g with $a = 1$, $b = 2$, and $c = 3$ over the pyramid Pyr

N	8	16	32	64
$E_N^{\mathrm{Tra}}(g)$	2.46763	2.14976	2.13358	2.05323

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Further Results on Continuous Random Variables via Fractional Integrals

Ibrahim Slimane, Zoubir Damani, Shilpi Jain, and Praveen Agarwal

Abstract In this paper, some new fractional weighted inequalities related to Čebyšev, Ostrowski, and Lupaş inequalities are established, and some of their applications for continuous random variables having the probability density function (p.d.f.) defined on a finite interval are derived. Furthermore, some upper bounds for fractional expectation and fractional variance are given.

1 Introduction

The well-known results of Čebyšev, Grüss, Ostrowski, and Lupaş inequalities have attracted much attention over the years, and many variants of these inequalities have appeared in the literature [1–6]. These inequalities are crucial due to their numerous applications in various areas of mathematics such as the applications on random variables via Fractional Calculus for which we would like to refer the reader to [7–11].

Motivated and inspired by the works mentioned above and the references therein, in this paper, we provide new fractional integral inequalities of Čebyšev, Ostrowski, and Lupaş type as well as applications for continuous random variables.

Let us initially recall the classical results for the Čebyšev functional for two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{R}$:

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$$C(f,g) := \frac{1}{b-a}\int_a^b f(\tau)g(\tau)d\tau - \frac{1}{b-a}\int_a^b f(\tau)d\tau \frac{1}{b-a}\int_a^b g(\tau)d\tau$$

In [12] , Čebyšev derived the following interesting result involving two absolutely continuous functions whose first derivatives are continuous and bounded:

$$|C(f,g)| \leq \frac{1}{12}(b-a)^2 ||f'||_\infty ||g'||_\infty, \tag{1}$$

where $||f'||_\infty := \sup_{t\in[a,b]} |f'(t)|$.

Another inequality for $C(f,g)$ was derived by Grüss [13], under the assumption that $m < f \leq M$ and $n < g \leq N$, namely,

$$|C(f,g)| \leq \frac{1}{4}(M-m)(N-n).$$

In 1970, Ostrowski [14] proved, among others, the following result that is—in a sense—a combination of the results by Čebyšev and Grüss:

$$|C(f,g)| \leq \frac{1}{8}(b-a)(M-m)||g'||_\infty. \tag{2}$$

Finally, a result by Lupaş [15] states that

$$|C(f,g)| \leq \frac{1}{\pi^2}(b-a)||f'||_2||g'||_2, \tag{3}$$

where f, g are absolutely continuous and $f', g' \in L_2[a,b]$.

In the following, we present some basic definitions.

2 Some Definitions

Definition 1 ([16]) The Riemann–Liouville fractional integral operator of order $\alpha > 0$, for a continuous function f on $[a,b]$, is defined as

$$J_a^\alpha [f(t)] = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau,\ \alpha > 0,\ a < t \leq b.$$

Definition 2 ([11]) The fractional expectation of order $\alpha > 0$, for a random variable X with a probability density function h defined on $[a,b]$, is given by

$$E_\alpha X := \frac{1}{\mathcal{Q}\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} \tau h(\tau)\, d\tau, \ \alpha > 0, \tag{4}$$

where $\mathcal{Q} := J_a^\alpha[h(b)]$.

For any continuous function v, the fractional expectation of order $\alpha > 0$ of $v(X)$ is defined by

$$E_\alpha v(X) := \frac{1}{\mathcal{Q}\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} v(\tau) h(\tau)\, d\tau, \alpha > 0. \tag{5}$$

Definition 3 ([11]) The fractional variance of order $\alpha > 0$, for X, is defined as

$$\mathrm{Var}_\alpha(X) = \frac{1}{\mathcal{Q}\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1}(\tau - E_\alpha(X))^2 h(\tau) d\tau.$$

Using the above definitions, the authors in [11] prove the following property for the fractional variance:

Theorem 1

$$\mathrm{Var}_\alpha(X) = E_\alpha(X^2) - E_\alpha(X)^2, \alpha > 0$$

3 Main Results

Theorem 2

$$0 \leq \mathrm{Var}_\alpha(X) \leq \frac{1}{2}(b-a), \alpha > 0$$

Proof Due to the following Grüss type inequality:

$$0 \leq \frac{\int_a^b p(\tau) g^2(\tau) d\tau}{\int_a^b p(\tau) d\tau} - \left(\frac{\int_a^b p(\tau) g(\tau) d\tau}{\int_a^b p(\tau) d\tau}\right)^2 \leq \frac{1}{4}(M-m)^2, \tag{6}$$

provided that p and g are measurable on $[a, b]$, and all the integrals in (6) exist and are finite,

$$\int_a^b p(\tau) d\tau > 0 \ \text{ and } \ m \leq g \leq M, \ \text{a.e., on } [a, b].$$

We set in (6),

$$p(\tau) = \frac{1}{\Gamma(\alpha)}(b-\tau)^{\alpha-1} f(\tau),\ g(\tau) = \tau - E_\alpha(X),\ \tau \in [a,b].$$

We observe that in this case $m = a - E_\alpha(X)$, $M = b - E_\alpha(X)$ from which we can derive the desired result.

Let us introduce a fractional weighted type Čebyšev functional:

$$C_{\alpha,s'}(f,g) := \frac{1}{J_a^\alpha[s'(b)]} J_a^\alpha[f(b)g(b)s'(b)] - \frac{1}{J_a^\alpha[s'(b)]} J_a^\alpha[f(b)s'(b)]$$
$$\frac{1}{J_a^\alpha[s'(b)]} J_a^\alpha[g(b)s'(b)],$$

where

$$\Im(t) := \frac{1}{\Gamma(\alpha)} \int (b-\tau)^{\alpha-1} s'(\tau) d\tau$$

is assumed to be absolutely continuous, and f, g are Lebesgue measurable on $[a,b]$ and such that the above integrals exist.

Theorem 3 *Let $\Im : [a,b] \to [\Im(a), \Im(b)]$ be a continuous strictly increasing function on]a,b[, and f, g are as above. One can verify that*

I: $m \le f(t) \le M\ \forall t \in [a,b]$.*
*I**: $g : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$.*

Additionally, $\frac{g'}{\Im'} \in L_\infty[a,b]$,

$$|C_{\alpha,s'}(f,g)| \le \frac{1}{8} J_a^\alpha[s'(b)](M-m) \left|\left| \frac{g'}{\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)} s'} \right|\right|_\infty. \tag{7}$$

The constant $\frac{1}{8}$ is the best possible.

Proof By (2), for the functions $f \circ \Im^{-1}$ and $g \circ \Im^{-1}$ on $[\Im(a), \Im(b)]$, we get

$$\left| \frac{1}{\Im(a) - \Im(b)} \int_{\Im(a)}^{\Im(b)} f \circ \Im^{-1}(u) g \circ \Im^{-1}(u) du \right.$$
$$\left. - \frac{1}{[\Im(a) - \Im(b)]^2} \int_{\Im(a)}^{\Im(b)} f \circ \Im^{-1}(u) du . \int_{\Im(a)}^{\Im(b)} g \circ \Im^{-1}(u) du \right| \tag{8}$$
$$\le \frac{1}{8} [\Im(a) - \Im(b)](M-m) ||(g \circ \Im^{-1})'||_\infty.$$

By the change of variable $t = \Im^{-1}(u)$, we can prove that

$$\frac{1}{\Im(a)-\Im(b)}\int_{\Im(a)}^{\Im(b)} f\circ\Im^{-1}(u)g\circ\Im^{-1}(u)du = \frac{1}{J_a^{\alpha}[s'(b)]}J_a^{\alpha}[f(b)g(b)s'(b)]$$

$$\frac{1}{\Im(a)-\Im(b)}\int_{\Im(a)}^{\Im(b)} f\circ\Im^{-1}(u)du = \frac{1}{J_a^{\alpha}[s'(b)]}J_a^{\alpha}[f(b)s'(b)]$$

$$\frac{1}{\Im(a)-\Im(b)}\int_{\Im(a)}^{\Im(b)} g\circ\Im^{-1}(u)du = \frac{1}{J_a^{\alpha}[s'(b)]}J_a^{\alpha}[g(b)s'(b)]$$

Also,

$$\left\|(g\circ\Im^{-1})'\right\|_{\infty} = \left\|\frac{g'}{\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}s'}\right\|_{\infty}$$

This completed the proof of the theorem.

Furthermore, let

$$W(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (b-\tau)^{\alpha-1}w(\tau)d\tau$$

be a continuous and strictly increasing function on $]a, b[$, where $w(x) : [a, b] \to \mathbb{R}_+^*$ is a continuous function.

Corollary 1 *If f, g satisfy the conditions I^*, I^{**}, and $\frac{g'}{w} \in L_\infty[a, b]$, then we have*

$$|C_{\alpha,w}(f, g)| \le \frac{1}{8}J_a^{\alpha}[w(b)](M-m)\left\|\frac{g'}{\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}w}\right\|_{\infty}. \tag{9}$$

As a particular case of the above corollary, we obtain the following:

Corollary 2 *If $w(x)$ is a continuous p.d.f. on $[a, b]$ of random variable X, we have*

$$\left|E_\alpha fg[X] - E_\alpha f[X]E_\alpha g[X]\right| \le \frac{1}{8}J_a^{\alpha}[w(b)](M-m)\left\|\frac{g'}{\frac{(b-X)^{\alpha-1}}{\Gamma(\alpha)}w}\right\|_{\infty}. \tag{10}$$

Also, for $\alpha = 1$, we have

$$\left|Efg[X] - Ef[X]Eg[X]\right| \le \frac{1}{8}(M-m)\left\|\frac{g'}{w}\right\|_{\infty}. \tag{11}$$

Theorem 4 *Let $\Im$ be as above and f, g be absolutely continuous on $[a, b]$ such that $\frac{f'}{\Im'}, \frac{g'}{\Im'} \in L_\infty[a, b]$. Then, we have*

$$|C_{\alpha,s'}(f, g)| \leq \frac{1}{12}\left(J_a^\alpha[s'(b)]\right)^2 \left\|\frac{f'}{\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}s'}\right\|_\infty \left\|\frac{g'}{\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}s'}\right\|_\infty. \tag{12}$$

Proof By making use of (1) for the function $f \circ \Im^{-1}$ and $g \circ \Im^{-1}$ on $[\Im(a), \Im(b)]$, we get the desired result.

Corollary 3 *Suppose that w is as in Corollary 4 and f, g are absolutely continuous on $[a, b]$, where $\frac{f'}{w}, \frac{g'}{w} \in L_\infty[a, b]$. Then, we have*

$$|C_{\alpha,w}(f, g)| \leq \frac{1}{12}\left(J_a^\alpha[w(b)]\right)^2 \left\|\frac{f'}{\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}w}\right\|_\infty \left\|\frac{g'}{\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}w}\right\|_\infty. \tag{13}$$

Consequently, if $w(x)$ is a continuous $p.d.f.$ on $[a, b]$ of random variable X, we derive the following result:

Corollary 4

$$\left|E_\alpha fg[X] - E_\alpha f[X]E_\alpha g[X]\right| \leq \frac{1}{12}\left(J_a^\alpha[w(b)]\right)^2 \left\|\frac{f'}{\frac{(b-X)^{\alpha-1}}{\Gamma(\alpha)}w}\right\|_\infty \left\|\frac{g'}{\frac{(b-X)^{\alpha-1}}{\Gamma(\alpha)}w}\right\|_\infty. \tag{14}$$

If $f = g$, then

$$\left|E_\alpha f^2[X] - E_\alpha f[X]^2\right| \leq \frac{1}{12}\left(J_a^\alpha[w(b)]\right)^2 \left\|\frac{f'}{\frac{(b-X)^{\alpha-1}}{\Gamma(\alpha)}w}\right\|_\infty^2. \tag{15}$$

Therefore, if $f = x$

$$\left|\mathrm{Var}_\alpha(X)\right| \leq \frac{1}{12}\left(J_a^\alpha[w(b)]\right)^2 \left\|\frac{1}{\frac{(b-X)^{\alpha-1}}{\Gamma(\alpha)}w}\right\|_\infty^2. \tag{16}$$

Corollary 5 *For $\alpha = 1$, we obtain the classical case*

$$\left|Efg[X] - Ef[X]Eg[X]\right| \leq \frac{1}{12}\left\|\frac{f'}{w}\right\|_\infty \left\|\frac{g'}{w}\right\|_\infty. \tag{17}$$

If $f = g$, then

$$\left|Ef^2[X] - Ef[X]^2\right| \leq \frac{1}{12}\left|\left|\frac{f'}{w}\right|\right|_\infty^2. \tag{18}$$

Therefore, if $f = x$, we obtain that

$$\left|\mathrm{Var}(X)\right| \leq \frac{1}{12}\left|\left|\frac{1}{w}\right|\right|_\infty^2. \tag{19}$$

Theorem 5 *Assume that $\Im$ is as above; $f, g : [a, b] \to \mathbb{R}$ are absolutely continuous on $[a, b]$ and $\frac{f'}{(\Im')^{1/2}}, \frac{g'}{(\Im')^{1/2}} \in L_2[a, b]$. Thus, we get*

$$|C_{\alpha,s'}(f, g)| \leq \frac{1}{\pi^2}\left(J_a^\alpha[s'(b)]\right)\left|\left|\frac{f'}{(\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}s')^{1/2}}\right|\right|_2\left|\left|\frac{g'}{(\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}s')^{1/2}}\right|\right|_2. \tag{20}$$

Proof Using Lupaş inequality (3) for the functions $f \circ \Im^{-1}$ and $g \circ \Im^{-1}$ on $[\Im(a), \Im(b)]$, we have

$$\begin{aligned}
&\left|\frac{1}{\Im(a) - \Im(b)}\int_{\Im(a)}^{\Im(b)} f \circ \Im^{-1}(u) g \circ \Im^{-1}(u) du\right. \\
&\left.-\frac{1}{[\Im(a) - \Im(b)]^2}\int_{\Im(a)}^{\Im(b)} f \circ \Im^{-1}(u) du. \int_{\Im(a)}^{\Im(b)} g \circ \Im^{-1}(u) du\right| \\
&\leq \frac{1}{\pi^2}[\Im(a) - \Im(b)]\left|\left|(f \circ \Im^{-1})'\right|\right|_2\left|\left|(g \circ \Im^{-1})'|\right|\right|_2.
\end{aligned} \tag{21}$$

We can also show that

$$\int_{\Im(a)}^{\Im(b)}\left|(f \circ \Im^{-1})'(u)\right|^2 du = \int_{\Im(a)}^{\Im(b)}\left|\frac{(f' \circ \Im^{-1})(u)}{(\Im' \circ \Im^{-1})(u)}\right|^2 du.$$

Thanks to the change of variable $t = \Im^{-1}(u)$, we derive that

$$\begin{aligned}
\int_{\Im(a)}^{\Im(b)}\left|\frac{(f' \circ \Im^{-1})(u)}{(\Im' \circ \Im^{-1})(u)}\right|^2 du &= \int_a^b \left|\frac{f'(t)}{\Im'(t)}\right|^2 \Im'(t) dt \\
&= \int_a^b \left|\frac{f'(t)}{[\Im'(t)]^{1/2}}\right|^2 dt.
\end{aligned}$$

Corollary 6 *Let $w(x) : [a, b] \to \mathbb{R}_+^*$ be a continuous function and f, g be absolutely continuous on $[a, b]$, where $\frac{f'}{(\Im')^{1/2}}, \frac{g'}{(\Im')^{1/2}} \in L_2[a, b]$. Then, we have that*

$$|C_{\alpha,w}(f,g)| \leq \frac{1}{\pi^2}\left(J_a^{\alpha}[w(b)]\right)\left|\left|\frac{f'}{(\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}w)^{1/2}}\right|\right|_2\left|\left|\frac{g'}{(\frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)}w)^{1/2}}\right|\right|_2. \quad (22)$$

From the above, we immediately deduce the following:

Corollary 7 *If w is a $p.d.f.$ on $[a,b]$ of random variable X, then*

$$\left|E_\alpha fg[X] - E_\alpha f[X]E_\alpha g[X]\right| \leq \frac{1}{\pi^2}\left(J_a^{\alpha}[w(b)]\right)\left|\left|\frac{f'}{(\frac{(b-X)^{\alpha-1}}{\Gamma(\alpha)}w)^{1/2}}\right|\right|_2\left|\left|\frac{g'}{(\frac{(b-X)^{\alpha-1}}{\Gamma(\alpha)}w)^{1/2}}\right|\right|_2. \quad (23)$$

If $f = g$, then

$$\left|E_\alpha f^2[X] - E_\alpha f[X]^2\right| \leq \frac{1}{\pi^2}\left(J_a^{\alpha}[w(b)]\right)\left|\left|\frac{f'}{(\frac{(b-X)^{\alpha-1}}{\Gamma(\alpha)}w)^{1/2}}\right|\right|_2^2. \quad (24)$$

If $f = x$, then

$$\left|\mathrm{Var}_\alpha(X)\right| \leq \frac{1}{\pi^2}\left(J_a^{\alpha}[w(b)]\right)\left|\left|\frac{1}{(\frac{(b-X)^{\alpha-1}}{\Gamma(\alpha)}w)^{1/2}}\right|\right|_2^2. \quad (25)$$

Corollary 8 *For $\alpha = 1$, we deduce the following classical result:*

$$\left|Efg[X] - Ef[X]Eg[X]\right| \leq \frac{1}{\pi^2}\left|\left|\frac{f'}{w^{1/2}}\right|\right|_2\left|\left|\frac{g'}{w^{1/2}}\right|\right|_2. \quad (26)$$

If $f = g$, then

$$\left|Ef^2[X] - Ef[X]^2\right| \leq \frac{1}{\pi^2}\left|\left|\frac{f'}{w^{1/2}}\right|\right|_2^2. \quad (27)$$

Therefore, if $f = x$, we get

$$\left|\mathrm{Var}(X)\right| \leq \frac{1}{\pi^2}\left|\left|\frac{1}{w^{1/2}}\right|\right|_2^2. \quad (28)$$

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Nonunique Fixed Points on Partial Metric Spaces Via Control Functions

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Abstract In this note, we aim to emphasize the significance of the nonunique fixed point results in an abstract space: partial metric space. Indeed, partial metric is a natural extension of the standard metric from the aspect of computer science. The presented results aim to cover and unify several results on the topic in the related literature. We also indicate the validity of the results by a concrete example.

1 Introduction and Preliminaries

The notion of partial metric is one of the most fascinating extensions of the concept of metric. The main characteristic property of a partial metric, proposed by Matthews [28], is on the self-distance (indistancy or reflexivity axiom). Despite the standard metric, in a partial metric, self-distance (the distance of a point to itself) needs not to be zero. At the first sight, nonzero self-distance can be seen as absurd and nonsense. On the other hand, the following example indicates that, surprisingly, this is a very interesting and reasonable case when we consider it in the framework of computer sciences.

One of the classical metric definitions on the class of all infinite sequences (let us denote with S_I) can be expressed as follows:

$$d : S_I \times S_I : [0, \infty) \text{ such that } d(x, y) = 2^{-\sup\{n | \forall i < n \text{ such that } x_i = y_i\}}. \tag{1}$$

It is obvious that $d(x, y)$ provides all axioms of standard metric on S_I. Now, we take "the point views of computers sciences" into account and reconsider the mentioned metric function by extending its domain with combining the class of all finite sequences (let us denote with S_F) with the class of all infinite sequences. In computer science programming, usage of the finite sequences is more

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reasonable than the infinite sequences when we regard the termination of a program. Programming with infinite sequence may cause to infinite loop, and hence a program creates a terrible fault that it is not terminated. After these rough discussions, we modify the metric above by keeping the rule same but on the extended domain: union of the class of finite sequence S_F and infinite sequences S_I. For simplicity, let us fix the letter, $S := S_F \cup S_I$, for the class of finite and infinite sequences. Now, the new distance function $\delta : S \times S \to [0, \infty)$ creates a new structure with the same definition

$$\delta : S \times S \to [0, \infty) \text{ such that } \delta(x, y) = 2^{-\sup\{n | \forall i < n \text{ such that } x_i = y_i\}}. \tag{2}$$

It is clear that δ is not a metric. Indeed, for the finite sequence $x = (x_1, x_2, \cdots, x_{19})$, the self-distance $\delta(x, y) = \dfrac{1}{2^{19}} \neq 0$. As it is seen, the example makes the idea reasonable and worthy.

Hereupon, the letters $\mathbb{R}_0^+$ and $\mathbb{N}_0$ are occupied to denote the set of nonnegative real numbers and the set of nonnegative integer numbers, respectively.

In what follows, we recollect the axiomatic definition of partial metric for the sake of completeness of the text.

Definition 1 (See, e.g., [28, 29]) A function $\delta : S \times S \to \mathbb{R}_0^+$ on a (non-empty) set S is named as a partial metric if the following axioms are fulfilled:

$(P1)$ $x = y \Leftrightarrow \delta(x, x) = \delta(y, y) = \delta(x, y)$,
$(P2)$ $\delta(x, x) \leq \delta(x, y)$,
$(P3)$ $\delta(x, y) = \delta(y, x)$,
$(P4)$ $\delta(x, y) \leq \delta(x, z) + \delta(z, y) - \delta(z, z)$,

for all $x, y, z \in S$. Here, the coupled letter (S, δ) is said to be a partial metric space.

Although, self-distance needs not to be zero, from $(P1)$ and $(P2)$, we observe that $\delta(x, y) = 0$ implies $x = y$ (reflexivity axiom).

Throughout the paper, we presume that S is a non-empty set endowed with a partial metric δ, and F is a self-mapping on a partial metric space (S, δ). Moreover, we shall use the letter d to denote a metric defined on S.

The basic and classical example of a partial metric is the following.

Example 1 (See, e.g., [28, 29]) Let $S = \mathbb{R}_0^+$ and δ be defined on S by $\delta(x, y) = \max\{x, y\}$ for all $x, y \in S$. Then, (S, δ) is a partial metric space.

Example 2 (See, e.g., [22, 32]) Consider function $\sigma_i : S \times S \to \mathbb{R}_0^+$ $(i \in \{1, 2, 3\})$ given by

$$\sigma_1(x, y) = d(x, y) + \delta(x, y),$$
$$\sigma_2(x, y) = d(x, y) + \max\{\nu(x), \nu(y)\},$$
$$\sigma_3(x, y) = d(x, y) + a,$$

where $\nu : S \to \mathbb{R}_0^+$ is an arbitrary function and $a \geq 0$. It is easy to see that all these three functions form partial metrics on S.

Example 3 (See [28, 29]*)* Let $S = \{[a, b] : a, b \in \mathbb{R},\ a \leq b\}$ and define $\delta([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then, (S, δ) is a partial metric space.

Example 4 (See [28]*)* Let $S = [0, 1] \cup [2, 3]$ and define $\delta : S \times S \to \mathbb{R}_0^+$ by

$$\delta(x, y) = \begin{cases} \max\{x, y\} & \text{if } \{x, y\} \cap [2, 3] \neq \emptyset, \\ |x - y| & \text{if } \{x, y\} \subset [0, 1]. \end{cases}$$

Then, (S, δ) is a partial metric space.

On account of the topology of a standard metric space, we are able to define corresponding topological notions in the setting of a partial metric space, for more details, see, e.g., [1–32]. In particular, we consider the open ball

$$O_p(x, \epsilon) = \{y \in S : \delta(x, y) < \delta(x, x) + \epsilon\},$$

and open cover $\left\{O_p(x, \epsilon) : x \in S, \epsilon > 0\right\}$ for all $x \in S$ and $\epsilon > 0$. Moreover, the topology τ_δ, induced by a partial metric δ, is classified as T_0 topology on S.

Definition 2 A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partial metric space (S, δ) converges to a point $x \in S$ ($x_n \to x$, in short) with respect to τ_δ if and only if $\delta(x, x) = \lim_{n\to\infty} \delta(x, x_n)$.

Despite the intensive similarity between the definitions and topologies of standard and partial metrics, the structure of partial metric spaces varies in many aspects. The most important difference between them is on the uniqueness of a limit. More precisely, the limit of a sequence in partial metric space is not necessarily unique. For instance, recon the sequence $\{\frac{1}{n^2+n+1}\}_{n\in\mathbb{N}}$ in the partial metric space, introduced in Example 1. It is easy to see that

$$\delta(m, m) = \lim_{n\to\infty} \delta(m, \frac{1}{n^2 + n + 1}) = m \qquad \text{for any integer } m.$$

As a result, limit of that sequence depends on the integer m; hence, it is not unique. To repair and fix this weakness of the partial metric, we add some certain condition so that we guarantee uniqueness of the limit of a sequence.

Lemma 1 (See, e.g., [22, 32]) *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in (S, δ) such that $x_n \to x$ and $x_n \to y$ with respect to τ_δ. If*

$$\lim_{n\to\infty} \delta(x_n, x_n) = \delta(x, x) = \delta(y, y),$$

then $x = y$.

In what follows, we underline the connection between the usual metric spaces and the partial metric spaces. On account of a partial metric (S, δ), we deduce the following functions $d_\delta, d_m^\delta, d_0 : S \times S \to \mathbb{R}_0^+$ with the following definitions

$$d_\delta(x, y) = 2\delta(x, y) - \delta(x, x) - \delta(y, y), \tag{3}$$

$$\begin{aligned} d_m^\delta(x, y) &= \max\{\delta(x, y) - \delta(x, x), \delta(x, y) - \delta(y, y)\}, \\ &= \delta(x, y) - \min\{\delta(x, x), \delta(y, y)\}, \end{aligned} \tag{4}$$

$$d_0(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \delta(x, y) & \text{otherwise}, \end{cases} \tag{5}$$

form standard metrics on S, for more details, see, e.g., [17, 29].

The following topological inclusions are well known and easy to check:

$$\tau_p \subseteq \tau_{d_\delta} = \tau_{d_\delta^m} \subseteq \tau_{d_0}.$$

Furthermore, the following equivalence will be useful later on:

$$\lim_{n\to\infty} d_\delta(x, x_n) = 0 \Leftrightarrow \delta(x, x) = \lim_{n\to\infty} \delta(x, x_n) = \lim_{n,m\to\infty} \delta(x_n, x_m). \tag{6}$$

We emphasize that for the given partial metric example in Example 1, the corresponding standard metrics d_δ and d_δ^m provide the Euclidean metrics on S.

The analog of the topological notions, such as, fundamental (Cauchy), completeness, in the setting of partial metric spaces is given below:

Definition 3 (See, e.g., [21, 28, 29])

1. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (S, δ) is called a fundamental (Cauchy) sequence in (S, δ) if $\lim_{n,m\to\infty} \delta(x_n, x_m)$ exists and is finite.
2. (S, δ) is called complete if every fundamental sequence $\{x_n\}_{n\in\mathbb{N}}$ converges with respect to τ_δ to a point $x \in S$ such that $\delta(x, x) = \lim_{n,m\to\infty} \delta(x_n, x_m)$.

In what follows, we shall give a characterization of fundamental sequence and completeness in the setting of partial metric spaces.

Lemma 2 (See [29])

1. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (S, δ) is a fundamental sequence in (S, δ) if and only if it is a fundamental sequence in the metric space (X, d_δ).*
2. *(S, δ) is complete if and only if the metric space (X, d_δ) is complete.*

We note that the considered partial metric in Examples 1, 3, and 4 provides the completeness of the corresponding abstract space.

In our context, the following characterization will be useful.

Lemma 3 (See, e.g., [27]) *A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* (S,δ) *is a fundamental (Cauchy) sequence in* (S,δ) *if and only if it satisfies the following condition:*

$(*)$ *for each* $\varepsilon > 0$, *there is* $n_0 \in \mathbb{N}$ *such that* $\delta(x_n, x_m) - \delta(x_n, x_n) < \varepsilon$ *whenever* $n_0 \le n \le m$.

Lemma 4 *Let* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *be sequences in* (S,δ) *such that* $x_n \to x$ *and* $y_n \to y$ *with respect to* τ_{d_δ}. *Then,*

$$\lim_{n\to\infty} \delta(x_n, y_n) = \delta(x, y).$$

For our purposes, we need to recall the following notion.

Definition 4 (cf. [11])

1. F is called orbitally continuous if

$$\lim_{i,j\to\infty} \delta(F^{n_i}x, F^{n_j}x) = \lim_{i\to\infty} \delta(F^{n_i}x, z) = \delta(z, z) \tag{7}$$

implies

$$\lim_{i,j\to\infty} \delta(FF^{n_i}x, FF^{n_j}x) = \lim_{i\to\infty} \delta(FF^{n_i}x, Fz) = \delta(Fz, Fz), \tag{8}$$

for each $x \in S$.

Equivalently, F is orbitally continuous provided that if $F^{n_i}x \to z$ with respect to τ_{d_δ}, then $F^{n_i+1}x \to Fz$ with respect to τ_{d_δ}, for each $x \in S$.

2. (S,δ) is called orbitally complete if every fundamental sequence of type $\{F^{n_i}x\}_{i\in\mathbb{N}}$ converges with respect to τ_{d_δ}, that is, if there is $z \in S$ such that

$$\lim_{i,j\to\infty} \delta(F^{n_i}x, F^{n_j}x) = \lim_{i\to\infty} \delta(F^{n_i}x, z) = \delta(z, z). \tag{9}$$

In this manuscript, we investigate the existence of a fixed point for certain mapping in the context of partial metric spaces without caring the uniqueness. More accurately, this paper is prepared as a typical nonunique fixed point result in the trend of the famous work of Ćirić [11]. The presented results not only extend and generalize the existing results in the literature but also unify some and enrich this trend. We shall also provide an example to indicate the advantages in usage of partial metric spaces rather than standard metric spaces.

2 The Results

In this section, we shall state and prove the main theorems of the paper.

From now on, we assume that all partial metric spaces (S,δ) are orbitally complete, and self-mapping F on (S,δ) is orbitally continuous. Regarding these assumptions, we shall avoid to put these assumptions to all statements of the following theorems and corollaries to keep away from the repetitions.

Now, we recall the definition of auxiliary functions that we shall use in the statements of our results.

A function $\varphi : [0,\infty) \to [0,\infty)$ is called a comparison function [10, 31] if it is increasing and $\varphi^n(t) \to 0$ as $n \to \infty$ for every $t \in [0,\infty)$, where φ^n is the n-th iterate of φ.

Let Φ be the family of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(p1) ϕ is nondecreasing;

(p2) $\displaystyle\sum_{n=1}^{+\infty} \phi^n(t) < \infty$ for all $t > 0$.

Then, a function $\phi \in \Phi$ is called (c)-comparison function.

More details and examples of comparison and (c)-comparison functions can be found in [31]. The following crucial lemma underlines the interesting properties of comparison functions.

Lemma 5 ([31]) *If $\phi : [0,\infty) \to [0,\infty)$ is a comparison function, then*

1. *each iterate ϕ^k of ϕ, $k \geq 1$ is also a comparison function;*
2. *ϕ is continuous at 0;*
3. *$\phi(t) < t$ for all $t > 0$.*

It is clear that if ϕ is a (c)-comparison function is a comparison function. Hence, the properties above are also valid for (c)-comparison functions.

Ćirić Type Nonunique Fixed Point Theorems

In what follows, we state and prove the first main result that is inspired from the work of Ćirić [11].

Theorem 1 *If there is $\phi \in \Phi$ such that*

$$\begin{aligned}\min\{\delta(Fx,Fy),\delta(x,Fx),\delta(y,Fy)\} - \min\{d_m^\delta(x,Fy), d_m^\delta(Fx,y)\} \\ \leq \phi(\delta(x,y) - |\delta(x,x) - \delta(y,y)|),\end{aligned} \tag{10}$$

for all $x, y \in S$, then for each $x_0 \in S$ the sequence $\{F^n x_0\}_{n\in\mathbb{N}_0}$ converges with respect to τ_{d_δ} to a fixed point of F.

Proof Take an arbitrary point $x_0 \in S$. We define the iterative sequence $\{x_n\}_{n\in\mathbb{N}_0}$ as follows:

$$x_{n+1} = Fx_n, \quad n \in \mathbb{N}_0.$$

If there exists $n_0 \in \mathbb{N}_0$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of F. Assume then that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}_0$.

Substituting $x = x_n$ and $y = x_{n+1}$ in (10), we find the inequality

$$\begin{aligned}
\min\{&\delta(x_{n+1}, x_{n+2}), \delta(x_n, x_{n+1}), \delta(x_{n+1}, x_{n+2})\} \\
&\qquad - \min\{d_m^\delta(x_n, x_{n+2}), d_m^\delta(x_{n+1}, x_{n+1})\} \\
&\qquad \leq \phi(\delta(x_n, x_{n+1}) - |\delta(x_n, x_n) - \delta(x_{n+1}, x_{n+1})|),
\end{aligned}$$

which imply that

$$\begin{aligned}
\min\{&\delta(x_n, x_{n+1}), \delta(x_{n+1}, x_{n+2})\} \\
&\leq \phi(\delta(x_n, x_{n+1}) - \delta(x_n, x_n) + \delta(x_{n+1}, x_{n+1})) \\
&\leq \phi(\delta(x_n, x_{n+1})) \\
&< \delta(x_n, x_{n+1}).
\end{aligned} \tag{11}$$

Suppose $\delta(x_{n_0}, x_{n_0+1}) \leq \delta(x_{n_0+1}, x_{n_0+2})$ for some $n_0 \in \mathbb{N}_0$. Then, the inequality above yields that

$$\delta(x_{n_0}, x_{n_0+1}) < \delta(x_{n_0}, x_{n_0+1}),$$

a contradiction.

Therefore, $\delta(x_n, x_{n+1}) \geq \delta(x_{n+1}, x_{n+2})$ for all $n \in \mathbb{N}_0$.

Hence, by (11), we get

$$\delta(x_{n+1}, x_{n+2}) \leq \phi(\delta(x_n, x_{n+1})) \leq \cdots \leq \phi^{n+1}(\delta(x_0, x_1)), \tag{12}$$

for any $n \in \mathbb{N}$. We shall show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (S, δ). Indeed, let $n, m \in \mathbb{N}_0$ with $n < m$. Then, by using (12) and ($P4$), we derive that

$$\begin{aligned}
\delta(x_n, x_m) - \delta(x_n, x_n) &\leq \delta(x_n, x_{n+1}) + \cdots + \delta(x_{m-1}, x_m) - \sum_{k=n+1}^{m-1} \delta(x_k, x_k) \\
&\leq \phi^n(\delta(x_0, x_1)) \cdots + \phi^{m-1}(\delta(x_0, x_1)) \\
&\leq \sum_{k=n}^{m-1} \phi^k(\delta(x_0, x_1)) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

As a result, the sequence $\{x_n\}_{n\in\mathbb{N}_0}$ satisfies condition $(*)$ of Lemma 3. Consequently, it is a Cauchy sequence in (S, δ). Since $x_n = F^n x_0$ for all n, and (S, δ) is F-orbitally complete, there is $z \in S$ such that $x_n \to z$ with respect to τ_{d_δ}. By the orbital continuity of F, we deduce that $x_n \to Fz$ with respect to τ_{d_δ}. Hence, $z = Fz$, which concludes the proof.

The following result is an immediate consequence of Theorem 1 by letting $\phi(t) = kt$ where $k \in (0, 1)$.

Corollary 1 *If there is* $k \in (0, 1)$ *such that*

$$\begin{aligned} \min\{\delta(Fx, Fy), \delta(x, Fx), \delta(y, Fy)\} - \min\{\delta(x, Fy), \delta(Fx, y)\} \\ \leq k(\delta(x, y) - |\delta(x, x) - \delta(y, y)|), \end{aligned} \tag{13}$$

for all $x, y \in S$, *then for each* $x_0 \in S$ *the sequence* $\{F^n x_0\}_{n\in\mathbb{N}_0}$ *converges to a fixed point of* F.

Regarding the monotonicity of the (c)-comparison function, we derive the following corollary:

Corollary 2 *If there is* $\phi \in \Phi$ *such that*

$$\begin{aligned} \min\{\delta(Fx, Fy), \delta(x, Fx), \delta(y, Fy)\} - \min\{\delta(x, Fy), \delta(Fx, y)\} \\ \leq \phi(\delta(x, y)), \end{aligned} \tag{14}$$

for all $x, y \in S$, *then for each* $x_0 \in S$ *the sequence* $\{F^n x_0\}_{n\in\mathbb{N}_0}$ *converges to a fixed point of* F.

The following result is an immediate consequence of Corollary 2 by letting $\phi(t) = kt$ where $k \in (0, 1)$.

Corollary 3 *If there is* $k \in (0, 1)$ *such that*

$$\begin{aligned} \min\{\delta(Fx, Fy), \delta(x, Fx), \delta(y, Fy)\} - \min\{\delta(x, Fy), \delta(Fx, y)\} \\ \leq k\delta(x, y), \end{aligned} \tag{15}$$

for all $x, y \in S$, *then for each* $x_0 \in S$ *the sequence* $\{F^n x_0\}_{n\in\mathbb{N}_0}$ *converges to a fixed point of* F.

Notice that each metric forms a partial metric, but the converse is not true. Thus, the following is the immediate consequence of Corollary 2.

Corollary 4 *If there is* $\phi \in \Phi$ *such that*

$$\begin{aligned} \min\{d(Fx, Fy), d(x, Fx), d(y, Fy)\} - \min\{d(x, Fy), d(Fx, y)\} \\ \leq \phi(d(x, y)), \end{aligned} \tag{16}$$

for all $x, y \in S$, *then for each* $x_0 \in S$ *the sequence* $\{F^n x_0\}_{n\in\mathbb{N}_0}$ *converges to a fixed point of* F.

The next result is belong to Ćirić [11] in the context of metric spaces that is derived from Corollary 4 by letting $\phi(t) = kt$ where $k \in (0, 1)$.

Corollary 5 ([11, Nonunique Fixed Point Theorem of Ćirić]) *If there is* $k \in (0, 1)$ *such that*

$$\min\{d(Fx, Fy), d(x, Fx), d(y, Fy)\} - \min\{d(x, Fy), d(Fx, y)\} \\ \leq kd(x, y), \tag{17}$$

for all $x, y \in S$, then for each $x_0 \in S$ the sequence $\{F^n x_0\}_{n \in \mathbb{N}_0}$ converges to a fixed point of F.

The following are examples where Theorem 1 can be applied but not Corollary 5 for the metrics d_δ and d_m^δ, and d_0, respectively.

Example 5 Let $S = \{0, 1, 2\}$ endowed with a partial metric $\delta(x, y) = \max\{x, y\}$ for all $x, y \in S$. Define $F : S \to X$ by $F0 = F1 = 0$ and $F2 = 1$. Since (S, δ) is complete, then it is F-orbitally complete. Moreover, it is obvious that F is orbitally continuous. An easy computation shows that

$$\min\{\delta(Fx, Fy), \delta(x, Fx), \delta(y, Fy)\} - \min\{d_m^\delta(x, Fy), d_m^\delta(Fx, y)\} \\ \leq \phi(\delta(x, y) - |\delta(x, x) - \delta(y, y)|),$$

for all $x, y \in S$ and for certain ϕ, e.g., by letting $\psi(t) = \frac{t}{3}$. So, the conditions of Theorem 1 are satisfied. However, there is no ϕ such that

$$\min\{d_\delta(T1, T2), d_\delta(1, T1), d_\delta(2, T2)\} - \min\{d_\delta(1, T2), d_\delta(T1, 2)\} \\ = 1 - 0 = 1 \leq \psi(d_p(1, 2)) < d_p(1, 2) = 1$$

is satisfied. Accordingly, Corollary 5 cannot be applied to the complete metric space (S, d_δ).

Achari Type Nonunique Fixed Point Theorems

The following theorem is based on the interesting result of Achari [3].

Theorem 2 *Suppose that there exists $\psi \in \Phi$ such that*

$$\tfrac{P(x,y) - Q(x,y)}{R(x,y)} \leq \psi(\delta(x, y)), \tag{18}$$

for all $x, y \in S$, where

$$P(x, y) = \min\{\delta(Fx, Fy)\delta(x, y), \delta(x, Fx)\delta(y, Fy)\},$$
$$Q(x, y) = \min\{d_m^\delta(x, Fx)d_m^\delta(x, Fy), d_m^\delta(y, Fy)d_m^\delta(Fx, y)\},$$
$$R(x, y) = \min\{\delta(x, Fx), \delta(y, Fy)\},$$

with $R(x, y) \neq 0$. Then, for each $x_0 \in S$, the sequence $\{F^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of F.

Proof For an arbitrary initial point $x_0 \in S$, we construct an iterative sequence $\{x_n\}_{n \in \mathbb{N}_0}$ as follows:

$$x_{n+1} = Fx_n, \quad n \in \mathbb{N}_0.$$

Without loss of generality, we suppose then that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}_0$. Indeed, if there exists $n_0 \in \mathbb{N}_0$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of F.

By letting $x = x_n$ and $y = x_{n+1}$ in (21) we find the inequality

$$\frac{P(x_n, x_{n+1}) - Q(x_n, x_{n+1})}{R(x_n, x_{n+1})} \leq \psi(\delta(x_n, x_{n+1})),$$

where

$$\begin{aligned}
P(x_n, x_{n+1}) &= \min\{\delta(Fx_n, Fx_{n+1})\delta(x_n, x_{n+1}), \delta(x_n, Fx_n)\delta(x_{n+1}, Fx_{n+1})\}, \\
&= \min\{\delta(x_{n+1}, x_{n+2})\delta(x_n, x_{n+1}), \delta(x_n, x_{n+1})\delta(x_{n+1}, x_{n+2})\},
\end{aligned}$$

$$\begin{aligned}
Q(x_n, x_{n+1}) &= \min\{d_m^\delta(x_n, Fx_n)d_m^\delta(x_n, Fx_{n+1}), d_m^\delta(x_{n+1}, Fx_{n+1})d_m^\delta(Fx_n, x_{n+1})\}, \\
&= \min\{d_m^\delta(x_n, x_{n+1})d_m^\delta(x_n, x_{n+2}), d_m^\delta(x_{n+1}, x_{n+2})d_m^\delta(x_{n+1}, x_{n+1})\}, \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
R(x_n, x_{n+1}) &= \min\{\delta(x_n, Fx_n), \delta(x_{n+1}, Fx_{n+1})\} \\
&= \min\{\delta(x_n, x_{n+1}), \delta(x_{n+1}, x_{n+2})\}.
\end{aligned}$$

Consequently, we derive that

$$\frac{\delta(x_{n+1}, x_{n+2})\delta(x_n, x_{n+1})}{\min\{\delta(x_n, x_{n+1}), \delta(x_{n+1}, x_{n+2})\}} \leq \psi(\delta(x_n, x_{n+1})). \tag{19}$$

Suppose for some n_0, we have $\delta(x_{n_0+1}, x_{n_0+2}) \geq \delta(x_{n_0}, x_{n_0+1})$. Then, the inequality above yields that

$$\frac{\delta(x_{n_0+1}, x_{n_0+2})\delta(x_{n_0}, x_{n_0+1})}{\delta(x_{n_0}, x_{n_0+1})} \leq \psi(\delta(x_{n_0}, x_{n_0+1})),$$

and hence

$$\delta(x_{n_0}, x_{n_0+1}) \leq \psi(\delta(x_{n_0}, x_{n_0+1})) < \delta(x_{n_0}, x_{n_0+1}),$$

a contradiction. Consequently, we deduce that $\delta(x_{n+1}, x_{n+2}) \leq \delta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and further, from (19) , we have

$$\delta(x_{n_0+1}, x_{n_0+2}) \leq \psi(\delta(x_{n_0}, x_{n_0+1})) \leq \cdots \leq \psi^{n+1}(\delta(x_0, x_1)),$$

for all $n \in \mathbb{N}$.

As a next step, we shall prove that the constructive sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in (S, δ). Suppose that $n, m \in \mathbb{N}_0$ with $n < m$. Then, by using $(P4)$, we find that

$$\begin{aligned}
\delta(x_n, x_m) &\le \delta(x_n, x_{n+1}) + \cdots + \delta(x_{m-1}, x_m) - \sum_{k=n}^{m-1} \delta(x_k, x_k) \\
&\le \psi^n(\delta(x_0, x_1)) \cdots + \psi^{m-1}(\delta(x_0, x_1)) \\
&\le \sum_{k=n}^{m-1} \psi^k(\delta(x_0, x_1)) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Accordingly, we get that

$$\lim_{n\to\infty} d_\delta(x, x_n) = 0 \Leftrightarrow 0 = \delta(x, x) = \lim_{n\to\infty} \delta(x, x_n) = \lim_{n,m\to\infty} \delta(x_n, x_m). \tag{20}$$

Consequently, it is a Cauchy sequence in (S, δ). On account of $x_n = F^n x_0$ for all n, and regarding the orbitally completeness of (S, δ), there is $z \in S$ such that $x_n \to z$ with respect to τ_{d_δ}. Taking the orbital continuity of F into account, we find that $x_n \to Fz$ with respect to τ_{d_δ}. Thus, $z = Fz$, which concludes the proof.

An immediate corollary of Theorem 7 is obtained by letting $\psi(t) = kt$ for $k \in [0, 1)$

Corollary 6 *Suppose that there exists $\psi \in \Phi$ such that*

$$\frac{P(x, y) - Q(x, y)}{R(x, y)} \le \psi(\delta(x, y)), \tag{21}$$

for all $x, y \in S$, where

$$\begin{aligned}
P(x, y) &= \min\{\delta(Fx, Fy)\delta(x, y), \delta(x, Fx)\delta(y, Fy)\}, \\
Q(x, y) &= \min\{d_m^\delta(x, Fx)d_m^\delta(x, Fy), d_m^\delta(y, Fy)d_m^\delta(Fx, y)\}, \\
R(x, y) &= \min\{\delta(x, Fx), \delta(y, Fy)\}.
\end{aligned}$$

with $R(x, y) \neq 0$. Then, for each $x_0 \in S$, the sequence $\{F^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of F.

The following is the famous theorem of Achari [3] in the setting of standard metric spaces.

Corollary 7 ([3, Nonunique fixed point of Achari]) *Suppose that there exists $k \in [0, 1)$ such that*

$$\frac{P(x,y)-Q(x,y)}{R(x,y)} \le kd(x, y), \tag{22}$$

for all $x, y \in S$, *where*

$$
\begin{aligned}
P(x, y) &= \min\{d(Fx, Fy)d(x, y), d(x, Fx)d(y, Fy)\},\\
Q(x, y) &= \min\{(x, Fx)\delta(x, Fy), \delta(y, Fy)\delta(Fx, y)\},\\
R(x, y) &= \min\{d(x, Fx), d(y, Fy)\},
\end{aligned}
$$

with $R(x, y) \neq 0$. *Then, for each* $x_0 \in S$, *the sequence* $\{F^n x_0\}_{n\in\mathbb{N}}$ *converges to a fixed point of* F.

Pachpatte Type Nonunique Fixed Point Theorems

Let Θ be the set of all functions $\vartheta \in \Phi$ with an additional condition

$$\varphi(t^2) \leq [\varphi(t)]^2 \text{ for all } t > 0.$$

Inspired from the renowned result of Pachpatte [30], we propose the following result.

Theorem 3 *Suppose that there exists* $\vartheta \in \Theta$ *such that*

$$m(x, y) - n(x, y) \leq \vartheta(\delta(x, Fx)\delta(y, Fy)), \tag{23}$$

for all $x, y \in S$, *where*

$$
\begin{aligned}
m(x, y) &= \min\{[\delta(Fx, Fy)]^2, \delta(x, y)\delta(Fx, Fy), [\delta(y, Fy)]^2\},\\
n(x, y) &= \min\{d_m^\delta(x, Fx)d_m^\delta(y, Fy), d_m^\delta(x, Fy)d_m^\delta(y, Fx)\}.
\end{aligned}
$$

Then, for each $x_0 \in S$, *the sequence* $\{F^n x_0\}_{n\in\mathbb{N}}$ *converges to a fixed point of* F.

Proof Fix initial point $x_0 \in S$, we set up a recursive sequence $\{x_n\}_{n\in\mathbb{N}_0}$ by the following definition:

$$x_{n+1} = Fx_n, \quad n \in \mathbb{N}_0.$$

We assume, without loss of generality, that the adjacent terms are distinct, that is, $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}_0$. In fact, if there exists $n_0 \in \mathbb{N}_0$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} forms a fixed point of F.

By letting $x = x_n$ and $y = x_{n+1}$ in (23), we derive the following inequality:

$$
\begin{aligned}
m(x_n, x_{n+1}) - n(x_n, x_{n+1}) &\leq \vartheta(\delta(x_n, Fx_n)\delta(x_{n+1}, Fx_{n+1})),\\
&= \vartheta(\delta(x_n, x_{n+1})\delta(x_{n+1}, x_{n+2})),
\end{aligned}
\tag{24}
$$

where

$$\begin{aligned} m(x_n, x_{n+1}) &= \min\{[\delta(Fx_n, Fx_{n+1})]^2, \delta(x_n, x_{n+1})\delta(Fx_n, Fx_{n+1}), [\delta(x_{n+1}, Fx_{n+1})]^2\} \\ &= \min\{[\delta(x_{n+1}, x_{n+2})]^2, \delta(x_n, x_{n+1})\delta(x_{n+1}, x_{n+2}), [\delta(x_{n+1}, x_{n+2})]^2\}, \end{aligned}$$

$$\begin{aligned} n(x_n, x_{n+1}) &= \min\{d_m^\delta(x_n, Fx_n)d_m^\delta(x_{n+1}, Fx_{n+1}), d_m^\delta(x_n, Fx_{n+1})d_m^\delta(x_{n+1}, Fx_n)\} \\ &= \min\{d_m^\delta(x_n, x_{n+1})d_m^\delta(x_{n+1}, x_{n+2}), d_m^\delta(x_n, x_{n+2})d_m^\delta(x_{n+1}, x_{n+1})\} \\ &= 0. \end{aligned}$$

Consequently, the inequality (24) turns into

$$\min\{[\delta(x_{n+1}, x_{n+2})]^2, \delta(x_n, x_{n+1})\delta(x_{n+1}, x_{n+2})\} \leq \vartheta(\delta(x_n, x_{n+1})\delta(x_{n+1}, x_{n+2})), \tag{25}$$

for all $n \in \mathbb{N}$. Suppose that $\delta(x_{n_0}, x_{n_0+1}) \leq \delta(x_{n_0+1}, x_{n_0+2})$ for some $n_0 \in \mathbb{N}$. Then, the inequality (25) becomes

$$\begin{aligned} \delta(x_{n_0}, x_{n_0+1})\delta(x_{n_0+1}, x_{n_0+2}) &\leq \vartheta(\delta(x_{n_0}, x_{n_0+1})\delta(x_{n_0+1}, x_{n_0+2})) \\ &< \delta(x_{n_0}, x_{n_0+1})\delta(x_{n_0+1}, x_{n_0+2}), \end{aligned} \tag{26}$$

a contradiction. Thus, we have $\delta(x_{n+1}, x_{n+2}) < \delta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Moreover, keeping the property of ϑ in mind, we derive, from inequality (26), that

$$\delta(x_n, x_{n+1})^2 \leq \vartheta(\delta(x_{n-1}, x_n)^2) < [\vartheta(\delta(x_{n-1}, x_n))]^2$$

$$\delta(x_n, x_{n+1}) \leq \vartheta^n(\delta(x_0, x_1)) \text{ for all } n \in \mathbb{N}. \tag{27}$$

In what follows, we indicate that the recursive sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in (S, δ). Consider $n, m \in \mathbb{N}_0$ with $n < m$. Then, by using $(P4)$, we find that

$$\begin{aligned} \delta(x_n, x_m) &\leq \delta(x_n, x_{n+1}) + \cdots + \delta(x_{m-1}, x_m) - \sum_{k=n}^{m-1} \delta(x_k, x_k) \\ &\leq \vartheta^n(\delta(x_0, x_1)) \cdots + \vartheta^{m-1}(\delta(x_0, x_1)) \\ &\leq \sum_{k=n}^{m-1} \vartheta^k(\delta(x_0, x_1)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Attendantly, we find that

$$\lim_{n\to\infty} d_\delta(x, x_n) = 0 \Leftrightarrow 0 = \delta(x, x) = \lim_{n\to\infty} \delta(x, x_n) = \lim_{n,m\to\infty} \delta(x_n, x_m). \tag{28}$$

As a result, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (S, δ). Keeping, $x_n = F^n x_0$ for all n, in mind, and regarding the orbitally completeness of (S, δ),we deduce that there is $z \in S$ such that $x_n \to z$ with respect to τ_{d_δ}. Employing the orbital continuity of F, we get that $x_n \to Fz$ with respect to τ_{d_δ}. So, $z = Fz$.

An immediate corollary of Theorem 3 is obtained by letting $\vartheta(t) = kt$ for $k \in [0, 1)$

Corollary 8 *Suppose that there exists* $k \in [0, 1)$ *such that*

$$m(x, y) - n(x, y) \leq k\delta(x, Fx)\delta(y, Fy),$$

for all $x, y \in S$, *where*

$$\begin{aligned} m(x, y) &= \min\{[\delta(Fx, Fy)]^2, \delta(x, y)\delta(Fx, Fy), [\delta(y, Fy)]^2\}, \\ n(x, y) &= \min\{d_m^\delta(x, Fx)d_m^\delta(y, Fy), d_m^\delta(x, Fy)d_m^\delta(y, Fx)\}. \end{aligned}$$

Then, for each $x_0 \in S$, *the sequence* $\{F^n x_0\}_{n\in\mathbb{N}}$ *converges to a fixed point of* F.

In what follows, we deduce the renowned result of Pachpatte [30] in the setting of standard metric spaces.

Corollary 9 ([30, Nonunique fixed point of Pachpatte]) *Suppose that there exists* $k \in [0, 1)$ *such that*

$$m(x, y) - n(x, y) \leq kd(x, Fx)d(y, Fy),$$

for all $x, y \in S$, *where*

$$\begin{aligned} m(x, y) &= \min\{[d(Fx, Fy)]^2, d(x, y)d(Fx, Fy), [d(y, Fy)]^2\}, \\ n(x, y) &= \min\{d(x, Fx)d(y, Fy), d(x, Fy)d(y, Fx)\}. \end{aligned}$$

Then, for each $x_0 \in S$, *the sequence* $\{F^n x_0\}_{n\in\mathbb{N}}$ *converges to a fixed point of* F.

Conclusion

It is possible to obtain more consequence of the obtained result by considering different type control functions. Notice also that all obtained results in the context of partial metric spaces are valid in the setting of standard metric spaces, either.

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Some New Refinement of Gauss–Jacobi and Hermite–Hadamard Type Integral Inequalities

Artion Kashuri and Rozana Liko

Abstract In this paper, the authors discover two interesting identities regarding Gauss–Jacobi and Hermite–Hadamard type integral inequalities. By using the first lemma as an auxiliary result, some new bounds with respect to Gauss–Jacobi type integral inequalities are established. Also, using the second lemma, some new estimates with respect to Hermite–Hadamard type integral inequalities via general fractional integrals are obtained. It is pointed out that some new special cases can be deduced from main results. Some applications to special means for different positive real numbers and new error estimates for the trapezoidal formula are provided as well. These results give us the generalizations, refinement and significant improvements of the new and previous known results. The ideas and techniques of this paper may stimulate further research.

1 Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$. For any subset $K \subseteq \mathbb{R}^n$, K° is the interior of K. The set of integrable functions on the interval $[a_1, a_2]$ is denoted by $L[a_1, a_2]$.

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1 *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a_1, a_2 \in I$ with $a_1 < a_2$. Then, the following inequality holds:*

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} f(x)dx \leq \frac{f(a_1) + f(a_2)}{2}. \tag{1}$$

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This inequality (1) *is also known as trapezium inequality.*

The trapezium type inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. For other recent results that generalize, improve and extend the inequality (1) through various classes of convex functions, interested readers are referred to [1–33, 35, 37, 38].
The Gauss–Jacobi type quadrature formula has the following:

$$\int_{a_1}^{a_2} (x - a_1)^p (a_2 - x)^q f(x) dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^{\star}|f|, \tag{2}$$

for certain $B_{m,k}$, γ_k and rest $R_m^{\star}|f|$, see [34].

Recently in [20], Liu obtained several integral inequalities for the left-hand side of (2). Also in [28], Özdemir et al. established several integral inequalities concerning the left-hand side of (2) via some kinds of convexity.
Let us recall some special functions and evoke some basic definitions as follows.

Definition 1 For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the k-gamma function is defined by

$$\Gamma_k(x) = \lim_{n \longrightarrow \infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}. \tag{3}$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt. \tag{4}$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha). \tag{5}$$

For $k = 1$, (4) gives integral representation of gamma function.

Definition 2 ([24]) Let $f \in L[a_1, a_2]$. Then, k-fractional integrals of order $\alpha, k > 0$ with $a_1 \geq 0$ are defined as

$$I_{a_1^+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{a_1}^{x} (x - t)^{\frac{\alpha}{k}-1} f(t) dt, \quad x > a_1$$

and

$$I_{a_2^-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{x}^{a_2} (t - x)^{\frac{\alpha}{k}-1} f(t) dt, \quad a_2 > x. \tag{6}$$

For $k = 1$, k-fractional integrals give Riemann–Liouville integrals.

Definition 3 ([36]) A set $S \subseteq \mathbb{R}^n$ is said to be an invex set with respect to the mapping $\eta : S \times S \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$.

The invex set S is also termed an η-connected set.

Definition 4 Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \longrightarrow \mathbb{R}^n$. A function $f : S \longrightarrow [0, +\infty)$ is said to be preinvex with respect to η, if for every $x, y \in S$ and $t \in [0, 1]$,

$$f\big(x + t\eta(y, x)\big) \leq (1 - t)f(x) + tf(y). \tag{7}$$

Also, let us define a function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty, \tag{8}$$

$$\frac{1}{A} \leq \frac{\varphi(s)}{\varphi(r)} \leq A \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{9}$$

$$\frac{\varphi(r)}{r^2} \leq B\frac{\varphi(s)}{s^2} \text{ for } s \leq r \tag{10}$$

$$\left|\frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2}\right| \leq C|r - s|\frac{\varphi(r)}{r^2} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2, \tag{11}$$

where $A, B, C > 0$ are independent of $r, s > 0$. If $\varphi(r)r^{\alpha}$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(r)}{r^{\beta}}$ is decreasing for some $\beta \geq 0$, then φ satisfies (8), (9), (10) and (11), see [31]. Therefore, we define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{a_1^+}I_{\varphi} f(x) = \int_{a_1}^{x} \frac{\varphi(x - t)}{x - t} f(t)dt, \quad x > a_1, \tag{12}$$

$$_{a_2^-}I_{\varphi} f(x) = \int_{x}^{a_2} \frac{\varphi(t - x)}{t - x} f(t)dt, \quad x < a_2. \tag{13}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann–Liouville fractional integral, k-Riemann–Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc., see [30].
Motivated by the above literatures, the main objective of this paper is to discover in Sects. 2 and 3 two interesting identities and to establish some new bounds regarding Gauss–Jacobi and Hermite–Hadamard type integral inequalities. By using in Sect. 2 the first lemma as an auxiliary result, some new bounds with respect to Gauss–Jacobi type integral inequalities will be given. Also, using in Sect. 3

the second lemma, some new estimates with respect to Hermite–Hadamard type integral inequalities via general fractional integrals will be obtained. It is pointed out that some new special cases will be deduced from main results. In Sect. 4, some applications to special means for different positive real numbers and new error estimates for the trapezoidal formula will be given. These results will give us the generalizations, refinement and significant improvements of the new and previous known results. The ideas and techniques of this paper may stimulate further research.

2 Some New Bounds of the Quadrature Formula of Gauss–Jacobi Type

Throughout this study, for brevity, we define

$$\Lambda^*_{m.n}(t) = \int_0^t \frac{\varphi\left(\eta(a_2, ma_1)\frac{x}{n+1}\right)}{\frac{x}{n+1}}dx < \infty, \quad \eta(a_2, ma_1) > 0.$$

For establishing some new bounds integral inequalities for Gauss–Jacobi type, we need the following lemma.

Lemma 1 *Suppose that* $n = 0, 1, 2, \ldots,$ *and* $m \in (0, 1]$ *be a fixed number. Let* $P = [ma_1, ma_1 + \eta(a_2, ma_1)] \subseteq \mathbb{R}$ *be an open* m*-invex subset. Assume that* $f : P \longrightarrow \mathbb{R}$ *be a continuous mapping on* P° *with respect to* $\eta : P \times P \longrightarrow \mathbb{R}$ *for* $\eta(a_2, ma_1) > 0$. *Then, for any fixed* $p, q > 0$, *we have*

$$\int_{ma_1}^{ma_1+\frac{\eta(a_2,ma_1)}{n+1}} \left[\Lambda^*_{m,n}\left(\frac{(n+1)(x-ma_1)}{\eta(a_2, ma_1)}\right)\right]^p$$

$$\times\left[\Lambda^*_{m,n}\left(\frac{m(n+1)a_1 + \eta(a_2, ma_1) - (n+1)x}{\eta(a_2, ma_1)}\right)\right]^q f(x)dx$$

$$= \frac{\eta(a_2, ma_1)}{n+1}\int_0^1 \left[\Lambda^*_{m,n}(t)\right]^p\left[\Lambda^*_{m,n}(1-t)\right]^q f\left(ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)\right)dt. \tag{14}$$

We denote

$$T^{p,q}_{f,\Lambda^*_{m,n}}(a_1, a_2) := \frac{\eta(a_2, ma_1)}{n+1} \tag{15}$$

$$\times\int_0^1 \left[\Lambda^*_{m,n}(t)\right]^p\left[\Lambda^*_{m,n}(1-t)\right]^q f\left(ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)\right)dt.$$

Proof Using (15) and changing the variable $x = ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)$, we have

$$T^{p,q}_{f,\Lambda^*_{m,n}}(a_1,a_2) = \frac{\eta(a_2,ma_1)}{n+1}\int_{ma_1}^{ma_1+\frac{\eta(a_2,ma_1)}{n+1}}\left[\Lambda^*_{m,n}\left(\frac{(n+1)(x-ma_1)}{\eta(a_2,ma_1)}\right)\right]^p$$

$$\times\left[\Lambda^*_{m,n}\left(\frac{m(n+1)a_1+\eta(a_2,ma_1)-(n+1)x}{\eta(a_2,ma_1)}\right)\right]^q f(x)\frac{(n+1)}{\eta(a_2,ma_1)}dx$$

$$= \int_{ma_1}^{ma_1+\frac{\eta(a_2,ma_1)}{n+1}}\left[\Lambda^*_{m,n}\left(\frac{(n+1)(x-ma_1)}{\eta(a_2,ma_1)}\right)\right]^p$$

$$\times\left[\Lambda^*_{m,n}\left(\frac{m(n+1)a_1+\eta(a_2,ma_1)-(n+1)x}{\eta(a_2,ma_1)}\right)\right]^q f(x)dx.$$

This completes the proof of the lemma.

Corollary 1 *Taking $n = 0$, $m = 1$, $\eta(a_2, ma_1) = a_2 - ma_1$ and $\varphi(x) = x$, in Lemma 1, we get the following identity:*

$$\int_{a_1}^{a_2}(x-a_1)^p(a_2-x)^q f(x)dx = (a_2-a_1)^{p+q+1}\int_0^1 t^p(1-t)^q f(a_1+t(a_2-a_1))dt. \tag{16}$$

With the help of Lemma 1, we have the following results.

Theorem 2 *Suppose that $n = 0, 1, 2, \ldots,$ and $m \in (0, 1]$ be a fixed number. Let $P = [ma_1, ma_1 + \eta(a_2, ma_1)] \subseteq \mathbb{R}$ be an open m-invex subset. Assume that $f : P \longrightarrow \mathbb{R}$ be a continuous mapping on P° with respect to $\eta : P \times P \longrightarrow \mathbb{R}$ for $\eta(a_2, ma_1) > 0$. If $|f|^{\frac{k}{k-1}}$ is preinvex mapping on P for $k > 1$, then for any fixed $p, q > 0$, we have*

$$\left|T^{p,q}_{f,\Lambda^*_{m,n}}(a_1,a_2)\right| \le \left(\frac{1}{2(n+1)}\right)^{\frac{k-1}{k}}\frac{\eta(a_2,ma_1)}{n+1}\sqrt[k]{A^{p,q}_{\Lambda^*_{m,n}}(k)} \tag{17}$$

$$\times\left[(2n+1)|f(ma_1)|^{\frac{k}{k-1}}+|f(a_2)|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}},$$

where

$$A^{p,q}_{\Lambda^*_{m,n}}(k) := \int_0^1\left[\Lambda^*_{m,n}(t)\right]^{kp}\left[\Lambda^*_{m,n}(1-t)\right]^{kq}dt.$$

Proof Since $|f|^{\frac{k}{k-1}}$ is preinvex mapping on P, combining with Lemma 1, Hölder inequality and properties of the modulus, we get

$$\left|T^{p,q}_{f,\Lambda^*_{m,n}}(a_1,a_2)\right| \le \frac{\eta(a_2,ma_1)}{n+1}$$

$$\times \int_0^1 \left[\Lambda^*_{m,n}(t)\right]^p \left[\Lambda^*_{m,n}(1-t)\right]^q \left| f\left(ma_1 + \frac{t}{n+1}\eta(a_2,ma_1)\right)\right| dt$$

$$\le \frac{\eta(a_2,ma_1)}{n+1}\left[\int_0^1 \left[\Lambda^*_{m,n}(t)\right]^{kp}\left[\Lambda^*_{m,n}(1-t)\right]^{kq} dt\right]^{\frac{1}{k}}$$

$$\times \times \left[\int_0^1 \left| f\left(ma_1 + \frac{t}{n+1}\eta(a_2,ma_1)\right)\right|^{\frac{k}{k-1}} dt\right]^{\frac{k-1}{k}}$$

$$\le \frac{\eta(a_2,ma_1)}{n+1}\sqrt[k]{A^{p,q}_{\Lambda^*_{m,n}}(k)}$$

$$\times\left[\int_0^1 \left(\left(1-\frac{t}{n+1}\right)|f(ma_1)|^{\frac{k}{k-1}} + \frac{t}{n+1}|f(a_2)|^{\frac{k}{k-1}}\right) dt\right]^{\frac{k-1}{k}}$$

$$= \left(\frac{1}{2(n+1)}\right)^{\frac{k-1}{k}} \frac{\eta(a_2,ma_1)}{n+1}\sqrt[k]{A^{p,q}_{\Lambda^*_{m,n}}(k)} \times \left[(2n+1)|f(ma_1)|^{\frac{k}{k-1}} + |f(a_2)|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}}.$$

So, the proof of this theorem is completed.

We point out some special cases of Theorem 2.

Corollary 2 *Under the assumption of Theorem 2 with $n = 0$ and $\varphi(t) = t$, we get*

$$\left|T^{p,q}_{f,\Lambda^*_1}(a_1,a_2)\right| \le \eta^{p+q+1}(a_2,ma_1)\sqrt[k]{\beta(kp+1,kq+1)} \tag{18}$$

$$\times\left[\frac{|f(ma_1)|^{\frac{k}{k-1}} + |f(a_2)|^{\frac{k}{k-1}}}{2}\right]^{\frac{k-1}{k}},$$

where $\Lambda^*_1 := \eta(a_2,ma_1)t$.

Corollary 3 *Under the assumption of Theorem 2 with $n = 0$ and $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, we get*

$$\left|T^{p,q}_{f,\Lambda^*_2}(a_1,a_2)\right| \le \frac{\eta^{\alpha(p+q)+1}(a_2,ma_1)}{\Gamma^{p+q}(\alpha+1)}\sqrt[k]{\beta(\alpha kp+1,\alpha kq+1)} \tag{19}$$

$$\times\left[\frac{|f(ma_1)|^{\frac{k}{k-1}} + |f(a_2)|^{\frac{k}{k-1}}}{2}\right]^{\frac{k-1}{k}},$$

where $\Lambda_2^* := \frac{\eta^{\alpha}(a_2, ma_1)}{\Gamma(\alpha+1)} t^{\alpha}$.

Corollary 4 *Under the assumption of Theorem 2 with $n = 0$ and $\varphi(t) = \frac{t^{\frac{\alpha}{k_1}}}{k_1 \Gamma_{k_1}(\alpha)}$, we get*

$$\left|T^{p,q}_{f,\Lambda_3^*}(a_1, a_2)\right| \leq \frac{\eta^{\frac{\alpha}{k_1}(p+q)+1}(a_2, ma_1)}{\left[k_1 \Gamma_{k_1}(\alpha + k_1)\right]^{p+q}} \sqrt[k]{\beta\left(\frac{\alpha k p}{k_1} + 1, \frac{\alpha k q}{k_1} + 1\right)} \tag{20}$$

$$\times \left[\frac{|f(ma_1)|^{\frac{k}{k-1}} + |f(a_2)|^{\frac{k}{k-1}}}{2}\right]^{\frac{k-1}{k}},$$

where $\Lambda_3^* := \frac{\eta^{\frac{\alpha}{k_1}}(a_2, ma_1)}{k_1 \Gamma_{k_1}(\alpha+k_1)} t^{\frac{\alpha}{k_1}}$.

Corollary 5 *Under the assumption of Theorem 2 with $n = 0$ and $\varphi(t) = t(ma_1 + \eta(a_2, ma_1) - t)^{\alpha-1}$ and $f(x)$ is symmetric to $x = ma_1 + \frac{\eta(a_2, ma_1)}{2}$, we get*

$$\left|T^{p,q}_{f,\Lambda_4^*}(a_1, a_2)\right| \leq \frac{\eta^{\frac{k-1}{k}(p+q)+1}(a_2, ma_1)}{\alpha^{p+q}} \sqrt[k]{C^{p,q,m}(\alpha, k)} \tag{21}$$

$$\times \left[\frac{|f(ma_1)|^{\frac{k}{k-1}} + |f(a_2)|^{\frac{k}{k-1}}}{2}\right]^{\frac{k-1}{k}},$$

where

$$C^{p,q,m}(\alpha, k) := \int_{ma_1}^{ma_1+\eta(a_2,ma_1)} \left[(ma_1 + \eta(a_2, ma_1))^{\alpha} - t^{\alpha}\right]^{kp} \tag{22}$$

$$\times \left[(ma_1 + \eta(a_2, ma_1))^{\alpha} - (2ma_1 + \eta(a_2, ma_1) - t)^{\alpha}\right]^{kq} dt$$

and $\Lambda_4^* := \frac{(ma_1+\eta(a_2,ma_1))^{\alpha} - (ma_1+(1-t)\eta(a_2,ma_1))^{\alpha}}{\alpha}$.

Theorem 3 *Suppose that $n = 0, 1, 2, \ldots,$ and $m \in (0, 1]$ be a fixed number. Let $P = [ma_1, ma_1 + \eta(a_2, ma_1)] \subseteq \mathbb{R}$ be an open m-invex subset. Assume that $f : P \longrightarrow \mathbb{R}$ be a continuous mapping on P° with respect to $\eta : P \times P \longrightarrow \mathbb{R}$ for $\eta(a_2, ma_1) > 0$. If $|f|^l$ is preinvex mapping on P for $l \geq 1$, then for any fixed $p, q > 0$, we have*

$$\left|T^{p,q}_{f,\Lambda_{m,n}^*}(a_1, a_2)\right| \leq \frac{\eta(a_2, ma_1)}{n+1}\left[A^{p,q}_{\Lambda_{m,n}^*}(1)\right]^{\frac{l-1}{l}} \times \sqrt[l]{B^{p,q}_{\Lambda_{m,n}^*}|f(ma_1)|^l + C^{p,q}_{\Lambda_{m,n}^*}|f(a_2)|^l}, \tag{23}$$

where

$$B^{p,q}_{\Lambda^*_{m,n}} := \int_0^1 \left(1 - \frac{t}{n+1}\right) \left[\Lambda^*_{m,n}(t)\right]^p \left[\Lambda^*_{m,n}(1-t)\right]^q dt,$$

$$C^{p,q}_{\Lambda^*_{m,n}} := \frac{1}{n+1} \int_0^1 t \left[\Lambda^*_{m,n}(t)\right]^p \left[\Lambda^*_{m,n}(1-t)\right]^q dt,$$

and $A^{p,q}_{\Lambda^*_{m,n}}(1)$ *is defined as in Theorem 2.*

Proof Since $|f|^l$ is preinvex mapping on P, combining with Lemma 1, the well-known power mean inequality and properties of the modulus, we get

$$\left|T^{p,q}_{f,\Lambda^*_{m,n}}(a_1, a_2)\right| \leq \frac{\eta(a_2, ma_1)}{n+1}$$

$$\times \int_0^1 \left[\Lambda^*_{m,n}(t)\right]^p \left[\Lambda^*_{m,n}(1-t)\right]^q \left| f\left(ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)\right)\right| dt$$

$$\leq \frac{\eta(a_2, ma_1)}{n+1} \left[\int_0^1 \left[\Lambda^*_{m,n}(t)\right]^p \left[\Lambda^*_{m,n}(1-t)\right]^q dt\right]^{\frac{l-1}{l}}$$

$$\times \left[\int_0^1 \left[\Lambda^*_{m,n}(t)\right]^p \left[\Lambda^*_{m,n}(1-t)\right]^q \left| f\left(ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)\right)\right|^l dt\right]^{\frac{1}{l}}$$

$$\leq \frac{\eta(a_2, ma_1)}{n+1} \left[A^{p,q}_{\Lambda^*_{m,n}}(1)\right]^{\frac{l-1}{l}}$$

$$\times \left[\int_0^1 \left[\Lambda^*_{m,n}(t)\right]^p \left[\Lambda^*_{m,n}(1-t)\right]^q \left(\left(1 - \frac{t}{n+1}\right)|f(ma_1)|^l + \frac{t}{n+1}|f(a_2)|^l\right) dt\right]^{\frac{1}{l}}$$

$$= \frac{\eta(a_2, ma_1)}{n+1} \left[A^{p,q}_{\Lambda^*_{m,n}}(1)\right]^{\frac{l-1}{l}} \times \sqrt[l]{B^{p,q}_{\Lambda^*_{m,n}}|f(ma_1)|^l + C^{p,q}_{\Lambda^*_{m,n}}|f(a_2)|^l}.$$

So, the proof of this theorem is completed.

We point out some special cases of Theorem 3.

Corollary 6 *Under the assumption of Theorem 3 with* $n = 0$ *and* $\varphi(t) = t$, *we get*

$$\left|T^{p,q}_{f,\Lambda^*_1}(a_1, a_2)\right| \leq \eta^{p+q+1}(a_2, ma_1) \times \beta^{\frac{l-1}{l}}(p+1, q+1) \tag{24}$$

$$\times \sqrt[l]{\beta(p+1, q+2)|f(ma_1)|^l + \beta(q+1, p+2)|f(a_2)|^l}.$$

Corollary 7 *Under the assumption of Theorem 3 with $n = 0$ and $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, we get*

$$\left|T^{p,q}_{f,\Lambda_2^*}(a_1, a_2)\right| \leq \frac{\eta^{\alpha(p+q)+1}(a_2, ma_1)}{\Gamma^{p+q}(\alpha+1)} \times \beta^{\frac{l-1}{l}}(\alpha p+1, \alpha q+1) \tag{25}$$

$$\times \sqrt[l]{\beta(\alpha p+1, \alpha q+2)|f(ma_1)|^l + \beta(\alpha q+1, \alpha p+2)|f(a_2)|^l}.$$

Corollary 8 *Under the assumption of Theorem 3 with $n = 0$ and $\varphi(t) = \frac{t^{\frac{\alpha}{k_1}}}{k_1 \Gamma_{k_1}(\alpha)}$, we get*

$$\left|T^{p,q}_{f,\Lambda_3^*}(a_1, a_2)\right| \leq \frac{\eta^{\frac{\alpha}{k_1}(p+q)+1}(a_2, ma_1)}{\left[k_1 \Gamma_{k_1}(\alpha+k_1)\right]^{p+q}} \times \beta^{\frac{l-1}{l}}\left(\frac{p\alpha}{k_1}+1, \frac{q\alpha}{k_1}+1\right). \tag{26}$$

$$\times \sqrt[l]{\beta\left(\frac{p\alpha}{k_1}+1, \frac{q\alpha}{k_1}+2\right)|f(ma_1)|^l + \beta\left(\frac{q\alpha}{k_1}+1, \frac{p\alpha}{k_1}+2\right)|f(a_2)|^l}.$$

Corollary 9 *Under the assumption of Theorem 3 with $n = 0$ and $\varphi(t) = t(ma_1 + \eta(a_2, ma_1) - t)^{\alpha-1}$ and $f(x)$ is symmetric to $x = ma_1 + \frac{\eta(a_2, ma_1)}{2}$, we get*

$$\left|T^{p,q}_{f,\Lambda_4^*}(a_1, a_2)\right| \leq \eta(a_2, ma_1)\left[\frac{C^{p,q,m}(\alpha, 1)}{\alpha^{p+q}}\right]^{\frac{l-1}{l}} \tag{27}$$

$$\times \sqrt[l]{D^{p,q,m}|f(ma_1)|^l + D^{q,p,m}|f(a_2)|^l},$$

where

$$D^{p,q,m} := \frac{1}{\alpha^{p+q}\eta^2(a_2, ma_1)} \tag{28}$$

$$\times \int_{ma_1}^{ma_1+\eta(a_2,ma_1)} (t-a_1)\left[(ma_1+\eta(a_2, ma_1))^\alpha - t^\alpha\right]^p$$

$$\times \left[(ma_1+\eta(a_2, ma_1))^\alpha - (2ma_1+\eta(a_2, ma_1)-t)^\alpha\right]^q dt.$$

3 Some New Refinement of Hermite–Hadamard Type via General Fractional Integral Inequalities

Theorem 4 *Suppose that $n = 0, 1, 2, \dots,$ and $m \in (0, 1]$ be a fixed number. Let $f : P = [ma_1, ma_1 + \eta(a_2, ma_1)] \longrightarrow \mathbb{R}$ be a preinvex function on P with $\eta(a_2, ma_1) > 0$, then the following inequalities for generalized fractional integral hold:*

$$f\left(ma_1 + \frac{\eta(a_2, ma_1)}{2}\right) \le \frac{1}{2\Lambda^*_{m,n}(1)} \tag{29}$$

$$\times\left[{}_{(ma_1)^+}I_\varphi f\left(ma_1 + \frac{\eta(a_2, ma_1)}{n+1}\right) + {}_{\left(ma_1 + \frac{\eta(a_2, ma_1)}{n+1}\right)^-}I_\varphi f(ma_1)\right] \le \frac{f(ma_1) + f(a_2)}{2}.$$

Proof For $t \in [0, 1]$, let $x = ma_1 + \dfrac{t}{n+1}\eta(a_2, ma_1)$ and $y = ma_1 + \left(1 - \frac{t}{n+1}\right)$ $\eta(a_2, ma_1)$. From the preinvexity of f, we get

$$f\left(ma_1 + \frac{\eta(a_2, ma_1)}{2}\right) = f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2},$$

i.e.,

$$2f\left(ma_1 + \frac{\eta(a_2, ma_1)}{2}\right) \le f\left(ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)\right) \tag{30}$$

$$+f\left(ma_1 + \left(1 - \frac{t}{n+1}\right)\eta(a_2, ma_1)\right).$$

Multiplying both sides of (30) by $\dfrac{\varphi\left(\eta(a_2, ma_1)\frac{t}{n+1}\right)}{\frac{t}{n+1}}$ and integrating the resulting inequality with respect to t over $(0, 1]$, we obtain

$$2f\left(ma_1 + \frac{\eta(a_2, ma_1)}{2}\right)\int_0^1 \frac{\varphi\left(\eta(a_2, ma_1)\frac{t}{n+1}\right)}{\frac{t}{n+1}}dt$$

$$\le \int_0^1 \frac{\varphi\left(\eta(a_2, ma_1)\frac{t}{n+1}\right)}{\frac{t}{n+1}} f\left(ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)\right)dt$$

$$+\int_0^1 \frac{\varphi\left(\eta(a_2, ma_1)\frac{t}{n+1}\right)}{\frac{t}{n+1}} f\left(ma_1 + \left(1 - \frac{t}{n+1}\right)\eta(a_2, ma_1)\right)dt.$$

Hence,

$$2f\left(ma_1+\frac{\eta(a_2,ma_1)}{2}\right)\int_0^1 \frac{\varphi\left(\eta(a_2,ma_1)\frac{t}{n+1}\right)}{\frac{t}{n+1}}dt$$

$$\leq \left[{}_{(ma_1)^+}I_\varphi f\left(ma_1+\frac{\eta(a_2,ma_1)}{n+1}\right)+{}_{\left(ma_1+\frac{\eta(a_2,ma_1)}{n+1}\right)^-}I_\varphi f(ma_1)\right].$$

So, the first inequality is proved.

To prove the other half of the inequality in (29), since f is preinvex, we have

$$f\left(ma_1+\frac{t}{n+1}\eta(a_2,ma_1)\right)+f\left(ma_1+\left(1-\frac{t}{n+1}\right)\eta(a_2,ma_1)\right) \tag{31}$$

$$\leq f(ma_1)+f(a_2).$$

Multiplying both sides of (31) by $\frac{\varphi\left(\eta(a_2,ma_1)\frac{t}{n+1}\right)}{\frac{t}{n+1}}$ and integrating the resulting inequality with respect to t over $(0, 1]$, we obtain

$$\left[{}_{(ma_1)^+}I_\varphi f\left(ma_1+\frac{\eta(a_2,ma_1)}{n+1}\right)+{}_{\left(ma_1+\frac{\eta(a_2,ma_1)}{n+1}\right)^-}I_\varphi f(ma_1)\right]$$

$$\leq \left[f(ma_1)+f(a_2)\right]\int_0^1 \frac{\varphi\left(\eta(a_2,ma_1)\frac{t}{n+1}\right)}{\frac{t}{n+1}}dt.$$

Therefore, the second inequality is proved. The proof of this theorem is complete.

We point out some special cases of Theorem 4.

Corollary 10 *Taking $n=0$, $m=1$ and $\eta(a_2,ma_1)=a_2-ma_1$ in Theorem 4, we get [[30], Theorem 5].*

Corollary 11 *If in Theorem 4, we get $n=0$ and $\varphi(t)=t$, then the inequalities in (29) become the inequalities*

$$f\left(ma_1+\frac{\eta(a_2,ma_1)}{2}\right)\leq\frac{1}{2\eta(a_2,ma_1)} \tag{32}$$

$$\times\left[I_{(ma_1)^+}f(ma_1+\eta(a_2,ma_1))+I_{(ma_1+\eta(a_2,ma_1))^-}f(ma_1)\right]\leq\frac{f(ma_1)+f(a_2)}{2},$$

where $I_{a_1^+}f$ and $I_{a_2^-}f$ are the classical Riemann integrals.

Corollary 12 *If in Theorem 4, we get $n = 0$ and $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then the inequalities in (29) become the inequalities*

$$f\left(ma_1 + \frac{\eta(a_2, ma_1)}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2\eta^\alpha(a_2, ma_1)} \tag{33}$$

$$\times\Big[J^\alpha_{(ma_1)^+} f(ma_1 + \eta(a_2, ma_1)) + J^\alpha_{(ma_1+\eta(a_2,ma_1))^-} f(ma_1)\Big] \leq \frac{f(ma_1) + f(a_2)}{2},$$

where $J^\alpha_{a_1^+} f$ and $J^\alpha_{a_2^-} f$ are the fractional Riemann integrals.

Corollary 13 *If in Theorem 4, we get $n = 0$ and $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then the inequalities in (29) become the inequalities*

$$f\left(ma_1 + \frac{\eta(a_2, ma_1)}{2}\right) \leq \frac{\Gamma_k(\alpha+k)}{2\eta^{\frac{\alpha}{k}}(a_2, ma_1)} \tag{34}$$

$$\times\Big[I^{\alpha,k}_{(ma_1)^+} f(ma_1 + \eta(a_2, ma_1)) + I^{\alpha,k}_{(ma_1+\eta(a_2,ma_1))^-} f(ma_1)\Big] \leq \frac{f(ma_1) + f(a_2)}{2}.$$

Corollary 14 *If in Theorem 4, we get $n = 0$ and $\varphi(t) = t(ma_1+\eta(a_2, ma_1)-t)^{\alpha-1}$ and $f(x)$ is symmetric to $x = ma_1 + \frac{\eta(a_2,ma_1)}{2}$, then the inequalities in (29) become the inequalities*

$$\begin{aligned} f\left(ma_1+\frac{\eta(a_2, ma_1)}{2}\right) &\leq \frac{\alpha}{(ma_1+\eta(a_2, ma_1))^\alpha-(ma_1)^\alpha} \times \int_{ma_1}^{ma_1+\eta(a_2,ma_1)} f(t)d_\alpha t \\ &\leq \frac{f(ma_1) + f(a_2)}{2}. \end{aligned} \tag{35}$$

Corollary 15 *If in Theorem 4, we get $n = 0$ and $\varphi(t) = \frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$, $\alpha \in (0, 1)$, then the inequalities in (29) become the inequalities*

$$f\left(ma_1 + \frac{\eta(a_2, ma_1)}{2}\right) \leq \frac{1-\alpha}{2(1-\exp(-D))} \tag{36}$$

$$\times\Big[\mathscr{I}^\alpha_{(ma_1)^+} f(ma_1+\eta(a_2, ma_1))+\mathscr{I}^\alpha_{(ma_1+\eta(a_2,ma_1))^-} f(ma_1)\Big] \leq \frac{f(ma_1) + f(a_2)}{2},$$

where $\mathscr{I}^{\alpha}_{a_1^+} f$ *and* $\mathscr{I}^{\alpha}_{a_2^-} f$ *are the right-side and left-side fractional integral operators with exponential kernel and* $D = \left(\frac{1-\alpha}{\alpha}\right)\eta(a_2, ma_1)$.

For establishing some new results regarding general fractional integrals, we need to prove the following lemma.

Lemma 2 *Suppose that* $n = 0, 1, 2, \ldots,$ *and* $m \in (0, 1]$ *be a fixed number. Let* $f : P = [ma_1, ma_1 + \eta(a_2, ma_1)] \longrightarrow \mathbb{R}$ *be a differentiable mapping on* $(ma_1, ma_1 + \eta(a_2, ma_1))$ *with* $\eta(a_2, ma_1) > 0$. *If* $f' \in L(P)$, *then the following identity for generalized fractional integrals holds:*

$$\frac{f(ma_1) + f\left(ma_1 + \frac{\eta(a_2,ma_1)}{n+1}\right)}{2(n+1)}$$

$$-\frac{1}{2\Lambda^*_{m,n}(1)} \times \left[{}_{(ma_1)^+}I_\varphi f\left(ma_1 + \frac{\eta(a_2, ma_1)}{n+1}\right) + {}_{\left(ma_1+\frac{\eta(a_2,ma_1)}{n+1}\right)^-}I_\varphi f(ma_1)\right]$$

$$= \frac{\eta(a_2, ma_1)}{2(n+1)^2\Lambda^*_{m,n}(1)} \tag{37}$$

$$\times \int_0^1 \left[\Lambda^*_{m,n}(t) - \Lambda^*_{m,n}(1-t)\right] f'\left(ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)\right) dt.$$

We denote

$$H_{f,\Lambda^*_{m,n}}(a_1, a_2) := \frac{\eta(a_2, ma_1)}{2(n+1)^2\Lambda^*_{m,n}(1)} \tag{38}$$

$$\times \int_0^1 \left[\Lambda^*_{m,n}(t) - \Lambda^*_{m,n}(1-t)\right] f'\left(ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)\right) dt.$$

Proof Integrating by parts (38) and changing the variable of integration, we have

$$H_{f,\Lambda^*_{m,n}}(a_1, a_2) = \frac{\eta(a_2, ma_1)}{2(n+1)^2\Lambda^*_{m,n}(1)}$$

$$\times \left\{ \int_0^1 \Lambda^*_{m,n}(t) f'\left(ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)\right) dt\right.$$

$$\left. - \int_0^1 \Lambda^*_{m,n}(1-t) f'\left(ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)\right) dt \right\}$$

$$= \frac{\eta(a_2, ma_1)}{2(n+1)^2 \Lambda^*_{m,n}(1)} \times \left\{ \frac{(n+1)\Lambda^*_{m,n}(t) f\left(ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)\right)}{\eta(a_2, ma_1)}\Bigg|_0^1 \right.$$

$$- \frac{n+1}{\eta(a_2, ma_1)} \int_0^1 \frac{\varphi\left(\eta(a_2, ma_1)\frac{t}{n+1}\right)}{\frac{t}{n+1}} f\left(ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)\right) dt$$

$$- \frac{(n+1)\Lambda^*_{m,n}(1-t) f\left(ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)\right)}{\eta(a_2, ma_1)}\Bigg|_0^1$$

$$\left. - \frac{n+1}{\eta(a_2, ma_1)} \int_0^1 \frac{\varphi\left(\eta(a_2, ma_1)\frac{(1-t)}{n+1}\right)}{\frac{1-t}{n+1}} f\left(ma_1 + \frac{t}{n+1}\eta(a_2, ma_1)\right) dt \right\}$$

$$= \frac{f(ma_1) + f\left(ma_1 + \frac{\eta(a_2, ma_1)}{n+1}\right)}{2(n+1)}$$

$$- \frac{1}{2\Lambda^*_{m,n}(1)} \times \left[{}_{(ma_1)^+}I_\varphi f\left(ma_1 + \frac{\eta(a_2, ma_1)}{n+1}\right) + {}_{\left(ma_1 + \frac{\eta(a_2, ma_1)}{n+1}\right)^-}I_\varphi f(ma_1) \right].$$

This completes the proof of the lemma.

Remark 1 Taking $n = 0,\ m = 1$ and $\eta(a_2, ma_1) = a_2 - ma_1$ in Lemma 2, we get

$$H_{f,\Lambda^*_{1,0}}(a_1, a_2) = \frac{f(a_1) + f(a_2)}{2} - \frac{1}{2\Lambda^*_{1,0}(1)} \times \left[{}_{a_1^+}I_\varphi f(a_2) + {}_{a_2^-}I_\varphi f(a_1) \right]. \tag{39}$$

Theorem 5 *Suppose that* $n = 0, 1, 2, \ldots,$ *and* $m \in (0, 1]$ *be a fixed number. Let* $f : P = [ma_1, ma_1 + \eta(a_2, ma_1)] \longrightarrow \mathbb{R}$ *be a differentiable mapping on* $(ma_1, ma_1 + \eta(a_2, ma_1))$ *with* $\eta(a_2, ma_1) > 0$. *If* $|f'|^q$ *is preinvex on* P *for* $q > 1$ *and* $p^{-1} + q^{-1} = 1$, *then the following inequality for generalized fractional integrals holds:*

$$\left|H_{f,\Lambda^*_{m,n}}(a_1, a_2)\right| \le \frac{\eta(a_2, ma_1)}{2^{1+\frac{1}{q}}(n+1)^{2+\frac{1}{q}}\Lambda^*_{m,n}(1)} \sqrt[p]{K_{\Lambda^*_{m,n}}(p)} \tag{40}$$

$$\times \sqrt[q]{(2n+1)|f'(ma_1)|^q + |f'(a_2)|^q},$$

where

$$K_{\Lambda^*_{m,n}}(p) := \int_0^1 \left|\Lambda^*_{m,n}(t) - \Lambda^*_{m,n}(1-t)\right|^p dt. \tag{41}$$

Proof From Lemma 2, the preinvexity of $|f'|^q$, Hölder inequality and properties of the modulus, we have

$$\left|H_{f,\Lambda^*_{m,n}}(a_1,a_2)\right| \leq \frac{\eta(a_2,ma_1)}{2(n+1)^2\Lambda^*(1)}$$

$$\times \int_0^1 \left|\Lambda^*_{m,n}(t)-\Lambda^*_{m,n}(1-t)\right| \left| f'\left(ma_1+\frac{t}{n+1}\eta(a_2,ma_1)\right)\right| dt$$

$$\leq \frac{\eta(a_2,ma_1)}{2(n+1)^2\Lambda^*(1)} \left(\int_0^1 \left|\Lambda^*_{m,n}(t)-\Lambda^*_{m,n}(1-t)\right|^p dt\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \left| f'\left(ma_1+\frac{t}{n+1}\eta(a_2,ma_1)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\eta(a_2,ma_1)}{2(n+1)^2\Lambda^*(1)} \sqrt[p]{K_{\Lambda^*_{m,n}}(p)} \left(\int_0^1 \left(\left(1-\frac{t}{n+1}\right)|f'(ma_1)|^q+\frac{t}{n+1}|f'(a_2)|^q\right) dt\right)^{\frac{1}{q}}$$

$$= \frac{\eta(a_2,ma_1)}{2^{1+\frac{1}{q}}(n+1)^{2+\frac{1}{q}}\Lambda^*_{m,n}(1)} \sqrt[p]{K_{\Lambda^*_{m,n}}(p)} \times \sqrt[q]{(2n+1)|f'(ma_1)|^q+|f'(a_2)|^q}.$$

The proof of this theorem is complete.

We point out some special cases of Theorem 5.

Corollary 16 *Taking $n=0,\ m=1$ and $\eta(a_2,ma_1)=a_2-ma_1$ in Theorem 5, we get*

$$\left|H_{f,\Lambda^*_{1,0}}(a_1,a_2)\right| \leq \frac{(a_2-a_1)}{2^{1+\frac{1}{q}}\Lambda^*_{1,0}(1)} \sqrt[p]{K_{\Lambda^*_{1,0}}(p)} \times \sqrt[q]{|f'(a_1)|^q+|f'(a_2)|^q}. \tag{42}$$

Corollary 17 *Taking $p=q=2$ in Theorem 5, we get*

$$\left|H_{f,\Lambda^*_{m,n}}(a_1,a_2)\right| \leq \frac{\eta(a_2,ma_1)}{2\sqrt{2(n+1)}(n+1)^2\Lambda^*_{m,n}(1)} \sqrt{K_{\Lambda^*_{m,n}}(2)} \tag{43}$$

$$\times\sqrt{(2n+1)|f'(ma_1)|^2+|f'(a_2)|^2}.$$

Corollary 18 *Taking $n=0,\ m=1,\ \eta(a_2,ma_1)=a_2-ma_1$ and $\varphi(t)=t$ in Theorem 5, we get [[7], Theorem 2.3].*

Corollary 19 *Taking $n=0,\ m=1,\ \eta(a_2,ma_1)=a_2-ma_1$ and $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 5, we get [[27], Theorem 8].*

Corollary 20 *Taking $n = 0$, $m = 1$, $\eta(a_2, ma_1) = a_2 - ma_1$ and $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 5, we get [[12], Theorem 8].*

Corollary 21 *Taking $n = 0$, $m = 1$, $\eta(a_2, ma_1) = a_2 - ma_1$ and $\varphi(t) = t(ma_1 + \eta(a_2, ma_1) - t)^{\alpha-1}$ and $f(x)$ is symmetric to $x = ma_1 + \frac{\eta(a_2,ma_1)}{2}$, in Theorem 5, we get*

$$\left|H_{f,\Lambda_4^*}(a_1, a_2)\right| \leq \frac{\sqrt[q]{\frac{\eta(a_2,ma_1)}{2}}}{\sqrt[p]{p\alpha+1} \times \left[(ma_1 + \eta(a_2, ma_1))^\alpha - (ma_1)^\alpha\right]} \tag{44}$$

$$\times \sqrt[p]{(ma_1)^{p\alpha+1} + (ma_1 + \eta(a_2, ma_1))^{p\alpha+1} - \frac{(2ma_1 + \eta(a_2, ma_1))^{p\alpha+1}}{2^{p\alpha}}}$$

$$\times \sqrt[q]{\frac{|f'(ma_1)|^q + |f'(a_2)|^q}{2}}.$$

Theorem 6 *Suppose that $n = 0, 1, 2, \ldots,$ and $m \in (0, 1]$ be a fixed number. Let $f : P = [ma_1, ma_1 + \eta(a_2, ma_1)] \longrightarrow \mathbb{R}$ be a differentiable mapping on $(ma_1, ma_1 + \eta(a_2, ma_1))$ with $\eta(a_2, ma_1) > 0$. If $|f'|^q$ is preinvex on P for $q \geq 1$, then the following inequality for generalized fractional integrals holds:*

$$\left|H_{f,\Lambda_{m,n}^*}(a_1, a_2)\right| \leq \frac{\eta(a_2, ma_1)}{2(n+1)^{2+\frac{1}{q}}\Lambda_{m,n}^*(1)}\left[K_{\Lambda_{m,n}^*}(1)\right]^{1-\frac{1}{q}} \tag{45}$$

$$\times \sqrt[q]{L_{\Lambda_{m,n}^*}|f'(ma_1)|^q + F_{\Lambda_{m,n}^*}|f'(a_2)|^q},$$

where

$$L_{\Lambda_{m,n}^*} := \int_0^1 (n+1-t)\left|\Lambda_{m,n}^*(t) - \Lambda_{m,n}^*(1-t)\right|dt, \tag{46}$$

$$F_{\Lambda_{m,n}^*} := \int_0^1 t\left|\Lambda_{m,n}^*(t) - \Lambda_{m,n}^*(1-t)\right|dt, \tag{47}$$

and $K_{\Lambda_{m,n}^*}(1)$ is defined as in Theorem 5.

Proof From Lemma 2, the preinvexity of $|f'|^q$, power mean inequality and properties of the modulus, we have

$$\left|H_{f,\Lambda^*_{m,n}}(a_1,a_2)\right| \leq \frac{\eta(a_2,ma_1)}{2(n+1)^2\Lambda^*_{m,n}(1)}$$

$$\times\int_0^1 \left|\Lambda^*_{m,n}(t)-\Lambda^*_{m,n}(1-t)\right|\left|f'\left(ma_1+\frac{t}{n+1}\eta(a_2,ma_1)\right)\right|dt$$

$$\leq \frac{\eta(a_2,ma_1)}{2(n+1)^2\Lambda^*_{m,n}(1)}\left(\int_0^1 \left|\Lambda^*_{m,n}(t)-\Lambda^*_{m,n}(1-t)\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left|\Lambda^*_{m,n}(t)-\Lambda^*_{m,n}(1-t)\right|\left|f'\left(ma_1+\frac{t}{n+1}\eta(a_2,ma_1)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\eta(a_2,ma_1)}{2(n+1)^2\Lambda^*_{m,n}(1)}\left[K_{\Lambda^*_{m,n}}(1)\right]^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left|\Lambda^*_{m,n}(t)-\Lambda^*_{m,n}(1-t)\right|\left(\left(1-\frac{t}{n+1}\right)\left|f'(ma_1)\right|^q+\frac{t}{n+1}\left|f'(a_2)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$=\frac{\eta(a_2,ma_1)}{2(n+1)^{2+\frac{1}{q}}\Lambda^*_{m,n}(1)}\left[K_{\Lambda^*_{m,n}}(1)\right]^{1-\frac{1}{q}}\times\sqrt[q]{L_{\Lambda^*_{m,n}}|f'(ma_1)|^q+F_{\Lambda^*_{m,n}}|f'(a_2)|^q}.$$

The proof of this theorem is complete.

We point out some special cases of Theorem 6.

Corollary 22 *Taking $n=0,\ m=1$ and $\eta(a_2,ma_1)=a_2-ma_1$ in Theorem 6, we get*

$$\left|H_{f,\Lambda^*_{1,0}}(a_1,a_2)\right| \leq \frac{(a_2-a_1)}{2\Lambda^*_{1,0}(1)}\left[K_{\Lambda^*_{1,0}}(1)\right]^{1-\frac{1}{q}}\times\sqrt[q]{L_{\Lambda^*_{1,0}}|f'(a_1)|^q+F_{\Lambda^*_{1,0}}|f'(a_2)|^q}. \tag{48}$$

Corollary 23 *Taking $q=1$ in Theorem 6, we get*

$$\left|H_{f,\Lambda^*_{m,n}}(a_1,a_2)\right| \leq \frac{\eta(a_2,ma_1)}{2(n+1)^3\Lambda^*_{m,n}(1)}\times\left[L_{\Lambda^*_{m,n}}|f'(ma_1)|+F_{\Lambda^*_{m,n}}|f'(a_2)|\right]. \tag{49}$$

Corollary 24 *Under the assumption of Theorem 6 with $n = 0$ and $\varphi(t) = t$, we get*

$$\left|H_{f,\Lambda_1^*}(a_1, a_2)\right| \le \frac{\eta(a_2, ma_1)}{2^{2+\frac{1}{q}}} \times \sqrt[q]{|f'(ma_1)|^q + |f'(a_2)|^q}. \tag{50}$$

Corollary 25 *Under the assumption of Theorem 6 with $n = 0$ and $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, we get*

$$\left|H_{f,\Lambda_2^*}(a_1, a_2)\right| \le \left(\frac{2^\alpha - 1}{2^{\alpha+1}}\right) \sqrt[q]{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)}} \eta(a_2, ma_1) \times \sqrt[q]{|f'(ma_1)|^q + |f'(a_2)|^q}. \tag{51}$$

Corollary 26 *Under the assumption of Theorem 6 with $n = 0$ and $\varphi(t) = \frac{t^{\frac{\alpha}{k_1}}}{k_1 \Gamma_{k_1}(\alpha)}$, we get*

$$\begin{aligned}\left|H_{f,\Lambda_3^*}(a_1, a_2)\right| \le \left(\frac{2^{\frac{\alpha}{k_1}} - 1}{2^{\frac{\alpha}{k_1}+1}}\right) \sqrt[q]{\frac{\Gamma_{k_1}(\alpha+k_1)}{\Gamma_{k_1}(\alpha+k_1+1)}} \eta(a_2, ma_1) \\ \times \sqrt[q]{|f'(ma_1)|^q + |f'(a_2)|^q}.\end{aligned} \tag{52}$$

Corollary 27 *Under the assumption of Theorem 6 with $n = 0$ and $\varphi(t) = t(ma_1 + \eta(a_2, ma_1) - t)^{\alpha-1}$ and $f(x)$ is symmetric to $x = ma_1 + \frac{\eta(a_2, ma_1)}{2}$, we get*

$$\left|H_{f,\Lambda_4^*}(a_1, a_2)\right| \le \frac{\eta(a_2, ma_1)}{2\overline{\Lambda_{m,0}^*}(1)} \left[\overline{K_{\Lambda_m^*}(1)}\right]^{1-\frac{1}{q}} \sqrt[q]{\overline{K_{\Lambda_m^*}}} \times \sqrt[q]{|f'(ma_1)|^q + |f'(a_2)|^q}, \tag{53}$$

where

$$\overline{\Lambda^{*m,0}(1)} := \frac{(ma_1 + \eta(a_2, ma_1))^\alpha}{\alpha},$$

$$\overline{K_{\Lambda_m^*}(1)} := \frac{2}{\alpha}\left[(ma_1 + \eta(a_2, ma_1))^{\alpha+1} - 2\left(ma_1 + \frac{\eta(a_2, ma_1)}{2}\right)^{\alpha+1} + (ma_1)^{\alpha+1}\right],$$

$$\overline{K_{\Lambda_m^*}} := \frac{1}{\alpha}\left[F_{11}^{(m)} - F_{12}^{(m)} + F_{21}^{(m)} - F_{22}^{(m)}\right],$$

and

$$F_{11}^{(m)} := \frac{1}{\eta^2(a_2, ma_1)}$$

$$\times\left\{\frac{(ma_1+\eta(a_2,ma_1))}{\alpha+1}\left[(ma_1+\eta(a_2,ma_1))^{\alpha+1}-\left(ma_1+\frac{\eta(a_2,ma_1)}{2}\right)^{\alpha+1}\right]\right.$$

$$\left.-\frac{1}{\alpha+2}\left[(ma_1+\eta(a_2,ma_1))^{\alpha+2}-\left(ma_1+\frac{\eta(a_2,ma_1)}{2}\right)^{\alpha+2}\right]\right\},$$

$$F_{12}^{(m)}:=\frac{1}{\eta^2(a_2,ma_1)}\left\{\frac{1}{\alpha+2}\left[\left(ma_1+\frac{\eta(a_2,ma_1)}{2}\right)^{\alpha+2}-(ma_1)^{\alpha+2}\right]\right.$$

$$\left.-\frac{ma_1}{\alpha+1}\left[\left(ma_1+\frac{\eta(a_2,ma_1)}{2}\right)^{\alpha+1}-(ma_1)^{\alpha+1}\right]\right\},$$

$$F_{21}^{(m)}:=\frac{1}{\eta^2(a_2,ma_1)}\left\{\frac{1}{\alpha+2}\left[(ma_1+\eta(a_2,ma_1))^{\alpha+2}-\left(ma_1+\frac{\eta(a_2,ma_1)}{2}\right)^{\alpha+2}\right]\right.$$

$$\left.-\frac{ma_1}{\alpha+1}\left[(ma_1+\eta(a_2,ma_1))^{\alpha+1}-\left(ma_1+\frac{\eta(a_2,ma_1)}{2}\right)^{\alpha+1}\right]\right\},$$

$$F_{22}^{(m)}:=\frac{1}{\eta^2(a_2,ma_1)}\left\{\frac{(ma_1+\eta(a_2,ma_1))}{\alpha+1}\left[\left(ma_1+\frac{\eta(a_2,ma_1)}{2}\right)^{\alpha+1}-(ma_1)^{\alpha+1}\right]\right.$$

$$\left.-\frac{1}{\alpha+2}\left[(ma_1+\eta(a_2,ma_1))^{\alpha+2}-(ma_1)^{\alpha+2}\right]\right\}.$$

4 Applications to Special Means

Consider the following special means for different real numbers α, β and $\alpha\beta \neq 0$:

1. The arithmetic mean:

$$A := A(\alpha,\beta) = \frac{\alpha+\beta}{2}.$$

2. The harmonic mean:

$$H := H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}.$$

3. The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln|\beta| - \ln|\alpha|}.$$

4. The generalized log-mean:

$$L_n := L_n(\alpha, \beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)}\right]^{\frac{1}{n}}; \quad n \in \mathbb{Z} \setminus \{-1, 0\}.$$

It is well known that L_n is monotonic nondecreasing over $n \in \mathbb{Z}$ with $L_{-1} := L$. In particular, we have the following inequality $H \leq L \leq A$. Now, using the theory results in Sect. 3, we give some applications to special means for different real numbers.

Proposition 1 *Let $m \in (0, 1]$ be a fixed number, $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$ and $\eta(a_2, ma_1) > 0$. Then, for $r \geq 2$, where $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds:*

$$\left|A\left((ma_1)^r, (ma_1 + \eta(a_2, ma_1))^r\right) - L_r\left(ma_1, ma_1 + \eta(a_2, ma_1)\right)\right|$$
$$\leq \frac{r}{2} \frac{\eta(a_2, ma_1)}{\sqrt[p]{p+1}} \times \sqrt[q]{A\left(|ma_1|^{q(r-1)}, |a_2|^{q(r-1)}\right)}. \tag{54}$$

Proof Applying Theorem 5 for $f(x) = x^r$ and $\varphi(t) = t$, one can obtain the result immediately.

Proposition 2 *Let $m \in (0, 1]$ be a fixed number, $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$ and $\eta(a_2, ma_1) > 0$. Then, for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds:*

$$\left|\frac{1}{H\left(ma_1, ma_1 + \eta(a_2, ma_1)\right)} - \frac{1}{L\left(ma_1, ma_1 + \eta(a_2, ma_1)\right)}\right| \leq \frac{\eta(a_2, ma_1)}{2\sqrt[p]{p+1}}$$
$$\times \frac{1}{\sqrt[q]{H\left((ma_1)^{2q}, a_2^{2q}\right)}}. \tag{55}$$

Proof Applying Theorem 5 for $f(x) = \dfrac{1}{x}$ and $\varphi(t) = t$, one can obtain the result immediately.

Proposition 3 *Let $m \in (0, 1]$ be a fixed number, $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, where $a_1 < a_2$ and $\eta(a_2, ma_1) > 0$. Then, for $r \geq 2$ and $q \geq 1$, the following inequality holds:*

$$\left|A\left((ma_1)^r, (ma_1+\eta(a_2,ma_1))^r\right) - L_r\left(ma_1, ma_1+\eta(a_2,ma_1)\right)\right|$$

$$\leq \frac{r}{2^{2+\frac{1}{q}}}\eta(a_2,ma_1)\times\sqrt[q]{A\left(|ma_1|^{q(r-1)}, |a_2|^{q(r-1)}\right)}. \quad (56)$$

Proof Applying Theorem 6 for $f(x) = x^r$ and $\varphi(t) = t$, one can obtain the result immediately.

Proposition 4 *Let $m \in (0, 1]$ be a fixed number, $a_1, a_2 \in \mathbb{R}\setminus\{0\}$, where $a_1 < a_2$ and $\eta(a_2, a_1) > 0$. Then, for $q \geq 1$, the following inequality holds:*

$$\left|\frac{1}{H\left(ma_1, ma_1+\eta(a_2,ma_1)\right)} - \frac{1}{L\left(ma_1, ma_1+\eta(a_2,ma_1)\right)}\right| \leq \frac{\eta(a_2,ma_1)}{2^{2+\frac{1}{q}}} \times \frac{1}{\sqrt[q]{H\left((ma_1)^{2q}, a_2^{2q}\right)}}. \quad (57)$$

Proof Applying Theorem 6 for $f(x) = \dfrac{1}{x}$ and $\varphi(t) = t$, one can obtain the result immediately.

Remark 2 Applying our Theorems 5 and 6 for appropriate choices of function $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}, \frac{t^{\frac{\alpha}{k_1}}}{k_1\Gamma_{k_1}(\alpha)}$; $\varphi(t) = t(ma_1 + \eta(a_2, ma_1) - t)^{\alpha-1}$, where $f(x)$ is symmetric to $x = ma_1 + \frac{\eta(a_2,ma_1)}{2}$ and $m \in (0, 1]$ is a fixed number, $\varphi(t) = \frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha \in (0, 1)$, such that $|f'|^q$ to be preinvex, we can deduce some new general fractional integral inequalities using above special means. The details are left to the interested reader.

Remark 3 Also, in Remark 2, if we choose $\eta(a_2, ma_1) = a_2 - ma_1$, where $m \in (0, 1]$ is a fixed number, we can deduce some new general fractional integral inequalities for convex functions using above special means. The details are left to the interested reader.

Next, we provide some new error estimates for the trapezoidal formula.

Let Q be the partition of the points $a_1 = x_0 < x_1 < \ldots < x_k = a_2$ of the interval $[a_1, a_2]$. Let us consider the following quadrature formula:

$$\int_{a_1}^{a_2} f(x)dx = T(f, Q) + E(f, Q),$$

where

$$T(f, Q) = \sum_{i=0}^{k-1}\frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)$$

is the trapezoidal version, and $E(f, Q)$ denotes their associated approximation error.

Proposition 5 *Let $f : [a_1, a_2] \longrightarrow \mathbb{R}$ be a differentiable function on (a_1, a_2), where $a_1 < a_2$. If $|f'|^q$ is convex on $[a_1, a_2]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\left|E(f, Q)\right| \leq \frac{1}{2^{\frac{q+1}{q}} \sqrt[p]{p+1}} \sum_{i=0}^{k-1} (x_{i+1} - x_i)^2 \times \sqrt[q]{|f'(x_i)|^q + |f'(x_{i+1})|^q}. \tag{58}$$

Proof Applying Theorem 5 for $n = 0$, $m = 1$, $\eta(a_2, ma_1) = a_2 - ma_1$ and $\varphi(t) = t$ on the subintervals $[x_i, x_{i+1}]$ $(i = 0, \ldots, k-1)$ of the partition Q, we have

$$\left|\frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x)dx\right|$$

$$\leq \frac{(x_{i+1} - x_i)}{2\sqrt[p]{p+1}} \left[\frac{|f'(x_i)|^q + |f'(x_{i+1})|^q}{2}\right]^{\frac{1}{q}}. \tag{59}$$

Hence, from (59), we get

$$\left|E(f, Q)\right| = \left|\int_{a_1}^{a_2} f(x)dx - T(f, Q)\right|$$

$$\leq \left|\sum_{i=0}^{k-1} \left\{\int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)\right\}\right|$$

$$\leq \sum_{i=0}^{k-1} \left|\left\{\int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)\right\}\right|$$

$$\leq \frac{1}{2^{\frac{q+1}{q}} \sqrt[p]{p+1}} \sum_{i=0}^{k-1} (x_{i+1} - x_i)^2 \times \sqrt[q]{|f'(x_i)|^q + |f'(x_{i+1})|^q}.$$

The proof of this proposition is complete.

Proposition 6 *Let $f : [a_1, a_2] \longrightarrow \mathbb{R}$ be a differentiable function on (a_1, a_2), where $a_1 < a_2$. If $|f'|^q$ is convex on $[a_1, a_2]$ for $q \geq 1$, then the following inequality holds:*

$$\left|E(f, Q)\right| \leq \frac{1}{2^{2+\frac{1}{q}}} \sum_{i=0}^{k-1} (x_{i+1} - x_i)^2 \times \sqrt[q]{|f'(x_i)|^q + |f'(x_{i+1})|^q}. \tag{60}$$

Proof The proof is analogous as to that of Proposition 5 but uses Theorem 6 for $\eta(a_2, ma_1) = a_2 - ma_1$ and $\varphi(t) = t$, where $n = 0$ and $m = 1$.

Remark 4 Applying our Theorems 5 and 6, where $n = 0$ and $m = 1$, for appropriate choices of function $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, $\frac{t^{\frac{\alpha}{k_1}}}{k_1\Gamma_{k_1}(\alpha)}$; $\varphi(t) = t(a_2 - t)^{\alpha-1}$, where $f(x)$ is symmetric to $x = \frac{a_1+a_2}{2}$, and $\varphi(t) = \frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha \in (0, 1)$, such that $|f'|^q$ to be convex, we can deduce some new general fractional integral inequalities using above ideas and techniques. The details are left to the interested reader.

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New Trapezium Type Inequalities for Preinvex Functions Via Generalized Fractional Integral Operators and Their Applications

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Abstract The authors have proved an identity for trapezium type inequalities of differentiable preinvex functions with respect to another function via generalized integral operator. The obtained results provide unifying inequalities of trapezium type. Various special cases have been identified. Also, some applications of presented results to special means and new error estimates for the trapezium formula have been analyzed. The ideas and techniques of this paper may stimulate further research in the field of integral inequalities.

1 Introduction

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1 *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $p_1, p_2 \in I$ with $p_1 < p_2$. Then, the following inequality holds:*

$$f\left(\frac{p_1 + p_2}{2}\right) \leq \frac{1}{p_2 - p_1}\int_{p_1}^{p_2} f(x)dx \leq \frac{f(p_1) + f(p_2)}{2}. \tag{1}$$

This inequality (1) *is also known as trapezium inequality.*

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (1) in the premises of newly invented definitions due to motivation of convex

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function. Interested readers see the references [1–6, 8, 10, 11, 13, 14, 18–24, 26–29, 31–33].

The aim of this paper is to establish trapezium type generalized integral inequalities for preinvex functions with respect to another function, some applications to special means, and new error bounds for the trapezium formula. Interestingly, the special cases of presented results are fractional integral inequalities. Therefore, it is important to summarize the study of fractional integrals. At start, let us recall some mathematical preliminaries and definitions that will be helpful for further study.

Definition 1 ([30]) A set $S \subseteq \mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta : S \times S \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$.

The invex set is also termed as an η-connected set.

Definition 2 ([25]) Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \longrightarrow \mathbb{R}^n$. A function $f : S \longrightarrow [0, +\infty)$ is said to be preinvex with respect to η, if for every $x, y \in S$ and $t \in [0, 1]$,

$$f\big(x + t\eta(y, x)\big) \leq (1 - t)f(x) + tf(y). \tag{2}$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

Definition 3 ([22]) Let $f \in L[p_1, p_2]$. Then k-fractional integrals of order $\alpha, k > 0$ with $p_1 \geq 0$ are defined by

$$I^{\alpha,k}_{p_1^+} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{p_1}^{x} (x - t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > p_1$$

and

$$I^{\alpha,k}_{p_2^-} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{x}^{p_2} (t - x)^{\frac{\alpha}{k}-1} f(t)dt, \quad p_2 > x, \tag{3}$$

where $\Gamma_k(\cdot)$ is the k-gamma function.

For $k = 1$, k-fractional integrals give Riemann–Liouville integrals. For $\alpha = k = 1$, k-fractional integrals give classical integrals.

Definition 4 ([15, 16]) Let $g : [p_1, p_2] \to \mathbb{R}$ be an increasing and positive monotone function on $[p_1, p_2]$, having a continuous derivative on (p_1, p_2). The left-sided fractional integral of f with respect to g on $[p_1, p_2]$ of order $\alpha > 0$ is defined by

$$I^{\alpha,g}_{p_1+} f\ (x) = \frac{1}{\Gamma\ (\alpha)} \int_{p_1}^{x} \frac{g'(u) f\ (u)}{[g(x) - g(u)]^{1-\alpha}} du,\ x > p_1, \tag{4}$$

provided that the integral exists. The right-sided fractional integral of f with respect to g on $[p_1, p_2]$ of order $\alpha > 0$ is defined by

$$I_{p_2-}^{\alpha,g} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{p_2} \frac{g'(u) f(u)}{[g(u) - g(x)]^{1-\alpha}} du, \; x < p_2, \tag{5}$$

provided that the integral exists.

Jleli and Samet in [10] proved the Hadamard type inequality for Riemann–Liouville fractional integral of a convex function f with respect to another function g.
Also in [26], Sarikaya and Ertuğral defined a function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty, \tag{6}$$

$$\frac{1}{A} \leq \frac{\varphi(s)}{\varphi(r)} \leq A \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2, \tag{7}$$

$$\frac{\varphi(r)}{r^2} \leq B \frac{\varphi(s)}{s^2} \text{ for } s \leq r, \tag{8}$$

$$\left| \frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2} \right| \leq C|r - s| \frac{\varphi(r)}{r^2} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2, \tag{9}$$

where $A, B, C > 0$ are independent of $r, s > 0$. If $\varphi(r)r^\alpha$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(r)}{r^\beta}$ is decreasing for some $\beta \geq 0$, then φ satisfies (6)–(9), see [27]. Therefore, the left-sided and right-sided generalized integral operators are defined as follows:

$${}_{p_1^+}I_\varphi f(x) = \int_{p_1}^x \frac{\varphi(x - t)}{x - t} f(t)dt, \quad x > p_1, \tag{10}$$

$${}_{p_2^-}I_\varphi f(x) = \int_x^{p_2} \frac{\varphi(t - x)}{t - x} f(t)dt, \quad x < p_2. \tag{11}$$

The most important feature of generalized integrals is that they produce Riemann–Liouville fractional integrals, k-Riemann–Liouville fractional integrals, Katugampola fractional integrals, conformable fractional integrals, Hadamard fractional integrals, etc., see [9, 12, 26].

Recently, Farid in [7] generalized the above integral by introducing an increasing and positive monotone function g on $[p_1, p_2]$, having continuous derivative on (p_1, p_2). The generalized fractional integral operator defined by Farid may be given as follows.

Definition 5 The left- and right-sided generalized fractional integral of a function f with respect to another function g may be given, respectively, as follows:

$$G^{\varphi,g}_{p_1+}f(x) = \int_{p_1}^{x} \frac{\varphi(g(x)-g(u))}{g(x)-g(u)} g'(u)f(u)\,du, \ x > p_1, \tag{12}$$

$$G^{\varphi,g}_{p_2-}f(x) = \int_{x}^{p_2} \frac{\varphi(g(u)-g(x))}{g(u)-g(x)} g'(u)f(u)\,du, \ x < p_2. \tag{13}$$

This operator generalizes the various fractional integrals of a function f with respect to another function g.

The following special cases are focused in our study.

(i) If we take $\varphi(u) = u$, then the operators (12) and (13) reduce to Riemann–Liouville integral of f with respect to function g.

$$I^{g}_{p_1+}f(x) = \int_{p_1}^{x} g'(u)f(u)\,du, \quad x > p_1, \tag{14}$$

$$I^{g}_{p_2-}f(x) = \int_{x}^{p_2} g'(u)f(u)\,du, \quad x < p_2. \tag{15}$$

If $g(u) = u$, then (14) and (15) will reduce to Riemann integral of f.

(ii) If we take $\varphi(u) = \frac{u^\alpha}{\Gamma(\alpha)}$, then the operators (12) and (13) reduce to Riemann–Liouville fractional integral of f with respect to function g.

$$I^{\varphi,g}_{p_1+}f(x) = \frac{1}{\Gamma(\alpha)} \int_{p_1}^{x} [g(x)-g(u)]^{\alpha-1} g'(u)f(u)\,du, \ x > p_1, \tag{16}$$

$$I^{\varphi,g}_{p_2-}f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{p_2} [g(u)-g(x)]^{\alpha-1} g'(u)f(u)\,du, \ x < p_2. \tag{17}$$

If $g(u) = u$, then (16) and (17) will reduce to left- and right-sided Riemann–Liouville fractional integrals of f, respectively.

(iii) If we take $\varphi(u) = \frac{u^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then the operators (12) and (13) reduce to k-Riemann–Liouville fractional integral of f with respect to function g.

$$I^{\varphi,g}_{p_1+,k}f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{p_1}^{x} [g(x)-g(u)]^{\frac{\alpha}{k}-1} g'(u)f(u)\,du, \ x > p_1, \tag{18}$$

$$I^{\varphi,g}_{p_2-,k}f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{x}^{p_2} [g(u)-g(x)]^{\alpha-1} g'(u)f(u)\,du, \ x < p_2. \tag{19}$$

If $g(u) = u$, then these operators in (18) and (19) reduce to k-fractional integral operators given in [22].

(iv) If we take $\varphi_g(u) = u(g(p_2) - u)^{\alpha-1}$ for $\alpha \in (0, 1)$, then the operator given in (12) and (13) reduces to conformable fractional integral operator of f with respect to a function g.

$$K_{p_1}^{\alpha,g} f(x) = \int_{p_1}^{x} [g(u)]^{\alpha-1} g'(u) f(u)\, du, \; x > p_1. \tag{20}$$

This operator (20) generalizes conformable fractional integral operator that was given by Khalil et al. in [17].

(v) If we take $\varphi(u) = \frac{u}{\alpha} \exp(-Au)$, where $A = \frac{1-\alpha}{\alpha}$ and $\alpha \in (0, 1)$, then the operators given in (12) and (13) reduce to fractional integral operator of f with respect to function g with exponential kernel.

$$J_{p_1+}^{\alpha,g} f(x) = \frac{1}{\alpha} \int_{p_1}^{x} \exp\left(-A(g(x) - g(u))\right) g'(u) f(u)\, du, \; x > p_1, \tag{21}$$

$$J_{p_2-}^{\alpha,g} f(x) = \frac{1}{\alpha} \int_{x}^{p_2} \exp\left(-A(g(x) - g(u))\right) g'(u) f(u)\, du, \; x < p_2. \tag{22}$$

Operators in (21) and (22) generalize fractional integral operator with exponential kernel that was introduced by Kirane and Torebek in [18].

Motivated by the above literature, the main objective of this paper is to discover in Section 2 an interesting identity in order to study some new bounds regarding trapezium type inequalities of differentiable preinvex functions with respect to another function via generalized integral operator. By using the established identity as an auxiliary result, some new estimates for trapezium type integral inequalities via generalized integrals are obtained. It is pointed out that some new fractional integral inequalities have been deduced from main results. In Section 3, some applications to special means and new error estimates for the trapezium formula are given. The ideas and techniques of this paper may stimulate further research in the field of integral inequalities.

2 Main Results

Throughout this study, let $P = [mp_1, mp_1 + \eta(p_2, mp_1)]$ be an invex subset with respect to $\eta : P \times P \longrightarrow \mathbb{R}$, where $p_1 < p_2$ and $m \in (0, 1]$. Also for all $t \in [0, 1]$, for brevity, we define

$$\Lambda_m^{\varphi,g}(t) := \int_0^t \frac{\varphi\left(g\left(mp_1 + u\eta(p_2, mp_1)\right) - g(mp_1)\right)}{g\left(mp_1 + u\eta(p_2, mp_1)\right) - g(mp_1)} \times g'\left(mp_1 + u\eta(p_2, mp_1)\right) du < \infty \tag{23}$$

and

$$\Delta_m^{\varphi,g}(t) := \int_t^1 \frac{\varphi\left(g\left(mp_1+\eta(p_2,mp_1)\right)-g\left(mp_1+u\eta(p_2,mp_1)\right)\right)}{g\left(mp_1+\eta(p_2,mp_1)\right)-g\left(mp_1+u\eta(p_2,mp_1)\right)} \tag{24}$$

$$\times g'\left(mp_1+u\eta(p_2,mp_1)\right)du < \infty,$$

where g is an increasing and positive monotone function on P, having continuous derivative on $P^\circ = (mp_1, mp_1+\eta(p_2,mp_1))$.
For establishing some new results regarding general fractional integrals, we need to prove the following lemma.

Lemma 1 *Let $f : P \longrightarrow \mathbb{R}$ be a differentiable mapping on P°. If $f' \in L(P)$, then the following identity for generalized fractional integrals hold:*

$$\frac{f(mp_1)+f\left(mp_1+\eta(p_2,mp_1)\right)}{2}$$

$$-\frac{1}{2\eta(p_2,mp_1)} \times \left[\frac{G^{\varphi,g}_{(mp_1)^+}f\left(mp_1+\eta(p_2,mp_1)\right)}{\Delta_m^{\varphi,g}(0)} + \frac{G^{\varphi,g}_{(mp_1+\eta(p_2,mp_1))^-}f(mp_1)}{\Lambda_m^{\varphi,g}(1)}\right]$$

$$= \frac{\eta(p_2,mp_1)}{2\Lambda_m^{\varphi,g}(1)} \times \int_0^1 \Lambda_m^{\varphi,g}(t) f'\left(mp_1+t\eta(p_2,mp_1)\right)dt \tag{25}$$

$$-\frac{\eta(p_2,mp_1)}{2\Delta_m^{\varphi,g}(0)} \times \int_0^1 \Delta_m^{\varphi,g}(t) f'\left(mp_1+t\eta(p_2,mp_1)\right)dt.$$

We denote

$$T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(p_1,p_2) := \frac{\eta(p_2,mp_1)}{2\Lambda_m^{\varphi,g}(1)} \times \int_0^1 \Lambda_m^{\varphi,g}(t) f'\left(mp_1+t\eta(p_2,mp_1)\right)dt \tag{26}$$

$$-\frac{\eta(p_2,mp_1)}{2\Delta_m^{\varphi,g}(0)} \times \int_0^1 \Delta_m^{\varphi,g}(t) f'\left(mp_1+t\eta(p_2,mp_1)\right)dt.$$

Proof Integrating by parts Eq. (26) and changing the variable of integration, we have

$$T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(p_1,p_2) = \frac{\eta(p_2,mp_1)}{2\Lambda_m^{\varphi,g}(1)}$$

$$\times \left\{ \frac{\Lambda_m^{\varphi,g}(t) f\left(mp_1+t\eta(p_2,mp_1)\right)}{\eta(p_2,mp_1)}\Bigg|_0^1 - \frac{1}{\eta(p_2,mp_1)} \right.$$

$$\times \int_0^1 \frac{\varphi\left(g\left(mp_1 + t\eta(p_2, mp_1)\right) - g(mp_1)\right)}{g\left(mp_1 + t\eta(p_2, mp_1)\right) - g(mp_1)}$$

$$\left.\times g'\left(mp_1 + t\eta(p_2, mp_1)\right) f\left(mp_1 + t\eta(p_2, mp_1)\right) dt\right\}$$

$$-\frac{\eta(p_2, mp_1)}{2\Delta_m^{\varphi,g}(0)} \times \left\{ \left.\frac{\Delta_m^{\varphi,g}(t) f\left(mp_1 + t\eta(p_2, mp_1)\right)}{\eta(p_2, mp_1)}\right|_0^1 \right.$$

$$-\frac{1}{\eta(p_2, mp_1)} \times \int_0^1 \frac{\varphi\left(g\left(mp_1 + \eta(p_2, mp_1)\right) - g\left(mp_1 + t\eta(p_2, mp_1)\right)\right)}{g\left(mp_1 + \eta(p_2, mp_1)\right) - g\left(mp_1 + t\eta(p_2, mp_1)\right)}$$

$$\left.\times g'\left(mp_1 + t\eta(p_2, mp_1)\right) f\left(mp_1 + t\eta(p_2, mp_1)\right) dt\right\}$$

$$= \frac{\eta(p_2, mp_1)}{2\Lambda_m^{\varphi,g}(1)}$$

$$\times \left\{ \frac{\Lambda_m^{\varphi,g}(1) f\left(mp_1 + \eta(p_2, mp_1)\right)}{\eta(p_2, mp_1)} - \frac{1}{\eta^2(p_2, mp_1)} \times G^{\varphi,g}_{(mp_1+\eta(p_2,mp_1))^-} f(mp_1) \right.$$

$$-\frac{\eta(p_2, mp_1)}{2\Delta_m^{\varphi,g}(0)}$$

$$\times \left\{ \frac{-\Delta_m^{\varphi,g}(0) f(mp_1)}{\eta(p_2, mp_1)} + \frac{1}{\eta^2(p_2, mp_1)} \times G^{\varphi,g}_{(mp_1)^+} f\left(mp_1 + \eta(p_2, mp_1)\right) \right.$$

$$= \frac{f(mp_1) + f\left(mp_1 + \eta(p_2, mp_1)\right)}{2}$$

$$-\frac{1}{2\eta(p_2, mp_1)} \times \left[\frac{G^{\varphi,g}_{(mp_1)^+} f\left(mp_1 + \eta(p_2, mp_1)\right)}{\Delta_m^{\varphi,g}(0)} + \frac{G^{\varphi,g}_{(mp_1+\eta(p_2,mp_1))^-} f(mp_1)}{\Lambda_m^{\varphi,g}(1)} \right].$$

This completes the proof of the lemma.

Remark 1 Taking $m = 1$, $\varphi(t) = g(t) = t$ and $\eta(p_2, mp_1) = p_2 - mp_1$ in Lemma 1, we get

$$T_f(p_1, p_2) := \frac{f(p_1) + f(p_2)}{2} - \frac{1}{p_2 - p_1} \int_{p_1}^{p_2} f(t)dt.$$

Theorem 2 *Let $f : P \longrightarrow \mathbb{R}$ be a differentiable mapping on P° and $\eta(p_2, mp_1) > 0$. If $|f'|^q$ is preinvex on P for $q > 1$ and $p^{-1} + q^{-1} = 1$, then the following inequalities for generalized fractional integrals hold:*

$$\left|T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(p_1, p_2)\right| \leq \frac{\eta(p_2, mp_1)}{2}\sqrt[q]{\frac{|f'(mp_1)|^q + |f'(p_2)|^q}{2}} \tag{27}$$
$$\times\left[\frac{\sqrt[p]{B_{\Lambda_m}^{\varphi,g}(p)}}{\Lambda_m^{\varphi,g}(1)} + \frac{\sqrt[p]{B_{\Delta_m}^{\varphi,g}(p)}}{\Delta_m^{\varphi,g}(0)}\right],$$

where

$$B_{\Lambda_m}^{\varphi,g}(p) := \int_0^1 \left[\Lambda_m^{\varphi,g}(t)\right]^p dt, \quad B_{\Delta_m}^{\varphi,g}(p) := \int_0^1 \left[\Delta_m^{\varphi,g}(t)\right]^p dt. \tag{28}$$

Proof From Lemma 1, preinvexity of $|f'|^q$, Hölder inequality and properties of the modulus, we have

$$\left|T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(p_1, p_2)\right| \leq \frac{\eta(p_2, mp_1)}{2\Lambda_m^{\varphi,g}(1)} \times \int_0^1 \Lambda_m^{\varphi,g}(t)\left|f'\left(mp_1 + t\eta(p_2, mp_1)\right)\right|dt$$
$$+\frac{\eta(p_2, mp_1)}{2\Delta_m^{\varphi,g}(0)} \times \int_0^1 \Delta_m^{\varphi,g}(t)\left|f'\left(mp_1 + t\eta(p_2, mp_1)\right)\right|dt$$
$$\leq \frac{\eta(p_2, mp_1)}{2\Lambda_m^{\varphi,g}(1)} \times \left(\int_0^1 \left[\Lambda_m^{\varphi,g}(t)\right]^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'\left(mp_1 + t\eta(p_2, mp_1)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{\eta(p_2, mp_1)}{2\Delta_m^{\varphi,g}(0)} \times \left(\int_0^1 \left[\Delta_m^{\varphi,g}(t)\right]^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'\left(mp_1 + t\eta(p_2, mp_1)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{\eta(p_2, mp_1)}{2\Lambda_m^{\varphi,g}(1)}\sqrt[p]{B_{\Lambda_m}^{\varphi,g}(p)} \times \left(\int_0^1 \left[(1-t)\left|f'(mp_1)\right|^q + t\left|f'(p_2)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\frac{\eta(p_2, mp_1)}{2\Delta_m^{\varphi,g}(0)}\sqrt[p]{B_{\Delta_m}^{\varphi,g}(p)} \times \left(\int_0^1 \left[(1-t)\left|f'(mp_1)\right|^q + t\left|f'(p_2)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$= \frac{\eta(p_2, mp_1)}{2}\sqrt[q]{\frac{|f'(mp_1)|^q + |f'(p_2)|^q}{2}} \times \left[\frac{\sqrt[p]{B_{\Lambda_m}^{\varphi,g}(p)}}{\Lambda_m^{\varphi,g}(1)} + \frac{\sqrt[p]{B_{\Delta_m}^{\varphi,g}(p)}}{\Delta_m^{\varphi,g}(0)}\right].$$

The proof of this theorem is complete.

We point out some special cases of Theorem 2.

Corollary 1 *Taking $p = q = 2$ in Theorem 2, we get*

$$\left|T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(p_1, p_2)\right| \le \frac{\eta(p_2, mp_1)}{2}\sqrt{\frac{|f'(mp_1)|^2 + |f'(p_2)|^2}{2}} \tag{29}$$

$$\times\left[\frac{\sqrt{B_{\Lambda_m}^{\varphi,g}(2)}}{\Lambda_m^{\varphi,g}(1)} + \frac{\sqrt{B_{\Delta_m}^{\varphi,g}(2)}}{\Delta_m^{\varphi,g}(0)}\right].$$

Corollary 2 *Taking $|f'| \le K$ in Theorem 2, we get*

$$\left|T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(p_1, p_2)\right| \le \frac{K\eta(p_2, mp_1)}{2} \times \left[\frac{\sqrt[p]{B_{\Lambda_m}^{\varphi,g}(p)}}{\Lambda_m^{\varphi,g}(1)} + \frac{\sqrt[p]{B_{\Delta_m}^{\varphi,g}(p)}}{\Delta_m^{\varphi,g}(0)}\right]. \tag{30}$$

Corollary 3 *Taking $\varphi(t) = t$ in Theorem 2, we get*

$$\left|T_{f,\Lambda_m^{g},\Delta_m^{g}}(p_1, p_2)\right| \le \frac{\sqrt[q]{\eta(p_2, mp_1)}}{2}\sqrt[q]{\frac{|f'(mp_1)|^q + |f'(p_2)|^q}{2}} \tag{31}$$

$$\times\left[\frac{\sqrt[p]{B_1^g(p)} + \sqrt[p]{B_2^g(p)}}{g(mp_1 + \eta(p_2, mp_1)) - g(mp_1)}\right],$$

where

$$B_1^g(p) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)} \left[g(t) - g(mp_1)\right]^p dt, \tag{32}$$

and

$$B_2^g(p) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)} \left[g(mp_1 + \eta(p_2, mp_1)) - g(t)\right]^p dt. \tag{33}$$

Corollary 4 *Taking $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 2, we get*

$$\left|T_{f,\Lambda_m^{g},\Delta_m^{g}}(p_1, p_2)\right| \le \frac{\sqrt[q]{\eta(p_2, mp_1)}}{2}\sqrt[q]{\frac{|f'(mp_1)|^q + |f'(p_2)|^q}{2}} \tag{34}$$

$$\times\left[\frac{\sqrt[p]{B_3^g(p,\alpha)} + \sqrt[p]{B_4^g(p,\alpha)}}{\left[g(mp_1 + \eta(p_2, mp_1)) - g(mp_1)\right]^\alpha}\right],$$

where

$$B_3^g(p,\alpha) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)} \big[g(t) - g(mp_1)\big]^{p\alpha} dt, \tag{35}$$

and

$$B_4^g(p,\alpha) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)} \big[g(mp_1+\eta(p_2,mp_1)) - g(t)\big]^{p\alpha} dt. \tag{36}$$

Corollary 5 *Taking $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2, we get*

$$\big|T_{f,\Lambda_m^g,\Delta_m^g}(p_1,p_2)\big| \le \frac{\sqrt[q]{\eta(p_2,mp_1)}}{2}\sqrt[q]{\frac{|f'(mp_1)|^q + |f'(p_2)|^q}{2}} \tag{37}$$

$$\times\left[\frac{\sqrt[p]{B_5^g(p,\alpha,k)} + \sqrt[p]{B_6^g(p,\alpha,k)}}{\big[g(mp_1+\eta(p_2,mp_1)) - g(mp_1)\big]^{\frac{\alpha}{k}}}\right],$$

where

$$B_5^g(p,\alpha,k) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)} \big[g(t) - g(mp_1)\big]^{\frac{p\alpha}{k}} dt, \tag{38}$$

and

$$B_6^g(p,\alpha,k) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)} \big[g(mp_1+\eta(p_2,mp_1)) - g(t)\big]^{\frac{p\alpha}{k}} dt. \tag{39}$$

Corollary 6 *Taking $\varphi_g(t) = t(g(mp_1+\eta(p_2,mp_1)) - t)^{\alpha-1}$ in Theorem 2, we get*

$$\big|T_{f,\Lambda_m^g,\Delta_m^g}(p_1,p_2)\big| \le \frac{\sqrt[q]{\eta(p_2,mp_1)}}{2\big[g^\alpha(mp_1+\eta(p_2,mp_1)) - g^\alpha(mp_1)\big]} \tag{40}$$

$$\times\sqrt[q]{\frac{|f'(mp_1)|^q + |f'(p_2)|^q}{2}} \times \left[\sqrt[p]{B_7^g(p)} + \sqrt[p]{B_8^g(p,\alpha)}\right],$$

where

$$B_7^g(p) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)} \big[g(t) - g(mp_1)\big]^{p} dt, \tag{41}$$

and

$$B_8^g(p,\alpha) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)} \Big[g^\alpha(mp_1+\eta(p_2,mp_1)) - g^\alpha(t)\Big]^p dt. \tag{42}$$

Corollary 7 *Taking $\varphi(t) = \frac{t}{\alpha}\exp(-At)$, where $A = \frac{1-\alpha}{\alpha}$, in Theorem 2, we get*

$$\begin{aligned} \left|T_{f,\Lambda_m^g,\Delta_m^g}(p_1,p_2)\right| &\le \frac{\eta(p_2,mp_1)}{2\Big\{1-\exp\Big[A\left(g(mp_1)-g(mp_1+\eta(p_2,mp_1))\right)\Big]\Big\}} \\ &\times \sqrt[q]{\frac{|f'(mp_1)|^q+|f'(p_2)|^q}{2}} \times \left[\sqrt[p]{B_9^g(p)} + \sqrt[p]{B_{10}^g(p)}\right], \end{aligned} \tag{43}$$

where

$$B_9^g(p) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)} \Big\{1-\exp\Big[A\left(g(mp_1)-g(t)\right)\Big]\Big\}^p dt, \tag{44}$$

and

$$B_{10}^g(p) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)} \Big\{1-\exp\Big[A\left(g(t)-g(mp_1+\eta(p_2,mp_1))\right)\Big]\Big\}^p dt. \tag{45}$$

Theorem 3 *Let $f : P \longrightarrow \mathbb{R}$ be a differentiable mapping on P° and $\eta(p_2,mp_1) > 0$. If $|f'|^q$ is preinvex on P for $q \ge 1$, then the following inequalities for generalized fractional integrals hold:*

$$\begin{aligned} \left|T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(p_1,p_2)\right| &\le \frac{\eta(p_2,mp_1)}{2\Lambda_m^{\varphi,g}(1)}\Big[B_{\Lambda_m}^{\varphi,g}(1)\Big]^{1-\frac{1}{q}} \\ &\times \sqrt[q]{C_{\Lambda_m}^{\varphi,g}|f'(mp_1)|^q + D_{\Lambda_m}^{\varphi,g}|f'(p_2)|^q} \\ &+ \frac{\eta(p_2,mp_1)}{2\Delta_m^{\varphi,g}(0)}\Big[B_{\Delta_m}^{\varphi,g}(1)\Big]^{1-\frac{1}{q}} \sqrt[q]{E_{\Delta_m}^{\varphi,g}|f'(mp_1)|^q + F_{\Delta_m}^{\varphi,g}|f'(p_2)|^q}, \end{aligned} \tag{46}$$

where

$$C_{\Lambda_m}^{\varphi,g} := \int_0^1 (1-t)\Lambda_m^{\varphi,g}(t)dt, \quad D_{\Lambda_m}^{\varphi,g} := \int_0^1 t\Lambda_m^{\varphi,g}(t)dt, \tag{47}$$

$$E_{\Delta_m}^{\varphi,g} := \int_0^1 (1-t)\Delta_m^{\varphi,g}(t)dt, \quad F_{\Delta_m}^{\varphi,g} := \int_0^1 t\Delta_m^{\varphi,g}(t)dt \tag{48}$$

and $B_{\Lambda_m}^{\varphi,g}(1)$, $B_{\Delta_m}^{\varphi,g}(1)$ are defined as in Theorem 2.

Proof From Lemma 1, preinvexity of $|f'|^q$, power mean inequality, and properties of the modulus, we have

$$\left|T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(p_1,p_2)\right| \le \frac{\eta(p_2,mp_1)}{2\Lambda_m^{\varphi,g}(1)} \times \int_0^1 \Lambda_m^{\varphi,g}(t)\left|f'(mp_1+t\eta(p_2,mp_1))\right|dt$$

$$+\frac{\eta(p_2,mp_1)}{2\Delta_m^{\varphi,g}(0)} \times \int_0^1 \Delta_m^{\varphi,g}(t)\left|f'(mp_1+t\eta(p_2,mp_1))\right|dt$$

$$\le \frac{\eta(p_2,mp_1)}{2\Lambda_m^{\varphi,g}(1)}\times\left(\int_0^1 \Lambda_m^{\varphi,g}(t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \Lambda_m^{\varphi,g}(t)\left|f'(mp_1+t\eta(p_2,mp_1))\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{\eta(p_2,mp_1)}{2\Delta_m^{\varphi,g}(0)}\times\left(\int_0^1 \Delta_m^{\varphi,g}(t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \Delta_m^{\varphi,g}(t)\left|f'(mp_1+t\eta(p_2,mp_1))\right|^q dt\right)^{\frac{1}{q}}$$

$$\le \frac{\eta(p_2,mp_1)}{2\Lambda_m^{\varphi,g}(1)}\left[B_{\Lambda_m}^{\varphi,g}(1)\right]^{1-\frac{1}{q}}\times\left(\int_0^1 \Lambda_m^{\varphi,g}(t)\left[(1-t)\left|f'(mp_1)\right|^q+t\left|f'(p_2)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\frac{\eta(p_2,mp_1)}{2\Delta_m^{\varphi,g}(0)}\left[B_{\Delta_m}^{\varphi,g}(1)\right]^{1-\frac{1}{q}}\times\left(\int_0^1 \Delta_m^{\varphi,g}(t)\left[(1-t)\left|f'(mp_1)\right|^q+t\left|f'(p_2)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$=\frac{\eta(p_2,mp_1)}{2\Lambda_m^{\varphi,g}(1)}\left[B_{\Lambda_m}^{\varphi,g}(1)\right]^{1-\frac{1}{q}}\sqrt[q]{C_{\Lambda_m}^{\varphi,g}|f'(mp_1)|^q+D_{\Lambda_m}^{\varphi,g}|f'(p_2)|^q}$$

$$+\frac{\eta(p_2,mp_1)}{2\Delta_m^{\varphi,g}(0)}\left[B_{\Delta_m}^{\varphi,g}(1)\right]^{1-\frac{1}{q}}\sqrt[q]{E_{\Delta_m}^{\varphi,g}|f'(mp_1)|^q+F_{\Delta_m}^{\varphi,g}|f'(p_2)|^q}.$$

The proof of this theorem is complete.

We point out some special cases of Theorem 3.

Corollary 8 *Taking $q=1$ in Theorem 3, we get*

$$\left|T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(p_1,p_2)\right| \le \frac{\eta(p_2,mp_1)}{2\Lambda_m^{\varphi,g}(1)}\left[C_{\Lambda_m}^{\varphi,g}|f'(mp_1)|+D_{\Lambda_m}^{\varphi,g}|f'(p_2)|\right] \tag{49}$$

$$+\frac{\eta(p_2,mp_1)}{2\Delta_m^{\varphi,g}(0)}\left[E_{\Delta_m}^{\varphi,g}|f'(mp_1)|+F_{\Delta_m}^{\varphi,g}|f'(p_2)|\right].$$

Corollary 9 *Taking $|f'|\le K$ in Theorem 3, we get*

$$\left|T_{f,\Lambda_m^{\varphi,g},\Delta_m^{\varphi,g}}(p_1,p_2)\right| \le \frac{K\eta(p_2,mp_1)}{2}\times\left[\frac{B_{\Lambda_m}^{\varphi,g}(1)}{\Lambda_m^{\varphi,g}(1)}+\frac{B_{\Delta_m}^{\varphi,g}(1)}{\Delta_m^{\varphi,g}(0)}\right]. \tag{50}$$

Corollary 10 *Taking $\varphi(t) = t$ in Theorem 3, we get*

$$\left|T_{f,\Lambda_m^g,\Delta_m^g}(p_1,p_2)\right| \le \frac{1}{2\sqrt[q]{\eta(p_2,mp_1)}\left[g(mp_1+\eta(p_2,mp_1))-g(mp_1)\right]} \tag{51}$$

$$\times\left\{\left[B_1^g(1)\right]^{1-\frac{1}{q}}\sqrt[q]{\left[B_1^g(1)\eta(p_2,mp_1)-C_1^g\right]|f'(mp_1)|^q+C_1^g|f'(p_2)|^q}\right.$$

$$\left.+\left[B_2^g(1)\right]^{1-\frac{1}{q}}\sqrt[q]{\left[B_2^g(1)\eta(p_2,mp_1)-E_1^g\right]|f'(mp_1)|^q+E_1^g|f'(p_2)|^q}\right\},$$

where

$$C_1^g := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)}(t-mp_1)(g(t)-g(mp_1))dt, \tag{52}$$

$$E_1^g := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)}(t-mp_1)(g(mp_1+\eta(p_2,mp_1))-g(t))dt, \tag{53}$$

and $B_1^g(1)$, $B_2^g(1)$ are defined as in Corollary 3 for value $p = 1$.

Corollary 11 *Taking $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 3, we get*

$$\left|T_{f,\Lambda_m^g,\Delta_m^g}(p_1,p_2)\right| \le \frac{1}{2\sqrt[q]{\eta(p_2,mp_1)}\left[g(mp_1+\eta(p_2,mp_1))-g(mp_1)\right]^\alpha} \tag{54}$$

$$\times\left\{\left[B_3^g(1,\alpha)\right]^{1-\frac{1}{q}}\sqrt[q]{\left[B_3^g(1,\alpha)\eta(p_2,mp_1)-C_1^g(\alpha)\right]|f'(mp_1)|^q+C_1^g(\alpha)|f'(p_2)|^q}\right.$$

$$\left.+\left[B_4^g(1,\alpha)\right]^{1-\frac{1}{q}}\sqrt[q]{\left[B_4^g(1,\alpha)\eta(p_2,mp_1)-E_1^g(\alpha)\right]|f'(mp_1)|^q+E_1^g(\alpha)|f'(p_2)|^q}\right\},$$

where

$$C_1^g(\alpha) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)}(t-mp_1)(g(t)-g(mp_1))^\alpha dt, \tag{55}$$

$$E_1^g(\alpha) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)}(t-mp_1)(g(mp_1+\eta(p_2,mp_1))-g(t))^\alpha dt, \tag{56}$$

and $B_3^g(1,\alpha)$, $B_4^g(1,\alpha)$ are defined as in Corollary 4 for value $p = 1$.

Corollary 12 *Taking* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ *in Theorem 3, we get*

$$\begin{aligned}\left|T_{f,\Lambda_m^g,\Delta_m^g}(p_1,p_2)\right| &\leq \frac{1}{2\sqrt[q]{\eta(p_2,mp_1)}\Big[g(mp_1+\eta(p_2,mp_1))-g(mp_1)\Big]^{\frac{\alpha}{k}}}\\ &\quad\times\left\{\Big[B_5^g(1,\alpha,k)\Big]^{1-\frac{1}{q}}\right.\end{aligned}\tag{57}$$

$$\times\sqrt[q]{\Big[B_5^g(1,\alpha,k)\eta(p_2,mp_1)-C_1^g(\alpha,k)\Big]|f'(mp_1)|^q+C_1^g(\alpha,k)|f'(p_2)|^q}$$

$$+\Big[B_6^g(1,\alpha,k)\Big]^{1-\frac{1}{q}}$$

$$\left.\times\sqrt[q]{\Big[B_6^g(1,\alpha,k)\eta(p_2,mp_1)-E_1^g(\alpha,k)\Big]|f'(mp_1)|^q+E_1^g(\alpha,k)|f'(p_2)|^q}\right\},$$

where

$$C_1^g(\alpha,k) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)}(t-mp_1)(g(t)-g(mp_1))^{\frac{\alpha}{k}}dt,\tag{58}$$

$$E_1^g(\alpha,k) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)}(t-mp_1)(g(mp_1+\eta(p_2,mp_1))-g(t))^{\frac{\alpha}{k}}dt,\tag{59}$$

and $B_5^g(1,\alpha,k)$, $B_6^g(1,\alpha,k)$ *are defined as in Corollary 5 for value* $p=1$.

Corollary 13 *Taking* $\varphi_g(t) = t(g(mp_1+\eta(p_2,mp_1))-t)^{\alpha-1}$ *in Theorem 3, we get*

$$\left|T_{f,\Lambda_m^g,\Delta_m^g}(p_1,p_2)\right| \leq \frac{\sqrt[q]{\alpha\eta^2(p_2,mp_1)}}{2\Big[g^\alpha(mp_1+\eta(p_2,mp_1))-g^\alpha(mp_1)\Big]}\tag{60}$$

$$\times\left\{\Big[B_7^g(1)\Big]^{1-\frac{1}{q}}\sqrt[q]{C_{\Lambda_m}^{\star g}(\alpha)|f'(mp_1)|^q+D_{\Lambda_m}^{\star g}(\alpha)|f'(p_2)|^q}\right.$$

$$\left.+\Big[B_8^g(1,\alpha)\Big]^{1-\frac{1}{q}}\sqrt[q]{E_{\Delta_m}^{\star g}(\alpha)|f'(mp_1)|^q+F_{\Delta_m}^{\star g}(\alpha)|f'(p_2)|^q}\right\},$$

where

$$C^{\star g}_{\Lambda_m}(\alpha) := \frac{B^g_7(1)}{\alpha\eta^2(p_2, mp_1)} - \frac{1}{\alpha\eta(p_2, mp_1)} \tag{61}$$

$$\times\left[\frac{g^\alpha(mp_1 + \eta(p_2, mp_1))}{2} + \frac{E^g_2(\alpha)}{\eta^2(p_2, mp_1)}\right],$$

$$D^{\star g}_{\Lambda_m}(\alpha) := \frac{1}{\alpha\eta(p_2, mp_1)}\left[\frac{g^\alpha(mp_1 + \eta(p_2, mp_1))}{2} + \frac{E^g_2(\alpha)}{\eta^2(p_2, mp_1)}\right], \tag{62}$$

$$E^{\star g}_{\Delta_m}(\alpha) := \frac{B^g_8(1, \alpha)}{\alpha\eta^2(p_2, mp_1)} - \frac{1}{\alpha\eta(p_2, mp_1)} \tag{63}$$

$$\times\left[\frac{g^\alpha(mp_1 + \eta(p_2, mp_1))}{2} - \frac{E^g_3(\alpha)}{\eta^2(p_2, mp_1)}\right],$$

$$F^{\star g}_{\Delta_m}(\alpha) := \frac{1}{\alpha\eta(p_2, mp_1)}\left[\frac{g^\alpha(mp_1 + \eta(p_2, mp_1))}{2} - \frac{E^g_3(\alpha)}{\eta^2(p_2, mp_1)}\right], \tag{64}$$

$$E^g_2(\alpha) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)} (t - mp_1) \tag{65}$$

$$\times\Big[g(mp_1) + g(mp_1 + \eta(p_2, mp_1)) - g(t)\Big]^\alpha dt,$$

$$E^g_3(\alpha) := \int_{mp_1}^{mp_1+\eta(p_2,mp_1)} (t - mp_1)g^\alpha(t)dt, \tag{66}$$

where $B^g_7(1)$ and $B^g_8(1, \alpha)$ are defined as in Corollary 6 for value $p = 1$.

Corollary 14 *Taking $\varphi(t) = \frac{t}{\alpha}\exp(-At)$, where $A = \frac{1-\alpha}{\alpha}$, in Theorem 3, we get*

$$\left|T_{f,\Lambda^g_m,\Delta^g_m}(p_1, p_2)\right| \leq \frac{(1-\alpha)\eta^{\frac{q+1}{q}}(p_2, mp_1)}{2\Big\{1 - \exp\Big[A\left(g(mp_1) - g(mp_1 + \eta(p_2, mp_1))\right)\Big]\Big\}} \tag{67}$$

$$\times\Bigg\{\Big[B^g_9(1)\Big]^{1-\frac{1}{q}}\sqrt[q]{\left(B^g_9(1) - \frac{D^{\diamond g}_{\Lambda_m}}{(1-\alpha)\eta(p_2, mp_1)}\right)|f'(mp_1)|^q + D^{\diamond g}_{\Lambda_m}|f'(p_2)|^q}$$

$$+\Big[B^g_{10}(1)\Big]^{1-\frac{1}{q}}\sqrt[q]{\left(B^g_{10}(1) - \frac{F^{\diamond g}_{\Delta_m}}{(1-\alpha)\eta(p_2, mp_1)}\right)|f'(mp_1)|^q + F^{\diamond g}_{\Delta_m}|f'(p_2)|^q}\Bigg\},$$

where

$$D^{\diamond g}_{\Lambda_m} := \frac{1}{\eta^2(p_2, mp_1)} \tag{68}$$

$$\times \int_{mp_1}^{mp_1+\eta(p_2,mp_1)} (t - mp_1)\Big[1 - \exp\left[A(g(mp_1) - g(t))\right]\Big]dt,$$

$$F^{\diamond g}_{\Delta_m} := \frac{1}{\eta^2(p_2, mp_1)}, \tag{69}$$

$$\times \int_{mp_1}^{mp_1+\eta(p_2,mp_1)} (t - mp_1)\Big[1 - \exp\left[A(g(t) - g(mp_1 + \eta(p_2, mp_1)))\right]\Big]dt,$$

and $B^g_9(1),\ B^g_{10}(1)$ *are defined as in Corollary 7 for value* $p = 1$.

Remark 2 Applying our Theorems 2 and 3 for appropriate choices of function $g(t) = t$; $g(t) = \ln t$, $\forall t > 0$; $g(t) = e^t$, etc., where $\varphi(t) = t$, $\frac{t^\alpha}{\Gamma(\alpha)}$, $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\varphi_g(t) = t(g(p_2) - t)^{\alpha-1}$ for $\alpha \in (0, 1)$; $\varphi(t) = \frac{t}{\alpha} \exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha \in (0, 1)$, we can deduce some new general fractional integral inequalities. The details are left to the interested reader.

3 Applications

Consider the following special means for different real numbers p_1, p_2 and $p_1 p_2 \neq 0$, as follows:

1. the arithmetic mean:

$$A := A(p_1, p_2) = \frac{p_1 + p_2}{2},$$

2. the harmonic mean:

$$H := H(p_1, p_2) = \frac{2}{\frac{1}{p_1} + \frac{1}{p_2}},$$

3. the logarithmic mean:

$$L := L(p_1, p_2) = \frac{p_2 - p_1}{\ln|p_2| - \ln|p_1|},$$

4. the generalized log mean:

$$L_r := L_r(p_1, p_2) = \left[\frac{p_2^{r+1} - p_1^{r+1}}{(r+1)(p_2 - p_1)}\right]^{\frac{1}{r}}; \quad r \in \mathbb{Z} \setminus \{-1, 0\}.$$

It is well known that L_r is monotonic nondecreasing over $r \in \mathbb{Z}$ with $L_{-1} := L$. In particular, we have the following inequality $H \leq L \leq A$. Now, using the theory results in Section 2, we give some applications to special means for different real numbers.

Proposition 1 *Let $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, where $p_1 < p_2$ and $\eta(p_2, mp_1) > 0$. Then for $r \in \mathbb{N}$ and $r \geq 2$, where $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds:*

$$\begin{aligned}&\left|A((mp_1)^r, (mp_1 + \eta(p_2, mp_1))^r) - L_r^r(mp_1, mp_1 + \eta(p_2, mp_1))\right| \\ &\quad\leq \frac{r\eta(p_2, mp_1)}{\sqrt[p]{p+1}} \times \sqrt[q]{A\left(|mp_1|^{q(r-1)}, |p_2|^{q(r-1)}\right)}. \qquad (70)\end{aligned}$$

Proof Taking $f(t) = t^r$ and $g(t) = \varphi(t) = t$, in Theorem 2, one can obtain the result immediately.

Proposition 2 *Let $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, where $p_1 < p_2$ and $\eta(p_2, mp_1) > 0$. Then for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{H(mp_1, mp_1 + \eta(p_2, mp_1))} - \frac{1}{L(mp_1, mp_1 + \eta(p_2, mp_1))}\right| \leq \frac{\eta(p_2, mp_1)}{\sqrt[p]{p+1}} \qquad (71)\\ &\quad\times \frac{1}{\sqrt[q]{H\left(|mp_1|^{2q}, |p_2|^{2q}\right)}}.\end{aligned}$$

Proof Taking $f(t) = \frac{1}{t}$ and $g(t) = \varphi(t) = t$, in Theorem 2, one can obtain the result immediately.

Proposition 3 *Let $p_1, p_2 \in \mathbb{R} \setminus \{0\}$, where $p_1 < p_2$ and $\eta(p_2, mp_1) > 0$. Then for $r \in \mathbb{N}$ and $r \geq 2$, where $q \geq 1$, the following inequality holds:*

$$\begin{aligned}&\left|A((mp_1)^r, (mp_1 + \eta(p_2, mp_1))^r) - L_r^r(mp_1, mp_1 + \eta(p_2, mp_1))\right| \\ &\quad\leq r2^{\frac{1-2q}{q}} \eta^{\frac{q-3}{q}}(p_2, mp_1) \qquad (72)\\ &\quad\times\left\{\sqrt[q]{\frac{\eta^3(p_2, mp_1)}{3}} \sqrt[q]{A\left(|mp_1|^{q(r-1)}, 2|p_2|^{q(r-1)}\right)} + \sqrt[q]{H(m, r, q, p_1, p_2)}\right\},\end{aligned}$$

where

$$H(m,r,q,p_1,p_2) := \left(\frac{\eta^3(p_2,mp_1)}{2} - P(m,p_1,p_2)\right)|mp_1|^{q(r-1)} \tag{73}$$

$$+P(m,p_1,p_2)|p_2|^{q(r-1)},$$

and

$$P(m,p_1,p_2) := \frac{(mp_1+\eta(p_2,mp_1))\eta^2(p_2,mp_1)}{2} \tag{74}$$

$$-\frac{(mp_1+\eta(p_2,mp_1))^3-(mp_1)^3}{3} + (mp_1)\frac{(mp_1+\eta(p_2,mp_1))^2-(mp_1)^2}{2}.$$

Proof Taking $f(t) = t^r$ and $g(t) = \varphi(t) = t$, in Theorem 3, one can obtain the result immediately.

Proposition 4 *Let* $p_1, p_2 \in \mathbb{R}\setminus\{0\}$, *where* $p_1 < p_2$ *and* $\eta(p_2, mp_1) > 0$. *Then for* $q \geq 1$, *the following inequality holds:*

$$\left|\frac{1}{H(mp_1, mp_1+\eta(p_2,mp_1))} - \frac{1}{L(mp_1, mp_1+\eta(p_2,mp_1))}\right|$$

$$\leq 2^{\frac{1-2q}{q}}\eta^{\frac{q-3}{q}}(p_2,mp_1) \tag{75}$$

$$\times\left\{\sqrt[q]{\frac{\eta^3(p_2,mp_1)}{3}}\frac{1}{\sqrt[q]{H\left(2|mp_1|^{2q},|p_2|^{2q}\right)}} + \sqrt[q]{G(m,q,p_1,p_2)}\right\},$$

where

$$G(m,q,p_1,p_2) := \frac{\frac{\eta^3(p_2,mp_1)}{2} - P(m,p_1,p_2)}{|mp_1|^{2q}} + \frac{P(m,p_1,p_2)}{|p_2|^{2q}}, \tag{76}$$

and $P(m, p_1, p_2)$ *is defined as in Proposition* 3.

Proof Taking $f(t) = \dfrac{1}{t}$ and $g(t) = \varphi(t) = t$, in Theorem 3, one can obtain the result immediately.

Remark 3 Applying our Theorems 2 and 3 for appropriate choices of function $g(t) = t$; $g(t) = \ln t$, $\forall t > 0$;, $g(t) = e^t$, etc., where $\varphi(t) = t$, $\frac{t^\alpha}{\Gamma(\alpha)}$, $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\varphi_g(t) = t(g(p_2)-t)^{\alpha-1}$ for $\alpha \in (0,1)$; $\varphi(t) = \frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha \in (0,1)$, such that $|f'|^q$ to be convex, we can deduce some new general fractional integral inequalities using special means. The details are left to the interested reader.

Next, we provide some new error estimates for the trapezium formula. Let Q be the partition of the points $p_1 = x_0 < x_1 < \ldots < x_k = p_2$ of the interval $[p_1, p_2]$. Let us consider the following quadrature formula:

$$\int_{p_1}^{p_2} f(x)dx = T(f, Q) + E(f, Q),$$

where

$$T(f, Q) = \sum_{i=0}^{k-1} \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)$$

is the trapezium version and $E(f, Q)$ denotes their associated approximation error.

Proposition 5 *Let $f : [p_1, p_2] \longrightarrow \mathbb{R}$ be a differentiable function on (p_1, p_2), where $p_1 < p_2$. If $|f'|^q$ is convex on $[p_1, p_2]$ for $q > 1$ and $p^{-1} + q^{-1} = 1$, then the following inequality holds:*

$$\left|E(f, Q)\right| \leq \frac{1}{\sqrt[q]{2}\sqrt[p]{p+1}} \times \sum_{i=0}^{k-1}(x_{i+1} - x_i)^2 \sqrt[q]{|f'(x_i)|^q + |f'(x_{i+1})|^q}. \tag{77}$$

Proof Applying Theorem 2 for $m = 1$, $\eta(p_2, mp_1) = p_2 - mp_1$ and $g(t) = \varphi(t) = t$ on the subintervals $[x_i, x_{i+1}]$ $(i = 0, \ldots, k-1)$ of the partition Q, we have

$$\left|\frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i}\int_{x_i}^{x_{i+1}} f(x)dx\right|$$
$$\leq \frac{(x_{i+1} - x_i)}{\sqrt[q]{2}\sqrt[p]{p+1}} \sqrt[q]{|f'(x_i)|^q + |f'(x_{i+1})|^q}. \tag{78}$$

Hence from (78), we get

$$\left|E(f, Q)\right| = \left|\int_{p_1}^{p_2} f(x)dx - T(f, Q)\right|$$

$$\leq \left|\sum_{i=0}^{k-1}\left\{\int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)\right\}\right|$$

$$\leq \sum_{i=0}^{k-1}\left|\left\{\int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)\right\}\right|$$

$$\leq \frac{1}{\sqrt[q]{2}\sqrt[p]{p+1}} \times \sum_{i=0}^{k-1}(x_{i+1} - x_i)^2 \sqrt[q]{|f'(x_i)|^q + |f'(x_{i+1})|^q}.$$

The proof of this proposition is complete.

Proposition 6 *Let $f : [p_1, p_2] \longrightarrow \mathbb{R}$ be a differentiable function on (p_1, p_2), where $p_1 < p_2$. If $|f'|^q$ is convex on $[p_1, p_2]$ for $q \geq 1$, then the following inequality holds:*

$$\left|E(f, Q)\right| \leq 2^{\frac{1-2q}{q}} \times \sum_{i=0}^{k-1} (x_{i+1} - x_i)^{\frac{2q-3}{q}} \tag{79}$$

$$\times \left\{ \sqrt[q]{\frac{(x_{i+1} - x_i)^3 \left(|f'(x_i)|^q + 2|f'(x_{i+1})|^q\right)}{6}} + \sqrt[q]{S_f(q, x_i, x_{i+1})} \right\},$$

where

$$S_f(q, x_i, x_{i+1}) := \left(\frac{(x_{i+1} - x_i)^3}{2} - P(x_i, x_{i+1}) \right) |f'(x_i)|^q \tag{80}$$

$$+P(x_i, x_{i+1})|f'(x_{i+1})|^q$$

and

$$P(x_i, x_{i+1}) := \frac{x_{i+1}(x_{i+1} - x_i)^2}{2} - \frac{x_{i+1}^3 - x_i^3}{3} + \frac{x_i(x_{i+1}^2 - x_i^2)}{2}. \tag{81}$$

Proof The proof is analogous as to that of Proposition 5 but uses Theorem 3.

Remark 4 Applying our Theorems 2 and 3 for appropriate choices of function $g(t) = t$; $g(t) = \ln t$, $\forall\, t > 0$; , $g(t) = e^t$, etc., where $\varphi(t) = t$, $\frac{t^{\alpha}}{\Gamma(\alpha)}$, $\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$; $\varphi_g(t) = t(g(p_2) - t)^{\alpha-1}$ for $\alpha \in (0, 1)$; $\varphi(t) = \frac{t}{\alpha} \exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha \in (0, 1)$, such that $|f'|^q$ to be convex, we can deduce some new bounds for the trapezium formula using above ideas and techniques. The details are left to the interested reader.

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New Trapezoid Type Inequalities for Generalized Exponentially Strongly Convex Functions

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Abstract By using a new general identity and introducing some very general new notions of generalized exponentially strongly convex functions, new trapezoid type inequalities are established. We apply these inequalities to provide approximations for the integral of a real valued function. Approximations for some new weighted means of two positive numbers are also obtained.

Mathematics Subject Classification: 26D15, 26A51

1 Introduction

In 2018, Awan et al. introduced the new notion of exponentially convex function:

Definition 1 ([1]) A function $f : [a, b] \to \mathbb{R}$ is called exponentially convex if

$$f(tx_1 + (1-t)x_2) \le t\frac{f(x_1)}{e^{rx_1}} + (1-t)\frac{f(x_2)}{e^{rx_2}}, \tag{1}$$

for $\forall x_1, x_2 \in [a, b]$, $\forall t \in [0, 1]$ and $r \in \mathbb{R}$.

In particular, if $r = 0$, then (1) reduces to convex function in the classical sense. Let $f : [a, b] \to \mathbb{R}$ be a convex function, then the inequality

$$f(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{2}$$

is known in the literature as the Hermite–Hadamard inequality (see, for instance, [2, 3]). In fact, the inequality (2) holds if and only if f is a convex function. The Hermite–Hadamard inequality provides approximations for integral mean of a real

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valued function f. The concept of convex function was extended in many directions and frameworks due to its numerous applications in optimization, variational methods, geometry, and artificial intelligence. Hence, the inequality (2) has also been extended and generalized for different classes of generalized convex functions (see [14] and the references therein).

In 2019, Mehreen and Anwar [8] extended the above Definition 1 by introducing the new notions of exponentially p-convex function and exponentially s-convex function in the second sense, respectively. In fact, they can be generalized uniformly as follows:

Definition 2 Let $[a, b] \subset (0, \infty)$. A function $f : [a, b] \to \mathbb{R}$ is called exponentially (α, s)-convex if

$$f((tx_1^{\alpha} + (1-t)x_2^{\alpha})^{1/\alpha}) \leq t^s \frac{f(x_1)}{e^{rx_1}} + (1-t)^s \frac{f(x_2)}{e^{rx_2}}, \tag{3}$$

for $\forall x_1, x_2 \in [a, b]$, $\forall t \in [0, 1]$, $s \in (0, 1]$, $\alpha \neq 0$ and $r \in \mathbb{R}$.

In particular, if $s = 1$, then (3) reduces to exponentially α-convex function in [8]; if $\alpha = 1$, then (3) reduces to exponentially s-convex function in [8]; if $r = 0$, $\alpha = 1$, then (3) reduces to s- convex function in [4].

Definition 3 Let $[a, b] \subset \mathbb{R} - \{0\}$. A function $f : [a, b] \to \mathbb{R}$ is called exponentially harmonically s-convex, if

$$f(\frac{x_1x_2}{tx_2 + (1-t)x_1}) \leq t^s \frac{f(x_1)}{e^{rx_1}} + (1-t)^s \frac{f(x_2)}{e^{rx_2}}, \tag{4}$$

for $\forall x_1, x_2 \in [a, b]$, $\forall t \in [0, 1]$, $s \in (0, 1]$ and $r \in \mathbb{R}$.

If $s = 1$, $r = 0$, then (4) reduces to harmonically convex function in [5].

$$F(\alpha, \beta, \gamma, z) = \frac{1}{B(\gamma - \alpha, \alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-zt)^{-\beta} dt \tag{5}$$

is the hypergeometric function, where $|z| < 1$, $\gamma > \alpha > 0$, and

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt, \quad , \alpha, \beta > 0$$

is the Beta function.

An interesting question in (2) was estimating the difference between the left and middle terms and between the right and middle terms. Such as

Theorem 1 ([10]) *Let $f \in BV[a, b]$, then*

$$|\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) du| \leq \frac{1}{2} V_a^b(f). \tag{6}$$

The constant $\frac{1}{2}$ *is the best possible.*

Theorem 2 ([6]) *Let* $|f^{'}|$ *is convex on* $[a, b]$, *then*

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du| \leq \frac{b-a}{8}(|f^{'}(a)| + |f^{'}(b)|). \tag{7}$$

Theorem 3 ([16]) *Let* $f \in L[a, b]$ *and* $|f^{'}|^q$ *is convex on* $[a, b]$, *then*

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du| \leq \frac{b-a}{4\times 2^{1/q}}(|f^{'}(a)|^q + |f^{'}(b)|^q)^{1/q}. \tag{8}$$

Theorem 4 ([26]) *Let* $|f^{'}|$ *is* h*-convex on* $[a, b]$, *that is,*

$$|f^{'}(ta+(1-t)b)| \leq h(t)|f^{'}(a)| + \frac{1-\lambda}{\lambda}h(1-t)|f^{'}(b)|,$$

then

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du| \leq \frac{b-a}{2}(\int_0^1 h(t)dt)(|f^{'}(a)| + |f^{'}(b)|). \tag{9}$$

Theorem 5 ([23]) *Let* $f^{'} \in L^p[a, b]$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du| \leq \frac{(b-a)^{1/q}}{2\times(q+1)^{1/q}}\|f^{'}\|_p. \tag{10}$$

Theorem 6 ([7]) *Let* $|f^{''}|^q$ *is* s*-convex on* $[a, b]$, *if* $q = 1$, *then*

$$|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du| \leq \frac{(b-a)^2}{2(s+2)(s+3)}(|f^{''}(a)| + |f^{''}(b)|); \tag{11}$$

if $1 < q < \infty$, *then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq \frac{(b-a)^2}{2\times 4^{1-(1/q)}(s+2)^{1/q}(s+3)^{1/q}}(|f^{''}(a)|^q + |f^{''}(b)|^q)^{1/q}. \tag{12}$$

Theorem 7 ([8] Theorem 3. 6, 3. 7, 3. 3, 3. 5) *Let* $|f^{'}|^q$ *is exponentially* s*-convex on* $[a, b]$. *If* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)}{2^{(1/p)+1}\{(s+1)(s+2)\}^{1/q}}(s+\frac{1}{2^s})^{1/q}\{(\frac{|f^{'}(a)|}{e^{ra}})^q+(\frac{|f^{'}(b)|}{e^{rb}})^q\}^{1/q}, \tag{13}$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)}{2(p+1)^{1/p}(s+1)^{1/q}}\{(\frac{|f^{'}(a)|}{e^{ra}})^q+(\frac{|f^{'}(b)|}{e^{rb}})^q\}^{1/q}. \tag{14}$$

If $p=1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)}{2(s+1)(s+2)}\{(3s+4)\frac{|f^{'}(a)|}{e^{ra}}+(s+4)\frac{|f^{'}(b)|}{e^{rb}}\} \tag{15}$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(b-a)}{2(s+1)(s+2)}(s+\frac{1}{2^s})\{\frac{|f^{'}(a)|}{e^{ra}}+\frac{|f^{'}(b)|}{e^{rb}}\}. \tag{16}$$

Remark 1 $\big(\frac{|f^{'}(a)|}{e^{ra}}\big)^q+\big(\frac{|f^{'}(b)|}{e^{rb}}\big)^q$ in (13) and (14) should be replaced by $\frac{|f^{'}(a)|^q}{e^{ra}}+\frac{|f^{'}(b)|^q}{e^{rb}}$, see Theorems 29 and 30 below.

The above (6)–(16) are called trapezoid type inequalities, and they have been developed for other types of functions and have wide applications in numerical analysis and in the theory of some special estimating error bounds for some means and quadrature rules, etc. (see [7–9, 12, 15, 23, 25, 30, 31] and the references therein).

The paper is categorized as follows:

In Sect. 2, we introduce some very general new notions of generalized exponentially strongly convex functions (or exponentially $(\alpha, \beta, \lambda, \lambda_1, \lambda_2, s_0, t, h_1, h)$-strongly convex functions). They unified and generalized many known and new classes of convex functions. In Sect. 3, by using a new general identity and the above convex functions, new trapezoid type inequalities are established. These trapezoid type inequalities provide the estimations of integral mean of a real valued function f and improve and generalize the corresponding results in [8]. In Sect. 5, approximations for some new weighted means of two positive numbers are also obtained.

2 Generalized Exponentially Strongly Convex Functions

Throughout this paper, let $h : [a, b] \to (0, \infty)$ $h_1 : (0, 1) \to (0, \infty)$ be given functions.

Definition 4 Let D be a α-convex set, $[a, b] \subset D$. A function $f : [a, b] \to \mathbb{R}$ is called exponentially (α, h_1)-convex if

$$f((tx_1^{\alpha} + (1-t)x_2^{\alpha})^{1/\alpha}) \leq h_1(t)\frac{f(x_1)}{e^{rx_1}} + h_1(1-t)\frac{f(x_2)}{e^{rx_2}}, \tag{17}$$

for $\forall x_1, x_2 \in [a, b], \forall t \in [0, 1], s \in (0, 1]$ and $r \in \mathbb{R}$.

In particular, if $D = (0, \infty)$, $h_1(t) = t^s$, then Definition 4 reduces to Definition 2.

Remark 2 ([11]) An interval D is said to be a α-convex set, if $(tx_1^{\alpha}+(1-t)x_2^{\alpha})^{1/\alpha} \in D$ for all $x_1, x_2 \in D, t \in [0, 1]$, where $\alpha = 2k + 1$ or $\alpha = \frac{n}{m}, n = 2r + 1, m = 2k + 1, k, r \in N$. If $D = (0, \infty)$, then $\alpha \in \mathbb{R} - \{0\}$.

Definition 5 Let D be a α-convex set, $[a, b] \subset D$. A function $f : [a, b] \to (0, \infty)$ is called $(\alpha, \beta, \lambda, \lambda_1, \lambda_2, s_0, t, h_1)$- convex if

$$f((\lambda x_1^{\alpha}+\lambda_1(1-\lambda)x_2^{\alpha})^{1/\alpha}) \leq \{h_1(t^{s_0})(f^{\beta}(x_1)+\lambda_2 h_1(1-t^{s_0})f^{\beta}(x_2)\}^{1/\beta}, \tag{18}$$

$\forall x_1, x_2 \in [a, b], \forall \lambda, \lambda_1, \lambda_2, s_0, t \in [0, 1]$, and $\beta > 0$.

If $\lambda_1 = \lambda_2 = \lambda_0, s_0 = 1$, then (18) reduces to $(\alpha, \beta, \lambda, \lambda_0, t, h_1)$- convex function in [18], that is,

$$f((\lambda x_1^{\alpha} + \lambda_0(1-\lambda)x_2^{\alpha})^{1/\alpha}) \leq \{h_1(t)f^{\beta}(x_1) + \lambda_0 h_1(1-t)f^{\beta}(x_2)\}^{1/\beta}, \tag{19}$$

$\forall x_1, x_2 \in [a, b], \forall \lambda, \lambda_0, t \in [0, 1]$, and $\beta > 0$.

In the above inequalities (18) and (19), the most innovative part consists of the fact that possibly different parameters $\alpha, \beta, \lambda, \lambda_1, \lambda_2, \lambda_0, t, s_0$, and h_1, are allowed, for example:

If $\beta = \lambda_0 = 1$, then (19) reduces to (α, h_1)-convex function in [18];
If $\alpha = \beta = 1, \lambda = t$, then (19) reduces to (h_1, λ_0)-convex function in [27];
If $\alpha = \beta = 1, \lambda = t, \lambda_0 = 1$, then (19) reduces to h_1-convex function in [27];
If $\alpha = \beta = 1, \lambda = t, h_1(t) = t$, then (19) reduces to λ_0-convex function in [27];
If $\alpha = \beta = 1, \lambda_0 = 1, h_1(t) = t$, then (19) reduces to (λ, t)-convex function in [2];

In (18), after replacing λ with t, let $\alpha = \beta = 1, \lambda_1 = \lambda_2 = \lambda$, we get

$$f(tx_1 + \lambda(1-t)x_2) \leq h_1(t^{s_0})f(x_1) + \lambda h_1(1-t^{s_0})f(x_2), \tag{20}$$

then (20) reduces to (s_0, λ, h_1)-convex function in [29].

If $h_1(t) = t$, then (20) reduces to (s_0, λ)-convex function in [28].

If $h_1(t) = 1$, then (20) reduces to (λ, P)-convex function in [28].
If $h_1(t) = t^s$, $s \in (0, 1]$, then (20) reduces to (s_0, λ, s)-convex function in [28].
If $h_1(t) = t^s$, $s \in (0, 1]$, $s_0 = 1$, then (20) reduces to (λ, s)-convex function in [28].
If $h_1(t) = t(1-t)$, $s_0 = 1$, then (20) reduces to (λ, tgs)-convex function in [28].
If $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $s_0 = 1$, then (20) reduces to (λ, MT)-convex function in [28].

In (18), after replacing λ with t, let $\alpha = \beta = 1, s_0 = 1, \lambda_1 = 1, \lambda_2 = \frac{1-\lambda}{\lambda}$, $\lambda \in (0, 1)$, then (18) reduces to h_1-convex function of the second sense in [26].

In (18), after replacing λ with t, let $\lambda_1 = \lambda_2 = \lambda, s_0 = 1, h_1(t) = t^s, 0 < |s| \leq 1$, then (18) is said to be a $(\alpha, \beta, \lambda, s)$-convex function:

$$f((tx_1^\alpha + \lambda(1-t)x_2^\alpha)^{1/\alpha}) \leq \{t^s f^\beta(x_1) + \lambda(1-t)^s f^\beta(x_2)\}^{1/\beta}. \tag{21}$$

If $s = \beta = 1, \lambda = 1$, then (21) reduces to α-convex function in [11].

If $\alpha = \beta = s = 1, \lambda = 1$, then (21) reduces to convex function in the classical sense.

If $\alpha = -1, \beta = 1, \lambda = 1$, then (21) reduces to harmonically s-convex function in [5].

If $\alpha = -1, s = \beta = 1, \lambda = 1$, then (21) reduces to harmonically convex function in [5].

If $\alpha = \beta = 1, 0 < s \leq 1, \lambda = 1$, then (21) reduces to s-convex function in [4].

If $s = 0, \alpha = \beta = 1, \lambda = 1$, then (21) reduces to P- function in [2].

If $s = -1, \alpha = \beta = 1, \lambda = 1$, then (21) reduces to Godunova-Levin function in [2].

If $-1 \leq s \leq 0, \alpha = \beta = 1, \lambda = 1$, then (21) reduces to Godunova-Levin s-convex function in [21].

If $\alpha = s = 1, \lambda = 1$, then (21) reduces to β-convex function in [2].

If $s = 1, \alpha = -1, \lambda = 1$, then (21) reduces to harmonically β-convex function in [13].

If $\alpha = 1, \beta = -1, 0 < |s| \leq 1$, then (21) is said to be a (AH, s)-convex function, where AH means the arithmetic-harmonic means.

If $\alpha = -1, \beta = 1, 0 < |s| \leq 1$, then (21) is said to be a (HA, s)-convex function.

If $\alpha = \beta = -2$, then (21) is said to be a (AS, s)- convex function, where AS means the arithmetic-square harmonic means.

If $\lambda = t, \lambda_0 = 0, \alpha = \beta = 1, h_1(t) = t$ in (19), then

$$f(tx_1) \leq tf(x_1),$$

we say that f is a star-shaped function (see [17]).

Definition 6 ([19]) A function $f : [a, b] \to \mathbb{R}$ is called strongly convex with modulus c if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) - ct(1-t)(x_1 - x_2)^2, \tag{22}$$

$\forall x_1, x_2 \in [a, b], \forall t \in [0, 1], c > 0.$

Strongly convex functions have been introduced by Polyak [19] and play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature (see, for instance, [20, 22] and the references therein).

In 2016, Adamek [24] generalized (21) to the following:

Definition 7 ([24]) A function $f : [a, b] \to \mathbb{R}$ is called h-strongly convex if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) - t(1-t)h(x_1 - x_2), \tag{23}$$

$\forall x_1, x_2 \in [a, b]$, $\forall t \in [0, 1]$, and $h : [a, b] \to [0, \infty)$.

In particular, if $h(x_1 - x_2) = c(x_1 - x_2)^2$, $c > 0$, then (23) reduces to (22).

In this section, we want to extend the above definitions to the following generalized exponentially strongly convex functions or exponentially $(\alpha, \beta, \lambda, \lambda_1, \lambda_2, s_0, t, h_1, h)$- strongly convex functions:

Definition 8 Let $(X, \|\cdot\|)$ denote the real normed linear spaces, D be a convex subset of X, and $h, h_1 : (0, \infty) \to (0, \infty)$ be given functions, in which h, h_1 is not identical to 0. A function $f : D \to (0, \infty)$ is called exponentially $(\alpha, \beta, \lambda, \lambda_1, \lambda_2, s_0, t, h_1, h)$-strongly convex if

$$f((\lambda\|x_1\|^\alpha + \lambda_1(1-\lambda)\|x_2\|^\alpha)^{1/\alpha}) \leq \{h_1(t^{s_0})(\frac{f(\|x_1\|)}{e^{r\|x_1\|}})^\beta$$
$$+\lambda_2 h_1(1-t^{s_0})(\frac{f(\|x_2\|)}{e^{r\|x_2\|}})^\beta\}^{1/\beta} - t(1-t)h(|x_1 - x_2|),$$

$\forall x_1, x_2 \in D$, $\forall \lambda, \lambda_1, \lambda_2, s_0, t \in [0, 1]$, $r \in \mathbb{R}$, α, β are real numbers and $\alpha, \beta \neq 0$.

In particular, if $X = \mathbb{R}^{\Bbbk}$, we get:

Definition 9 Let D be a α-convex set of $\mathbb{R}^{\Bbbk}$, $[a, b] \subset D$. A function $f : [a, b] \to (0, \infty)$ is called exponentially $(\alpha, \beta, \lambda, \lambda_1, \lambda_2, s_0, t, h_1, h)$-strongly convex if

$$f((\lambda x_1^\alpha + \lambda_1(1-\lambda)x_2^\alpha)^{1/\alpha}) \leq \{h_1(t^{s_0})(\frac{f(x_1)}{e^{rx_1}})^\beta + \lambda_2 h_1(1-t^{s_0})(\frac{f(x_2)}{e^{rx_2}})^\beta\}^{1/\beta}$$
$$-t(1-t)h(|x_1 - x_2|), \tag{24}$$

$\forall x_1, x_2 \in [a, b]$, $\forall \lambda, \lambda_1, \lambda_2, s_0, t \in [0, 1]$, $r \in \mathbb{R}$, α and β are the real numbers, and $\alpha, \beta \neq 0$.

After replacing λ with t, let $\lambda_1 = \lambda_2 = \lambda$, $h_1(t) = t^s$, $s_0 = 1$ in (24), that is,

$$f((tx_1^\alpha + \lambda(1-t)x_2^\alpha)^{1/\alpha}) \leq \{t^s(\frac{f(x_1)}{e^{rx_1}})^\beta + \lambda(1-t)^s(\frac{f(x_2)}{e^{rx_2}})^\beta\}^{1/\beta}$$
$$-t(1-t)h(|x_1 - x_2|), \tag{25}$$

$\forall x_1, x_2 \in [a, b]$, $\forall \lambda, s, t \in [0, 1]$, $r \in \mathbb{R}$, α and β are the real numbers, and $\alpha, \beta \neq 0$, then (25) is said to exponentially $(\alpha, \beta, \lambda, s, h)$-strongly convex function.

If $\alpha = -1$, then (25) is said to be an exponentially harmonically (β, λ, s, h)-strongly convex function.

If $\alpha = -1, \lambda = 1$, then (25) is said to be an exponentially harmonically (β, s, h)-strongly convex function;

If $\alpha = -1, \beta = \lambda = 1$, then (25) is said to be an exponentially harmonically (s, h)-strongly convex function. With condition $s = 1$, (25) is said to be an exponentially harmonically h-strongly convex function.

If $r = 0$, then (25) is said to be a $(\alpha, \beta, \lambda, s, h)$- strongly convex function.

If $r = 0, \alpha = 1$, then (25) is said to be a (β, λ, s, h)-strongly convex function.

If $r = 0, h = 0$, then (25) reduces to (21).

If $r = 0, \alpha = \beta = \lambda = s = 1$, then (25) reduces to (23).

If $\alpha = 1, \lambda = 1$, then (25) is said to be an exponentially (β, s, h)-strongly convex function.

If $\alpha = 1, \beta = \lambda = 1$, then (25) is said to be an exponentially (s, h)-strongly convex function.

If $\alpha = 1, \beta = \lambda = s = 1$, then (25) is said to be an exponentially h-strongly convex function.

Hence, Definitions 8 and 9 are very general notions of convex functions. They unified and generalized many known and new classes of convex functions.

3 Main Results

Lemma 1 *Let D be a α-convex set, $[a, b] \subset D$. If $f'' \in L[a, b]$, then*

$$\frac{(\lambda b^\alpha - a^\alpha)^2}{\alpha^2} \int_0^1 t(1-t)(ta^\alpha + \lambda(1-t)b^\alpha)^{4/\alpha} f''((ta^\alpha + \lambda(1-t)b^\alpha)^{1/\alpha})dt$$
$$= a^{2\alpha+2} f(a) + (\lambda^{1/\alpha} b)^{2\alpha+2} f(\lambda^{1/\alpha} b) + \frac{1}{\alpha(\lambda b^\alpha - a^\alpha)} \int_a^{\lambda^{1/\alpha} b} f(u)w(u)du, \tag{26}$$

where

$$w(u) = 2(\alpha+1)(2\alpha+3)(a^\alpha + \lambda b^\alpha)u^{2\alpha+1} - 3(\alpha+1)(3\alpha+2)u^{3\alpha+1}$$
$$- \lambda(\alpha+2)(\alpha+3)a^\alpha b^\alpha u^{\alpha+1}. \tag{27}$$

Proof Setting $u = (ta^\alpha + \lambda(1-t)b^\alpha)^{1/\alpha}$, and by integration by parts, we get

$$\int_0^1 t(1-t)(ta^\alpha + \lambda(1-t)b^\alpha)^{4/\alpha} f''((ta^\alpha + \lambda(1-t)b^\alpha)^{1/\alpha})dt$$

$$= \frac{\alpha}{(\lambda b^\alpha - a^\alpha)^3} \int_a^{\lambda^{1/\alpha} b} (u^\alpha - \lambda b^\alpha)(a^\alpha - u^\alpha) u^{\alpha+3} f''(u) du$$

$$= \frac{-\alpha}{(\lambda b^\alpha - a^\alpha)^3} \int_a^{\lambda^{1/\alpha} b} \{(2\alpha + 3)(a^\alpha + \lambda b^\alpha) u^{2\alpha+2}$$

$$-3(\alpha + 1) u^{3\alpha+2} - \lambda(\alpha + 3) a^\alpha b^\alpha u^{\alpha+2}\} f'(u) du$$

$$= \frac{\alpha^2}{(\lambda b^\alpha - a^\alpha)^2} \{(\lambda^{1/\alpha} b)^{2\alpha+2} f(\lambda^{1/\alpha} b) + a^{2\alpha+2} f(a)$$

$$+ \frac{1}{\alpha(\lambda b^\alpha - a^\alpha)} \int_a^{\lambda 1/\alpha b} f(u) w(u) du\}. \tag{28}$$

Multiplying both sides of (28) by $\frac{(\lambda b^\alpha - a^\alpha)^2}{\alpha^2}$, we get required equality (26).

In particular, if $\alpha = -1$, then by Lemma 1, we get:

Corollary 1 *Let* $[a, b] \subset \mathbb{R} - \{0\}$, $f'' \in L[a, b]$, *then*

$$\frac{f(a) + f(\lambda^{-1} b)}{2} - \frac{1}{\lambda^{-1} b - a} \int_a^{\lambda^{-1} b} f(u) du$$
$$= \frac{(b - \lambda a)^2}{2a^2 b^2} \int_0^1 t(1-t)[ta^{-1} + \lambda(1-t) b^{-1}]^{-4} f''((ta^{-1} + \lambda(1-t) b^{-1})^{-1}) dt. \tag{29}$$

If $\lambda = 1$, then by Corollary 1, we get:

Corollary 2 *Let* $[a, b] \subset \mathbb{R} - \{0\}$, $f'' \in L[a, b]$, *then*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(u) du$$
$$= \frac{(b - a)^2}{2a^2 b^2} \int_0^1 t(1 - t)[ta^{-1} + (1 - t) b^{-1}]^{-4} f''((ta^{-1} + (1 - t) b^{-1})^{-1}) dt. \tag{30}$$

Lemma 2 ([25]) *Let* $f : [a, b] \to \mathbb{R}$. *If* $f'' \in L[a, b]$, *then*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(u) du$$
$$= \frac{(b - a)^2}{2} \int_0^1 t(1 - t) f''(ta + (1 - t) b) dt. \tag{31}$$

Lemma 3 ([32]) *Let $f:[a,b]\to\mathbb{R}$. If $f^{'}\in L[a,b], \alpha\neq 0$, then*

$$\frac{f(a)+f(b)}{2}-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b\frac{f(u)}{u^{1-\alpha}}du$$
$$=\frac{(b^{\alpha}-a^{\alpha})}{2\alpha}\int_0^1\frac{1-2t}{(ta^{\alpha}+(1-t)b^{\alpha})^{1-(1/\alpha)}}f^{'}((ta^{\alpha}+(1-t)b^{\alpha})^{1/\alpha})dt. \quad (32)$$

Taking $\alpha=1$ in (32), we get the following identity in [6]:

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du$$
$$=\frac{(b-a)}{2}\int_0^1(1-2t)f^{'}(ta+(1-t)b)dt.$$

Theorem 8 *Let $[a,b]\subset(0,\infty)$, $f^{''}\in L[a,b]$, and $|f^{''}|^p$ is exponentially $(\alpha,\beta,\lambda,s,h)$-strongly convex on $[a,b]$, $1\leq p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, and for $p=1$, define $q=\infty$, $\frac{1}{\infty}=0$, then*

$$|a^{2\alpha+2}f(a)+(\lambda^{1/\alpha}b)^{2\alpha+2}f(\lambda^{1/\alpha}b)+\frac{1}{\alpha(\lambda b^{\alpha}-a^{\alpha})}\int_a^{\lambda^{1/\alpha}b}f(u)w(u)du|$$
$$\leq\frac{(\lambda b^{\alpha}-a^{\alpha})^2}{6^{1/q}\alpha^2}(\lambda^{1/\alpha}b)^4\{F(2,-\frac{4}{\alpha},4,1-\frac{1}{\lambda}(\frac{a}{b})^{\alpha})\}^{1/q}$$
$$\times\{C_{\beta}\frac{\beta^2}{(s+2\beta)(s+3\beta)}[F(2+\frac{s}{\beta},-\frac{4}{\alpha},4+\frac{s}{\beta},1-\frac{1}{\lambda}(\frac{a}{b})^{\alpha})\frac{|f^{''}(a)|^p}{e^{ra}}$$
$$+\lambda^{1/\beta}F(2,-\frac{4}{\alpha},4+\frac{s}{\beta},1-\frac{1}{\lambda}(\frac{a}{b})^{\alpha})\frac{|f^{''}(b)|^p}{e^{rb}}]$$
$$-\frac{1}{30}F(3,-\frac{4}{\alpha},6,1-\frac{1}{\lambda}(\frac{a}{b})^{\alpha})h(b-a)\}^{1/p}, \quad (33)$$

where $w(u)$ is defined by (27) and

$$C_{\beta}=\begin{cases}1, & \beta\geq 1,\\ 2^{(1/\beta)-1}, & 0<\beta<1.\end{cases}$$

Proof By Lemma 1 and using the Hölder inequality, we have

$$|a^{2\alpha+2}f(a)+(\lambda^{1/\alpha}b)^{2\alpha+2}f(\lambda^{1/\alpha}b)+\frac{1}{\alpha(\lambda b^{\alpha}-a^{\alpha})}\int_a^{\lambda^{1/\alpha}b}f(u)w(u)du|$$

$$\leq \frac{(\lambda b^\alpha - a^\alpha)^2}{\alpha^2} \int_0^1 t(1-t)(ta^\alpha + \lambda(1-t)b^\alpha)^{4/\alpha} |f''((ta^\alpha + \lambda(1-t)b^\alpha)^{1/\alpha})| dt$$

$$\leq \frac{(\lambda b^\alpha - a^\alpha)^2}{\alpha^2} \times I_1^{1/q} \times I_2^{1/p}, \tag{34}$$

where

$$I_1 = \int_0^1 t(1-t)(ta^\alpha + \lambda(1-t)b^\alpha)^{4/\alpha} dt,$$

$$I_2 = \int_0^1 t(1-t)(ta^\alpha + \lambda(1-t)b^\alpha)^{4/\alpha} |f''((ta^\alpha + \lambda(1-t)b^\alpha)^{1/\alpha})|^p dt.$$

By (5), we get

$$I_1 = (\lambda^{1/\alpha} b)^4 \int_0^1 t(1-t)[1 - (1 - \frac{1}{\lambda}(\frac{a}{b})^\alpha)t]^{4/\alpha} dt$$

$$= \frac{(\lambda^{1/\alpha} b)^4}{6} F(2, -\frac{4}{\alpha}, 4, 1 - \frac{1}{\lambda}(\frac{a}{b})^\alpha). \tag{35}$$

By using the exponentially $(\alpha, \beta, \lambda, s, h)$-strongly convexity of $|f''|^p$ on $[a, b]$ and C_p-inequality (see [2]), we obtain

$$I_2 \leq (\lambda^{1/\alpha} b)^4 \int_0^1 t(1-t)[1 - (1 - \frac{1}{\lambda}(\frac{a}{b})^\alpha)t]^{4/\alpha}$$

$$\times \{[t^s(\frac{|f''(a)|^p}{e^{ra}})^\beta + \lambda(1-t)^s(\frac{|f''(b)|^p}{e^{rb}})^\beta]^{1/\beta} - t(1-t)h(b-a)\} dt$$

$$\leq (\lambda^{1/\alpha} b)^4 \int_0^1 t(1-t)[1 - (1 - \frac{1}{\lambda}(\frac{a}{b})^\alpha)t]^{4/\alpha}$$

$$\times \{C_\beta[t^{s/\beta} \frac{|f''(a)|^p}{e^{ra}} + \lambda^{1/\beta}(1-t)^{s/\beta} \frac{|f''(b)|^p}{e^{rb}}] - t(1-t)h(b-a)\} dt$$

$$= (\lambda^{1/\alpha} b)^4 \{C_\beta[I_3 \frac{|f''(a)|^p}{e^{ra}} + \lambda^{1/\beta} I_4 \frac{|f''(b)|^p}{e^{rb}}] - I_5 h(b-a)\}, \tag{36}$$

where

$$I_3 = \int_0^1 t^{1+(s/\beta)}(1-t)[1 - (1 - \frac{1}{\lambda}(\frac{a}{b})^\alpha)t]^{4/\alpha} dt,$$

$$I_4 = \int_0^1 t(1-t)^{1+(s/\beta)}[1 - (1 - \frac{1}{\lambda}(\frac{a}{b})^\alpha)t]^{4/\alpha} dt,$$

$$I_5 = \int_0^1 t^2(1-t)^2[1 - (1 - \frac{1}{\lambda}(\frac{a}{b})^\alpha)t]^{4/\alpha} dt.$$

By (5), we get

$$I_3 = \frac{\beta^2}{(s+2\beta)(s+3\beta)} F(2+\frac{s}{\beta}, -\frac{4}{\alpha}, 4+\frac{s}{\beta}, 1-\frac{1}{\lambda}(\frac{a}{b})^{\alpha}), \tag{37}$$

$$I_4 = \frac{\beta^2}{(s+2\beta)(s+3\beta)} F(2, -\frac{4}{\alpha}, 4+\frac{s}{\beta}, 1-\frac{1}{\lambda}(\frac{a}{b})^{\alpha}), \tag{38}$$

$$I_5 = \frac{1}{30} F(3, -\frac{4}{\alpha}, 6, 1-\frac{1}{\lambda}(\frac{a}{b})^{\alpha}). \tag{39}$$

A combination of (34)–(39) gives the required result.

Taking $\alpha = -1$ in (33), we get the following:

Theorem 9 *Let $[a, b] \subset (0, \infty)$, $f'' \in L[a, b]$, and $|f''|^p$ is exponentially harmonically (β, λ, s, h)-strongly convex on $[a, b]$, $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and for $p = 1$, define $q = \infty$, $\frac{1}{\infty} = 0$, then*

$$\begin{aligned}
&|\frac{f(a)+f(\lambda^{-1}b)}{2} - \frac{1}{\lambda^{-1}b-a}\int_a^{\lambda^{-1}b} f(u)du| \le \frac{b^{2/p}(b-\lambda a)^2}{2\times 6^{1/q}\lambda^{2(2-(1/q))}a^{2/p}} \\
&\le \{C_\beta \frac{\beta^2}{(s+2\beta)(s+3\beta)}[F(2+\frac{s}{\beta}, 4, 4+\frac{s}{\beta}, 1-\frac{b}{\lambda a})\frac{|f''(a)|^p}{e^{ra}} \\
&+\lambda^{1/\beta} F(2, 4, 4+\frac{s}{\beta}, 1-\frac{b}{\lambda a})\frac{|f''(b)|^p}{e^{rb}}] \\
&-\frac{1}{30} F(3, 4, 6, 1-\frac{b}{\lambda a}) h(b-a)\}^{1/p}.
\end{aligned} \tag{40}$$

Taking $\beta = 1$ in (40), we get the following:

Theorem 10 *Let $[a, b] \subset (0, \infty)$, $f'' \in L[a, b]$, and $|f''|^p$ is exponentially harmonically (β, s, h)-strongly convex on $[a, b]$, $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and for $p = 1$, define $q = \infty$, $\frac{1}{\infty} = 0$, then*

$$\begin{aligned}
&|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^{b} f(u)du| \le \frac{b^{2/p}(b-a)^2}{2\times 6^{1/q}a^{2/p}} \\
&\times\{C_\beta \frac{\beta^2}{(s+2\beta)(s+3\beta)}[F(2+\frac{s}{\beta}, 4, 4+\frac{s}{\beta}, 1-\frac{b}{a})\frac{|f''(a)|^p}{e^{ra}} \\
&+F(2, 4, 4+\frac{s}{\beta}, 1-\frac{b}{a})\frac{|f''(b)|^p}{e^{rb}}] \\
&-\frac{1}{30} F(3, 4, 6, 1-\frac{b}{a}) h(b-a)\}^{1/p}.
\end{aligned} \tag{41}$$

Taking $\beta = 1$ in (41), we get the following:

Theorem 11 *Let* $[a, b] \subset (0, \infty)$, $f'' \in L[a, b]$, *and* $|f''|^p$ *is exponentially harmonically* (s, h)*-strongly convex on* $[a, b]$, $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *and for* $p = 1$, *define* $q = \infty$, $\frac{1}{\infty} = 0$, *then*

$$\begin{aligned}
&|\frac{1}{2}[f(a) + f(b)] - \frac{1}{b-a}\int_a^b f(u)du| \\
&\leq \frac{b^{2/p}(b-a)^2}{2 \times 6^{1/q} a^{2/p}} \times \{\frac{1}{(s+2)(s+3)} \\
&\times [F(2+s, 4, 4+s, 1-\frac{b}{a})\frac{|f''(a)|^p}{e^{ra}} \\
&+F(2, 4, 4+s, 1-\frac{b}{a})\frac{|f''(b)|^p}{e^{rb}}] - \frac{1}{30}F(3, 4, 6, 1-\frac{b}{a})h(b-a)\}^{1/p}. \quad (42)
\end{aligned}$$

Taking $s = 1, h = 0$ in (42), we get

$$\begin{aligned}
&|\frac{1}{2}[f(a) + f(b)] - \frac{1}{b-a}\int_a^b f(u)du| \\
&\leq \frac{1}{12}\{G(a, b)(\frac{|f''(a)|^p}{e^{ra}}) + G(b, a)(\frac{|f''(b)|^p}{e^{rb}})\}^{1/p},
\end{aligned}$$

where $G(a, b)$ is defined by

$$G(a, b) = \frac{a\{2b^3 + 3ab^2 - 6a^2b + a^3 - 6ab^2 \log(\frac{b}{a})\}}{(b-a)^{2(2-p)}}.$$

Similarly, by different decompositions of the integrand and the Hölder inequality, under the same conditions, we can get different interesting results:

Example 1 By using the Hölder inequality,

$$\begin{aligned}
&\int_0^1 t(1-t)(ta^{-1} + \lambda(1-t)b^{-1})^{-4}|f''((ta^{-1} + \lambda(1-t)b^{-1})^{-1})|dt \\
&\leq \{\int_0^1 t^q(1-t)^q(ta^{-1} + \lambda(1-t)b^{-1})^{-4q}dt\}^{1/q} \\
&\times\{\int_0^1 |f''((ta^{-1} + \lambda(1-t)b^{-1})^{-1})|^p dt\}^{1/p}, \quad (43)
\end{aligned}$$

and the exponentially harmonically (β, λ, s, h)-strongly convexity of $|f''|^p$ on $[a, b]$, we get:

Theorem 12 *Under the assumptions of Theorem 9, by Corollary 1, we have*

$$
\begin{aligned}
&|\frac{1}{2}[f(a)+f(\lambda^{-1}b)]-\frac{1}{\lambda^{-1}b-a}\int_a^{\lambda^{-1}b}f(u)du| \\
&\le \frac{b^2(b-\lambda a)^2}{2\lambda^4a^2}\{B(q+1,q+1)F(q+1,4q,2q+2,1-\frac{b}{\lambda a})\}^{1/q} \\
&\times\{C_\beta(\frac{\beta}{\beta+s})[\frac{|f^{''}(a)|^p}{e^{ra}}+\lambda^{1/\beta}\frac{|f^{''}(b)|^p}{e^{rb}}]-\frac{1}{6}h(b-a)\}^{1/p}. \qquad (44)
\end{aligned}
$$

Theorem 13 *Under the assumptions of Theorem 11, we have*

$$
\begin{aligned}
&|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^{b}f(u)du| \\
&\le \frac{b^2(b-a)^2}{2a^2}\{B(q+1,q+1)F(q+1,4q,2q+2,1-\frac{b}{a})\}^{1/q} \\
&\times\{\frac{1}{1+s}[\frac{|f^{''}(a)|^p}{e^{ra}}+\frac{|f^{''}(b)|^p}{e^{rb}}]-\frac{1}{6}h(b-a)\}^{1/p}. \qquad (45)
\end{aligned}
$$

Example 2 By using the Hölder inequality,

$$
\begin{aligned}
&\int_0^1 t(1-t)(ta^{-1}+\lambda(1-t)b^{-1})^{-4}|f^{''}((ta^{-1}+\lambda(1-t)b^{-1})^{-1})|dt \\
&\le \{\int_0^1 t^q dt\}^{1/q}\times\{\int_0^1(1-t)^p(ta^{-1}+\lambda(1-t)b^{-1})^{-4p} \\
&\times|f^{''}((ta^{-1}+\lambda(1-t)b^{-1})^{-1})|^p dt\}^{1/p}, \qquad (46)
\end{aligned}
$$

and the exponentially harmonically (β,λ,s,h)-strongly convexity of $|f^{''}|^p$ on $[a,b]$, we get:

Theorem 14 *Under the assumptions of Theorem 9, by Corollary 1, we have*

$$
\begin{aligned}
&|\frac{1}{2}[f(a)+f(\lambda^{-1}b)]-\frac{1}{\lambda^{-1}b-a}\int_a^{\lambda^{-1}b}f(u)du| \\
&\le \frac{b^2(b-\lambda a)^2}{2\times\lambda^4a^2}(\frac{1}{q+1})^{1/q} \\
&\times\{C_\beta[B(1+\frac{s}{\beta},p+1)F(1+\frac{s}{\beta},4p,p+\frac{s}{\beta}+2,1-\frac{b}{\lambda a})\frac{|f^{''}(a)|^p}{e^{ra}} \\
&+\lambda^{1/\beta}\frac{\beta}{s+\beta(1+p)}F(1,4p,p+\frac{s}{\beta}+2,1-\frac{b}{\lambda a})\frac{|f^{''}(b)|^p}{e^{rb}}]
\end{aligned}
$$

$$-\frac{1}{p+2}F(1,4p,p+3,1-\frac{b}{\lambda a})h(b-a)\}^{1/p}. \tag{47}$$

Theorem 15 *Under the assumptions of Theorem 11, we have*

$$\begin{aligned}
&|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(u)du| \\
&\leq \frac{b^2(b-a)^2}{2a^2}(\frac{1}{q+1})^{1/q}\{B(s+1,p+1)F(s+1,4p,p+s+2,1 \\
&\quad -(\frac{b}{a}))(\frac{|f^{''}(a)|^p}{e^{ra}}) \\
&+\frac{1}{s+p+1}F(1,4p,p+s+2,1-(\frac{b}{a}))\frac{|f^{''}(b)|^p}{e^{rb}} \\
&-\frac{1}{p+2}F(1,4p,p+3,1-\frac{b}{a})h(b-a)\}^{1/p}.
\end{aligned} \tag{48}$$

Example 3 By using the Hölder inequality,

$$\begin{aligned}
&\int_0^1 t(1-t)(ta^{-1}+\lambda(1-t)b^{-1})^{-4}|f^{''}((ta^{-1}+\lambda(1-t)b^{-1})^{-1})|dt \\
&\leq \{\int_0^1(1-t)^q dt\}^{1/q}\times\{\int_0^1 t^p(ta^{-1}+\lambda(1-t)b^{-1})^{-4p} \\
&\times|f^{''}((ta^{-1}+\lambda(1-t)b^{-1})^{-1})|^p dt\}^{1/p},
\end{aligned} \tag{49}$$

and the exponentially harmonically (β,λ,s,h)-strongly convexity of $|f^{''}|^p$ on $[a,b]$, we get:

Theorem 16 *Under the assumptions of Theorem 9, by Corollary 1, we have*

$$\begin{aligned}
&|\frac{1}{2}[f(a)+f(\lambda^{-1}b)]-\frac{1}{\lambda^{-1}b-a}\int_a^{\lambda^{-1}b} f(u)du| \\
&\leq \frac{b^2(b-\lambda a)^2}{2\times\lambda^4 a^2}\{\frac{1}{q+1}\}^{1/q} \\
&\{C_\beta[\frac{\beta}{s+\beta(p+1)}F(p+1+\frac{s}{\beta},4p,p+\frac{s}{\beta}+2,1-\frac{b}{\lambda a})\frac{|f^{''}(a)|^p}{e^{ra}} \\
&+\lambda^{1/\beta}B(p+1,\frac{s}{\beta}+1)F(p+1,4p,p+\frac{s}{\beta}+2,1-\frac{b}{\lambda a})\frac{|f^{''}(b)|^p}{e^{rb}}] \\
&-\frac{1}{(p+2)(p+3)}F(p+2,4p,p+4,1-\frac{b}{\lambda a})h(b-a)\}^{1/p}.
\end{aligned} \tag{50}$$

Theorem 17 *Under the assumptions of Theorem 11, we have*

$$
\begin{aligned}
&|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(u)du| \\
&\leq \frac{b^2(b-a)^2}{2a^2}(\frac{1}{q+1})^{1/q} \\
&\times\{\frac{1}{s+p+1}F(p+s+1,4p,p+s+2,1-\frac{b}{a})\frac{|f''(a)|^p}{e^{ra}} \\
&+B(p+1,s+1)F(p+1,4p,p+s+2,1-\frac{b}{a})\frac{|f''(b)|^p}{e^{rb}} \\
&-\frac{1}{(p+2)(p+3)}F(p+2,4p,p+4,1-\frac{b}{a})h(b-a)\}^{1/p}.
\end{aligned}
\tag{51}
$$

Example 4 By using the Hölder inequality,

$$
\begin{aligned}
&\int_0^1 t(1-t)(ta^{-1}+\lambda(1-t)b^{-1})^{-4}|f''((ta^{-1}+\lambda(1-t)b^{-1})^{-1})|dt \\
&\leq \{\int_0^1 t^q(1-t)^q dt\}^{1/q} \\
&\times\{\int_0^1 (ta^{-1}+\lambda(1-t)b^{-1})^{-4p}|f''((ta^{-1}+\lambda(1-t)b^{-1})^{-1})|^p dt\}^{1/p},
\end{aligned}
\tag{52}
$$

and the exponentially harmonically (β,λ,s,h)-strongly convexity of $|f''|^p$ on $[a,b]$, we get:

Theorem 18 *Under the assumptions of Theorem 9, by Corollary 1, we have*

$$
\begin{aligned}
&|\frac{1}{2}[f(a)+f(\lambda^{-1}b)]-\frac{1}{\lambda^{-1}b-a}\int_a^{\lambda^{-1}b} f(u)du| \\
&\leq \frac{b^2(b-\lambda a)^2}{2\times\lambda^4 a^2}\{B(q+1,q+1)\}^{1/q} \\
&\times\{C_\beta\frac{\beta}{s+\beta}[F(\frac{s}{\beta}+1,4p,\frac{s}{\beta}+2,1-\frac{b}{\lambda a})\frac{|f''(a)|^p}{e^{ra}} \\
&+\lambda^{1/\beta}F(1,4p,\frac{s}{\beta}+2,1-\frac{b}{\lambda a})\frac{|f''(b)|^p}{e^{rb}}] \\
&-\frac{1}{6}F(2,4p,4,1-\frac{b}{\lambda a})h(b-a)\}^{1/p}.
\end{aligned}
\tag{53}
$$

Theorem 19 *Under the assumptions of Theorem 11, we have*

$$
\begin{aligned}
&|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(u)du| \\
&\leq \frac{b^2(b-a)^2}{2a^2}\{B(q+1,q+1)\}^{1/q} \\
&\times\{\frac{1}{s+1}[F(s+1,4p,s+2,1-\frac{b}{a})\frac{|f''(a)|^p}{e^{ra}} \\
&+F(1,4p,s+2,1-\frac{b}{a})\frac{|f''(b)|^p}{e^{rb}}] \\
&-\frac{1}{6}F(2,4p,4,1-\frac{b}{a})h(b-a)\}^{1/p}. \qquad (54)
\end{aligned}
$$

Example 5 By using the Hölder inequality,

$$
\begin{aligned}
&\int_0^1 t(1-t)(ta^{-1}+\lambda(1-t)b^{-1})^{-4}|f''((ta^{-1}+\lambda(1-t)b^{-1})^{-1})|dt \\
&\leq \{\int_0^1 t^q(1-t)^q(ta^{-1}+\lambda(1-t)b^{-1})^{-2q}dt\}^{1/q} \\
&\times\{\int_0^1 (ta^{-1}+\lambda(1-t)b^{-1})^{-2p}|f''((ta^{-1}+\lambda(1-t)b^{-1})^{-1})|^p dt\}^{1/p}, \qquad (55)
\end{aligned}
$$

and the exponentially harmonically (β,λ,s,h)-strongly convexity of $|f''|^p$ on $[a,b]$, we get:

Theorem 20 *Under the assumptions of Theorem 9, by Corollary 1, we have*

$$
\begin{aligned}
&|\frac{1}{2}[f(a)+f(\lambda^{-1}b)]-\frac{1}{\lambda^{-1}b-a}\int_a^{\lambda^{-1}b} f(u)du| \\
&\leq \frac{b^2(b-\lambda a)^2}{2\times\lambda^4a^2}\{B(q+1,q+1)F(q+1,2q,2q+2,1-\frac{b}{\lambda a})\}^{1/q} \\
&\times\{C_\beta\frac{\beta}{s+\beta}[F(\frac{s}{\beta}+1,2p,\frac{s}{\beta}+2,1-\frac{b}{\lambda a})\frac{|f''(a)|^p}{e^{ra}} \\
&+\lambda^{1/\beta}F(1,2p,\frac{s}{\beta}+2,1-\frac{b}{\lambda a})\frac{|f''(b)|^p}{e^{rb}}] \\
&-\frac{1}{6}F(2,2p,4,1-\frac{b}{\lambda a})h(b-a)\}^{1/p}. \qquad (56)
\end{aligned}
$$

Theorem 21 *Under the assumptions of Theorem 11, we have*

$$
\begin{aligned}
&|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(u)du| \\
&\quad\le \frac{b^2(b-a)^2}{2a^2}\{B(q+1,q+1)F(q+1,2q,2q+2,1-\frac{b}{a})\}^{1/q} \\
&\quad\times\{\frac{1}{s+1}[F(s+1,2p,s+2,1-\frac{b}{a})\frac{|f''(a)|^p}{e^{ra}}+F(1,2p,s+2,1-\frac{b}{a})\frac{|f''(b)|^p}{e^{rb}}] \\
&\quad-\frac{1}{6}F(2,2p,4,1-\frac{b}{a})h(b-a)\}^{1/p}. \qquad (57)
\end{aligned}
$$

Example 6 By using the Hölder inequality,

$$
\begin{aligned}
&\int_0^1 t(1-t)(ta^{-1}+\lambda(1-t)b^{-1})^{-4}|f''((ta^{-1}+\lambda(1-t)b^{-1})^{-1})|dt \\
&\quad\le\{\int_0^1 t^q(1-t)^q(ta^{-1}+\lambda(1-t)b^{-1})^{-3q}dt\}^{1/q} \\
&\quad\times\{\int_0^1(ta^{-1}+\lambda(1-t)b^{-1})^{-p}|f''((ta^{-1}+\lambda(1-t)b^{-1})^{-1})|^p dt\}^{1/p}, \qquad (58)
\end{aligned}
$$

and the exponentially harmonically (β,λ,s,h)-strongly convexity of $|f''|^p$ on $[a,b]$, we get:

Theorem 22 *Under the assumptions of Theorem 9, by Corollary 1, we have*

$$
\begin{aligned}
&|\frac{1}{2}[f(a)+f(\lambda^{-1}b)]-\frac{1}{\lambda^{-1}b-a}\int_a^{\lambda^{-1}b} f(u)du| \\
&\quad\le\frac{b^2(b-\lambda a)^2}{2\times\lambda^4a^2}\{B(q+1,q+1)F(q+1,3q,2q+2,1-\frac{b}{\lambda a})\}^{1/q} \\
&\quad\times\{C_\beta\frac{\beta}{\beta+s}[F(1+\frac{s}{\beta},p,\frac{s}{\beta}+2,1-\frac{b}{\lambda a})\frac{|f''(a)|^p}{e^{ra}} \\
&\quad+\lambda^{1/\beta}F(1,p,\frac{s}{\beta}+2,1-\frac{b}{\lambda a})\frac{|f''(b)|^p}{e^{rb}}] \\
&\quad-\frac{1}{6}F(2,p,4,1-\frac{b}{\lambda a})h(b-a)\}^{1/p}. \qquad (59)
\end{aligned}
$$

Theorem 23 *Under the assumptions of Theorem 11, we have*

$$
|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(u)du|
$$

$$\leq \frac{b^2(b-a)^2}{2a^2}\{B(q+1,q+1)F(q+1,3q,2q+2,1-\frac{b}{a})\}^{1/q}$$
$$\times\{\frac{1}{s+1}[F(s+1,p,s+2,1-\frac{b}{a})\frac{|f^{''}(a)|^p}{e^{ra}}+F(1,p,s+2,1-\frac{b}{a})\frac{|f^{''}(b)|^p}{e^{rb}}]$$
$$-\frac{1}{6}F(2,p,4,1-\frac{b}{a})h(b-a)\}^{1/p}. \tag{60}$$

Example 7 By using the Hölder inequality,

$$\int_0^1 t(1-t)[ta^{-1}+\lambda(1-t)b^{-1}]^{-4}|f^{''}((ta^{-1}+\lambda(1-t)b^{-1})^{-1})|dt$$
$$\leq \{\int_0^1 [ta^{-1}+\lambda(1-t)b^{-1}]^{-4q}dt\}^{1/q}$$
$$\times\{\int_0^1 t^p(1-t)^p|f^{''}((ta^{-1}+\lambda(1-t)b^{-1})^{-1})|^p dt\}^{1/p}, \tag{61}$$

and the exponentially harmonically (β, λ, s, h)-strongly convexity of $|f^{''}|^p$ on $[a, b]$, we get:

Theorem 24 *Under the assumptions of Theorem 9, if $q \neq 1/4$, then by Corollary 1, we have*

$$|\frac{1}{2}[f(a)+f(\lambda^{-1}b)]-\frac{1}{\lambda^{-1}b-a}\int_a^{\lambda^{-1}b} f(u)du|$$
$$\leq \frac{(b-\lambda a)^{1+(1/p)}}{2(ab)^{1+(1/p)}}\{\frac{(\lambda^{-1}b)^{4q-1}-a^{4q-1}}{4q-1}\}^{1/q}$$
$$\times\ \{C_\beta B(p+1,p+1+\frac{s}{\beta})[\frac{|f^{''}(a)|^p}{e^{ra}}+\lambda^{1/\beta}\frac{|f^{''}(b)|^p}{e^{rb}}]$$
$$-\ B(p+2,p+2)h(b-a)\}^{1/p}. \tag{62}$$

Theorem 25 *Under the assumptions of Theorem 11, if $q \neq 1/4$, then by Corollary 1, we have*

$$|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{1}{2}(\frac{1}{a}-\frac{1}{b})^{1+(\frac{1}{p})}\{\frac{b^{4q-1}-a^{4q-1}}{4q-1}\}^{1/q}$$
$$\times\{B(p+1,p+s+1)\}^{1/p}\{(\frac{|f^{''}(a)|^p}{e^{ra}}+\frac{|f^{''}(b)|^p}{e^{rb}}$$
$$-B(p+2,p+2)h(b-a)\}^{1/p}. \tag{63}$$

In particular, if $p = s = 1$, then by (63), we have

$$|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{1}{4}(\frac{1}{a}-\frac{1}{b})^2\{\frac{1}{6}[\frac{|f^{''}(a)|}{e^{ra}} + \frac{|f^{''}(b)|}{e^{rb}}] - \frac{1}{15}h(b-a)\}. \tag{64}$$

Example 8 By Lemma 2 and using the exponentially (β, s, h)-strongly convexity of $|f^{''}|^p$ on $[a, b]$, we have:

Theorem 26 *Let $f^{''} \in L[a, b]$, and $|f^{''}|$ is exponentially (β, s, h)-strongly convex on $[a, b]$, then*

$$|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{(b-a)^2}{2}\{\frac{\beta^2 C_\beta}{(s+2\beta)(s+3\beta)}(\frac{|f^{''}(a)|}{e^{ra}} + \frac{|f^{''}(b)|}{e^{rb}}) - \frac{1}{30}h(b-a)\}.$$

In particular, if $\beta = 1$, that is, $|f^{''}|$ is exponentially (s, h)-strongly convex on $[a, b]$, then

$$|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{(b-a)^2}{2}\{\frac{1}{(s+2)(s+3)}[\frac{|f^{''}(a)|}{e^{ra}} + \frac{|f^{''}(b)|}{e^{rb}}] - \frac{1}{30}h(b-a)\}. \tag{65}$$

If $s = 1$, in (65), that is, $|f^{''}|$ is exponentially h- strongly convex on $[a, b]$, then

$$|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{(b-a)^2}{12}\{\frac{1}{2}[\frac{|f^{''}(a)|}{e^{ra}} + \frac{|f^{''}(b)|}{e^{rb}}] - \frac{1}{5}h(b-a)\}. \tag{66}$$

Proof By Lemma 2 and using the exponentially (β, s, h)-strongly convexity of $|f^{''}|$ on $[a, b]$, we have

$$|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)|f^{''}(ta+(1-t)b)|dt$$

$$\leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)\{[t^s(\frac{|f''(a)|}{e^{ra}})^\beta+(1-t)^s(\frac{|f''(b)|}{e^{rb}})^\beta]^{1/\beta}-t(1-t)h(a-b)\}dt$$

$$\leq \frac{(b-a)^2}{2}\{C_\beta[(\int_0^1 t^{1+(s/\beta)}(1-t)dt)\frac{|f''(a)|}{e^{ra}}+(\int_0^1 t(1-t)^{1+(s/\beta)}dt)\frac{|f''(b)|}{e^{rb}}]$$

$$-(\int_0^1 t^2(1-t)^2dt)h(b-a)\}$$

$$= \frac{(b-a)^2}{2}\{\frac{\beta^2 C_\beta}{(s+2\beta)(s+3\beta)}(\frac{|f''(a)|}{e^{ra}}+\frac{|f''(b)|}{e^{rb}})-\frac{1}{30}h(b-a)\}.$$

The proof is completed.

Theorem 27 *Let $[a,b] \subset (0,\infty)$, $f'' \in L[a,b]$, and $|f''|^p$ is exponentially (β,s,h)-strongly convex on $[a,b]$, $1 \leq p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$, and for $p=1$, define $q=\infty$, $\frac{1}{\infty}=0$, then*

$$|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(u)du|$$

$$\leq \frac{(b-a)^2}{2(q+1)^{1/q}}\{C_\beta[B(p+1,\frac{s}{\beta}+1)\frac{|f''(a)|^p}{e^{ra}}+\frac{\beta}{s+\beta(p+1)}\frac{|f''(b)|^p}{e^{rb}}]$$

$$-\frac{1}{(p+2)(p+3)}h(b-a)\}^{1/p}.$$

In particular, if $\beta=1$, that is, $|f''|^p$ is exponentially (s,h)-strongly convex on $[a,b]$, then

$$|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(u)du|$$

$$\leq \frac{(b-a)^2}{2(q+1)^{1/q}}\{B(p+1,s+1)\frac{|f''(a)|^p}{e^{ra}}+\frac{1}{s+p+1}$$

$$\times\frac{|f''(b)|^p}{e^{rb}}-\frac{1}{(p+2)(p+3)}h(b-a)\}^{1/p}. \tag{67}$$

If $s=1$ in (67), that is, $|f''|^p$ is exponentially h-strongly convex on $[a,b]$, then

$$|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(u)du|$$

$$\leq \frac{(b-a)^2}{2(p+2)(q+1)^{1/q}}\{\frac{1}{p+1}\frac{|f''(a)|^p}{e^{ra}}+\frac{|f''(b)|^p}{e^{rb}}-\frac{1}{p+3}h(b-a)\}^{1/p}. \tag{68}$$

If $p = 1$ *in* (68)*, then*

$$|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(u)du| \\ \leq \frac{(b-a)^2}{24}\{2 \times \frac{|f''(a)|}{e^{ra}} + 4 \times \frac{|f''(b)|}{e^{rb}} - h(b-a)\}. \tag{69}$$

Proof By Lemma 2 and using the exponentially (β, s, h)-strongly convexity of $|f''|^p$ on $[a, b]$, we have

$$|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(u)du| \\ \leq \frac{(b-a)^2}{2}\int_0^1 t(1-t)|f''(ta+(1-t)b)|dt \\ \leq \frac{(b-a)^2}{2}(\int_0^1 t^q dt)^{1/q}\{\int_0^1 (1-t)^p|f''(ta+(1-t)b)|^p dt\}^{1/p} \\ \leq \frac{(b-a)^2}{2(q+1)^{1/q}}\{\int_0^1 (1-t)^p[(t^s(\frac{|f''(a)|^p}{e^{ra}})^\beta + (1-t)^s(\frac{|f''(b)|^p}{e^{rb}})^\beta)^{1/\beta} \\ -t(1-t)h(b-a)]dt\}^{1/p} \\ \leq \frac{(b-a)^2}{2(q+1)^{1/q}}\{C_\beta[(\int_0^1 t^{s/\beta}(1-t)^p dt)\frac{|f''(a)|^p}{e^{ra}} \\ +(\int_0^1 (1-t)^{p+(s/\beta)}dt)\frac{|f''(b)|^p}{e^{rb}}] - (\int_0^1 t(1-t)^{p+1}dt)h(b-a)\}^{1/p} \\ \leq \frac{(b-a)^2}{2(q+1)^{1/q}}\{C_\beta[B(p+1, \frac{s}{\beta}+1)\frac{|f''(a)|^p}{e^{ra}} + \frac{\beta}{s+\beta(p+1)}\frac{|f''(b)|^p}{e^{rb}}] \\ -\frac{1}{(p+2)(p+3)}h(b-a)\}^{1/p}.$$

The proof is completed.

Theorem 28 *Under the assumptions of Theorem* 27*, we have*

$$|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(u)du| \\ \leq \frac{(b-a)^2}{2}\{B(q+1, q+1)\}^{1/q}\{\frac{\beta C_\beta}{s+\beta}(\frac{|f''(a)|^p}{e^{ra}} + \frac{|f''(b)|^p}{e^{rb}}) \\ -\frac{1}{6}h(b-a)\}^{1/p}.$$

In particular, if $\beta = 1$*, that is,* $|f^{''}|^p$ *is exponentially* (s, h)*-strongly convex on* $[a, b]$*, then*

$$|\frac{1}{2}[f(a) + f(b)] - \frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{(b-a)^2}{2}\{B(q+1, q+1)\}^{1/q}\{\frac{1}{s+1}(\frac{|f^{''}(a)|^p}{e^{ra}} + \frac{|f^{''}(b)|^p}{e^{rb}})$$
$$-\frac{1}{6}h(b-a)\}^{1/p}. \tag{70}$$

If $s = 1$ *in* (70)*, that is,* $|f^{''}|^p$ *is exponentially* h*-strongly convex on* $[a, b]$*, then*

$$|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{(b-a)^2}{4}\{B(q+1, q+1)\}^{1/q}\{\frac{|f^{''}(a)|^p}{e^{ra}}+\frac{|f^{''}(b)|^p}{e^{rb}}-\frac{1}{3}h(b-a)\}^{1/p}. \tag{71}$$

If $p = 1$ *in* (70)*, then*

$$|\frac{1}{2}[f(a) + f(b)] - \frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{(b-a)^2}{2}\{\frac{1}{s+1}[\frac{|f^{''}(a)|}{e^{ra}} + \frac{|f^{''}(b)|}{e^{rb}}] - \frac{1}{6}h(b-a)\}. \tag{72}$$

Proof By Lemma 2 and using the exponentially (β, s, h)-strongly convexity of $|f^{''}|^p$ on $[a, b]$, we have

$$|\frac{1}{2}[f(a) + f(b)] - \frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{(b-a)^2}{2}(\int_0^1 t^q(1-t)^q dt)^{1/q}\{\int_0^1 |f^{''}(ta + (1-t)b)|^p dt\}^{1/p}$$
$$\leq \frac{(b-a)^2}{2}\{B(q+1, q+1)\}^{1/q}\{\int_0^1 [(t^s(\frac{|f^{''}(a)|^p}{e^{ra}})^\beta + (1-t)^s(\frac{|f^{''}(b)|^p}{e^{rb}})^\beta)^{1/\beta}$$
$$-t(1-t)h(b-a)]dt\}^{1/p}$$
$$\leq \frac{(b-a)^2}{2}\{B(q+1, q+1)\}^{1/q}\{C_\beta[(\int_0^1 t^{s/\beta}dt)\frac{|f^{''}(a)|^p}{e^{ra}}$$
$$+(\int_0^1 (1-t)^{s/\beta}dt)\frac{|f^{''}(b)|^p}{e^{rb}}] - (\int_0^1 t(1-t)dt)h(b-a)\}^{1/p}$$

$$\leq \frac{(b-a)^2}{2}\{B(q+1,q+1)\}^{1/q}\{\frac{\beta C_\beta}{s+\beta}(\frac{|f^{''}(a)|^p}{e^{ra}}+\frac{|f^{''}(b)|^p}{e^{rb}})$$
$$-\frac{1}{6}h(b-a)\}^{1/p}.$$

The proof is completed.

Example 9 By Lemma 3 and using the exponentially (α, β, s, h)-strongly convexity of $|f^{''}|^p$ on $[a, b]$, we can prove analogously

Theorem 29 *Let $[a,b] \subset (0,\infty)$, $f^{'} \in L[a,b]$, and $|f^{'}|^p$ is exponentially (β, s, h)-strongly convex on $[a,b]$, $1 \leq p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$, and for $p=1$, define $q=\infty$, $\frac{1}{\infty}=0$.*

(i) If $\alpha \neq 0, 1$, then

$$|\frac{1}{2}[f(a)+f(b)]-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b \frac{f(u)}{u^{1-\alpha}}du|$$
$$\leq \frac{b^{1-\alpha}|b^\alpha-a^\alpha|}{2|\alpha|(q+1)^{1/q}}\{\frac{\beta C_\beta}{\beta+s}[F(1+\frac{s}{\beta},(1-\frac{1}{\alpha})p,2+\frac{s}{\beta},1-(\frac{a}{b})^\alpha)$$
$$\times\frac{|f^{'}(a)|^p}{e^{ra}}+F(1,(1-\frac{1}{\alpha})p,2+\frac{s}{\beta},1-(\frac{a}{b})^\alpha)\frac{|f^{'}(b)|^p}{e^{rb}}]$$
$$-\frac{1}{6}F(2,(1-\frac{1}{\alpha})p,4,1-(\frac{a}{b})^\alpha)h(b-a)\}^{1/p}.$$

(ii) If $\alpha = 1$, then

$$|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{(b-a)}{2(q+1)^{1/q}}\{(\frac{\beta C_\beta}{(s+\beta)})(\frac{|f^{'}(a)|^p}{e^{ra}}+\frac{|f^{'}(b)|^p}{e^{rb}})-\frac{1}{6}h(b-a)\}^{1/p}. \quad (73)$$

In particular, if $\beta = 1$, that is, $|f^{'}|^p$ is exponentially (s, h)-strongly convex on $[a, b]$, then

$$|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(u)du|$$
$$\leq \frac{(b-a)}{2(q+1)^{1/q}}\{\frac{1}{s+1}(\frac{|f^{'}(a)|^p}{e^{ra}}+\frac{|f^{'}(b)|^p}{e^{rb}})-\frac{1}{6}h(b-a)\}^{1/p}. \quad (74)$$

If $p = 1$ in (74), then

$$
\begin{aligned}
&|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(u)du| \\
&\leq \frac{(b-a)}{2}\{\frac{1}{s+1}[\frac{|f^{'}(a)|}{e^{ra}} + \frac{|f^{'}(b)|}{e^{rb}}] - \frac{1}{6}h(b-a)\}.
\end{aligned}
\tag{75}
$$

Remark 3 When $h = 0$, (74) reduces to Theorem 3.7 in [8].

Theorem 30 *Under the assumptions of Theorem 29, we have:*

(i) If $p = 1$, then

$$
\begin{aligned}
&|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(u)du| \\
&\leq \frac{b-a}{2}\{\frac{\beta C_\beta}{(s+\beta)(s+2\beta)}(s+\frac{\beta}{2^{s/\beta}}) \\
&\times[\frac{|f^{'}(a)|}{e^{ra}} + \frac{|f^{'}(b)|}{e^{rb}}] - \frac{1}{16}h(b-a)\}.
\end{aligned}
\tag{76}
$$

In particular, if $\beta = 1$, that is, $|f^{'}|$ is exponentially (s, h)-strongly convex on $[a, b]$, then

$$
\begin{aligned}
&|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(u)du| \\
&\leq \frac{b-a}{2}\{\frac{1}{(s+1)(s+2)}(s+\frac{1}{2^{s}}) \\
&\times[\frac{|f^{'}(a)|}{e^{ra}} + \frac{|f^{'}(b)|}{e^{rb}}] - \frac{1}{16}h(b-a)\}.
\end{aligned}
\tag{77}
$$

(ii) If $1 < p < \infty$, then

$$
\begin{aligned}
&|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(u)du| \\
&\leq \frac{(b-a)}{2^{(1/q)+1}}\{\frac{\beta C_\beta}{(s+\beta)(s+2\beta)}(s+\frac{\beta}{2^{s/\beta}}) \\
&\times[\frac{|f^{'}(a)|^p}{e^{ra}} + \frac{|f^{'}(b)|^p}{e^{rb}}] - \frac{1}{16}h(b-a)\}^{1/p}.
\end{aligned}
\tag{78}
$$

In particular, if $\beta = 1$, that is, $|f^{'}|^p$ is exponentially (s, h)-strongly convex on $[a, b]$, then

$$
\begin{aligned}
&|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(u)du| \\
&\leq \frac{(b-a)}{2^{(1/q)+1}}\{\frac{1}{(s+1)(s+2)}(s+\frac{1}{2^s}) \\
&\times[\frac{|f^{'}(a)|^p}{e^{ra}}+\frac{|f^{'}(b)|^p}{e^{rb}}]-\frac{1}{16}h(b-a)\}^{1/p}. \quad (79)
\end{aligned}
$$

Remark 4 Equations (76) and (78) improve and generalize the corresponding results of Theorems 3.3 and 3.6 in [8].

4 Approximations for the Integral of *f*

Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$. By applying the trapezoidal rule:

$$
\int_a^b f(x)dx = \frac{f(a)+f(b)}{2}(b-a) - \frac{(b-a)^3}{12}f^{"}(\zeta), \ \ \zeta \in [a,b],
$$

one obtain

$$
\int_a^b f(x)dx = S_n(f) + R_n(f), \quad (80)
$$

where

$$
S_n(f) = \sum_{k=1}^{n} \frac{f(x_{k-1})+f(x_k)}{2}(x_k - x_{k-1}) \quad (81)
$$

and

$$
R_n(f) = -\sum_{k=1}^{n} \frac{(x_k - x_{k-1})^3}{12} f^{"}(\zeta_k), \ \ \zeta_k \in [x_{k-1}, x_k]. \quad (82)
$$

The remainder term $R_n(f)$ represents the error in approximating $\int_a^b f(x)dx$ by $S_n(f)$. If $f^{"} \in L^{\infty}[a, b]$, then the remainder term is given by

$$
\left|\int_a^b f(x)dx - S_n(f)\right| = |R_n(f)| \leq \frac{\|f^{"}\|_{\infty}}{12}\sum_{k=1}^{n}(x_k - x_{k-1})^3. \quad (83)
$$

Taking $x_k - x_{k-1} = \frac{b-a}{n}$ in (83), we get a classical result in numerical analysis (see [33] P. 885.):

$$\left|\int_a^b f(x)dx - S_n(f)\right| = |R_n(f)| \le \frac{\|f^{''}\|_\infty}{12n^2}(b-a)^3.$$

Using some results in the above section, under different conditions, we can get new approximations for the integral of f. In what follows, let

$$M_{j,p} = \max\{\frac{|f^j(x_k)|}{e^{(rx_k)/p}} : 1 \le k \le n\},\ \ j = 1, 2. \tag{84}$$

Theorem 31 *Let* $[a, b] \subset (0, \infty)$*,* $f^{''} \in L[a, b]$*, and* $1 \le p < \infty$*,* $\frac{1}{p} + \frac{1}{q} = 1$*, and for* $p = 1$*, define* $q = \infty$*,* $\frac{1}{\infty} = 0$*. If* $|f^{''}|^p$ *is exponentially s-convex on* $[a, b]$*, then in* (80)*, for every partition* P *of* $[a, b]$*, we get:*

(i) If $1 < p < \infty$*, then*

$$|R_n(f)| \le \frac{M_{2,p}}{2(q+1)^{1/q}}\{B(p+1, s+1) + \frac{1}{s+p+1}\}^{1/p} \sum_{k=1}^{n}(x_k - x_{k-1})^3. \tag{85}$$

If $x_k - x_{k-1} = \frac{b-a}{n}$*, then* (85) *reduces to*

$$|R_n(f)| \le \frac{M_{2,p}(b-a)^3}{2n^2(q+1)^{1/q}}\{B(p+1, s+1) + \frac{1}{s+p+1}\}^{1/p}.$$

(ii) If $p = 1$*, then*

$$|R_n(f)| \le \frac{M_{2,1}}{(s+2)(s+3)} \sum_{k=1}^{n}(x_k - x_{k-1})^3. \tag{86}$$

If $x_k - x_{k-1} = \frac{b-a}{n}$*, then* (86) *reduces to*

$$|R_n(f)| \le \frac{M_{2,1}(b-a)^3}{n^2(s+2)(s+3)}.$$

Proof By letting $h = 0$ in (67), we get

$$\begin{aligned}
&\left|\frac{f(x_{k-1}) + f(x_k)}{2}(x_k - x_{k-1}) - \int_{x_{k-1}}^{x_k} f(u)du\right| \\
&\quad \le \frac{(x_k - x_{k-1})^3}{2(q+1)^{1/q}}\{B(p+1, s+1)\frac{|f^{''}(x_{k-1})|^p}{e^{rx_{k-1}}} + \frac{1}{s+p+1}\frac{|f^{''}(x_k)|^p}{e^{rx_k}}\}^{1/p} \\
&\quad \le \frac{M_{2,p}(x_k - x_{k-1})^3}{2(q+1)^{1/q}}\{B(p+1, s+1) + \frac{1}{s+p+1}\}^{1/p}.
\end{aligned}$$

Summing over k from 1 to n, we get

$$\left|\int_a^b f(u)du-S_n(f)\right| \leq \frac{M_{2,p}}{2(q+1)^{1/q}}\{B(p+1,s+1)+\frac{1}{s+p+1}\}^{1/p}\sum_{k=1}^{n}(x_k-x_{k-1})^3.$$

By letting $h=0$ in (65) and similar arguments, we get (86). The proof is completed.

By letting $h=0$ in (70), (72), and similar arguments, we get

Theorem 32 *Under the assumptions of Theorem 31, we have*

(i) If $1<p<\infty$, *then*

$$|R_n(f)| \leq \frac{M_{2,p}}{(s+1)^{1/p}}\{B(q+1,q+1)\}^{1/q}\sum_{k=1}^{n}(x_k-x_{k-1})^3. \tag{87}$$

If $x_k-x_{k-1}=\frac{b-a}{n}$, *then* (87) *reduces to*

$$|R_n(f)| \leq \frac{M_{2,p}(b-a)^3}{n^2(s+1)^{1/p}}\{B(q+1,q+1)\}^{1/q}.$$

(ii) If $p=1$, *then*

$$|R_n(f)| \leq \frac{M_{2,1}}{s+1}\sum_{k=1}^{n}(x_k-x_{k-1})^3. \tag{88}$$

If $x_k-x_{k-1}=\frac{b-a}{n}$, *then* (88) *reduces to*

$$|R_n(f)| \leq \frac{M_{2,1}(b-a)^3}{n^2(s+1)}.$$

By letting $h=0$ *in* (74), (75), *and similar arguments, we get:*

Theorem 33 *Let* $[a,b]\subset(0,\infty)$, $f^{'}\in L[a,b]$, *and* $1\leq p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, *and for* $p=1$, *define* $q=\infty$, $\frac{1}{\infty}=0$. *If* $|f^{'}|^p$ *is exponentially* s*-convex on* $[a,b]$, *then in* (80), *for every partition* P *of* $[a,b]$, *we get:*

(i) If $1<p<\infty$, *then*

$$|R_n(f)| \leq \frac{M_{1,p}}{(q+1)^{1/q}(s+1)^{1/p}}\sum_{k=1}^{n}(x_k-x_{k-1})^2. \tag{89}$$

If $x_k-x_{k-1}=\frac{b-a}{n}$, *then* (89) *reduces to*

$$|R_n(f)| \leq \frac{M_{1,p}(b-a)^2}{n^2(q+1)^{1/q}(s+1)^{1/p}}.$$

(ii) If $p = 1$, then

$$|R_n(f)| \leq \frac{M_{1,1}}{s+1} \sum_{k=1}^{n} (x_k - x_{k-1})^2. \tag{90}$$

If $x_k - x_{k-1} = \frac{b-a}{n}$, then (90) reduces to

$$|R_n(f)| \leq \frac{M_{1,1}(b-a)^2}{n^2(s+1)}.$$

By letting $h = 0$ in (79), (77), and similar arguments, we get

Theorem 34 *Under the assumptions of Theorem 33, we have*

(i) If $1 < p < \infty$, then

$$|R_n(f)| \leq \frac{M_{1,p}}{2^{(2/q)}} \{\frac{1}{(s+1)(s+2)}(s + \frac{1}{2^s})\}^{1/p} \sum_{k=1}^{n} (x_k - x_{k-1})^2. \tag{91}$$

If $x_k - x_{k-1} = \frac{b-a}{n}$, then (91) reduces to

$$|R_n(f)| \leq \frac{M_{1,p}(b-a)^2}{2^{(2/q)}n^2} \{\frac{1}{(s+1)(s+2)}(s + \frac{1}{2^s})\}^{1/p}.$$

(ii) If $p = 1$, then

$$|R_n(f)| \leq \frac{M_{1,1}}{(s+1)(s+2)}(s + \frac{1}{2^s}) \sum_{k=1}^{n} (x_k - x_{k-1})^2. \tag{92}$$

If $x_k - x_{k-1} = \frac{b-a}{n}$, then (92) reduces to

$$|R_n(f)| \leq \frac{M_{1,1}(b-a)^2}{n^2(s+1)(s+2)}(s + \frac{1}{2^s}).$$

5 Approximations for Some New Means

We consider the means for $0 < a < b$ as follows:

(1) Defining the new weighted mean

$$K_{p,q}(a,b) = \{\frac{a^p + \omega(ab)^{p/2} + b^p}{(\omega+1)a^q + b^q}\}^{\frac{1}{p-q}}, \tag{93}$$

where the weight $\omega \geq 0$, $p \neq q$, and p and q cannot both be zero. If we take $q = 0$, then (93) reduces to the Heron mean:

$$H_p(a, b) = \{\frac{a^p + \omega(ab)^{p/2} + b^p}{\omega + 2}\}^{1/p}. \quad (94)$$

Taking $\omega = 0$ in (94), we get the power mean :

$$M_p(a, b) = (\frac{a^p + b^p}{2})^{1/p}.$$

Taking $\omega = 0$, $q = p - 1$ in (93), we get the Lehmer mean:

$$L_p(a, b) = \frac{a^p + b^p}{a^{p-1} + b^{p-1}}. \quad (95)$$

In particular,

$$L_0(a, b) = \frac{2ab}{a + b}$$

is the harmonic mean;

$$L_1(a, b) = A(a, b) = \frac{a + b}{2}$$

is the arithmetic mean;

$$L_2(a, b) = \frac{a^2 + b^2}{a + b}$$

is the inverse harmonic mean of the first kind;

$$L_3(a, b) = \frac{a^3 + b^3}{a^2 + b^2}$$

is the inverse harmonic mean of the second kind.

(2) The logarithmic mean

$$L(a, b) = \frac{b - a}{\log b - \log a}.$$

(3) The Stolarsky mean

$$S_p(a, b) = \{\frac{b^p - a^p}{p(b - a)}\}^{1/(p-1)},\ p \neq 0, 1.$$

(4) The identric mean

$$\lim_{p\to 1} S_p(a,b) = I(a,b) = \frac{1}{e}\Big(\frac{b^b}{a^a}\Big)^{\frac{1}{b-a}}.$$

Theorem 35 *Let* $1 \le p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$, *and*

$$\Delta_s = |A(a^{s+2}, b^{s+2}) - S_{s+3}^{s+2}(a,b)|,$$

then

$$\Delta_s \le \frac{(s+1)(s+2)(b-a)^2}{12}\{\frac{1}{2}(\frac{a^s}{e^{ra}}+\frac{b^s}{e^{rb}}) - \frac{1}{5}h(b-a)\}; \tag{96}$$

$$\Delta_s \le \frac{(s+1)(s+2)(b-a)^2}{24}\{\frac{2a^s}{e^{ra}}+\frac{4b^s}{e^{rb}} - h(b-a)\}; \tag{97}$$

$$\Delta_s \le \frac{(s+1)(s+2)(b-a)^2}{2}\{\frac{1}{s+1}(\frac{a^s}{e^{ra}}+\frac{b^s}{e^{rb}}) - \frac{1}{6}h(b-a)\}; \tag{98}$$

$$\begin{aligned}\Delta_s \le &\frac{(s+1)(s+2)(b-a)^2(ab)^{s+2}}{2}\\ &\times(B(q+1,q+1))^{1/q}\times S_{1-(s+4)p}^{-(s+4)}(a,b).\end{aligned} \tag{99}$$

In particular, if $s=1$, *that is,*

$$\Delta_1 = |A(a^3,b^3) - S_4^3(a,b)|.$$

then

$$\Delta_1 \le \frac{(b-a)^2}{2}\{\frac{1}{2}(\frac{a}{e^{ra}}+\frac{b}{e^{rb}}) - \frac{1}{5}h(b-a)\};$$

$$\Delta_1 \le \frac{(b-a)^2}{4}\{\frac{2a}{e^{ra}}+\frac{4b}{e^{rb}} - h(b-a)\};$$

$$\Delta_1 \le 3(b-a)^2\{\frac{a}{e^{ra}}+\frac{b}{e^{rb}} - \frac{1}{6}h(b-a)\};$$

$$\Delta_1 \le 3a^3b^3(b-a)^2\{B(q+1,q+1)\}^{1/q}S_{1-5p}^{-5}(a,b).$$

Proof Let $f(x) = \frac{x^{s+2}}{(s+1)(s+2)}$, $s \ne -1,-2$, then $f''(x) = x^s$. Using (66), (69), and (72), we get (96), (97), and (98), respectively. To prove (99) holds, we apply the integral representation of S_p:

$$S_{p+1}(a,b) = \{\int_0^1 [tb + (1-t)a]^p dt\}^{1/p},$$

and Corollary 2, we get

$$\begin{aligned}
&\frac{1}{(s+1)(s+2)}|A(a^{s+2}, b^{s+2}) - S_{s+3}^{s+2}(a,b)| \\
&\quad = \frac{(b-a)^2(ab)^{2+s}}{2}\int_0^1 t(1-t)[tb+(1-t)a]^{-(4+s)}dt \\
&\quad \le \frac{(b-a)^2(ab)^{2+s}}{2}\{\int_0^1 t^q(1-t)^q dt\}^{1/q}\{\int_0^1 [tb+(1-t)a]^{-(4+s)p}dt\}^{1/p} \\
&\quad = \frac{(b-a)^2(ab)^{2+s}}{2}\{B(q+1,q+1)\}^{1/q} S_{1-(4+s)p}^{-(4+s)}(a,b),
\end{aligned}$$

which implies that (99) holds. The proof is completed.

Theorem 36 *Let* $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *and*

$$\Delta_2(p) = |A(a^{(1/p)+2}, b^{(1/p)+2}) - S_{(1/p)+3}^{(1/p)+2}(a,b)|.$$

(i) If $1 < p < \infty$, *then*

$$\begin{aligned}
\Delta_2(p) \le &\frac{(p+1)(2p+1)(b-a)^2}{2p^2(p+2)(q+1)^{1/q}} \\
&\{\frac{1}{p+1}\frac{a}{e^{ra}} + \frac{b}{e^{rb}} - \frac{1}{p+3}h(b-a)\}^{1/p};
\end{aligned} \tag{100}$$

$$\begin{aligned}
\Delta_2(p) \le &\frac{(p+1)(2p+1)(b-a)^2}{4p^2}\{B(q+1,q+1)\}^{1/q} \\
&\{\frac{a}{e^{ra}} + \frac{b}{e^{rb}} - \frac{1}{3}h(b-a)\}^{1/p};
\end{aligned} \tag{101}$$

$$\Delta_2(p) \le \frac{(p+1)(2p+1)(b-a)}{2 \times p^2(q+1)^{1/q}}\{\frac{1}{s+1}(\frac{a}{e^{ra}} + \frac{b}{e^{rb}}) - \frac{1}{6}h(b-a)\}^{1/p}. \tag{102}$$

(ii) If $p = 1$, *then*

$$\Delta_2(1) \leq \frac{(b-a)^2}{4}(\frac{2a}{e^{ra}} + \frac{4b}{e^{rb}} - h(b-a)\};$$

$$\Delta_2(1) \leq \frac{3(b-a)^2}{2}\{\frac{a}{e^{ra}} + \frac{b}{e^{rb}} - \frac{1}{3}h(b-a)\};$$

$$\Delta_2(1) \leq 3(b-a)\{\frac{1}{s+1}(\frac{a}{e^{ra}} + \frac{b}{e^{rb}}) - \frac{1}{6}h(b-a)\}.$$

Proof Let $f(x) = \frac{p^2}{(p+1)(2p+1)}x^{(1/p)+2}$, $x > 0$, then $f^{'}(x) = \frac{p}{p+1}x^{(1/p)+1}$, $|f^{''}(x)|^p = x$. Using (68), (71), and (74), respectively, we get the required results.

Theorem 37 *Let*

$$\Delta_3(p) = |A(e^{(p+1)a}, e^{(p+1)b}) - L(e^a, e^b)S^p_{p+1}(e^a, e^b)|, \;\; p \neq -1, 0.$$

Then

$$\Delta_3(p) \leq \frac{(b-a)^2}{24}\{(p+1)^2(2e^{(p+1-r)a} + 4e^{(p+1-r)b}) - h(b-a)\}; \tag{103}$$

$$\Delta_3(p) \leq \frac{(b-a)^2}{2}\{\frac{(p+1)^2}{s+1}(e^{(p+1-r)a} + e^{(p+1-r)b}) - \frac{1}{6}h(b-a)\}; \tag{104}$$

$$\Delta_3(p) \leq \frac{(b-a)}{2}\{\frac{p+1}{s+1}[e^{(p+1-r)a} + e^{(p+1-r)b}] - \frac{1}{6}h(b-a)\}. \tag{105}$$

Proof Let $f(x) = e^{(p+1)x}$, then $|f^{''}(x)| = (p+1)^2 e^{(p+1)x}$. Using (69), (72), and (75), respectively, we get the required results.

Theorem 38 *Let* $1 < p < \infty$, *and*

$$\Delta_4 = |\log G(a,b) - \log I(a,b)|.$$

Then

$$\Delta_4 \leq \frac{(b-a)^2}{12}\{\frac{1}{2}(a^{-2p}e^{-ra} + b^{-2p}e^{-rb}) - \frac{1}{5}h(b-a)\};$$

$$\Delta_4 \leq \frac{(b-a)^2}{24}\{2a^{-2}e^{-ra} + 4b^{-2}e^{-rb} - h(b-a)\};$$

$$\Delta_4 \leq \frac{(b-a)^2}{2}\{\frac{1}{s+1}[a^{-2}e^{-ra} + b^{-2}e^{-rb}] - \frac{1}{6}h(b-a)\};$$

$$\Delta_4 \leq \frac{(b-a)}{2}\{\frac{1}{s+1}[a^{-1}e^{-ra} + b^{-1}e^{-rb}] - \frac{1}{6}h(b-a)\}.$$

Proof Let $f(x) = \log x,\ x > 0$, then $f^{'}(x) = x^{-1},\ f^{''}(x) = -x^{-2}$. Using (66), (69), (72), and (75), respectively, we get the required results.

Theorem 39 *Let* $1 < p < \infty$*, and*

$$\Delta_5 = |A(a^p, b^p) - S^p_{p+1}(a, b) + \omega[A(a^{p/2}, b^{p/2}) - S^{(p/2)}_{(p/2)+1}(a, b)]|.$$

Then

$$\Delta_5 \le \frac{(b-a)^2}{24}\{\frac{2p}{e^{ra}}[(p-1)a^{p-2} + \frac{\omega}{2}(\frac{p}{2} - 1)a^{(p/2)-2}] + \frac{4p}{e^{rb}}[(p-1)b^{p-2} + \frac{\omega}{2}(\frac{p}{2} - 1)b^{(p/2)-2}] - (\omega + 2)h(b-a)\};$$

$$\Delta_5 \le \frac{(b-a)^2}{2}\{\frac{1}{s+1}[\frac{p}{e^{ra}}((p-1)a^{p-2} + \frac{\omega}{2}(\frac{p}{2} - 1)a^{(p/2)-2}) + \frac{p}{e^{rb}}((p-1)b^{p-2} + \frac{\omega}{2}(\frac{p}{2} - 1)b^{(p/2)-2})] - \frac{1}{6}(\omega + 2)h(b-a)\};$$

$$\Delta_5 \le \frac{(b-a)}{2}\{\frac{1}{s+1}[\frac{p}{e^{ra}}(a^{p-1} + \frac{\omega}{2}a^{(p/2)-1}) + \frac{p}{e^{rb}}(b^{p-1} + \frac{\omega}{2}b^{(p/2)-1})] - \frac{1}{6}(\omega + 2)h(b-a)\}.$$

Proof Let $f(x) = \frac{x^p + \omega x^{p/2} + 1}{\omega + 2},\ x > 0$, then $f^{'}(x) = \frac{p}{\omega+2}(x^{p-1} + \frac{\omega}{2}x^{(p/2)-1})$, $f^{''}(x) = \frac{p}{\omega+2}[(p-1)x^{p-2} + \frac{\omega}{2}(\frac{p}{2} - 1)x^{(p/2)-2}]$. Using (69), (72), and (75), respectively, we get the required results.

Taking $\omega = 0$ in Theorem 39, we get the following results similar to Theorem 35:

Theorem 40 *Let* $1 < p < \infty$*, and*

$$\Delta_6 = |A(a^p, b^p) - S^p_{p+1}(a, b)|,$$

then $\Delta_6 \le \frac{(b-a)^2}{12}\{p(p-1)[\frac{a^{p-2}}{e^{ra}} + \frac{2b^{p-2}}{e^{rb}}] - h(b-a)\}$; $\Delta_6 \le \frac{(b-a)^2}{2}\{\frac{p(p-1)}{s+1}[\frac{a^{p-2}}{e^{ra}} + \frac{b^{p-2}}{e^{rb}}] - \frac{1}{3}h(b-a)\}$; $\Delta_6 \le \frac{(b-a)}{2}\{\frac{p}{s+1}[\frac{a^{p-1}}{e^{ra}} + \frac{b^{p-1}}{e^{rb}}] - \frac{1}{3}h(b-a)\}$.

Remark 5 Kuang [2] introduced the following double weight mean:

$$K(\omega_1, \omega_2, p) = \left(\frac{\omega_1(a^p + b^p) + 2\omega_2(ab)^{p/2}}{2(\omega_1 + \omega_2)}\right)^{1/p}, \tag{106}$$

where $p \neq 0, \omega_1, \omega_2 \geq 0, \omega_1 + \omega_2 > 0$.

It is easy to note that $K(1, \frac{\omega}{2}, p) = H_p(a, b)$. Hence, by replacing $H_p(a, b)$ with $K(\omega_1, \omega_2, p)$, we can get some results similar to Theorem 39.

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Additive-Quadratic ρ-Functional Equations in β-Homogeneous Normed Spaces

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Abstract Let $M_1 f(x, y) := \frac{3}{4} f(x+y) - \frac{1}{4} f(-x-y) + \frac{1}{4} f(x-y) + \frac{1}{4} f(y-x) - f(x) - f(y)$ and $M_2 f(x, y) := 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y)$. We solve the additive-quadratic ρ-functional inequalities

$$\|M_1 f(x, y)\| \leq \|\rho M_2 f(x, y)\|, \tag{1}$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{2}$, and

$$\|M_2 f(x, y)\| \leq \|\rho M_1 f(x, y)\|, \tag{2}$$

where ρ is a fixed complex number with $|\rho| < 1$. Using the direct method, we prove the Hyers–Ulam stability of the additive-quadratic ρ-functional inequalities (1) and (2) in β-homogeneous complex Banach spaces.

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1 Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [13] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [2] for additive mappings and by Rassias [10] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias' theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [12] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 4, 5, 8, 9, 14, 15]).

Definition 1 Let X be a linear space. A nonnegative-valued function $\|\cdot\|$ is an F-norm if it satisfies the following conditions:

(FN_1) $\|x\| = 0$ if and only if $x = 0$;
(FN_2) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all λ with $|\lambda| = 1$;
(FN_3) $\|x+y\| \le \|x\| + \|y\|$ for all $x, y \in X$;
(FN_4) $\|\lambda_n x\| \to 0$ provided $\lambda_n \to 0$;
(FN_5) $\|\lambda x_n\| \to 0$ provided $x_n \to 0$.

Then, $(X, \|\cdot\|)$ is called an F^*-space. An F-space is a complete F^*-space.

An F-norm is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbf{C}$ (see [11]).

In Section 2, we solve the additive-quadratic ρ-functional inequality (1) and prove the Hyers–Ulam stability of the additive-quadratic ρ-functional inequality (1) in β_2-homogeneous complex Banach space. In Section 3, we solve the additive-quadratic ρ-functional inequality (2) and prove the Hyers–Ulam stability of the additive-quadratic ρ-functional inequality (2) in β_2-homogeneous complex Banach space.

Throughout this paper, let β_1 and β_2 be positive real numbers with $\beta_1 \le 1$ and $\beta_2 \le 1$. Assume that X is a β_1-homogeneous real or complex normed space with norm $\|\cdot\|$ and that Y is a β_2-homogeneous complex Banach space with norm $\|\cdot\|$.

2 Additive-Quadratic ρ-Functional Inequality (1) in β-Homogeneous Complex Banach Spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < \frac{1}{2}$.

We solve and investigate the additive-quadratic ρ-functional inequality (1) in complex normed spaces.

Lemma 1 (i) *If a mapping $f : X \to Y$ satisfies $M_1 f(x, y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*

(ii) *If a mapping $f : X \to Y$ satisfies $M_2 f(x, y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*

Proof (i)

$$M_1 f_o(x, y) = f_o(x + y) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. So, f_o is the Cauchy additive mapping.

$$M_1 f_e(x, y) = \frac{1}{2} f_e(x + y) + \frac{1}{2} f_e(x - y) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. So, f_o is the quadratic mapping.

(ii)

$$M_2 f_o(x, y) = 2 f_o\left(\frac{x + y}{2}\right) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0, 0) = 0$, $f(0) = 0$ and f_o is the Cauchy additive mapping.

$$M_2 f_e(x, y) = 2 f_e\left(\frac{x + y}{2}\right) + 2 f_e\left(\frac{x - y}{2}\right) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0, 0) = 0$, $f(0) = 0$ and f_e is the quadratic mapping.

Therefore, the mapping $f : X \to Y$ is the sum of the Cauchy additive mapping and the quadratic mapping.

Lemma 2 (i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_1 f(x, y)\| \le \|\rho M_2 f(x, y)\| \tag{3}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

(ii) *If an even mapping $f : X \to Y$ satisfies (3), then $f : X \to Y$ is quadratic.*

Proof (i) Assume that $f : X \to Y$ satisfies (3).

Since f is an odd mapping, $f(0) = 0$.

Letting $y = x$ in (3), we get

$$\|f(2x) - 2f(x)\| \leq 0,$$

and so $f(2x) = 2f(x)$ for all $x \in X$. Thus,

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \tag{4}$$

for all $x \in X$.

It follows from (3) and (4) that

$$\begin{aligned}\|f(x+y) - f(x) - f(y)\| &\leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \\ &= |\rho|^{\beta_2}\|f(x+y) - f(x) - f(y)\|,\end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get

$$\|f(0)\| \leq \|2\rho f(0)\|.$$

So, $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\left\|\frac{1}{2}f(2x) - 2f(x)\right\| \leq 0,$$

and so $f(2x) = 4f(x)$ for all $x \in X$. Thus,

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{5}$$

for all $x \in X$.

It follows from (3) and (5) that

$$\left\|\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y)\right\|$$

$$\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\|$$
$$= |\rho|^{\beta_2} \left\| \frac{1}{2} f(x+y) + \frac{1}{2} f(x-y) - f(x) - f(y) \right\|,$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

We prove the Hyers–Ulam stability of the additive-quadratic ρ-functional inequality (3) in β-homogeneous complex Banach spaces for an odd mapping case.

Theorem 1 *Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|M_1 f(x, y)\| \leq \|\rho M_2 f(x, y)\| + \theta(\|x\|^r + \|y\|^r) \tag{6}$$

for all $x, y \in X$. Then, there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2^{\beta_1 r} - 2^{\beta_2}} \|x\|^r \tag{7}$$

for all $x \in X$.

Proof Letting $y = x$ in (6), we get

$$\|f(2x) - 2f(x)\| \leq 2\theta\|x\|^r \tag{8}$$

for all $x \in X$. So,

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \frac{2}{2^{\beta_1 r}} \theta \|x\|^r$$

for all $x \in X$. Hence,

$$\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| \leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\|$$
$$\leq \frac{2}{2^{\beta_1 r}} \sum_{j=l}^{m-1} \frac{2^{\beta_2 j}}{2^{\beta_1 r j}} \theta \|x\|^r \tag{9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (9) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So, one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an odd mapping, A is an odd mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (9), we get (7).

It follows from (6) that

$$\begin{aligned}\|A(x+y) - A(x) - A(y)\| &= \lim_{n\to\infty} \left\|2^n \left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty} \left\|2^n \rho \left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\quad + \lim_{n\to\infty} \frac{2^{\beta_2 n}}{2^{\beta_1 rn}} \theta(\|x\|^r + \|y\|^r) \\ &= \left\|\rho \left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|\end{aligned}$$

for all $x, y \in X$. So,

$$\|A(x+y) - A(x) - A(y)\| \le \left\|\rho \left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|$$

for all $x, y \in X$. By Lemma 2, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (7). Then, we have

$$\begin{aligned}\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le \frac{4\theta}{2^{\beta_1 r} - 2^{\beta_2}} \frac{2^{\beta_2 q}}{2^{\beta_1 qr}} \|x\|^r,\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So, we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A, as desired.

Theorem 2 *Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying (6). Then, there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^{\beta_2} - 2^{\beta_1 r}} \|x\|^r \tag{10}$$

for all $x \in X$.

Proof It follows from (8) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \le \frac{2}{2^{\beta_2}} \theta \|x\|^r$$

for all $x \in X$. Hence,

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| &\le \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\| \\ &\le \frac{2}{2^{\beta_2}} \sum_{j=l}^{m-1} \frac{2^{\beta_1 r j}}{2^{\beta_2 j}} \theta \|x\|^r \end{aligned} \tag{11}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (11) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So, one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (11), we get (10).

The rest of the proof is similar to the proof of Theorem 1.

Now, we prove the Hyers–Ulam stability of the additive-quadratic ρ-functional inequality (3) in β-homogeneous complex Banach spaces for an even mapping case.

Theorem 3 *Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (6). Then, there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2 \cdot 2^{\beta_2} \theta}{2^{\beta_1 r} - 4^{\beta_2}} \|x\|^r \tag{12}$$

for all $x \in X$.

Proof Letting $y = x$ in (6), we get

$$\left\| \frac{1}{2} f(2x) - 2 f(x) \right\| \le 2\theta \|x\|^r \tag{13}$$

for all $x \in X$. So,

$$\left\| f(x) - 4 f\left(\frac{x}{2}\right) \right\| \le \frac{2 \cdot 2^{\beta_2} \theta}{2^{\beta_1 r}} \|x\|^r$$

for all $x \in X$. Hence,

$$\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \leq \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|$$

$$\leq \frac{2 \cdot 2^{\beta_2}}{2^{\beta_1 r}} \sum_{j=l}^{m-1} \frac{4^{\beta_2 j}}{2^{\beta_1 r j}} \theta \|x\|^r \tag{14}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (14) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So, one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an even mapping, Q is an even mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (14), we get (12).

It follows from (6) that

$$\begin{aligned}
&\left\|\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right\| \\
&= \lim_{n\to\infty} \left\|4^n \left(\frac{1}{2}f\left(\frac{x+y}{2^n}\right) + \frac{1}{2}f\left(\frac{x-y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\
&\leq \lim_{n\to\infty} \left\|4^n \rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\
&\qquad + \lim_{n\to\infty} \frac{4^{\beta_2 n}}{2^{\beta_1 r n}} \theta(\|x\|^r + \|y\|^r) \\
&= \left\|\rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. So,

$$\begin{aligned}
&\left\|\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right\| \\
&\qquad \leq \left\|\rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. By Lemma 2, the mapping $Q : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (12). Then, we have

$$\begin{aligned}\|Q(x)-T(x)\| &= \left\|4^q Q\left(\frac{x}{2^q}\right)-4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\leq \left\|4^q Q\left(\frac{x}{2^q}\right)-4^q f\left(\frac{x}{2^q}\right)\right\|+\left\|4^q T\left(\frac{x}{2^q}\right)-4^q f\left(\frac{x}{2^q}\right)\right\| \\ &\leq \frac{2\cdot 2^{\beta_2}\theta}{2^{\beta_1 r}-4^{\beta_2}}\frac{4^{\beta_2 q}}{2^{\beta_1 q r}}\|x\|^r,\end{aligned}$$

which tends to zero as $q\to\infty$ for all $x\in X$. So, we can conclude that $Q(x)=T(x)$ for all $x\in X$. This proves the uniqueness of Q, as desired.

Theorem 4 *Let $r<\frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f: X\to Y$ be an even mapping satisfying $f(0)=0$ and* (6)*. Then, there exists a unique quadratic mapping $Q: X\to Y$ such that*

$$\|f(x)-Q(x)\|\leq\frac{2\cdot 2^{\beta_2}\theta}{4^{\beta_2}-2^{\beta_1 r}}\|x\|^r \tag{15}$$

for all $x\in X$.

Proof It follows from (13) that

$$\left\|f(x)-\frac{1}{4}f(2x)\right\|\leq\frac{2\theta}{2^{\beta_2}}\|x\|^r$$

for all $x\in X$. Hence,

$$\begin{aligned}\left\|\frac{1}{4^l}f(2^l x)-\frac{1}{4^m}f(2^m x)\right\| &\leq\sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(2^j x\right)-\frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\leq\frac{2\theta}{2^{\beta_2}}\sum_{j=l}^{m-1}\frac{2^{\beta_1 r}}{4^{\beta_2 j}}\|x\|^r \end{aligned}\tag{16}$$

for all nonnegative integers m and l with $m>l$ and all $x\in X$. It follows from (16) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x\in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So, one can define the mapping $Q: X\to Y$ by

$$Q(x):=\lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x\in X$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (16), we get (15).

The rest of the proof is similar to the proof of Theorem 7.

Remark 1 If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a β-homogeneous real Banach space, then all the assertions in this section remain valid.

3 Additive-Quadratic ρ-Functional Inequality (2) in β-Homogeneous Complex Banach Spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < 1$.

We solve and investigate the additive-quadratic ρ-functional inequality (2) in β-homogeneous complex normed spaces.

Lemma 3 (i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_2 f(x, y)\| \le \|\rho M_1 f(x, y)\| \tag{17}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

(ii) *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and (17), then $f : X \to Y$ is quadratic.*

Proof (i) Assume that $f : X \to Y$ satisfies (17).

Letting $y = 0$ in (17), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \le 0, \tag{18}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$ for all $x \in X$.

It follows from (17) and (18) that

$$\begin{aligned}\|f(x+y) - f(x) - f(y)\| &= \left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \\ &\le |\rho|^{\beta_2}\|f(x+y) - f(x) - f(y)\|,\end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (17).

Letting $y = 0$ in (17), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le 0, \tag{19}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (17) and (19) that

$$\begin{aligned}
&\left\| \frac{1}{2} f(x+y) + \frac{1}{2} f(x-y) - f(x) - f(y) \right\| \\
&\quad = \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\| \\
&\quad \le |\rho|^{\beta_2} \left\| \frac{1}{2} f(x+y) + \frac{1}{2} f(x-y) - f(x) - f(y) \right\|,
\end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

We prove the Hyers–Ulam stability of the additive-quadratic ρ-functional inequality (17) in β-homogeneous complex Banach spaces for an odd mapping case.

Theorem 5 *Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying*

$$\|M_2 f(x, y)\| \le \|\rho M_1 f(x, y)\| + \theta(\|x\|^r + \|y\|^r) \tag{20}$$

for all $x, y \in X$. Then, there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2^{\beta_1 r}\theta}{2^{\beta_1 r} - 2^{\beta_2}} \|x\|^r \tag{21}$$

for all $x \in X$.

Proof Letting $y = 0$ in (20), we get

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| = \left\| 2f\left(\frac{x}{2}\right) - f(x) \right\| \le \theta\|x\|^r \tag{22}$$

for all $x \in X$. So,

$$\begin{aligned}
\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| &\le \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\
&\le \sum_{j=l}^{m-1} \frac{2^{\beta_2 j}}{2^{\beta_1 r j}} \theta\|x\|^r
\end{aligned} \tag{23}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (23) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So, one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an odd mapping, A is an odd mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (23), we get (21).

The rest of the proof is similar to the proof of Theorem 1.

Theorem 6 *Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (20). *Then, there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2^{\beta_1 r}\theta}{2^{\beta_2} - 2^{\beta_1 r}} \|x\|^r \tag{24}$$

for all $x \in X$.

Proof It follows from (22) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \le \frac{2^{\beta_1 r}}{2^{\beta_2}} \theta \|x\|^r$$

for all $x \in X$. Hence,

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| &\le \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\| \\ &\le \sum_{j=l+1}^{m} \frac{2^{\beta_1 r j}}{2^{\beta_2 j}} \theta \|x\|^r \end{aligned} \tag{25}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (25) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So, one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (25), we get (24).

The rest of the proof is similar to the proof of Theorem 1.

Now, we prove the Hyers–Ulam stability of the additive-quadratic ρ-functional inequality (17) in β-homogeneous complex Banach spaces for an even mapping case.

Theorem 7 *Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (20). Then, there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2^{\beta_1 r}\theta}{2^{\beta_1 r} - 4^{\beta_2}}\|x\|^r \tag{26}$$

for all $x \in X$.

Proof Letting $y = 0$ in (20), we get

$$\left\| f(x) - 4f\left(\frac{x}{2}\right)\right\| = \left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \theta\|x\|^r \tag{27}$$

for all $x \in X$. So,

$$\begin{aligned}\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{m-1}\frac{4^{\beta_2 j}}{2^{\beta_1 r j}}\theta\|x\|^r \end{aligned}\tag{28}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (28) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So, one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an even mapping, Q is an even mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (28), we get (26).

The rest of the proof is similar to the proof of Theorem 3

Theorem 8 *Let $r < \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (20). Then, there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2^{\beta_1 r}\theta}{4^{\beta_2} - 2^{\beta_1 r}}\|x\|^r \tag{29}$$

for all $x \in X$.

Proof It follows from (27) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{2^{\beta_1 r}}{4^{\beta_2}} \theta \|x\|^r$$

for all $x \in X$. Hence,

$$\begin{aligned} \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j} f\left(2^j x\right) - \frac{1}{4^{j+1}} f\left(2^{j+1} x\right) \right\| \\ &\leq \sum_{j=l+1}^{m} \frac{2^{\beta_1 r j}}{4^{\beta_2 j}} \theta \|x\|^r \end{aligned} \tag{30}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (30) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So, one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (30), we get (29).

The rest of the proof is similar to the proof of Theorem 3.

Remark 2 If ρ is a real number such that $-1 < \rho < 1$ and Y is a β-homogeneous real Banach space, then all the assertions in this section remain valid.

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Stability of Bi-additive s-Functional Inequalities and Quasi-multipliers

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Abstract Park et al. (Rocky Mt J Math **49**, 593–607 (2019)) solved the following bi-additive s-functional inequalities:

$$\begin{aligned} &\|f(x+y, z-w)+f(x-y, z+w)-2f(x, z)+2f(y, w)\| \\ &\quad \le \left\| s\left(2f\left(\frac{x+y}{2}, z-w\right)+2f\left(\frac{x-y}{2}, z+w\right)-2f(x, z)+2f(y, w)\right)\right\|, \end{aligned} \tag{1}$$

$$\begin{aligned} &\left\|2f\left(\frac{x+y}{2}, z-w\right)+2f\left(\frac{x-y}{2}, z+w\right)-2f(x, z)+2f(y, w)\right\| \\ &\quad \le \|s\left(f(x+y, z-w)+f(x-y, z+w)-2f(x, z)+2f(y, w)\right)\|, \end{aligned} \tag{2}$$

where s is a fixed nonzero complex number with $|s| < 1$. Using the direct method, we prove the Hyers–Ulam stability of quasi-multipliers on Banach algebras, associated with the bi-additive s-functional inequalities (1) and (2).

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1 Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [19] concerning the stability of group homomorphisms. The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [17] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

Gilányi [10] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \le \|f(x + y)\|, \tag{3}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x + y) + f(x - y).$$

See also [18]. Fechner [8] and Gilányi [11] proved the Hyers–Ulam stability of the functional inequality (3).

Park [14, 15] defined additive ρ-functional inequalities and proved the Hyers–Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [3, 5–7]).

The notion of a quasi-multiplier is a generalization of the notion of a multiplier on a Banach algebra, which was introduced by Akemann and Pedersen [1] for C^*-algebras. McKennon [13] extended the definition to a general complex Banach algebra with bounded approximate identity as follows.

Definition 1 ([13]) Let A be a complex Banach algebra. A $\mathbf{C}$-bilinear mapping $P : A \times A \to A$ is called a *quasi-multiplier* on A if P satisfies

$$P(xy, zw) = xP(y, z)w$$

for all $x, y, z, w \in A$.

This paper is organized as follows: In Sections 2 and 3, we prove the Hyers–Ulam stability of the bi-additive s-functional inequalities (1) and (2) in complex Banach spaces by using the direct method. In Section 4, we investigate quasi-multipliers on Banach algebras associated with the bi-additive s-functional inequalities (1) and (2).

Throughout this paper, let X be a complex normed space and Y be a complex Banach space. Let A be a complex Banach algebra. Assume that s is a fixed nonzero complex number with $|s| < 1$.

2 Bi-additive s-Functional Inequality (1)

In [16], Park solved the bi-additive s-functional inequality (1) in complex normed spaces.

Lemma 1 ([16, Lemma 2.1]) *If a mapping $f : X^2 \to Y$ satisfies $f(0,z) = f(x,0) = 0$ and*

$$\begin{aligned}\|f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w)\| \quad (4)\\ \leq \left\|s\left(2f\left(\frac{x+y}{2},z-w\right)+2f\left(\frac{x-y}{2},z+w\right)-2f(x,z)+2f(y,w)\right)\right\|\end{aligned}$$

for all $x, y, z, w \in X$, then $f : X^2 \to Y$ is bi-additive.

Using the direct method, we prove the Hyers–Ulam stability of the bi-additive s-functional inequality (4) in complex Banach spaces.

Theorem 1 *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying*

$$\Psi(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \quad (5)$$

for all $x, y \in X$. Let $f : X^2 \to Y$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and

$$\begin{aligned}\|f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w)\| \quad (6)\\ \leq \left\|s\left(2f\left(\frac{x+y}{2},z-w\right)+2f\left(\frac{x-y}{2},z+w\right)-2f(x,z)+2f(y,w)\right)\right\|\\ +\varphi(x,y)\varphi(z,w)\end{aligned}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $P : X^2 \to Y$ such that

$$\|f(x,z)-P(x,z)\| \leq \frac{1}{2}\Psi(x,x)\varphi(z,0) \quad (7)$$

for all $x, z \in X$.

Proof Letting $w = 0$ and $y = x$ in (6), we get

$$\|f(2x, z) - 2f(x, z)\| \le \varphi(x, x)\varphi(z, 0) \tag{8}$$

for all $x, z \in X$.

It follows from (8) that

$$\left\| f(x, z) - 2f\left(\frac{x}{2}, z\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right)\varphi(z, 0)$$

for all $x, z \in X$. Hence,

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}, z\right) - 2^m f\left(\frac{x}{2^m}, z\right)\right\| &\le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}, z\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}, z\right)\right\| \\ &\le \frac{1}{2}\sum_{j=l+1}^{m} 2^j \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}\right)\varphi(z, 0)\end{aligned} \tag{9}$$

for all nonnegative integers m and l with $m > l$ and all $x, z \in X$. It follows from (9) that the sequence $\{2^k f(\frac{x}{2^k}, z)\}$ is Cauchy for all $x, z \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k}, z)\}$ converges. So one can define the mapping $P : X^2 \to Y$ by

$$P(x, z) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}, z\right)$$

for all $x, z \in X$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (9), we get (7).

It follows from (5) and (6) that

$$\begin{aligned}&\|P(x + y, z - w) + P(x - y, z + w) - 2P(x, z) + 2P(y, w)\| \\ &= \lim_{n\to\infty}\left\|2^n\left(f\left(\frac{x+y}{2^n}, z-w\right) + f\left(\frac{x-y}{2^n}, z + w\right) - 2f\left(\frac{x}{2^n}, z\right) + 2f\left(\frac{y}{2^n}, w\right)\right)\right\| \\ &\le \lim_{n\to\infty}\left\|2^n s\left(2f\left(\frac{x+y}{2^{n+1}}, z-w\right) + 2f\left(\frac{x-y}{2^{n+1}}, z+w\right) - 2f\left(\frac{x}{2^n}, z\right) + 2f\left(\frac{y}{2^n}, w\right)\right)\right\| \\ &\qquad + \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n}, \frac{x}{2^n}\right)\varphi(z, 0) \\ &\le \left\|s\left(2P\left(\frac{x+y}{2}, z - w\right) + 2P\left(\frac{x-y}{2}, z + w\right) - 2P(x, z) + 2P(y, w)\right)\right\|\end{aligned}$$

for all $x, y, z, w \in X$. So

$$\|P(x + y, z - w) + P(x - y, z + w) - 2P(x, z) + 2P(y, w)\|$$

$$\leq \left\| s\left(2P\left(\frac{x+y}{2}, z-w\right) + 2P\left(\frac{x-y}{2}, z+w\right) - 2P(x,z) + 2P(y,w)\right)\right\|$$

for all $x, y, z, w \in X$. By Lemma 1, the mapping $P : X^2 \to Y$ is bi-additive.

Now, let $T : X^2 \to Y$ be another bi-additive mapping satisfying (7). Then we have

$$\begin{aligned}\|P(x,z) - T(x,z)\| &= \left\|2^q P\left(\frac{x}{2^q}, z\right) - 2^q T\left(\frac{x}{2^q}, z\right)\right\| \\ &\leq \left\|2^q P\left(\frac{x}{2^q}, z\right) - 2^q f\left(\frac{x}{2^q}, z\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}, z\right) - 2^q f\left(\frac{x}{2^q}, z\right)\right\| \\ &\leq 2^q \Phi\left(\frac{x}{2^q}, \frac{x}{2^q}\right)\varphi(z, 0),\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x, z \in X$. So we can conclude that $P(x,z) = T(x,z)$ for all $x, z \in X$. This proves the uniqueness of P, as desired.

Corollary 1 *Let $r > 1$ and θ be nonnegative real numbers and $f : X^2 \to Y$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and*

$$\begin{aligned}&\|f(x+y, z-w) + f(x-y, z+w) - 2f(x,z) + 2f(y,w)\| \\ &\quad \leq \left\| s\left(2f\left(\frac{x+y}{2}, z-w\right) + 2f\left(\frac{x-y}{2}, z+w\right) - 2f(x,z) + 2f(y,w)\right)\right\| \\ &\quad +\theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r)\end{aligned} \tag{10}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $A : X^2 \to Y$ such that

$$\|f(x,z) - A(x,z)\| \leq \frac{2\theta}{2^r - 2}\|x\|^r\|z\|^r$$

for all $x, z \in X$.

Proof The proof follows from Theorem 1 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in X$.

Theorem 2 *Let $\varphi : X^2 \to [0, \infty)$ be a function satisfying*

$$\Psi(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\left(2^j x, 2^j y\right) < \infty \tag{11}$$

for all $x, y \in X$. Let $f : X^2 \to Y$ be a mapping satisfying (6) and $f(x,0) = f(0,z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $P : X^2 \to Y$ such that

$$\|f(x,z) - P(x,z)\| \le \frac{1}{2}\Psi(x,x)\,\varphi(z,0)$$

for all $x, z \in X$.

Proof It follows from (8) that

$$\left\| f(x,z) - \frac{1}{2} f(2x,z) \right\| \le \frac{1}{2}\varphi(x,x)\varphi(z,0)$$

for all $x, z \in X$.

The rest of the proof is similar to the proof of Theorem 1.

Corollary 2 *Let* $r < 1$ *and* θ *be nonnegative real numbers and* $f : X^2 \to Y$ *be a mapping satisfying* (10) *and* $f(x,0) = f(0,z) = 0$ *for all* $x, z \in X$. *Then there exists a unique bi-additive mapping* $P : X^2 \to Y$ *such that*

$$\|f(x,z) - P(x,z)\| \le \frac{2\theta}{2-2^r}\|x\|^r\|z\|^r$$

for all $x, z \in X$.

Proof The proof follows from Theorem 2 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in X$.

3 Bi-additive s-Functional Inequality (2)

In [16], Park solved the bi-additive s-functional inequality (2) in complex normed spaces.

Lemma 2 ([16, Lemma 3.1]) *If a mapping* $f : X^2 \to Y$ *satisfies* $f(0,z) = f(x,0) = 0$ *and*

$$\begin{aligned} &\left\| 2f\left(\frac{x+y}{2}, z-w\right) + 2f\left(\frac{x-y}{2}, z+w\right) - 2f(x,z) + 2f(y,w) \right\| \\ &\le \|s\,(f(x+y,z-w) + f(x-y,z+w) - 2f(x,z) + 2f(y,w))\| \end{aligned} \tag{12}$$

for all $x, y, z, w \in X$, *then* $f : X^2 \to Y$ *is bi-additive.*

Using the direct method, we prove the Hyers–Ulam stability of the bi-additive s-functional inequality (12) in complex Banach spaces.

Theorem 3 *Let* $\varphi : X^2 \to [0,\infty)$ *be a function satisfying* (5). *Let* $f : X^2 \to Y$ *be a mapping satisfying* $f(x,0) = f(0,z) = 0$ *and*

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2}, z-w\right)+2f\left(\frac{x-y}{2}, z+w\right)-2f(x,z)+2f(y,w)\right\| \quad (13)\\&\quad\le \|s\left(f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w)\right)\|\\&\quad+\varphi(x,y)\varphi(z,w)\end{aligned}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $P : X^2 \to Y$ such that

$$\|f(x,z)-P(x,z)\| \le \frac{1}{4}\Psi(2x,0)\varphi(z,0) \quad (14)$$

for all $x, z \in X$, where Ψ is given in the statement of Theorem 1.

Proof Letting $y = w = 0$ in (13), we get

$$\left\|4f\left(\frac{x}{2}, z\right)-2f(x,z)\right\| \le \varphi(x,0)\varphi(z,0) \quad (15)$$

for all $x, z \in X$.

It follows from (15) that

$$\left\|f(x,z)-2f\left(\frac{x}{2}, z\right)\right\| \le \frac{1}{2}\varphi(x,0)\varphi(z,0)$$

for all $x, z \in X$.

The rest of the proof is similar to the proof of Theorem 1.

Corollary 3 *Let $r > 1$ and θ be nonnegative real numbers and $f : X^2 \to Y$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and*

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2}, z-w\right)+2f\left(\frac{x-y}{2}, z+w\right)-2f(x,z)+2f(y,w)\right\| \quad (16)\\&\quad\le \|s\left(f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w)\right)\|\\&\quad+\theta(\|x\|^r+\|y\|^r)(\|z\|^r+\|w\|^r)\end{aligned}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $P : X^2 \to Y$ such that

$$\|f(x,z)-P(x,z)\| \le \frac{2^{r-1}\theta}{2^r-2}\|x\|^r\|z\|^r$$

for all $x, z \in X$.

Proof The proof follows from Theorem 3 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r+\|y\|^r)$ for all $x, y \in X$.

Theorem 4 *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying* (11). *Let $f : X^2 \to Y$ be a mapping satisfying* (13) *and $f(x,0) = f(0,z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $P : X^2 \to Y$ such that*

$$\|f(x,z) - P(x,z)\| \le \frac{1}{4}\Psi\,(2x,0)\,\varphi(z,0)$$

for all $x, z \in X$, where Ψ is given in the statement of Theorem 2.

Proof It follows from (15) that

$$\left\| f(x,z) - \frac{1}{2}f(2x,z)\right\| \le \frac{1}{4}\varphi(2x,0)\varphi(z,0)$$

for all $x, z \in X$.

The rest of the proof is similar to the proofs of Theorems 1 and 3.

Corollary 4 *Let $r < 1$ and θ be nonnegative real numbers and $f : X^2 \to Y$ be a mapping satisfying* (16) *and $f(x,0) = f(0,z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $P : X^2 \to Y$ such that*

$$\|f(x,z) - P(x,z)\| \le \frac{2^{r-1}\theta}{2-2^r}\|x\|^r\,\|z\|^r$$

for all $x, z \in X$.

Proof The proof follows from Theorem 4 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in X$.

4 Quasi-multipliers in Banach Algebras

In this section, we investigate quasi-multipliers on complex Banach algebras associated with the bi-additive s-functional inequalities (4) and (12).

Lemma 3 ([4, Lemma 2.1]) *Let $f : X^2 \to Y$ be a bi-additive mapping such that $f(\lambda x, \mu z) = \lambda\mu f(x,z)$ for all $x, z \in X$ and $\lambda, \mu \in S^1 := \{\nu \in \boldsymbol{C} \ : \ |\nu| = 1\}$. Then f is $\boldsymbol{C}$-bilinear.*

Theorem 5 *Let $\varphi : A^2 \to [0,\infty)$ be a function satisfying*

$$\Psi(x,y) := \sum_{j=1}^{\infty} 2^j\varphi\left(\frac{x}{2^j},\frac{y}{2^j}\right) < \infty \tag{17}$$

for all $x, y \in A$. Let $f : A^2 \to A$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and

$$\begin{aligned}
&\|f(\lambda(x+y),\mu(z-w))+f(\lambda(x-y),\mu(z+w))-2\lambda\mu f(x,z)+2\lambda\mu f(y,w)\| \\
&\le \left\| s\left(2f\left(\frac{x+y}{2},z-w\right)+2f\left(\frac{x-y}{2},z+w\right)-2f(x,z)+2f(y,w)\right)\right\| \qquad (18)\\
&\quad +\varphi(x,y)\varphi(z,w)
\end{aligned}$$

*for all $\lambda,\mu\in S^1$ and all $x,y,z,w\in A$. Then there exists a unique **C**-bilinear mapping $P: A^2\to A$ such that*

$$\|f(x,z)-P(x,z)\|\le\frac{1}{2}\Psi(x,x)\varphi(z,0) \qquad (19)$$

for all $x,z\in A$.

If, in addition, the mapping $f: A^2\to A$ satisfies

$$\|f(xy,zw)-xf(y,z)w\|\le\varphi(x,y)^2\varphi(z,w)^2 \qquad (20)$$

for all $x,y,z,w\in A$, then the mapping $P: A^2\to A$ is a quasi-multiplier.

Furthermore, if, in addition, the mapping $f: A^2\to A$ satisfies $f(2x,z)=2f(x,z)$ for all $x,z\in A$, then the mapping $f: A^2\to A$ is a quasi-multiplier.

Proof Let $\lambda=\mu=1$ in (18). By Theorem 1, there is a unique bi-additive mapping $P: A^2\to A$ satisfying (19) defined by

$$P(x,z):=\lim_{n\to\infty}2^n f\left(\frac{x}{2^n},z\right)$$

for all $x,z\in A$.

Letting $y=x$ and $w=0$ in (18), we get

$$\|f(2\lambda x,\mu z)-2\lambda\mu f(x,z)\|\le\varphi(x,0)\varphi(z,0)$$

for all $x,z\in A$ and all $\lambda,\mu\in S^1$. So

$$\begin{aligned}
\|P(2\lambda x,\mu z)-2\lambda\mu P(x,z)\| &= \lim_{n\to\infty}2^n\left\|f\left(2\lambda\frac{x}{2^n},\mu z\right)-2\lambda\mu f\left(\frac{x}{2^n},z\right)\right\| \\
&\le \lim_{n\to\infty}2^n\varphi\left(\frac{x}{2^n},\frac{x}{2^n}\right)\varphi(z,0)\le\lim_{n\to\infty}\frac{2^nL^n}{2^n}\varphi(x,x)\varphi(z,0)=0
\end{aligned}$$

for all $x,z\in A$ and all $\lambda,\mu\in S^1$. Hence, $P(2\lambda x,\mu z)=2\lambda\mu P(x,z)$ and so $P(\lambda x,\mu z)=\lambda\mu P(x,z)$ for all $x,z\in A$ and all $\lambda,\mu\in S^1$. By Lemma 3, the bi-additive mapping $P: A^2\to A$ is **C**-bilinear.

It follows from (20) that

$$\begin{aligned}
\|P(xy,zw)-xP(y,z)w\| &= \lim_{n\to\infty}4^n\left\|f\left(\frac{xy}{2^n\cdot 2^n},zw\right)-\frac{x}{2^n}f\left(\frac{y}{2^n},z\right)w\right\| \\
&\le \lim_{n\to\infty}4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)^2\varphi(z,w)^2=0
\end{aligned}$$

for all $x, y, z, w \in A$. Thus,

$$P(xy, zw) = xP(y, z)w$$

for all $x, y, z, w \in A$.

If $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then we can easily show that $P(x, z) = f(x, z)$ for all $x, z \in A$. Hence, the mapping $f : A^2 \to A$ is a quasi-multiplier.

Corollary 5 *Let $r > 2$ and θ be nonnegative real numbers, and $f : A^2 \to A$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned} &\|f(\lambda(x+y), \mu(z-w)) + f(\lambda(x-y), \mu(z+w)) - 2\lambda\mu f(x, z) + 2\lambda\mu f(y, w)\| \\ &\le \left\| s\left(2f\left(\frac{x+y}{2}, z-w\right) + 2f\left(\frac{x-y}{2}, z+w\right) - 2f(x, z) + 2f(y, w)\right)\right\| \\ &\quad + \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \end{aligned} \tag{21}$$

for all $\lambda, \mu \in S^1$ and all $x, y, z, w \in A$. Then there exists a unique $\mathbf{C}$-bilinear mapping $P : A^2 \to A$ such that

$$\|f(x, z) - P(x, z)\| \le \frac{2\theta}{2^r - 2}\|x\|^r\|z\|^r$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies

$$\|f(xy, zw) - xf(y, z)w\| \le \theta^2(\|x\|^r + \|y\|^r)^2(\|z\|^r + \|w\|^r)^2 \tag{22}$$

for all $x, y, z, w \in A$, then the mapping $P : A^2 \to A$ is a quasi-multiplier.

Furthermore, if, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-multiplier.

Proof The proof follows from Theorem 5 by taking $\varphi(x, y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in X$.

Theorem 6 *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\left(2^j x, 2^j y\right) < \infty \tag{23}$$

for all $x, y \in A$. Let $f : A^2 \to A$ be a mapping satisfying (18) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbf{C}$-bilinear mapping $P : A^2 \to A$ such that

$$\|f(x, z) - P(x, z)\| \le \frac{1}{2}\Psi(x, x)\varphi(z, 0)$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (20)*, then the mapping $P : A^2 \to A$ is a quasi-multiplier.*

Furthermore, if, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-multiplier.

Proof The proof is similar to the proof of Theorem 5.

Corollary 6 *Let $r < 1$ and θ be nonnegative real numbers, and $f : A^2 \to A$ be a mapping satisfying* (21) *and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbf{C}$-bilinear mapping $P : A^2 \to A$ such that*

$$\|f(x,z) - P(x,z)\| \leq \frac{2\theta}{2-2^r}\|x\|^r\|z\|^r$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (22)*, then the mapping $P : A^2 \to A$ is a quasi-multiplier.*

Furthermore, if, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-multiplier.

Proof The proof follows from Theorem 6 by taking $\varphi(x, y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in X$.

Similarly, we can obtain the following results.

Theorem 7 *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying* (17)*. Let $f : A^2 \to A$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned}\left\|2f\left(\lambda\frac{x+y}{2}, \mu(z-w)\right)+2f\left(\lambda\frac{x-y}{2}, \mu(z+w)\right)-2\lambda\mu f(x,z)+2\lambda\mu f(y,w)\right\| \\ \leq \|s\left(f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w)\right)\| \qquad (24)\\ +\varphi(x,y)\varphi(z,w)\end{aligned}$$

for all $\lambda, \mu \in S^1$ and all $x, y, z, w \in A$. Then there exists a unique $\mathbf{C}$-bilinear mapping $P : A^2 \to A$ such that

$$\|f(x,z) - P(x,z)\| \leq \frac{1}{4}\Psi(2x,0)\varphi(z,0),$$

for all $x \in A$, where Ψ is given in the statement of Theorem 5.

If, in addition, the mapping $f : A^2 \to A$ satisfies (20)*, then the mapping $P : A^2 \to A$ is a quasi-multiplier.*

Furthermore, if, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-multiplier.

Corollary 7 *Let $r > 2$ and θ be nonnegative real numbers, and $f : A^2 \to A$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\left\|2f\left(\lambda\frac{x+y}{2}, \mu(z-w)\right)+2f\left(\lambda\frac{x-y}{2}, \mu(z+w)\right)-2\lambda\mu f(x, z)+2\lambda\mu f(y, w)\right\|$$
$$\leq \|s\,(f(x+y, z-w)+f(x-y, z+w)-2f(x, z)+2f(y, w))\| \tag{25}$$
$$+\theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r)$$

*for all $\lambda, \mu \in S^1$ and all $x, y, z, w \in A$. Then there exists a unique **C**-bilinear mapping $P : A^2 \to A$ such that*

$$\|f(x, z) - P(x, z)\| \leq \frac{2^{r-1}\theta}{2^r - 2}\|x\|^r\|z\|^r$$

for all $x \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (22), then the mapping $P : A^2 \to A$ is a quasi-multiplier.

Furthermore, if, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-multiplier.

Proof The proof follows from Theorem 7 by taking $\varphi(x, y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in X$.

Theorem 8 *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying (23). Let $f : A \to A$ be a mapping satisfying (24) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique **C**-bilinear mapping $P : A^2 \to A$ such that*

$$\|f(x, z) - P(x, z)\| \leq \frac{1}{4}\Psi(2x, 0)\varphi(z, 0)$$

for all $x, z \in A$, where Ψ is given in the statement of Theorem 6.

If, in addition, the mapping $f : A^2 \to A$ satisfies (20), then the mapping $P : A^2 \to A$ is a quasi-multiplier.

Furthermore, if, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-multiplier.

Corollary 8 *Let $r < 1$ and θ be nonnegative real numbers, and $f : A \to A$ be a mapping satisfying (25) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique **C**-bilinear mapping $P : A^2 \to A$ such that*

$$\|f(x, z) - P(x, z)\| \leq \frac{2^{r-1}\theta}{2 - 2^r}\|x\|^r\|z\|^r$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (22), then the mapping $P : A^2 \to A$ is a quasi-multiplier.

Furthermore, if, in addition, the mapping $f : A^2 \to A$ *satisfies* $f(2x, z) = 2f(x, z)$ *for all* $x, z \in A$*, then the mapping* $f : A^2 \to A$ *is a quasi-multiplier.*

Proof The proof follows from Theorem 8 by taking $\varphi(x, y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in X$.

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On the Stability of Some Functional Equations and s-Functional Inequalities

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Abstract In this work, the Hyers–Ulam type stability and the hyperstability of the following functional equations

$$f(x+y) + f(x-y) = f(2x) + f(y) + f(-y),$$
$$f(ax+y) + f(ax-y) = f(ax) + af(x),$$
$$f(ax+y) + f(ax-y) = f(ax) + af(x) + f(y) + f(-y)$$

are proved. We also introduce and solve some s-functional inequalities, and we prove their Hyers–Ulam stabilities.

1 Introduction

The functional equation (ξ) is called stable if any function g satisfying the equation (ξ) *approximately* is near to true solution of (ξ). S. M. Ulam in 1940 [16] introduced the stability of homomorphisms between two groups. More precisely, he proposed the following problem: Given a group $(G_1, .)$, a metric group $(G_2, *, d)$ and a positive number ϵ, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(x.y), f(x) * f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$? If this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, D. H. Hyers [7] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. T. Aoki [1] and Th.M. Rassias [14] provided a generalization

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of the Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded. During the last decades, several stability problems of functional equations have been investigated by several mathematicians. A large list of references concerning the stability of functional equations can be found in [2, 4, 5, 8–13, 15].

In this paper, we deal with the following functional equations:

$$f(x+y)+f(x-y)=f(2x)+f(y)+f(-y), \tag{1}$$

$$f(ax+y)+f(ax-y)=f(ax)+af(x), \tag{2}$$

$$f(ax+y)+f(ax-y)=f(ax)+af(x)+f(y)+f(-y). \tag{3}$$

2 Solutions of Functional Equations (1), (2) and (3)

Theorem 1 *Let X and Y be vector spaces. A function $f : X \to Y$ satisfies (1) if and only if f is additive.*

Proof Let f satisfy (1). Letting $x = 0$ in (1), we get $f(0) = 0$. Letting $y = x$ in (1), we infer that f is odd. Therefore, (1) implies $f(x+y)+f(x-y) = f(2x)$. Replacing x by $\frac{x+y}{2}$ and y by $\frac{x-y}{2}$ in the last equation, we get f is additive.

Conversely, if f is additive, it is easy to check that f satisfies (1).

Theorem 2 *Let X and Y be vector spaces. If functions $f, g : X \to Y$ satisfy*

$$f(x+y)+f(x-y)=f(2x)+g(y)+g(-y), \quad x, y \in X, \tag{4}$$

then $f - f(0)$ is additive and $g(x)+g(-x) = f(0)$ for all $x \in X$.

Proof Letting $x = 0$ in (4), we get $f(y) + f(-y) = f(0) + g(y) + g(-y)$ for all $y \in X$. Therefore, f satisfies $f(x+y)+f(x-y) = f(2x)+f(y)+f(-y)-f(0)$ for all $x, y \in X$. It is easy to see that $f-f(0)$ satisfies (1). Then, $f-f(0)$ is additive by Theorem 1.

Letting $y = x$ in (4), we infer that $g(x) + g(-x) = f(0)$ for all $x \in X$.

Theorem 3 *Let X and Y be vector spaces. If functions $f, g : X \to Y$ satisfy*

$$f(x+y)+f(x-y)=g(2x)+g(y)+g(-y), \quad x, y \in X, \tag{5}$$

then there exist an additive function $A : X \to Y$ and a quadratic function $Q : X \to Y$ such that $f = A + Q + f(0)$ and $g = A + \frac{1}{2}Q + g(0)$.

Proof Letting $y = 0$ in (5), we get $g(2x) = 2f(x) - 2g(0)$ for all $x \in X$. Letting $x = 0$ in (5), we get $g(y)+g(-y) = f(y)+f(-y)-g(0)$ for all $y \in X$. Therefore, $3g(0) = 2f(0)$, and f satisfies

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y)-2f(0), \quad x,y\in X. \tag{6}$$

It is easy to see that $F=f-f(0)$ satisfies $F(x+y)+F(x-y)=2F(x)+F(y)+F(-y)$ for all $x,y\in X$. Then, F has the form $F=A+Q$, where $A:X\to Y$ is additive and $Q:X\to Y$ is quadratic (see [3]). This proves that $f=A+Q+f(0)$. Since $g(2x)-g(0)=2F(x)$, we get $g(x)=g(0)+A(x)+\frac{1}{2}Q(x)$ for all $x\in X$.

Theorem 4 *Let X and Y be vector spaces. If functions $f,g,h:X\to Y$ satisfy*

$$f(x+y)+f(x-y)=h(x)+g(y)+g(-y), \quad x,y\in X, \tag{7}$$

then there exist an additive function $A:X\to Y$ and a quadratic function $Q:X\to Y$ such that $f=A+Q+f(0)$, $h=2A+2Q+h(0)$, and $g_e=Q+g(0)$, where g_e is the even part of g.

Proof Letting $y=0$ in (7), we get $h(x)=2f(x)-2g(0)$ for all $x\in X$. Letting $x=0$ in (7), we get $g(y)+g(-y)=f(y)+f(-y)-h(0)$ for all $y\in X$. Therefore, $2f(0)=2g(0)+h(0)$, and f satisfies

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y)-2f(0), \quad x,y\in X.$$

It is easy to see that $F=f-f(0)$ satisfies $F(x+y)+F(x-y)=2F(x)+F(y)+F(-y)$ for all $x,y\in X$. Then, F has the form $F=A+Q$, where $A:X\to Y$ is additive and $Q:X\to Y$ is quadratic. This proves that $f=A+Q+f(0)$. Since $h(x)-h(0)=2F(x)$, we get $h(x)=h(0)+2A(x)+2Q(x)$ for all $x\in X$. On the other hand, we have

$$\begin{aligned} g(y)+g(-y) &= f(y)+f(-y)-h(0) \\ &= F(y)+F(-y)+2f(0)-h(0) \\ &= 2Q(y)+2g(0), \quad y\in X, \end{aligned}$$

which completes the proof.

Theorem 5 *Let X and Y be vector spaces, and let $a\neq 0,1$. If a function $f:X\to Y$ satisfies (2), then f is additive.*

Proof Letting $x=y=0$ in (2), we get $f(0)=0$. If we put $x=0$ in (2), we infer that f is odd. Letting $y=0$ in (2), we obtain $f(ax)=af(x)$ for all $x\in X$. Therefore, f satisfies $f(ax+y)+f(ax-y)=2f(ax)$ for all $x\in X$. Replacing x by x/a in the last equation, we get $f(x+y)+f(x-y)=2f(x)$ for all $x\in X$. This shows f is additive.

Theorem 6 *Let X and Y be vector spaces, and let $a\neq 0$. If functions $f,g:X\to Y$ satisfy $f(0)=0$ and*

$$f(ax+y)+f(ax-y)=g(x)+af(x), \quad x,y\in X, \tag{8}$$

then f and g are additive.

Proof Letting $y = 0$ in (8), we get $2f(ax) = g(x) + af(x)$. Therefore, f satisfies $f(ax + y) + f(ax - y) = 2f(ax)$ for all $x \in X$. Replacing x by x/a in the last equation, we get $f(x + y) + f(x - y) = 2f(x)$ for all $x \in X$. Then, f is additive. It follows from $2f(ax) = g(x) + af(x)$ that

$$\begin{aligned} g(x + y) &= 2f(ax + ay) - af(x + y) \\ &= [2f(ax) - af(x)] + [2f(ay) - af(y)] \\ &= g(x) + g(y), \quad x, y \in X. \end{aligned}$$

Therefore, g is additive.

Theorem 7 *Let X and Y be vector spaces, and let $a \in \mathbb{Z} \setminus \{0, \pm 1\}$. If a function $f : X \to Y$ satisfies (3), then f is additive.*

Proof We may suppose that $f \neq 0$. Letting $x = y = 0$ in (3), we get $f(0) = 0$. If we put $y = 0$ in (3), we obtain $f(ax) = af(x)$ for all $x \in X$. Therefore, f satisfies

$$f(ax + y) + f(ax - y) = 2f(ax) + f(y) + f(-y), \quad x, y \in X. \tag{9}$$

Replacing x by x/a in (9), we get

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y), \quad x, y \in X. \tag{10}$$

We claim that if f is even, then $f = 0$. If f is even, it follows from (10) that $f(ax) = a^2 f(x)$ for all $x \in X$. On the other hand, we have $f(ax) = af(x)$ for all $x \in X$. Hence, $a^2 = a$, which is a contradiction. Since f_e (the even part of f) satisfies in (3), we infer that $f_e = 0$ and f is odd. Therefore, (10) implies that f is additive.

Theorem 8 *Let X and Y be vector spaces, and let $a \in \mathbb{Z} \setminus \{0\}$. If functions $f, g : X \to Y$ satisfy*

$$f(ax + y) + f(ax - y) = g(x) + f(y) + f(-y), \quad x, y \in X, \tag{11}$$

then there exist a quadratic function $Q : X \to Y$ and an additive function $A : X \to Y$ such that $f = Q + A + f(0)$ and $g = 2[a^2 Q + aA]$.

Proof Letting $y = 0$ in (11), we get $2f(ax) = g(x) + 2f(0)$ for all $x \in X$. Therefore, f satisfies

$$f(ax+y)+f(ax-y) = 2f(ax)+f(y)+f(-y)-2f(0), \quad x, y \in X. \tag{12}$$

Replacing x by x/a in (12), we get

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y)-2f(0), \quad x, y \in X. \tag{13}$$

Therefore, the function $h : f - f(0)$ satisfies $h(x+y)+h(x-y) = 2h(x)+h(y)+h(-y)$ for all $x, y \in X$. Then, there exist a quadratic function $Q : X \to Y$ and an additive function $A : X \to Y$ such that $h = Q + A$. Hence, $f = Q + A + f(0)$ and $g = 2[a^2 Q + aA]$.

3 Some s-Functional Inequalities

Lemma 1 *Let X be a vector space and Y be a normed space, and let s be a complex number with $2|s|^2 < 1$. If a function $f : X \to Y$ satisfies*

$$\begin{aligned} &\|f(x+y)-f(x)-f(y)\| \\ &\leqslant \|s[f(x+y)+f(x-y)-f(2x)-f(y)-f(-y)]\|, \end{aligned} \tag{14}$$

for all $x, y \in X$, then f is additive.

Proof Letting $x = y = 0$ in (14), we get $f(0) = 0$. Letting $y = -x$ in (14) and using $f(0) = 0$, we get $\|f(x) + f(-x)\| \leqslant |s|\|f(x) + f(-x)\|$ for all $x \in X$. Since $|s| < 1$, we infer that f is odd. Therefore, (14) means

$$\begin{aligned} &\|f(x+y)-f(x)-f(y)\| \\ &\leqslant \|s[f(x+y)+f(x-y)-f(2x)]\|, \end{aligned} \tag{15}$$

for all $x, y \in X$. Letting $y = x$ in (15), we get $f(2x) = 2f(x)$ for all $x \in X$. It follows from (15) that

$$\begin{aligned} &\|f(x+y)+f(x-y)-f(2x)\| \\ &\leqslant \|s[f(2x)+f(2y)-f(2x+2y)]\| \\ &\leqslant 2\|s[f(x)+f(y)-f(x+y)]\|, \end{aligned}$$

for all $x, y \in X$. Therefore,

$$\|f(x+y)-f(x)-f(y)\| \leqslant 2|s|^2\|f(x+y)-f(x)-f(y)\|, \quad x, y \in X.$$

Since $2|s|^2 < 1$, we get $f(x+y) - f(x) - f(y) = 0$ for all $x, y \in X$. This proves that f is additive.

Lemma 2 *Let X be a vector space and Y be a normed space, and let s be a complex number with $|s| < 1$. If a function $f : X \to Y$ satisfies*

$$\begin{aligned} &\|f(x+y)+f(x-y)-f(2x)-f(y)-f(-y)\| \\ &\leqslant \|s[f(x+y)-f(x)-f(y)]\|, \end{aligned} \tag{16}$$

for all $x, y \in X$, then f is additive.

Proof Letting $x = y = 0$ in (16), we get $f(0) = 0$. Letting $y = -x$ in (16) and using $f(0) = 0$, we get $\|f(x) + f(-x)\| \leqslant |s|\|f(x) + f(-x)\|$ for all $x \in X$. Since $|s| < 1$, we infer that f is odd. Therefore, (16) means

$$\begin{aligned} &\|f(x+y) + f(x-y) - f(2x)\| \\ &\leqslant \|s[f(x+y) - f(x) - f(y)]\|, \end{aligned} \tag{17}$$

for all $x, y \in X$. Letting $y = 0$ in (17), we get $f(2x) = 2f(x)$ for all $x \in X$. It follows from (17) that

$$\begin{aligned} 2\|f(x) + f(y) - f(x+y)\| &= \|f(2x) + f(2y) - f(2x+2y)\| \\ &\leqslant \|s[f(2x) - f(x+y) - f(x-y)]\| \\ &\leqslant |s|^2\|f(x) + f(y) - f(x+y)\|, \end{aligned}$$

for all $x, y \in X$. Since $|s| < 1$, we get $f(x+y) - f(x) - f(y) = 0$ for all $x, y \in X$. This proves that f is additive.

Theorem 9 *Let X be a normed space and Y be a Banach space. Suppose that s is a complex number with $2|s|^2 < 1$. Let a function $f : X \to Y$ satisfy*

$$\begin{aligned} &\|f(x+y) - f(x) - f(y)\| \\ &\leqslant \|s[f(x+y) + f(x-y) - f(2x) - f(y) - f(-y)]\| \\ &\quad + \varepsilon(\|x\|^r + \|y\|^r), \quad x, y \in X, \end{aligned} \tag{18}$$

for some nonnegative real numbers $r < 1$ and ε. Then, there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leqslant \frac{2\varepsilon}{(1-|s|)(2-2^r)}\|x\|^r, \quad x \in X. \tag{19}$$

Proof Letting $y = -x$ in (18), we get

$$\begin{aligned} &\|f(0) - f(x) - f(-x)\| \\ &\leqslant \|s[f(0) - f(x) - f(-x)]\| + 2\varepsilon\|x\|^r, \quad x \in X. \end{aligned}$$

Therefore, we have

$$\|f(0) - f(x) - f(-x)\| \leqslant \frac{2\varepsilon}{1-|s|}\|x\|^r, \quad x \in X. \tag{20}$$

Letting $y = x$ in (18) and using (20), we get

$$\|f(2x) - 2f(x)\| \leqslant \frac{2\varepsilon}{1-|s|}\|x\|^r, \quad x \in X. \tag{21}$$

Replacing x by $2^n x$ in (21) and dividing the resulting inequality by 2^{n+1}, we obtain

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}\right\| \leqslant \left(\frac{2^r}{2}\right)^n \frac{\varepsilon}{1-|s|}\|x\|^r, \quad x \in X,\ n \in \mathbb{N}.$$

Hence,

$$\begin{aligned}\left\|\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\right\| &= \left\|\sum_{k=m}^{n-1}\left[\frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k}\right]\right\| \\ &\leqslant \sum_{k=m}^{n-1}\left\|\frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k}\right\| \\ &\leqslant \frac{\varepsilon\|x\|^r}{1-|s|}\sum_{k=m}^{n-1}\left(\frac{2^r}{2}\right)^k, \quad x \in X,\end{aligned} \tag{22}$$

for all nonnegative integers m and n with $n > m$. It follows from (22) that the sequence $\{\frac{f(2^n x)}{2^n}\}_n$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{\frac{f(2^n x)}{2^n}\}_n$ converges. So, one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{f(2^n x)}{2^n}, \quad x \in X.$$

Moreover, letting $m = 0$ and passing the limit $n \to \infty$ (22), we get (19). We now show A is additive. It follows from the definition of A and (18) that

$$\begin{aligned}&\|A(x+y) - A(x) - A(y)\| \\ &= \lim_{n\to\infty}\frac{1}{2^n}\left\|f(2^n(x+y)) - f(2^n x) - f(2^n y)\right\| \\ &\leqslant \lim_{n\to\infty}\frac{1}{2^n}\left\|s[f(2^n x+2^n y)+f(2^n x-2^n y)-f(2^{n+1}x)-f(2^n y)-f(-2^n y)]\right\| \\ &\quad + \lim_{n\to\infty}\left(\frac{2^r}{2}\right)^n\varepsilon(\|x\|^r + \|y\|^r) \\ &= \|s[A(x+y) + A(x-y) - A(2x) - A(y) - A(-y)]\|, \quad x, y \in X.\end{aligned}$$

By Lemma 1, we infer that A is additive. Finally, it remains to prove the uniqueness of the additive mapping A. Assume that $T : X \to Y$ be another additive mapping satisfying (19). Then, we have

$$\begin{aligned}\|A(x)-T(x)\| &= \frac{1}{2^n}\Big\|A(2^n x)-T(2^n x)\Big\| \\ &\leqslant \frac{1}{2^n}\Big\|A(2^n x)-f(2^n x)\Big\| + \frac{1}{2^n}\Big\|f(2^n x)-T(2^n x)\Big\| \\ &\leqslant \Big(\frac{2^r}{2}\Big)^n \frac{4\varepsilon}{(1-|s|)(2-2^r)}\|x\|^r,\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So, we can conclude that $A = T$, and this proves the uniqueness of A, as desired.

Theorem 10 *Let X be a normed space and Y be a Banach space. Suppose that s is a complex number with $2|s|^2 < 1$. Let a function $f : X \to Y$ satisfy (18) for some nonnegative real numbers $r > 1$ and ε. Then, there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \leqslant \frac{2\varepsilon}{(1-|s|)(2^r-2)}\|x\|^r, \quad x \in X. \tag{23}$$

Proof A similar argument as in the proof of Theorem 9 yields the inequality (21). Replacing x by $x/2^{n+1}$ in (21) and multiplying the resulting inequality by 2^n, we obtain

$$\Big\|2^{n+1}f\Big(\frac{x}{2^{n+1}}\Big)-2^n f\Big(\frac{x}{2^n}\Big)\Big\| \leqslant \Big(\frac{2}{2^r}\Big)^{n+1}\frac{\varepsilon}{1-|s|}\|x\|^r, \quad x \in X,\ n \in \mathbb{N}.$$

Hence,

$$\begin{aligned}\Big\|2^n f\Big(\frac{x}{2^n}\Big)-2^m f\Big(\frac{x}{2^m}\Big)\Big\| &= \Big\|\sum_{k=m}^{n-1}\Big[2^{k+1}f\Big(\frac{x}{2^{k+1}}\Big)-2^k f\Big(\frac{x}{2^k}\Big)\Big]\Big\| \\ &\leqslant \sum_{k=m}^{n-1}\Big\|2^{k+1}f\Big(\frac{x}{2^{k+1}}\Big)-2^k f\Big(\frac{x}{2^k}\Big)\Big\| \\ &\leqslant \frac{\varepsilon\|x\|^r}{1-|s|}\sum_{k=m}^{n-1}\Big(\frac{2}{2^r}\Big)^{k+1}, \quad x \in X,\end{aligned} \tag{24}$$

for all nonnegative integers m and n with $n > m$. It follows from (24) that the sequence $\{2^n f(\frac{x}{2^n})\}_n$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^n f(\frac{x}{2^n})\}_n$ converges. So, one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f\Big(\frac{x}{2^n}\Big), \quad x \in X.$$

Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (24), we get (23). Since the rest of the proof is similar to the proof of Theorem 9, we omit the rest of the proof.

Remark 1 By using Gajda's function (see [6]), we infer that Theorems 9 and 10 are false for $r = 1$.

Theorem 11 *Let X be a normed space and Y be a Banach space. Suppose that s is a complex number with $|s| < 1$. Let a function $f : X \to Y$ satisfy*

$$\begin{aligned}
&\|f(x+y) + f(x-y) - f(2x) - f(y) - f(-y)\| \\
&\leqslant \|s[f(x+y) - f(x) - f(y)]\| \\
&\quad + \varepsilon(\|x\|^r + \|y\|^r), \quad x, y \in X,
\end{aligned} \tag{25}$$

for some nonnegative real numbers $r < 1$ and ε. Then, there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leqslant \frac{\varepsilon}{2 - 2^r}\|x\|^r, \quad x \in X. \tag{26}$$

Proof Letting $x = y = 0$ in (25), we get $f(0) = 0$. If we let $y = 0$ in (25), we have

$$\|f(2x) - 2f(x)\| \leqslant \varepsilon\|x\|^r, \quad x \in X. \tag{27}$$

Replacing x by $2^n x$ in (27) and dividing the resulting inequality by 2^{n+1}, we obtain

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}\right\| \leqslant \frac{\varepsilon}{2}\left(\frac{2^r}{2}\right)^n \|x\|^r, \quad x \in X, \ n \in \mathbb{N}.$$

Hence,

$$\begin{aligned}
\left\|\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\right\| &= \left\|\sum_{k=m}^{n-1}\left[\frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k}\right]\right\| \\
&\leqslant \sum_{k=m}^{n-1}\left\|\frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k}\right\| \\
&\leqslant \frac{\varepsilon\|x\|^r}{2}\sum_{k=m}^{n-1}\left(\frac{2^r}{2}\right)^k, \quad x \in X,
\end{aligned} \tag{28}$$

for all nonnegative integers m and n with $n > m$. It follows from (28) that the sequence $\{\frac{f(2^n x)}{2^n}\}_n$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{\frac{f(2^n x)}{2^n}\}_n$ converges. So, one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \quad x \in X.$$

Moreover, letting $m = 0$ and passing the limit $n \to \infty$ (28), we get (26). Since the rest of the proof is similar to the proof of Theorem 9, we omit the rest of the proof.

Theorem 12 *Let X be a normed space and Y be a Banach space. Suppose that s is a complex number with $|s| < 1$. Let a function $f : X \to Y$ satisfy (25) for some nonnegative real numbers $r > 1$ and ε. Then, there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leqslant \frac{\varepsilon}{2^r - 2}\|x\|^r, \quad x \in X. \tag{29}$$

Proof A similar argument as in the proof of Theorem 11 yields the inequality (27). Replacing x by $x/2^{n+1}$ in (21) and multiplying the resulting inequality by 2^n, we obtain

$$\left\|2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right)\right\| \leqslant \frac{\varepsilon}{2}\left(\frac{2}{2^r}\right)^{n+1}\|x\|^r, \quad x \in X,\ n \in \mathbb{N}.$$

Hence,

$$\begin{aligned}\left\|2^n f\left(\frac{x}{2^n}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &= \left\|\sum_{k=m}^{n-1}\left[2^{k+1} f\left(\frac{x}{2^{k+1}}\right) - 2^k f\left(\frac{x}{2^k}\right)\right]\right\| \\ &\leqslant \sum_{k=m}^{n-1}\left\|2^{k+1} f\left(\frac{x}{2^{k+1}}\right) - 2^k f\left(\frac{x}{2^k}\right)\right\| \\ &\leqslant \frac{\varepsilon\|x\|^r}{2}\sum_{k=m}^{n-1}\left(\frac{2}{2^r}\right)^{k+1}, \quad x \in X,\end{aligned} \tag{30}$$

for all nonnegative integers m and n with $n > m$. It follows from (30) that the sequence $\{2^n f\left(\frac{x}{2^n}\right)\}_n$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^n f\left(\frac{x}{2^n}\right)\}_n$ converges. So, one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right), \quad x \in X.$$

Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (30), we get (29). Since the rest of the proof is similar to the proof of Theorem 9, we omit the proof.

Remark 2 By using Gajda's function (see [6]), we infer that Theorems 11 and 12 are false for $r = 1$.

Theorem 13 *Let X be a normed space and Y be a Banach space. Suppose that s is a complex number with $|s|^2 < 2$ and that $\varphi : X \times X \to [0, +\infty)$ is a function such* *that $\varphi(x,0)=0$ for all $x \in X$ and satisfies one of the following conditions:*

$$\lim_{n\to\infty}\frac{\varphi(2^nx,2^ny)}{2^n}=0, \quad or \quad \lim_{n\to\infty}2^n\varphi\Big(\frac{x}{2^n},\frac{y}{2^n}\Big)=0, \quad x,y\in X.$$

Let a function $f: X\to Y$ satisfy

$$\begin{aligned}&\|f(x+y)+f(x-y)-f(2x)-f(y)-f(-y)\|\\&\leqslant \|s[f(x+y)-f(x)-f(y)]\|+\varphi(x,y), \quad x,y\in X.\end{aligned} \tag{31}$$

Then, f is additive.

Proof Letting $x=y=0$ in (33), we get $f(0)=0$. If we let $y=0$ in (33), we have $f(2x)=2f(x)$ for all $x\in X$. Two cases arise: If $\lim_{n\to\infty}\frac{\varphi(2^nx,2^ny)}{2^n}=0$, then

$$\begin{aligned}&\|f(x+y)+f(x-y)-f(2x)-f(y)-f(-y)\|\\&=\frac{1}{2^n}\|f(2^nx+2^ny)+f(2^nx-2^ny)-f(2^{n+1}x)-f(2^ny)-f(-2^ny)\|\\&\leqslant \frac{1}{2^n}\|s[f(2^nx+2^ny)-f(2^nx)-f(2^ny)]\|+\frac{1}{2^n}\varphi(2^nx,2^ny)\\&=\|s[f(x+y)-f(x)-f(y)]\|+\frac{1}{2^n}\varphi(2^nx,2^ny), \quad x,y\in X,\ n\in\mathbb{N}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|f(x+y)+f(x-y)-f(2x)-f(y)-f(-y)\|\\&\leqslant \|s[f(x+y)-f(x)-f(y)]\|+\frac{1}{2^n}\varphi(2^nx,2^ny), \quad x,y\in X,\ n\in\mathbb{N}.\end{aligned}$$

Letting $n\to+\infty$ in the above inequality, we obtain (16). Hence, we conclude that f is additive by Lemma 2.

If $\lim_{n\to\infty}2^n\varphi\Big(\frac{x}{2^n},\frac{y}{2^n}\Big)=0$, then

$$\begin{aligned}&\|f(x+y)+f(x-y)-f(2x)-f(y)-f(-y)\|\\&=2^n\Big\|f\Big(\frac{x}{2^n}+\frac{y}{2^n}\Big)+f\Big(\frac{x}{2^n}-\frac{y}{2^n}\Big)-f\Big(\frac{2x}{2^n}\Big)-f\Big(\frac{y}{2^n}\Big)-f\Big(-\frac{y}{2^n}\Big)\Big\|\\&\leqslant 2^n\Big\|s\Big[f\Big(\frac{x}{2^n}+\frac{y}{2^n}\Big)-f\Big(\frac{x}{2^n}\Big)-f\Big(\frac{y}{2^n}\Big)\Big]\Big\|+2^n\varphi\Big(\frac{x}{2^n},\frac{x}{2^n}\Big)\\&=\|s[f(x+y)-f(x)-f(y)]\|+2^n\varphi\Big(\frac{x}{2^n},\frac{x}{2^n}\Big), \quad x,y\in X,\ n\in\mathbb{N}.\end{aligned}$$

Therefore,

$$\|f(x+y)+f(x-y)-f(2x)-f(y)-f(-y)\|$$
$$\leqslant \|s[f(x+y)-f(x)-f(y)]\|+2^n\varphi\Big(\frac{x}{2^n},\frac{x}{2^n}\Big),\quad x,y\in X,\ n\in\mathbb{N}.$$

Letting $n\to+\infty$ in the above inequality, we obtain (16). Hence, we conclude that f is additive by Lemma 2.

Corollary 1 *Let X be a normed space and Y be a Banach space. Suppose that s is a complex number with $|s|<1$ and that α, β and γ are nonnegative real numbers satisfy one of the following conditions:*

$$\beta>0,\ \alpha+\beta,\gamma\in(0,1),\quad \text{or}\quad \beta>0,\ \alpha+\beta,\gamma\in(1,+\infty).$$

Let a function $f:X\to Y$ satisfy

$$\begin{aligned}&\|f(x+y)+f(x-y)-f(2x)-f(y)-f(-y)\|\\ &\leqslant \|s[f(x+y)-f(x)-f(y)]\|\\ &\quad+\varepsilon\|x\|^{\alpha}\|y\|^{\beta}+\theta\|y\|^{\gamma},\quad x,y\in X,\end{aligned}$$

for some nonnegative constants θ and ε. Then, f is additive.

Lemma 3 *Let X be a vector space and Y be a normed space, and let s be a complex number with $|s|<1$. If a function $f:X\to Y$ satisfies*

$$\begin{aligned}&\|f(x+y)+f(x-y)-f(2x)\|\\ &\leqslant \|s[f(x+y)-f(x)-f(y)]\|,\end{aligned}\tag{32}$$

for all $x,y\in X$, then f is additive.

Proof Letting $x=y=0$ in (32), we get $f(0)=0$. Letting $y=0$ in (32) and using $f(0)=0$, we get $f(2x)=2f(x)$ for all $x\in X$. It follows from (32) that

$$\begin{aligned}2\|f(x)+f(y)-f(x+y)\|&=\|f(2x)+f(2y)-f(2x+2y)\|\\ &\leqslant \|s[f(2x)-f(x+y)-f(x-y)]\|\\ &\leqslant |s|^2\|f(x)+f(y)-f(x+y)\|,\end{aligned}$$

for all $x,y\in X$. Since $|s|<1$, we get $f(x+y)-f(x)-f(y)=0$ for all $x,y\in X$. This proves that f is additive.

Applying a similar method given in the proof of Theorem 13, we obtain the following:

Theorem 14 *Let X be a normed space and Y be a Banach space. Suppose that s is a complex number with $|s| < 1$ and that $\varphi : X \times X \to [0, +\infty)$ is a function such that $\varphi(x, 0) = 0$ for all $x \in X$ and satisfies one of the following conditions:*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0, \quad \textit{or} \quad \lim_{n\to\infty} 2^n \varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) = 0, \quad x, y \in X.$$

Let a function $f : X \to Y$ satisfy

$$\begin{aligned} &\|f(x+y) + f(x-y) - f(2x)\| \\ &\leqslant \|s[f(x+y) - f(x) - f(y)]\| + \varphi(x, y), \quad x, y \in X. \end{aligned} \tag{33}$$

Then, f is additive.

Corollary 2 *Let X be a normed space and Y be a Banach space. Suppose that s is a complex number with $|s| < 1$ and that α, β and γ are nonnegative real numbers satisfy one of the following conditions:*

$$\beta > 0,\ \alpha + \beta, \gamma \in (0, 1), \quad \textit{or} \quad \beta > 0,\ \alpha + \beta, \gamma \in (1, +\infty).$$

Let a function $f : X \to Y$ satisfy

$$\begin{aligned} &\|f(x+y) + f(x-y) - f(2x)\| \\ &\leqslant \|s[f(x+y) - f(x) - f(y)]\| \\ &\quad + \varepsilon\|x\|^\alpha \|y\|^\beta + \theta\|y\|^\gamma, \quad x, y \in X, \end{aligned}$$

for some nonnegative constants θ and ε. Then, f is additive.

Lemma 4 *Let X be a vector space and Y be a normed space, and let s be a complex number with $2|s|^2 < 1$. If a function $f : X \to Y$ satisfies*

$$\begin{aligned} &\|f(x+y) - f(x) - f(y)\| \\ &\leqslant \|s[f(x+y) + f(x-y) - f(2x)]\|, \end{aligned} \tag{34}$$

for all $x, y \in X$, then f is additive.

Proof Letting $x = y = 0$ in (34), we get $f(0) = 0$. Letting $y = x$ in (34) and using $f(0) = 0$, we get $f(2x) = 2f(x)$ for all $x \in X$. It follows from (34) that

$$\begin{aligned} \|f(2x) - f(x+y) - f(x-y)\| &\leqslant |s|\|f(2x) + f(2y) - f(2x+2y)\| \\ &\leqslant 2|s|^2\|f(x+y) + f(x-y) - f(2x)\|, \end{aligned}$$

for all $x, y \in X$. Since $2|s|^2 < 1$, we get $f(x+y) + f(x-y) - f(2x) = 0$ for all $x, y \in X$. This proves that f is additive.

Proposition 1 *Let X be a vector space and Y be a normed space, and let s be a complex number with $2|s|^2 < 1$. Suppose that a function $f : X \to Y$ satisfies*

$$\begin{aligned} &\|f(x+y) - f(x) - f(y)\| \\ &\leqslant \|s[f(x+y) + f(x-y) - f(2x)]\| + \varepsilon\|x-y\|^r, \end{aligned} \tag{35}$$

for all $x, y \in X$, where ε and r are nonnegative real number with $r \neq 1$. Then, f is additive.

Proof Letting $x = y = 0$ in (35), we get $f(0) = 0$. Letting $y = x$ in (35) and using $f(0) = 0$, we get $f(2x) = 2f(x)$ for all $x \in X$. Two cases arise: If $r < 1$, it follows from (35) that

$$\begin{aligned} &\|f(x+y) - f(x) - f(y)\| \\ &= \frac{1}{2^n}\|f(2^n x + 2^n y) - f(2^n x) - f(2^n y)\| \\ &\leqslant \frac{|s|}{2^n}\|f(2^n x + 2^n y) + f(2^n x - 2^n y) - f(2^{n+1}x)\| + \frac{1}{2^n}\|2^n x - 2^n y\|^r \\ &= \|s[f(x+y) + f(x-y) - f(2x)]\| + \frac{1}{2^n}\|2^n x - 2^n y\|^r, \quad x, y \in X,\ n \in \mathbb{N}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|f(x+y) - f(x) - f(y)\| \\ &\leqslant \|s[f(x+y) + f(x-y) - f(2x)]\| + \frac{1}{2^n}\|2^n x - 2^n y\|^r, \quad x, y \in X,\ n \in \mathbb{N}. \end{aligned}$$

Letting $n \to +\infty$ in the above inequality, we get (34). Hence, f is additive by Lemma 4. If $r > 1$, it follows from (35) that

$$\begin{aligned} &\|f(x+y) - f(x) - f(y)\| \\ &= 2^n\left\|f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right\| \\ &\leqslant 2^n|s|\left\|f\left(\frac{x+y}{2^n}\right) + f(\left(\frac{x-y}{2^n}\right)) - f\left(\frac{2x}{2^n}\right)\right\| + 2^n\left\|\frac{x-y}{2^n}\right\|^r \\ &= \|s[f(x+y) + f(x-y) - f(2x)]\| + 2^n\left\|\frac{x-y}{2^n}\right\|^r, \quad x, y \in X,\ n \in \mathbb{N}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|f(x+y) - f(x) - f(y)\| \\ &\leqslant \|s[f(x+y) + f(x-y) - f(2x)]\| + 2^n\left\|\frac{x-y}{2^n}\right\|^r, \quad x, y \in X,\ n \in \mathbb{N}. \end{aligned}$$

Letting $n \to +\infty$ in the above inequality, we get (34). Hence, f is additive by Lemma 4.

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Stability of the Cosine–Sine Functional Equation on Amenable Groups

Ajebbar Omar and Elqorachi Elhoucien

Abstract In this paper, we establish the stability of the functional equation

$$f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y)$$

on amenable groups.

1 Introduction

The stability problem of functional equations goes back to 1940 when Ulam [14] proposed a question concerning the stability of group homomorphisms. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers's theorem was generalized by Aoki [3] for additive mappings and Rassias [10] for linear mappings by considering an unbounded Cauchy difference. The stability problem of several functional equations has been extensively investigated by a number of authors. An account on further progress and developments in this field can be found in [5, 7, 8].

In this paper, we investigate the stability of the trigonometric functional equation

$$f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y), \; x, y \in G \tag{1}$$

on amenable groups.

The continuous solutions of the trigonometric functional equations

$$f(xy) = f(x)g(y) + g(x)f(y), \; x, y \in G \tag{2}$$

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and

$$f(xy) = f(x)f(y) - g(x)g(y),\ x, y \in G \tag{3}$$

are obtained by Poulsen and Stetkær [9], where G is a topological group that need not be abelian. Regular solutions of (2) and (3) were described by Aczél [1] on abelian groups. Chung et al. [4] solved the functional equation (1) on groups. Recently, Ajebbar and Elqorachi [2] obtained the solutions of the functional equation (1) on a semigroup generated by its squares. The stability properties of the functional equations (2) and (3) have been obtained by Székelyhidi [13] on amenable groups.

The aim of the present paper is to extend the Székelyhidi's results [13] to the functional equation (1).

2 Definitions and Preliminaries

Throughout this paper, G denotes a semigroup (a set with an associative composition) or a group. We denote by $\mathscr{B}(G)$ the linear space of all bounded complex-valued functions on G. We call $a : G \to C$ additive provided that $a(xy) = a(x) + a(y)$ for all $x, y \in G$ and call $m : G \to C$ multiplicative provided that $m(xy) = m(x)m(y)$ for all $x, y \in G$.

Let $\mathscr{V}$ be a linear space of complex-valued functions on G. We say that the functions $f_1, \cdots, f_n : G \to C$ are linearly independent modulo $\mathscr{V}$ if $\lambda_1 f_1 + \cdots + \lambda_n f_n \in \mathscr{V}$ implies that $\lambda_1 = \cdots = \lambda_n = 0$ for any $\lambda_1, \cdots, \lambda_n \in C$. We say that the linear space $\mathscr{V}$ is two-sided invariant if $f \in \mathscr{V}$ implies that the functions $x \mapsto f(xy)$ and $x \mapsto f(yx)$ belong to $\mathscr{V}$ for any $y \in G$.

Notice that the linear space $\mathscr{B}(G)$ is two-sided invariant.

3 Basic Results

Throughout this section, G denotes a semigroup and $\mathscr{V}$ a two-sided invariant linear space of complex-valued functions on G.

Lemma 1 *Let $f, g, h : G \to C$ be functions. Suppose that f, g and h are linearly independent modulo $\mathscr{V}$. If the function*

$$x \mapsto f(xy) - f(x)g(y) - g(x)f(y) - h(x)h(y)$$

belongs to $\mathscr{V}$ for all $y \in G$, then there exist two functions $\varphi_1, \varphi_2 \in \mathscr{V}$ such that

$$\psi(x, y) = \varphi_1(x)f(y) + \varphi_2(x)h(y) \tag{4}$$

for all $x, y \in G$*, where*

$$\psi(x, y) := f(xy) - f(x)g(y) - g(x)f(y) - h(x)h(y) \tag{5}$$

for all $x, y \in G$.

Proof We use a similar computation as one of the proofs of [13, Lemma 2.1].

Since the functions f, g and h are linearly independent modulo $\mathcal{V}$ so are f and h, then f and h are linearly independent. Then, there exist $y_0, z_0 \in G$ such that $f(y_0)h(z_0) - f(z_0)h(y_0) \neq 0$, which implies that $f(y_0)h(z_0) \neq 0$ or $f(z_0)h(y_0) \neq 0$. We can finally assume that $f(y_0) \neq 0$ and $h(z_0) \neq 0$. By applying (5) to the pair (x, y_0), we derive

$$g(x) = \alpha_0\, f(x) + \alpha_1\, h(x) + \alpha_2\, f(xy_0) - \alpha_2\, \psi(x, y_0) \tag{6}$$

for all $x \in G$, where $\alpha_0 := -f(y_0)^{-1}g(y_0) \in C$, $\alpha_1 := -f(y_0)^{-1}h(y_0) \in C$ and $\alpha_2 := f(y_0)^{-1} \in C$ are constants. Similarly, by applying (5) to the pair (x, z_0), we get that

$$h(x) = \beta_0\, f(x) + \beta_1\, g(x) + \beta_2\, f(xz_0) - \beta_2\, \psi(x, z_0) \tag{7}$$

for all $x \in G$, where $\beta_0 := -h(z_0)^{-1}g(z_0) \in C$, $\beta_1 := -h(z_0)^{-1}f(z_0) \in C$ and $\beta_2 := h(z_0)^{-1} \in C$ are constants.

Let $x \in G$ be arbitrary. Substituting (7) into (6), we obtain

$$\begin{aligned} g(x) &= \alpha_0\, f(x) + \alpha_1\, [\beta_0\, f(x) + \beta_1\, g(x) + \beta_2\, f(xz_0) - \beta_2\, \psi(x, z_0)] \\ &\quad + \alpha_2\, f(xy_0) - \alpha_2\, \psi(x, y_0) \\ &\quad = (\alpha_0 + \alpha_1\, \beta_0)\, f(x) + \alpha_1\, \beta_1\, g(x) + \alpha_1\, \beta_2\, f(xz_0) - \alpha_1\, \beta_2\, \psi(x, z_0) \\ &\quad + \alpha_2\, f(xy_0) - \alpha_2\, \psi(x, y_0). \end{aligned}$$

So that

$$\begin{aligned} (1 - \alpha_1\beta_1)\, g(x) &= (\alpha_0 + \alpha_1\beta_0)\, f(x) + \alpha_1\beta_2\, f(xz_0) - \alpha_1\beta_2\, \psi(x, z_0) \\ &\quad + \alpha_2\, f(xy_0) - \alpha_2\, \psi(x, y_0). \end{aligned} \tag{8}$$

Since $f(y_0)h(z_0) - f(z_0)h(y_0) \neq 0$ and $f(y_0)h(z_0) \neq 0$, we get that $\alpha_1\, \beta_1 \neq 1$. So, x being arbitrary, we derive from (8) that there exist $\gamma_0,\ \gamma_1,\ \gamma_2 \in C$ such that

$$g(x) = \gamma_0\, f(x) + \gamma_1\, f(xy_0) + \gamma_2\, f(xz_0) - \gamma_1\, \psi(x, y_0) - \gamma_2\, \psi(x, z_0) \tag{9}$$

for all $x \in G$. Similarly, we prove that there exist $\delta_0,\ \delta_1,\ \delta_2 \in C$ such that

$$h(x) = \delta_0\, f(x) + \delta_1\, f(xy_0) + \delta_2\, f(xz_0) - \delta_1\, \psi(x, y_0) - \delta_2\, \psi(x, z_0) \tag{10}$$

for all $x \in G$. Let $x, y, z \in G$ be arbitrary. In the following, we compute $f(xyz)$ first as $f((xy)z)$ and then as $f(x(yz))$. By applying (5) to the pair (xy, z), and taking (9) and (10) into account, we obtain

$$\begin{aligned}
f((xy)z) &= f(xy)\, g(z) + g(xy)\, f(z) + h(xy)\, h(z) + \psi(xy, z)\\
&= [f(x)g(y) + g(x)\, f(y) + h(x)h(y) + \psi(x, y)]g(z)\\
&+[\gamma_0\, f(xy)+\gamma_1\, f(xyy_0)+\gamma_2\, f(xyz_0)-\gamma_1\, \psi(xy, y_0)-\gamma_2\, \psi(xy, z_0)]f(z)\\
&+[\delta_0\, f(xy)+\delta_1\, f(xyy_0)+\delta_2\, f(xyz_0)-\delta_1\, \psi(xy, y_0)-\delta_2\, \psi(xy, z_0)]h(z)\\
&+ \psi(xy, z)\\
&= [f(x)g(y) + g(x)f(y) + h(x)h(y) + \psi(x, y)]g(z)\\
&+ \gamma_0\, [f(x)g(y) + g(x)f(y) + h(x)h(y) + \psi(x, y)]f(z)\\
&+ \gamma_1\, [f(x)g(yy_0) + g(x)f(yy_0) + h(x)h(yy_0) + \psi(x, yy_0)]f(z)\\
&+ \gamma_2\, [f(x)g(yz_0) + g(x)f(yz_0) + h(x)h(yz_0) + \psi(x, yz_0)]f(z)\\
&+ \delta_0\, [f(x)g(y) + g(x)f(y) + h(x)h(y) + \psi(x, y)]h(z)\\
&+ \delta_1\, [f(x)g(yy_0) + g(x)f(yy_0) + h(x)h(yy_0) + \psi(x, yy_0)]h(z)\\
&+ \delta_2\, [f(x)g(yz_0) + g(x)f(yz_0) + h(x)h(yz_0) + \psi(x, yz_0)]h(z)\\
&-[\gamma_1\, \psi(xy, y_0)+\gamma_2\, \psi(xy, z_0)]f(z) - [\delta_1\, \psi(xy, y_0) + \delta_2\, \psi(xy, z_0)]h(z)\\
&+ \psi(xy, z).
\end{aligned}$$

So that

$$\begin{aligned}
f((xy)z) &= f(x)[g(y)g(z) + \gamma_0\, g(y)f(z) + \gamma_1\, g(yy_0)f(z) + \gamma_2\, g(yz_0)f(z)\\
&+ \delta_0\, g(y)h(z) + \delta_1\, g(yy_0)h(z) + \delta_2\, g(yz_0)h(z)]\\
&+ g(x)[f(y)g(z) + \gamma_0\, f(y)f(z) + \gamma_1\, f(yy_0)f(z) + \gamma_2\, f(yz_0)f(z)\\
&+ \delta_0\, f(y)h(z) + \delta_1\, f(yy_0)h(z) + \delta_2\, f(yz_0)h(z)]\\
&+ h(x)[h(y)g(z) + \gamma_0\, h(y)f(z) + \gamma_1\, h(yy_0)f(z) + \gamma_2\, h(yz_0)f(z)\\
&+ \delta_0\, h(y)h(z) + \delta_1\, h(yy_0)h(z) + \delta_2\, h(yz_0)h(z)]\\
&+ [\gamma_0\, \psi(x, y) + \gamma_1\, \psi(x, yy_0) + \gamma_2\, \psi(x, yz_0) - \gamma_1\, \psi(xy, y_0)\\
&- \gamma_2\, \psi(xy, z_0)]f(z) + \psi(x, y)g(z) + [\delta_0\, \psi(x, y) + \delta_1\, \psi(x, yy_0)\\
&+ \delta_2\, \psi(x, yz_0) - \delta_1\, \psi(xy, y_0) - \delta_2\, \psi(xy, z_0)]h(z) + \psi(xy, z).
\end{aligned} \tag{11}$$

On the other hand, by applying (5) to the pair (x, yz), we get that

$$f(x(yz)) = f(x)g(yz) + g(x)f(yz) + h(x)h(yz) + \psi(x, yz). \tag{12}$$

Now, let $y,\ z \in G$ be arbitrary. By assumption, the functions

$$x \mapsto \psi(x, y),\ x \mapsto \psi(x, yy_0),\ x \mapsto \psi(x, yz_0),\ x \mapsto \psi(x, yz)$$

belong to $\mathscr{V}$. Moreover, since the linear space $\mathscr{V}$ is two-sided invariant, the functions

$$x \mapsto \psi(xy, y_0),\ x \mapsto \psi(xy, z_0),\ x \mapsto \psi(xy, z)$$

belong to $\mathscr{V}$. Hence, by using (11), (12) and the fact that f, g and h are linearly independent modulo $\mathscr{V}$, we get that

$$f(yz) = f(y)g(z) + [\gamma_0 f(y) + \gamma_1 f(yy_0) + \gamma_2 f(yz_0)]f(z) \tag{13}$$

$$+[\delta_0 f(y) + \delta_1 f(yy_0) + \delta_2 f(yz_0)]h(z).$$

From (9), (10) and (13), we get

$$\begin{aligned} f(yz) &= f(y)g(z) + [g(y) + \gamma_1 \psi(y, y_0) + \gamma_2 \psi(y, z_0)]f(z) \\ &\quad + [h(y) + \delta_1 \psi(y, y_0) + \delta_2 \psi(y, z_0)]h(z) \\ &= f(y)g(z) + g(y)f(z) + h(y)h(z) + [\gamma_1 \psi(y, y_0) + \gamma_2 \psi(y, z_0)]f(z) \\ &\quad + [\delta_1 \psi(y, y_0) + \delta_2 \psi(y, z_0)]h(z). \end{aligned}$$

Hence, by using (5), we obtain

$$\psi(y, z) = [\gamma_1 \psi(y, y_0) + \gamma_2 \psi(y, z_0)]f(z) + [\delta_1 \psi(y, y_0) + \delta_2 \psi(y, z_0)]h(z).$$

So, y and z being arbitrary, we deduce (4) by putting

$$\varphi_1(x) = \gamma_1 \psi(x, y_0) + \gamma_2 \psi(x, z_0)$$

and

$$\varphi_2(x) = \delta_1 \psi(x, y_0) + \delta_2 \psi(x, z_0)$$

for all $x \in G$. This completes the proof of Lemma 1.

Lemma 2 *Let $f, g, h : G \to C$ be functions. Suppose that f and h are linearly independent modulo $\mathscr{V}$ and $g \in \mathscr{V}$. If the function*

$$x \mapsto f(xy) - f(x)g(y) - g(x)f(y) - h(x)h(y)$$

belongs to $\mathscr{V}$ for all $y \in G$, then g is multiplicative.

Proof Let $y, z \in G$ be arbitrary. By using the same computation as one of the proofs of Lemma 1, we obtain from (11) and (12), with the same notations, the following identity:

$$\begin{aligned}
&f(x)g(yz) + g(x)f(yz) + h(x)h(yz) + \psi(x, yz)\\
&= f(x)[g(y)g(z) + \gamma_0\, g(y)f(z) + \gamma_1\, g(yy_0)f(z) + \gamma_2\, g(yz_0)f(z) + \delta_0\, g(y)h(z)\\
&+\delta_1\, g(yy_0)h(z)+\delta_2\, g(yz_0)h(z)]+g(x)[f(y)g(z)+\gamma_0\, f(y)f(z)+\gamma_1\, f(yy_0)f(z)\\
&+\gamma_2\, f(yz_0)f(z)+\delta_0\, f(y)h(z)+\delta_1\, f(yy_0)h(z)+\delta_2\, f(yz_0)h(z)]+h(x)[h(y)g(z)\\
&+ \gamma_0\, h(y)f(z) + \gamma_1\, h(yy_0)f(z) + \gamma_2\, h(yz_0)f(z) + \delta_0\, h(y)h(z) + \delta_1\, h(yy_0)h(z)\\
&+ \delta_2\, h(yz_0)h(z)] + [\gamma_0\, \psi(x, y) + \gamma_1\, \psi(x, yy_0) + \gamma_2\, \psi(x, yz_0) - \gamma_1\, \psi(xy, y_0)\\
&- \gamma_2\, \psi(xy, z_0)]f(z) - \psi(x, y)g(z) + [\delta_0\, \psi(x, y) + \delta_1\, \psi(x, yy_0) + \delta_2\, \psi(x, yz_0)\\
&- \delta_1\, \psi(xy, y_0) - \delta_2\, \psi(xy, z_0)]h(z) + \psi(xy, z)
\end{aligned}$$

for all $x \in G$. So that

$$\begin{aligned}
&f(x)[g(y)g(z) + \gamma_0\, g(y)f(z) + \gamma_1\, g(yy_0)f(z) + \gamma_2\, g(yz_0)f(z) + \delta_0\, g(y)h(z)\\
&+ \delta_1\, g(yy_0)h(z) + \delta_2\, g(yz_0)h(z) - g(yz)] + h(x)[h(y)g(z) + \gamma_0\, h(y)f(z)\\
&+ \gamma_1\, h(yy_0)f(z) + \gamma_2\, h(yz_0)f(z) + \delta_0\, h(y)h(z) + \delta_1\, h(yy_0)h(z)\\
&+ \delta_2\, h(yz_0)h(z) - h(yz)]\\
&= - g(x)[f(y)g(z)+\gamma_0\, f(y)f(z)+\gamma_1\, f(yy_0)f(z)+\gamma_2\, f(yz_0)f(z)+\delta_0\, f(y)h(z)\\
&+ \delta_1\, f(yy_0)h(z) + \delta_2\, f(yz_0)h(z) - f(yz)] - [\gamma_0\, \psi(x, y) + \gamma_1\, \psi(x, yy_0)\\
&+ \gamma_2\, \psi(x, yz_0) - \gamma_1\, \psi(xy, y_0) - \gamma_2\, \psi(xy, z_0)]f(z)\\
&- [\delta_0\, \psi(x, y) + \delta_1\, \psi(x, yy_0) + \delta_2\, \psi(x, yz_0) - \delta_1\, \psi(xy, y_0) - \delta_2\, \psi(xy, z_0)]h(z)\\
&- \psi(xy, z) + \psi(x, yz)
\end{aligned} \tag{14}$$

for all $x \in G$. Since $g \in \mathscr{V}$, the function $x \mapsto \psi(x, t)$ belongs to $\mathscr{V}$ for all $t \in G$ and $\mathscr{V}$ is a two-sided invariant linear space of complex-valued functions on G, we get that the right-hand side of the identity (14) belongs to $\mathscr{V}$ as a function of x, so does the left-hand side of (14). Since f and h are linearly independent modulo $\mathscr{V}$, we get that

$$\begin{aligned}
&g(y)g(z) + \gamma_0\, g(y)f(z) + \gamma_1\, g(yy_0)f(z) + \gamma_2\, g(yz_0)f(z) + \delta_0\, g(y)h(z)\\
&+ \delta_1\, g(yy_0)h(z) + \delta_2\, g(yz_0)h(z) - g(yz) = 0.
\end{aligned} \tag{15}$$

So, y and z being arbitrary, we get that

$$\begin{aligned} g(yz) - g(y)g(z) &= [\gamma_0\, g(y) + \gamma_1\, g(yy_0) + \gamma_2\, g(yz_0)]f(z) \\ &\quad + [\delta_0\, g(y) + \delta_1\, g(yy_0) + \delta_2\, g(yz_0)]h(z) \end{aligned} \tag{16}$$

for all $y, z \in G$. Now, let $y \in G$ be arbitrary. Since $g \in \mathcal{V}$ and $\mathcal{V}$ is a two-sided invariant linear space of complex-valued functions on G, we derive from (16) that the function

$$z \mapsto [\gamma_0\, g(y) + \gamma_1\, g(yy_0) + \gamma_2\, g(yz_0)]f(z) + [\delta_0\, g(y) + \delta_1\, g(yy_0) + \delta_2\, g(yz_0)]h(z)$$

belongs to $\mathcal{V}$. Hence, seen that f and h are linearly independent modulo $\mathcal{V}$, we get that $\gamma_0\, g(y) + \gamma_1\, g(yy_0) + \gamma_2\, g(yz_0) = 0$ and $\delta_0\, g(y) + \delta_1\, g(yy_0) + \delta_2\, g(yz_0) = 0$. Substituting this back into (16), we obtain $g(yz) = g(y)g(z)$ for all $z \in G$. So, y being arbitrary, we deduce that g is multiplicative. This completes the proof of Lemma 2.

Lemma 3 *Let $f, g, h : G \to C$ be functions. Suppose that f and h are linearly dependent modulo $\mathcal{V}$. If the function*

$$x \mapsto f(xy) - f(x)g(y) - g(x)f(y) - h(x)h(y)$$

belongs to $\mathcal{V}$ for all $y \in G$, then we have one of the following possibilities:

(1) *$f = 0$, g is arbitrary and $h \in \mathcal{V}$;*
(2) *$f, g, h \in \mathcal{V}$;*
(3) *$g + \frac{\lambda^2}{2} f = m - \lambda\,\varphi$, $h - \lambda\, f = \varphi$, where $\lambda \in C$ is a constant, $\varphi \in \mathcal{V}$ and $m : G \to C$ is a multiplicative function such that $m \in \mathcal{V}$;*
(4) *$f = \alpha\, m - \alpha\, b$, $g = \frac{1-\alpha\,\lambda^2}{2}\, m + \frac{1+\alpha\,\lambda^2}{2}\, b - \lambda\,\varphi$, $h = \alpha\lambda\, m - \alpha\lambda\, b + \varphi$, where $\alpha, \lambda \in C$ are constants, $m : G \to C$ is a multiplicative function and $b, \varphi \in \mathcal{V}$;*
(5) *$f = f_0$, $g = g_0 - \frac{\lambda^2}{2} f_0 - \lambda\,\varphi$, $h = \lambda\, f_0 + \varphi$, where $\lambda \in C$ is a constant, $\varphi \in \mathcal{V}$ and $f_0, g_0 : G \to C$ satisfy the sine addition law*

$$f_0(xy) = f_0(x)g_0(y) + g_0(x)f_0(y), \;\; x, y \in G.$$

Proof Let ψ be the function defined in (5). If $f = 0$, then g is arbitrary and the function $x \mapsto h(x)h(y)$ belongs to $\mathcal{V}$ for all y in G. Hence, $h \in \mathcal{V}$. The result occurs in (1) of Lemma 3. In what follows, we assume that $f \neq 0$. We have the following cases:

Case 1: $h \in \mathcal{V}$. Then, the function $x \mapsto h(x)h(y)$ belongs to $\mathcal{V}$ for all y in G. So, the function $x \mapsto f(xy) - f(x)g(y) - g(x)f(y)$ belongs to $\mathcal{V}$ for all y in G. Hence, according to [13, Lemma 2.2] and taking into account that $f \neq 0$, we get that one of the following possibilities holds:

(i) $f, g, h \in \mathcal{V}$, which occurs in (2) of Lemma 3.
(ii) $g = m$ and $h = \varphi$, where $\varphi \in \mathcal{V}$ and $m : G \to C$ is a multiplicative function such that $m \in \mathcal{V}$. This is the result (3) of Lemma 3 for $\lambda = 0$.
(iii) $f = \alpha\, m - \alpha\, b$, $g = \frac{1}{2}m + \frac{1}{2}b$, $h = \varphi$, where $\alpha \in C$ is a constant, $m : G \to C$ is a multiplicative function and $b, \varphi \in \mathcal{V}$. This is the result (4) of Lemma 3 for $\lambda = 0$.
(iv) $f(xy) = f(x)g(y) + g(x)f(y)$ for all $x, y \in G$ and $h = \varphi$, where $\varphi \in \mathcal{V}$, which is the result (5) of Lemma 3 for $\lambda = 0$.

Case 2: $h \notin \mathcal{V}$. If $f \in \mathcal{V}$, then the function $x \mapsto f(xy)$ belongs to $\mathcal{V}$ for all $y \in G$, because the linear space $\mathcal{V}$ is two-sided invariant. As the function $x \mapsto \psi(x, y)$ belongs to $\mathcal{V}$ for all $y \in G$, we get that the function $x \mapsto g(x)f(y) + h(x)h(y)$ belongs to $\mathcal{V}$ for all $y \in G$. Since $h \notin \mathcal{V}$, we have $h \neq 0$. We derive that there exist a constant $\alpha \in C \setminus \{0\}$ and a function $k \in \mathcal{V}$ such that

$$h = \alpha\, g + k, \tag{17}$$

so that

$$\begin{aligned}
\psi(x, y) &= f(xy) - f(x)g(y) - g(x)f(y) - (\alpha\, g(x) + k(x))(\alpha\, g(y) + k(y)) \\
&= f(xy) - f(x)g(y) - g(x)f(y) - \alpha^2\, g(x)g(y) - \alpha\, g(x)k(y) - \alpha\, k(x)g(y) \\
&\quad - k(x)k(y) \\
&= f(xy) - k(x)k(y) - g(x)[f(y) + \alpha^2\, g(y) + \alpha\, k(y)] - g(y)[f(x) + \alpha\, k(x)] \\
&= f(xy) - k(x)k(y) - g(x)[f(y) + \alpha\, h(y)] - g(y)[f(x) + \alpha\, k(x)]
\end{aligned}$$

for all $x, y \in G$. Since the functions $x \mapsto f(xy)$, $x \mapsto k(x)k(y)$, $x \mapsto g(y)[f(x) + \alpha\, k(x)]$ and $x \mapsto \psi(x, y)$ belong to $\mathcal{V}$ for all $y \in G$, we derive from the identity above that the function $x \mapsto g(x)[f(y) + \alpha\, h(y)]$ belongs to $\mathcal{V}$ for all $y \in G$, which implies that $g \in \mathcal{V}$ or $f(y) + \alpha\, h(y) = 0$ for all $y \in G$. Hence, since $\alpha \in C \setminus \{0\}$, we get that $g \in \mathcal{V}$ or $h = -\dfrac{1}{\alpha} f$. So, taking (17) into account, we get that $h \in \mathcal{V}$; which contradicts the assumption on h, and hence $f \notin \mathcal{V}$. As f and h are linearly dependent modulo $\mathcal{V}$, we infer that there exist a constant $\lambda \in C \setminus \{0\}$ and a function $\varphi \in \mathcal{V}$ such that

$$h = \lambda\, f + \varphi. \tag{18}$$

So, we get from (5) that

$$
\begin{aligned}
\psi(x, y) &= f(xy) - f(x)g(y) - g(x)f(y) - (\lambda f(x) + \varphi(x))(\lambda f(y) + \varphi(y)) \\
&= f(xy) - f(x)g(y) - g(x)f(y) - \lambda^2 f(x)f(y) - \lambda f(x)\varphi(y) - \lambda \varphi(x)f(y) \\
&\quad - \varphi(x)\varphi(y) \\
&= f(xy) - \varphi(x)\varphi(y) - f(x)[g(y) + \frac{\lambda^2}{2} f(y) + \lambda \varphi(y)] \\
&\quad - [g(x) + \frac{\lambda^2}{2} f(x) + \lambda \varphi(x)]f(y),
\end{aligned}
$$

for all $x, y \in G$, which implies that

$$\psi(x, y) + \varphi(x)\varphi(y) = f(xy) - f(x)\phi(y) - \phi(x)f(y) \tag{19}$$

for all $x, y \in G$, where

$$\phi := g + \frac{\lambda^2}{2} f + \lambda \varphi. \tag{20}$$

Since $\varphi \in \mathcal{V}$ and the function $x \mapsto \psi(x, y)$ belongs to $\mathcal{V}$ for all $y \in G$, we get from (19) that the function

$$x \mapsto f(xy) - f(x)\phi(y) - \phi(x)f(y)$$

belongs to $\mathcal{V}$ for all $y \in G$. Moreover, $\mathcal{V}$ is a two-sided invariant linear space of complex-valued function. Hence, according to [13, Lemma 2.2] and taking into account that $f, h \notin \mathcal{V}$, we have one of the following possibilities:

(i) $\phi = m$, where $m \in \mathcal{V}$ is multiplicative. Then, we get, from (20) and (18), that $g + \frac{\lambda^2}{2} f = m - \lambda \varphi$ and $h - \lambda f = \varphi$, where $\varphi \in \mathcal{V}$. The result occurs in (3) of Lemma 3.

(ii) $f = \alpha m - \alpha b$, $\phi = \frac{1}{2}m + \frac{1}{2}b$, where $m : G \to C$ is multiplicative, $b : G \to C$ is in $\mathcal{V}$ and $\alpha \in C$ is a constant. Taking (20) and (18) into account, we obtain, respectively,

$$
\begin{aligned}
g &= \frac{1}{2}m + \frac{1}{2}b - \frac{\lambda^2}{2}(\alpha m - \alpha b) - \lambda \varphi \\
&= \frac{1 - \alpha \lambda^2}{2} m + \frac{1 + \alpha \lambda^2}{2} b - \lambda \varphi
\end{aligned}
$$

and

$$h = \alpha\lambda m - \alpha\lambda b + \varphi.$$

So, the result (4) of Lemma 3 holds.

(iii) $f(xy) = f(x)\phi(y) + \phi(x)f(y)$ for all $x, y \in G$. The result (5) of Lemma 3 holds easily by using the identities (18) and (20). This completes the proof of Lemma 3.

Lemma 4 *Let $f, g, h : G \to C$ be functions. Suppose that f and h are linearly independent modulo $\mathscr{V}$. If the functions*

$$x \mapsto f(xy) - f(x)g(y) - g(x)f(y) - h(x)h(y)$$

and

$$x \mapsto f(xy) - f(yx)$$

belong to $\mathscr{V}$ for all $y \in G$, then we have one of the following possibilities:

(1) $f = -\lambda^2 f_0 + \lambda^2 \varphi$, $g = \frac{1+\rho^2}{2} f_0 + \rho g_0 + \frac{1-\rho^2}{2} \varphi$, $h = \lambda\rho f_0 + \lambda g_0 - \lambda\rho \varphi$, *where $\lambda \in C \setminus \{0\}$ and $\rho \in C$ are constants, $\varphi \in \mathscr{V}$ and $f_0, g_0 : G \to C$ satisfy the cosine addition law*

$$f_0(xy) = f_0(x)f_0(y) - g_0(x)g_0(y)$$

for all $x, y \in G$;

(2)

$$f(xy) - \lambda^2 M(xy) = (f(x) - \lambda^2 M(x))m(y) + m(x)(f(y) - \lambda^2 M(y)) + \lambda^2 m(xy) + \psi(x, y)$$

for all $x, y \in G$,

$$g = \frac{1}{2}\beta^2 f + \beta h + m$$

and

$$\beta f + h = \lambda M - \lambda m,$$

where $\beta \in C$ and $\lambda \in C \setminus \{0\}$ are constants, $m, M : G \to C$ are multiplicative functions such that $m \in \mathscr{V}$, $M \notin \mathscr{V}$ and ψ is the function defined in (5);

(3)

$$f(xy) = f(x)m(y) + m(x)f(y) + H(x)H(y) + \psi(x, y),$$

$$g = \frac{1}{2}\beta^2 f + \beta h + m$$

and

$$H(xy)-m(x)H(y)-H(x)m(y) = \eta_1\,\psi(x,y)+\eta_2\,m(x)L_1(y)+\eta_3\,m(x)L_2(y)$$
$$+\eta_4\,\psi(x,l_1(y)) + \eta_5\,\psi(x,l_2(y)) + \eta_6\,L_1(xy) + \eta_7\,L_2(xy)$$

for all $x, y \in G$*, where* $\beta, \eta_1, \cdots, \eta_7 \in C$ *are constants,* $m : G \to C$ *is a multiplicative function in* $\mathscr{V}$*,* $L_1, L_2 \in \mathscr{V}$*,* $l_1, l_2 : G \to G$ *are mappings,* $H = \beta f + h$ *and* ψ *is the function defined in* (5)*;*

(4) $f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y)$ *for all* $x, y \in G$*.*

Proof We split the discussion into the cases of f, g and h are linearly dependent modulo $\mathscr{V}$ and f, g and h are linearly independent modulo $\mathscr{V}$.
Case A: f, g and h are linearly dependent modulo $\mathscr{V}$. Since f and h are linearly independent modulo $\mathscr{V}$, we get that there exist a function $\varphi \in \mathscr{V}$ and two constants $\alpha, \beta \in C$ such that

$$g = \alpha\, f + \beta\, h + \varphi. \tag{21}$$

By substituting (21) into (5), we obtain

$$\begin{aligned}\psi(x,y) &= f(xy) - f(x)[\alpha\, f(y) + \beta\, h(y) + \varphi(y)] - [\alpha\, f(x) + \beta\, h(x) + \varphi(x)]f(y)\\ &\quad - h(x)h(y)\\ &= f(xy) - 2\alpha\, f(x)f(y) - f(x)\varphi(y) - \varphi(x)f(y) - \beta\, f(x)h(y) - \beta\, h(x)f(y)\\ &\quad - h(x)h(y),\end{aligned}$$

for all $x, y \in G$, which implies that

$$\begin{aligned}\psi(x,y) = f(xy) - (2\alpha - \beta^2) f(x)f(y) - f(x)\varphi(y) - \varphi(x)f(y)\\ - [\beta\, f(x) + h(x)][\beta\, f(y) + h(y)]\end{aligned} \tag{22}$$

for all $x, y \in G$. We have the following subcases:
Subcase A.1: $2\alpha \neq \beta^2$. Let $x, y \in G$ be arbitrary, and let $\delta \in C \setminus \{0\}$ such that

$$\delta^2 = -(2\alpha - \beta^2). \tag{23}$$

Multiplying both sides of (22) by $-\delta^2$ and then adding $\varphi(xy) - \varphi(x)\varphi(y)$ to both sides of the identity obtained, we derive

$$\begin{aligned}&-\delta^2\,\psi(x,y) + \varphi(xy) - \varphi(x)\varphi(y) = -\delta^2\, f(xy) + \varphi(xy) - [\delta^4\, f(x)f(y)\\ &- \delta^2\, f(x)\varphi(y) - \delta^2\,\varphi(x)f(y) + \varphi(x)\varphi(y)] + \delta^2\,[\beta\, f(x) + h(x)][\beta\, f(y) + h(y)].\end{aligned}$$

So, x and y being arbitrary, we get from the identity above that

$$-\delta^2\,\psi(x,y)+\varphi(xy)-\varphi(x)\varphi(y)=f_0(xy)-f_0(x)f_0(y)+g_0(x)g_0(y), \tag{24}$$

for all $x, y \in G$, where

$$f_0 := -\delta^2\, f + \varphi \tag{25}$$

and

$$g_0 := \delta\,(\beta\, f + h). \tag{26}$$

Let y be arbitrary. As $\varphi \in \mathscr{V}$, the function $x \mapsto \varphi(x)\varphi(y)$ belongs to $\mathscr{V}$, and since the linear space $\mathscr{V}$ is two-sided invariant, we get that the function $x \mapsto \varphi(xy)$ belongs to $\mathscr{V}$. Moreover, by assumption the function $x \mapsto \psi(x, y)$ belongs to $\mathscr{V}$. Hence, the left-hand side of the identity (24) belongs to $\mathscr{V}$ as a function of x. So that the function

$$x \mapsto f_0(xy) - f_0(x)f_0(y) + g_0(x)g_0(y)$$

belongs to $\mathscr{V}$. On the other hand, by using (25), we have

$$f_0(xy) - f_0(yx) = -\delta^2\,(f(xy) - f(yx)) + \varphi(xy) - \varphi(yx)$$

for all $x \in G$. So, y being arbitrary, the function $x \mapsto f_0(xy) - f_0(yx)$ belongs to $\mathscr{V}$ for all $y \in G$ because the functions $x \mapsto f(xy) - f(yx)$ and $x \mapsto \varphi(xy) - \varphi(yx)$ do. Moreover, f_0 and g_0 are linearly independent modulo $\mathscr{V}$ because f and h are. Hence, we get, according to [13, Lemma 1], that

$$f_0(xy) = f_0(x)f_0(y) - g_0(x)g_0(y)$$

for all $x, y \in G$. By putting $\lambda = \dfrac{1}{\delta}$, we get, from (25), that

$$f = -\lambda^2\, f_0 + \lambda^2\varphi. \tag{27}$$

By putting $\rho = \beta\lambda$, we get, from (26), that $h = \lambda\, g_0 - \beta\,(-\lambda^2\, f_0 + \lambda^2\varphi)$, which implies that

$$h = \lambda\rho\, f_0 + \lambda\, g_0 - \lambda\rho\,\varphi. \tag{28}$$

So, we derive from (21), (27) and (28) that

$$\begin{aligned} g &= \alpha\,(-\lambda^2 f_0 + \lambda^2 \varphi) + \beta\,(\lambda\rho\, f_0 + \lambda\, g_0 - \lambda\rho\,\varphi) + \varphi \\ &= (-\alpha\lambda^2 + \beta\lambda\rho)\, f_0 + \beta\lambda\, g_0 + (\alpha\lambda^2 - \beta\lambda\rho + 1)\,\varphi \\ &= (-\alpha\lambda^2 + \rho^2)\, f_0 + \rho\, g_0 + (\alpha\lambda^2 - \rho^2 + 1)\,\varphi. \end{aligned}$$

Using (23), we find, by elementary computations, that $\alpha\lambda^2 = \frac{1}{2}\rho^2 - \frac{1}{2}$. Hence, from the identity above, we get that

$$g = \frac{1+\rho^2}{2}\, f_0 + \rho\, g_0 + \frac{1-\rho^2}{2}\,\varphi.$$

The result obtained in this case occurs in (1) of Lemma 4.
Subcase A.2: $2\,\alpha = \beta^2$. In this case, the identity (22) becomes

$$\psi(x, y) = f(xy) - f(x)\varphi(y) - \varphi(x) f(y) - H(x)H(y) \tag{29}$$

for all $x, y \in G$, where

$$H := \beta\, f + h. \tag{30}$$

Since f and h are linearly independent modulo $\mathscr{V}$ so are f and H. Moreover, $\varphi \in \mathscr{V}$. Hence, according to Lemma 2, there exists a multiplicative function $m : G \to C$ in $\mathscr{V}$ such that $\varphi = m$. So, the identities (21) and (29) become, respectively,

$$g = \frac{1}{2}\beta^2\, f + \beta\, h + m. \tag{31}$$

and

$$\psi(x, y) = f(xy) - f(x)m(y) - m(x) f(y) - H(x)H(y) \tag{32}$$

for all $x, y \in G$. We use similar computations to the ones in the proof of [4, Theorem]. Let $x, y, z \in G$ be arbitrary. First, we compute $f(xyz)$ as $f(x(yz))$ and then as $f((xy)z)$. From (32), we get that

$$\begin{aligned} f(x(yz)) &= f(x)m(yz) + m(x) f(yz) + H(x)H(yz) + \psi(x, yz) \\ &= f(x)m(yz) + m(x)[f(y)m(z) + m(y) f(z) + H(y)H(z) + \psi(y, z)] \\ &\quad + H(x)H(yz) + \psi(x, yz), \end{aligned}$$

so that

$$\begin{aligned} f(x(yz)) &= f(x)m(yz) + m(xz) f(y) + m(xy) f(z) + m(x)H(y)H(z) \\ &\quad + m(x)\psi(y, z) + H(x)H(yz) + \psi(x, yz). \end{aligned} \tag{33}$$

On the other hand,

$$\begin{aligned} f((xy)z) &= f(xy)m(z) + m(xy)f(z) + H(xy)H(z) + \psi(xy, z) \\ &= [f(x)m(y) + m(x)f(y) + H(x)H(y) + \psi(x, y)]m(z) + m(xy)f(z) \\ &+ H(xy)H(z) + \psi(xy, z), \end{aligned}$$

and hence

$$\begin{aligned} f((xy)z) = f(x)m(yz) + m(xz)f(y) + m(xy)f(z) + H(x)H(y)m(z) \\ + m(z)\psi(x, y) + H(xy)H(z) + \psi(xy, z). \end{aligned} \tag{34}$$

From (33) and (34), we get that

$$\begin{aligned} &H(x)[H(yz)-H(y)m(z)-m(y)H(z)]-H(z)[H(xy)-m(x)H(y)-H(x)m(y)] \\ &= m(z)\psi(x, y) - m(x)\psi(y, z) + \psi(xy, z) - \psi(x, yz), \end{aligned} \tag{35}$$

for all $x, y, z \in G$. Since f and H are linearly independent modulo $\mathcal{V}$, they are, in particular, linearly independent. So, there exist $z_1, z_2 \in G$ such that

$$f(z_1)H(z_2) - f(z_2)H(z_1) \neq 0. \tag{36}$$

Let $x, y \in G$ be arbitrary. By putting $z = z_1$ and then $z = z_2$ in (35), we get, respectively,

$$H(x)k_i(y) - H(z_i)[H(xy) - H(x)m(y) - m(x)H(y)] = \psi_i(x, y), \tag{37}$$

where

$$k_i(y) := H(yz_i) - H(y)m(z_i) - m(y)H(z_i)$$

and

$$\psi_i(x, y) := m(z_i)\psi(x, y) - m(x)\psi(y, z_i) - \psi(x, yz_i) + \psi(xy, z_i) \tag{38}$$

for $i = 1, 2$. Multiplying both sides of (37) by $f(z_2)$ for $i = 1$, and by $f(z_1)$ for $i = 2$, and then subtracting the identities obtained, we get that

$$H(x)k_3(y)+[f(z_1)H(z_2)-f(z_2)H(z_1)][H(xy)-H(x)m(y)-m(x)H(y)]=\psi_3(x, y), \tag{39}$$

where

$$k_3(y) := f(z_2)k_1(y) - f(z_1)k_2(y)$$

and

$$\psi_3(x, y) := f(z_2)\psi_1(x, y) - f(z_1)\psi_2(x, y). \tag{40}$$

So, x and y being arbitrary, we get, taking (36) and (39) into account, that

$$H(xy) - H(x)m(y) - m(x)H(y) = H(x)k(y) + \Phi(x, y) \tag{41}$$

for all $x, y \in G$, where

$$k(x) := -[f(z_1)H(z_2) - f(z_2)H(z_1)]^{-1}k_3(x)$$

and

$$\Phi(x, y) := [f(z_1)H(z_2) - f(z_2)H(z_1)]^{-1}\psi_3(x, y) \tag{42}$$

for all $x, y \in G$. Substituting (41) into (35), we get that

$$\begin{aligned} &H(x)[H(y)k(z) + \Phi(y, z)] - H(z)[H(x)k(y) + \Phi(x, y)] \\ &= m(z)\psi(x, y) - m(x)\psi(y, z) + \psi(xy, z) - \psi(x, yz), \end{aligned}$$

which implies that

$$\begin{aligned} H(x)[H(y)k(z) - H(z)k(y) + \Phi(y, z)] = H(z)\Phi(x, y) + m(z)\psi(x, y) \\ - m(x)\psi(y, z) + \psi(xy, z) - \psi(x, yz) \end{aligned} \tag{43}$$

for all $x, y, z \in G$. Now, let $y, z \in G$ be arbitrary. Since $\mathscr{V}$ is a two-sided invariant linear space of complex-valued functions on G, and the functions $x \mapsto m(x)$ and $x \mapsto \psi(x, y)$ belong to $\mathscr{V}$, we deduce from (38), (40) and (42) that the functions $x \mapsto \Phi(x, y)$ and $x \mapsto \psi_i(x, y)$ belong to $\mathscr{V}$ for $i = 1, 2, 3$. Hence, the right-hand side of (43) belongs to $\mathscr{V}$ as a function of x. It follows that the left-hand side of (43) belongs to $\mathscr{V}$ as a function of x. As f and H are linearly independent modulo $\mathscr{V}$, we derive, from (43), that $H(y)k(z) - H(z)k(y) + \Phi(y, z) = 0$. So, y and z being arbitrary, we get that

$$H(z)k(x) = H(x)k(z) + \Phi(x, z) \tag{44}$$

for all $x, z \in G$.
On the other hand, we deduce from (36) that $f(z_1)H(z_2) \neq 0$ or $f(z_2)H(z_1) \neq 0$, so we can assume, without loss of generality, that $H(z_1) \neq 0$. Replacing z by z_1 in the identity (44), we derive that

$$k(x) = \gamma\, H(x) + \Phi_1(x) \tag{45}$$

for all $x \in G$, where $\gamma := H(z_1)^{-1}k(z_1)$ and

$$\Phi_1(x) := H(z_1)^{-1}\Phi(x, z_1) \tag{46}$$

for all $x \in G$. From (41) and (45) we get that

$$H(xy) = H(x)m(y)+m(x)H(y)+\gamma H(x)H(y)+H(x)\Phi_1(y)+\Phi(x, y) \tag{47}$$

for all $x, y \in G$. Since the functions m and $x \mapsto \Phi(x, y)$ belong to $\mathcal{V}$ for all $y \in G$, we get, from (47), that the function

$$x \mapsto H(xy) - H(x)[m(y) + \Phi_1(y) + \gamma H(y)] \tag{48}$$

belongs to $\mathcal{V}$ for all $y \in G$. As $H \notin \mathcal{V}$, we get from (48), according to [12, Theorem], that there exists a multiplicative function $M : G \to C$ such that

$$m + \Phi_1 + \gamma H = M. \tag{49}$$

We have the following subcases:

Case A.2.1: $\gamma \neq 0$. Putting $\lambda = \dfrac{1}{\gamma} \in C \setminus \{0\}$, we obtain from (49) the identity

$$H = \lambda M - \lambda m - \lambda \Phi_1. \tag{50}$$

Let $x, y \in G$ be arbitrary. Since m and M are multiplicative, we get from the identity above that $H(xy) - H(yx) = \lambda \Phi_1(yx) - \lambda \Phi_1(xy)$. Taking (47) into account, we get that $H(x)\Phi_1(y) - H(y)\Phi_1(x) + \Phi(x, y) - \Phi(y, x) = \lambda \Phi_1(yx) - \lambda \Phi_1(xy)$. So, x and y being arbitrary, we obtain

$$H(x)\Phi_1(y) = H(y)\Phi_1(x) + \Phi(y, x) - \Phi(x, y) + \lambda \Phi_1(yx) - \lambda \Phi_1(xy) \tag{51}$$

for all $x, y \in G$. Now, let y be arbitrary. As seen earlier, the functions Φ_1 and $x \mapsto \Phi(x, y) - \Phi(y, x)$ belong to $\mathcal{V}$. So, $\mathcal{V}$ being a two-sided invariant linear space of complex-valued functions on G, we get from (51) that the function $x \mapsto H(x)\Phi_1(y)$ belongs to $\mathcal{V}$. Taking into account that f and H are linearly independent, we get $\Phi_1(y) = 0$. So, y being arbitrary, we obtain $\Phi_1 = 0$. Hence, using (50), we get that

$$H = \lambda M - \lambda m. \tag{52}$$

Substituting this back into (32), we get, by an elementary computation, that

$$\begin{aligned} f(xy) - \lambda^2 M(xy) &= (f(x) - \lambda^2 M(x))m(y) + m(x)(f(y) - \lambda^2 M(y)) \\ &\quad + \lambda^2 m(xy) + \psi(x, y), \end{aligned} \tag{53}$$

for all $x, y \in G$. We conclude from (30), (31), (52) and (53) that the result (2) of Lemma 4 holds.

Case A.2.2: $\gamma = 0$. Let $y \in G$ be arbitrary. The identity (45) implies that $k = \Phi_1$. Hence, we derive from (44) that

$$H(x)\Phi_1(y) = H(y)\Phi_1(x) - \Phi(x, y),$$

for all $x \in G$. Since the function $x \mapsto \Phi(x, y)$ belongs to $\mathscr{V}$, we get, taking the identity above and (46) into account, that the function $x \mapsto H(x)\Phi_1(y)$ belongs to $\mathscr{V}$. As f and H are linearly independent modulo $\mathscr{V}$, we infer that $\Phi_1(y) = 0$. So, y being arbitrary, we get that $\Phi_1 = 0$. Hence, the identity (47) becomes

$$H(xy) = m(x)H(y) + H(x)m(y) + \Phi(x, y). \tag{54}$$

On the other hand, by using (38), (40) and (42) with the same notations, we derive that there exist $\eta_i \in C$ with $i = 1, \cdots, 7$ such that
$\Phi(x, y) = \eta_1\, \psi(x, y) + \eta_2\, m(x)\psi(y, z_1) + \eta_3\, m(x)\psi(y, z_2) + \eta_4\, \psi(x, yz_1) + \eta_5\, \psi(x, yz_2) + \eta_6\, \psi(xy, z_1) + \eta_7\, \psi(xy, z_2)$
$x, y \in G$. We get that

$$\begin{aligned}\Phi(x, y) = \eta_1\, \psi(x, y) &+ \eta_2\, m(x)L_1(y) + \eta_3\, m(x)L_2(y) + \eta_4\, \psi(x, l_1(y)) \\ &+ \eta_5\, \psi(x, l_2(y)) + \eta_6\, L_1(xy) + \eta_7\, L_2(xy)\end{aligned} \tag{55}$$

for all $x, y \in G$, where

$$L_i(x) := \psi(x, z_i)$$

for $i = 1, 2$ and for all $x \in G$, and $l_i : G \to G$ is defined for $i = 1, 2$ by $l_i(x) = xz_i$ for all $x \in G$. Hence, we get, from (54) and (51), the identity

$$\begin{aligned}H(xy) - m(x)H(y) - H(x)m(y) &= \eta_1\, \psi(x, y) + \eta_2\, m(x)L_1(y) + \eta_3\, m(x)L_2(y) \\ + \eta_4\, \psi(x, l_1(y)) + \eta_5\, \psi(x, l_2(y)) &+ \eta_6\, L_1(xy) + \eta_7\, L_2(xy)\end{aligned} \tag{56}$$

for all $x, y \in G$.

We conclude from (30), (31), (32) and (56) that the result (3) of Lemma 4 holds.

Case B: f, g and h are linearly independent modulo $\mathscr{V}$. Then, according to Lemma 1, there exist two functions $\varphi_1, \varphi_2 \in \mathscr{V}$ satisfying (4), where ψ is the function defined in (5). Let $y \in G$ be arbitrary. Since the functions $x \mapsto \psi(x, y)$ and $x \mapsto f(xy) - f(yx)$ belong to $\mathscr{V}$ by assumption, so does the function $x \mapsto \psi(y, x)$. Seeing that $\psi(y, x) = \varphi_1(y)f(x) + \varphi_2(y)h(x)$, and that f and h are linearly independent modulo $\mathscr{V}$, we get that $\varphi_1(y) = \varphi_2(y) = 0$. So, y being arbitrary, we deduce that $\psi(x, y) = 0$ for all $x, y \in G$. Then, the result (4) of Lemma 4 holds. This completes the proof of Lemma 4.

4 Stability of Equation (1) on Amenable Groups

Throughout this section, G is an amenable group with an identity element that we denote e. We will extend the Székelyhidi's results [13, Theorem 2.3], about the stability of the functional equation (2), to the functional equation (1).

Theorem 1 *Let $f, g, h : G \to C$ be functions. The function*

$$(x, y) \mapsto f(xy) - f(x)g(y) - g(x)f(y) - h(x)h(y)$$

is bounded if and only if one of the following assertions holds:

(1) $f = 0$*, g is arbitrary and $h \in \mathscr{B}(G)$;*
(2) $f, g, h \in \mathscr{B}(G)$*;*
(3)

$$\begin{cases} f = a\,m + \varphi, \\ g = (1 - \frac{\lambda^2}{2}\,a)m - \lambda\,b - \frac{\lambda^2}{2}\,\varphi, \\ h = \lambda\,a\,m + b + \lambda\,\varphi, \end{cases}$$

where $\lambda \in C$ is a constant, $a : G \to C$ is an additive function, $m : G \to C$ is a bounded multiplicative function and $b, \varphi : G \to C$ are two bounded functions;

(4)

$$\begin{cases} f = \alpha\,m - \alpha\,b, \\ g = \frac{1-\alpha\lambda^2}{2}\,m + \frac{1+\alpha\lambda^2}{2}\,b - \lambda\,\varphi, \\ h = \alpha\lambda\,m - \alpha\lambda\,b + \varphi, \end{cases}$$

where $\alpha, \lambda \in C$ are two constants, $m : G \to C$ is a multiplicative function and $b, \varphi : G \to C$ are two bounded functions;

(5)

$$\begin{cases} f = f_0, \\ g = g = g_0 - \frac{\lambda^2}{2}\,f_0 - \lambda\,b, \\ h = \lambda\,f_0 + b, \end{cases}$$

where $\lambda \in C$ is a constant, $b : G \to C$ is a bounded function and $f_0, g_0 : G \to C$ are functions satisfying the sine addition law

$$f_0(xy) = f_0(x)g_0(y) + g_0(x)f_0(y), \;\; x, y \in G;$$

(6)

$$\begin{cases} f = -\lambda^2 f_0 + \lambda^2 b, \\ g = \frac{1+\rho^2}{2} f_0 + \rho g_0 + \frac{1-\rho^2}{2} b, \\ h = \lambda\rho f_0 + \lambda g_0 - \lambda\rho b, \end{cases}$$

where $\rho \in C$ and $\lambda \in C \setminus \{0\}$ are two constants, $b : G \to C$ is a bounded function and $f_0, g_0 : G \to C$ are functions satisfying the cosine addition law

$$f_0(xy) = f_0(x)f_0(y) - g_0(x)g_0(y), \ \ x, y \in G;$$

(7)

$$\begin{cases} f = \lambda^2 M + am + b, \\ g = \beta\lambda(1 - \frac{1}{2}\beta\lambda)M + (1 - \beta\lambda)m - \frac{1}{2}\beta^2 a m - \frac{1}{2}\beta^2 b, \\ h = \lambda(1 - \beta\lambda)M - \lambda m - \beta a m - \beta b, \end{cases}$$

where $\beta \in C$ and $\lambda \in C \setminus \{0\}$ are two constants, $m, M : G \to G$ are two multiplicative functions such that m is bounded, $a : G \to C$ is an additive function and $b : G \to C$ is a bounded function;

(8)

$$\begin{cases} f = \frac{1}{2}a^2 m + \frac{1}{2}a_1 m + b, \\ g = -\frac{1}{4}\beta^2 a^2 m + \beta a m - \frac{1}{4}\beta^2 a_1 m + m - \frac{1}{2}\beta^2 b, \\ h = -\frac{1}{2}\beta a^2 m + a m - \frac{1}{2}\beta a_1 m - \beta b, \end{cases}$$

where $\beta \in C$ is a constant, $m : G \to C$ is a nonzero bounded multiplicative function, $a, a_1 : G \to C$ are two additive functions such that $a \neq 0$ and $b : G \to C$ is a bounded function;

(9) *$g = -\frac{1}{2}\beta^2 f + (1 + \beta a)m + \beta b$ and $h = -\beta f + a m + b$, where $\beta \in C$ is a constant and $a : G \to C$ is an additive function, $m : G \to C$ is a nonzero bounded multiplicative function and $b : G \to C$ is a bounded function such that the function*

$$(x, y) \mapsto f(xy)m((xy)^{-1}) - \frac{1}{2}a^2(xy) - (f(x)m(x^{-1}) - \frac{1}{2}a^2(x))$$
$$- (f(y)m(y^{-1}) - \frac{1}{2}a^2(y)) - a(x)b(y)m(y^{-1}) - a(y)b(x)m(x^{-1})$$

is bounded;

(10) *$f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y)$ for all $x, y \in G$.*

Proof First, we prove the necessity. Applying Lemma 3(1), Lemma 3(2), Lemma 3(4), Lemma 3(5), Lemma 4(1) and Lemma 4(4) with $\mathscr{V} = \mathscr{B}(G)$, we get that

either one of the conditions (1), (2), (4), (5), (6) and (10) in Theorem 1 is satisfied or we have one of the following cases:
Case A:

$$g + \frac{\lambda^2}{2} f = m - \lambda b$$

and

$$h - \lambda f = b,$$

where $\lambda \in C$ is a constant, $b : G \to C$ is a bounded function and $m : G \to C$ is a bounded multiplicative function. From (5) and the identities above, we obtain, by an elementary computation,

$$g = -\frac{\lambda^2}{2} f + m - \lambda b, \tag{57}$$

$$h = \lambda f + b \tag{58}$$

and

$$f(xy) - f(x)m(y) - m(x)f(y) = \psi(x, y) + b(x)b(y) \tag{59}$$

for all $x, y \in G$. If $m \neq 0$, then, by multiplying both sides of (59) by $m((xy)^{-1})$, and using the fact that m is a bounded multiplicative function, and that the functions b and ψ are bounded, we get that the function $(x, y) \mapsto f(xy)m((xy)^{-1}) - f(x)m(x^{-1}) - f(y)m(y^{-1})$ is bounded. Notice that we have the same result if $m = 0$. So, according to Hyers's theorem [11, Theorem 3.1], there exist an additive function $a : G \to C$ and a function $\varphi_0 \in \mathscr{B}(G)$ such that $f(x)m(x^{-1}) - a(x) = b_0(x)$ for all $x \in G$. Then, by putting $\varphi = m\,\varphi_0$, we get that $f = a\,m + \varphi$ with $\varphi \in \mathscr{B}(G)$. Substituting this back into (57) and (58), we obtain, by an elementary computation, that $g = (1 - \frac{\lambda^2}{2} a)m - \lambda b - \frac{\lambda^2}{2} \varphi$ and $h = \lambda\, a\, m + b + \lambda\, \varphi$. So, the result (3) of Theorem 1 holds.
Case B:

$$f(xy) - \lambda^2 M(xy) = (f(x) - \lambda^2 M(x))m(y) + m(x)(f(y) - \lambda^2 M(y)) \\ + \lambda^2 m(xy) + \psi(x, y)$$

for all $x, y \in G$,

$$g = \frac{1}{2}\beta^2 f + \beta h + m$$

and

$$\beta f + h = \lambda M - \lambda m,$$

where $\beta \in C$ and $\lambda \in C \setminus \{0\}$ are constants, $m, M : G \to C$ are multiplicative functions such that $m \in \mathscr{B}(G)$, $M \notin \mathscr{B}(G)$ and ψ is the function defined in (5). If $m \neq 0$, then, by multiplying both sides of the first identity above by $m((xy)^{-1})$ and using that m is multiplicative, we get that

$$
\begin{aligned}
&(f(xy) - \lambda^2 M(xy))m((xy)^{-1})\\
&= (f(x)-\lambda^2 M(x))m(x^{-1})+(f(y)-\lambda^2 M(y))m(y^{-1})+\lambda^2+m((xy)^{-1})\psi(x, y)
\end{aligned}
$$

for all $x, y \in G$. Since the functions m and ψ are bounded, then we get from the identity above that the function

$$
\begin{aligned}
&(x, y) \mapsto (f(xy) - \lambda^2 M(xy))m((xy)^{-1}) - (f(x) - \lambda^2 M(x))m(x^{-1})\\
&- (f(y) - \lambda^2 M(y))m(y^{-1})
\end{aligned}
$$

is bounded. Notice that we have the same result if $m = 0$. So, according to Hyers's theorem [11, Theorem 3.1], there exist an additive function $a : G \to C$ and a function $b_0 \in \mathscr{B}(G)$ such that

$$(f(x) - \lambda^2 M(x))m(x^{-1}) - a(x) = b_0(x)$$

for all $x \in G$. By putting $b = m\, b_0$, we derive that

$$f = \lambda^2 M + a\, m + b$$

with $b \in \mathscr{B}(G)$. As $g = \frac{1}{2}\beta^2 f + \beta h + m$ and $\beta f + h = \lambda M - \lambda m,$, we obtain

$$
\begin{aligned}
h &= -\beta(\lambda^2 M + a\, m + b) + \lambda M - \lambda m\\
&= \lambda(1 - \beta\lambda)M - \lambda m - \beta\, a\, m - \beta\, b
\end{aligned}
$$

and

$$
\begin{aligned}
g &= \frac{1}{2}\beta^2(\lambda^2 M + a\, m + b) + \beta(\lambda(1 - \beta\lambda)M - \lambda m - \beta\, a\, m - \beta\, b) + m\\
&= \beta\lambda(1 - \frac{1}{2}\beta\lambda)M + (1 - \beta\lambda)m - \frac{1}{2}\beta^2\, a\, m - \frac{1}{2}\beta^2\, b.
\end{aligned}
$$

The result occurs in (7) of Theorem 1.
Case C:

$$f(xy) = f(x)m(y) + m(x)f(y) + H(x)H(y) + \psi(x, y),$$

$$H(xy) - H(x)m(y) - m(x)H(y) = \eta_1\,\psi(x, y) + \eta_2\,m(x)L_1(y) + \eta_3\,m(x)L_2(y)$$
$$+ \eta_4\,\psi(x, l_1(y)) + \eta_5\,\psi(x, l_2(y)) + \eta_6\,L_1(xy) + \eta_7\,L_2(xy)$$

for all $x, y \in G$,

$$g = \frac{1}{2}\beta^2 f + \beta h + m$$

and

$$H = \beta f + h,$$

and where $\beta, \eta_1, \cdots, \eta_7 \in C$ are constants, $m : G \to C$ is a bounded multiplicative function, $L_1, L_2 \in \mathscr{B}(G)$, $l_1, l_2 : G \to G$ are mappings and ψ is the function defined in (5).

If $H \in \mathscr{B}(G)$, then f and h are linearly dependent modulo $\mathscr{B}(G)$. So, according to Lemma 3, one of the assertions (1)–(5) of Theorem 1 holds.

In what follows, we assume that $H \notin \mathscr{B}(G)$. Since the functions m, L_1, L_2 and ψ are bounded, we get from the second identity above that the function

$$(x, y) \mapsto H(xy) - H(x)m(y) - m(x)H(y)$$

is bounded. Hence, $m \neq 0$ because $H \notin \mathscr{B}(G)$. Then, according to [13, Theorem 2.3] and taking the assumption on H into account, we have one of the following subcases:

Subcase C.1: $H = a\,m + b$, where $a : G \to C$ is additive and $b \in \mathscr{B}(G)$. Then, $\beta f + h = a\,m + b$, which implies that

$$h = -\beta f + a\,m + b.$$

Moreover, since $g = \frac{1}{2}\beta^2 f + \beta h + m$, we get that

$$g = -\frac{1}{2}\beta^2 f + m + \beta\,a\,m + \beta\,b.$$

Let $x, y \in G$ be arbitrary. By using the first identity in the present case, we get that

$$\psi(x, y) = f(xy) - f(x)m(y) - m(x)f(y) - (a(x)m(x) + b(x))(a(y)m(y) + b(y))$$
$$= f(xy) - f(x)m(y) - m(x)f(y) - a(x)a(y)m(xy) - m(x)a(x)b(y)$$
$$- m(y)a(y)b(x) - b(x)b(y).$$

Since m is a nonzero multiplicative function on the group G, we have $m(xy) = m(x)m(y) \neq 0$ and $m((xy)^{-1}) = m(x^{-1})m(y^{-1}) = (m(x))^{-1}(m(y))^{-1}$. Hence, by multiplying both sides of the identity above by $m((xy)^{-1})$, we get that

$$m((xy)^{-1})[\psi(x,y)b(x)b(y)] = f(xy)m((xy)^{-1}) - f(x)m(x^{-1}) - f(y)m(y^{-1})$$
$$- a(x)a(y) - a(x)b(y)m(y^{-1}) - a(y)b(x)m(x^{-1})$$
$$= (f(xy)m((xy)^{-1}) - \frac{1}{2}a^2(xy)) - (f(x)m(x^{-1}) - \frac{1}{2}a^2(x))$$
$$- (f(y)m(y^{-1}) - \frac{1}{2}a^2(y)) - a(x)b(y)m(y^{-1}) - a(y)b(x)m(x^{-1}).$$

So, x and y being arbitrary and the functions m, b and ψ being bounded, we deduce that the function

$$(x,y) \mapsto f(xy)m((xy)^{-1}) - \frac{1}{2}a^2(xy) - (f(x)m(x^{-1}) - \frac{1}{2}a^2(x))$$
$$- (f(y)m(y^{-1}) - \frac{1}{2}a^2(y)) - a(x)b(y)m(y^{-1}) - a(y)b(x)m(x^{-1})$$

is bounded. The result occurs in (9) of the list of Theorem 1.
Subcase C.2: $H(xy) = H(x)m(y) + H(y)m(x)$ for all $x, y \in G$. Since m is a nonzero multiplicative function on the group G, we have $m(x) \neq 0$ for all $x \in G$. Then, in view of $H \notin \mathscr{B}(G)$, we get from the last functional equation that there exists a nonzero additive function $a : G \to C$ such that $H = a\,m$. Substituting this back in the first identity in the present case and proceeding exactly as in Subcase C.1, we get that the function

$$(x,y) \mapsto 2f(xy)m((xy)^{-1}) - a^2(xy) - (2f(x)m(x^{-1}) - a^2(x))$$
$$- (2f(y)m(y^{-1}) - a^2(y))$$

is bounded. Hence, according to Hyers's theorem [11, Theorem 3.1], there exist an additive function $a_1 : G \to C$ and a function $b_0 \in \mathscr{B}(G)$ such that $2f(x)m(x^{-1}) - a^2(x) = a_1(x) + b_0(x)$ for all $x, y \in G$. So, by putting $b = \frac{1}{2}m\,b_0$, we deduce that $b \in \mathscr{B}(G)$ because $m, b_0 \in \mathscr{B}(G)$ and

$$f = \frac{1}{2}a^2\,m + \frac{1}{2}a_1\,m + b. \tag{60}$$

Since $H = \beta\,f + h$ and $g = \frac{1}{2}\beta^2\,f + \beta\,h + m$, we get, by using (60) and an elementary computation, that $g = -\frac{1}{4}\beta^2\,a^2\,m + \beta\,a\,m - \frac{1}{4}\beta^2\,a_1\,m + m - \frac{1}{2}\beta^2\,b$ and $h = -\frac{1}{2}\beta\,a^2\,m + a\,m - \frac{1}{2}\beta\,a_1\,m - \beta\,b$. The result occurs in (8) of the list of Theorem 1.

Conversely, we check by elementary computations that if one of the assertions (1)–(10) in Theorem 1 is satisfied, then the function $(x,y) \mapsto f(xy) - f(x)g(y) - g(x)f(y) - h(x)h(y)$ is bounded. This completes the proof of Theorem 1.

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Introduction to Halanay Lemma, via Weakly Picard Operator Theory

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Abstract In this paper, we present an introduction to Halanay lemma, from the weakly Picard operator theory point of view.

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1 Introduction

For $\alpha > \beta > 0$, $h > 0$ and $t_0 \in \mathbb{R}$, let $x \in C([t_0 - h, \infty[)$ with $x\big|_{[t_0,\infty[} \in C^1([t_0, \infty[)$ be a solution of the following inequation:

$$x'(t) \leq -\alpha x(t) + \beta \max_{\theta \in [t-h,t]} x(\theta),\, t \in [t_0, \infty[.$$

By Halanay's lemma ([14]), there exists $k > 0$, $\gamma > 0$ such that

$$x(t) \leq ke^{-\gamma(t-t_0)},\ \text{for all } t \in [t_0 - h, \infty[.$$

There exist some proofs of Halanay's Lemma ([9, 12, 14],...) and a large number of papers on Hanalay's lemma and Halanay-type lemma ([3, 4, 7, 9, 13, 15, 17, 18, 29, 30], ...).

The aim of this paper is to present an introduction to Halanay's lemma from the weakly Picard operator theory point of view.

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2 Preliminaries

In this section, we will present several basic notions and results that are important for a good understanding of our main theorems.

The Operator $\max\limits_I$ *on a Space of Continuous Functions*

Let $t_0 \in \mathbb{R}$ and $a, b \in C([t_0, +\infty[)$ be two mappings such that $a(t) < b(t),\ \forall\, t \geq t_0$. Let $a_0 := \inf\{a(t) \mid t \geq t_0\}$. We suppose that $-\infty < a_0 < t_0$. Let $I(t) = [a(t), b(t)]$, for each $t \geq t_0$. Now we consider the operator

$$\max_I : C([a_0, +\infty[) \to C([t_0, +\infty[),$$

defined by

$$\max_I(x)(t) := \max_{\theta \in I(t)} x(\theta).$$

Lemma 1 *For the above* $\max\limits_I$ *operator, we have the following properties:*

(i) *the* $\max\limits_I$ *operator is increasing;*

(ii) $\max\limits_I(kx) = k \max(x),\ \forall\, k \in \mathbb{R}_+^*, \forall\, x \in C([a_0, +\infty[)$*;*

(iii) $|\max\limits_I(x)(t) - \max\limits_I(y)(t)| \leq \max\limits_I(|x - y|)(t),\ \forall\, x, y \in C([a_0, +\infty[)$.

For examples of $\max\limits_I$ operators, see [1, 2, 8, 9, 11, 14, 20, 21],...

Functional Differential Equations with Maxima

Let $f \in C([t_0, +\infty[\times \mathbb{R}^2)$. We consider the functional differential equation

$$x'(t) = f(t, x(t), \max_I(x)(t)),\ t \geq t_0. \tag{1}$$

By a solution of this equation, we understand a function

$$x \in C([a_0, +\infty[) \cap C^1([t_0, +\infty[) := \{x \in C([a_0, +\infty[) \,\big|\, x|_{[t_0, +\infty[} \in C^1([t_0, +\infty[)\},$$

which satisfies (1).

The Cauchy problem for (1) is the following: given $\varphi \in C([a_0, t_0])$, the problem is to study the solution x of (1), for which

$$x\big|_{[a_0,t_0]} = \varphi.$$

For this Cauchy problem, see [1, 2, 5, 9, 14, 16, 19, 24],...

Halanay Functional Differential Equation with Maxima

In [14] (pp. 378–380), Halanay has considered the following equation:

$$x'(t) = -\alpha x(t) + \beta \max_{\theta \in [t-h,t]} x(\theta), \ t \geq t_0, \tag{2}$$

where $\alpha > \beta > 0$, $h > 0$ and $t_0 \in \mathbb{R}$.

In equation (2), the interval function is $I(t) = [t-h, t]$. In this case,

$$a(t) = t - h, \ b(t) = t \text{ and } a_0 = t_0 - h.$$

In his work [14], Halanay remarked that there exists $\gamma > 0$ such that the function $x \in C([t_0-h, +\infty[)$ given by $x(t) := e^{-\gamma(t-t_0)}$ is a solution of (2) on $[t_0-h, +\infty[$.

Weakly Picard Operators on $(X, \to)$

We will present now the basic notion related to the weakly Picard operator theory in the general context of an L-space in the sense of Fréchet.

Let X be a nonempty set. Denote by $\Delta(X)$ the diagonal of $X \times X$. We also denote by $s(X) := \{(x_n)_{n\in N} | x_n \in X, \ n \in N\}$ the set of all sequences in X.

Let $c(X) \subset s(X)$ a subset of $s(X)$ and $\mathrm{Lim} : c(X) \to X$ an operator. By definition, the triple $(X, c(X), \mathrm{Lim})$ is called an L-space ([10]) if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in N$, then $(x_n)_{n\in N} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in N} = x$.

(ii) If $(x_n)_{n\in N} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in N} = x$, then for all subsequences, $(x_{n_i})_{i\in N}$, of $(x_n)_{n\in N}$ we have that $(x_{n_i})_{i\in N} \in c(X)$ and $\mathrm{Lim}(x_{n_i})_{i\in N} = x$.

By definition, an element of $c(X)$ is a convergent sequence and $x := \mathrm{Lim}(x_n)_{n\in N}$ is the limit of this sequence. In this case, we can also write $x_n \to x$ as $n \to +\infty$.

Throughout this paper, we denote an L-space by $(X, \to)$.

Recall now the following important abstract concept.

Definition 1 (I.A. Rus [26]) Let $(X, \to)$ be an L-space. An operator $A : X \to X$ is, by definition, a weakly Picard operator (briefly WPO) if:

(i) $F_A \neq \emptyset$;
(ii) for each $x \in X$, the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges to an element $x^*(x) \in F_A$ as $n \to \infty$.

In particular, if $F_A = \{x^*\}$, then A is said to be a Picard operator (briefly PO).

If $(X, \rightarrow)$ is an L-space and A is a weakly Picard operator, then the following set retraction can be defined

$$A^\infty : X \rightarrow F_A,\ A^\infty(x) := \text{Lim}(A^n(x))_{n \in N}.$$

Let $(X, \preceq)$ be an ordered set and $A : X \rightarrow X$. Then we denote by

$$(LF)_A := \{x \in X : x \preceq A(x)\} \text{ respectively } (UF)_A := \{x \in X : A(x) \preceq x\},$$

the lower (respectively upper) fixed point set of A.

Let $(X, \rightarrow)$ be an L-space and $\preceq$ be an order relation on X. If the following implication holds

$$x_n \preceq y_n, \text{ for all } n \in \mathbb{N}, x_n \rightarrow x \in X, y_n \rightarrow y \in X \Rightarrow x \preceq y,$$

then the triple $(X, \rightarrow, \preceq)$ is called an ordered L-space.

The following results were given in [26] and [21]. See also [23, 27, 28]

Theorem 1 (Abstract Gronwall Lemma) *Let $(X, \rightarrow, \preceq)$ be an ordered L-space and $A : X \rightarrow X$ be an operator. Suppose that:*

(i) *A is a PO (we denote by x_A^* its unique fixed point);*
(ii) *A is increasing.*

Then:

(a) *$u \preceq x_A^*$, for every $u \in (LF)_A$;*
(b) *$x_A^* \preceq v$, for every $v \in (UF)_A$.*

In the case of weakly Picard operators, the following Gronwall-type lemma holds.

Theorem 2 *Let $(X, \rightarrow, \preceq)$ be an ordered L-space and $A : X \rightarrow X$ be an operator. Suppose that:*

(i) *A is a WPO;*
(ii) *A is increasing.*

Then:

(a) *for each $x \in X$ with $x \preceq A(x) \ \Rightarrow \ x \preceq A^\infty(x)$;*
(b) *for each $x \in X$ with $x \succeq A(x) \ \Rightarrow \ x \succeq A^\infty(x)$.*

In the same setting, the following abstract comparison theorems take place.

Theorem 3 (Abstract Gronwall-Comparison Lemma) *Let $(X, \rightarrow, \preceq)$ be an ordered L-space and $A, B : X \rightarrow X$ be two operators. Suppose that:*

(i) *A and B are POs (we denote by x_A^*, respectively, x_B^* their unique fixed point);*
(ii) *A is increasing;*
(iii) $A(x) \preceq B(x)$, *for every* $x \in X$.

Then, for each $x \in X$ *with* $x \preceq A(x) \;\Rightarrow\; x \preceq x_B^*$.

In the case of weakly Picard operators, the following theorems hold.

Theorem 4 *Let* $(X, \to, \preceq)$ *be an ordered L-space and* $A, B : X \to X$ *be two operators. Suppose that:*

(i) *A and B are WPOs;*
(ii) *A is increasing;*
(iii) $A(x) \preceq B(x)$, *for every* $x \in X$.

Then, the following conclusions hold:

(a) *for every* $x, y \in X$ *with* $x \preceq y \;\Rightarrow\; A^\infty(x) \preceq B^\infty(y)$;
(b) *if, additionally, the operator B is increasing, then for each* $x \in X$ *such that* $x \succeq A(x) \;\Rightarrow\; x \succeq B^\infty(x)$.

Theorem 5 (Abstract Comparison Lemmna) *Let* $(X, \to, \preceq)$ *be an ordered L-space and* $A, B, C : X \to X$ *be three operators. Suppose that:*

(i) *A, B, and C are WPOs;*
(ii) *the operator B is increasing;*
(iii) $A(x) \preceq B(x) \preceq C(x)$, *for every* $x \in X$.

Then, for every $x, y, z \in X$ *with* $x \preceq y \preceq z \;\Rightarrow\; A^\infty(x) \preceq B^\infty(y) \preceq C^\infty(z)$.

In particular, if (X, d) is a metric space and $\to$ is the metric convergence, the following concrete lemmas hold.

Theorem 6 *Let* $(X, d, \preceq)$ *be an ordered and complete metric space and* $A : X \to X$ *be an operator with closed graph. Suppose that:*

(i) *A is a graphic contraction, i.e., there exists* $\alpha \in [0, 1[$ *such that*

$$d(A(x), A^2(x)) \le \alpha d(x, A(x)), \text{ for every } x \in X;$$

(ii) *A is increasing.*

Then:

(a) *A is a WPO;*
(b) *for each* $x \in X$ *with* $x \preceq A(x) \;\Rightarrow\; x \preceq A^\infty(x)$;
(c) *for each* $x \in X$ *with* $x \succeq A(x) \;\Rightarrow\; x \succeq A^\infty(x)$.

Theorem 7 *Let* $(X, d, \preceq)$ *be an ordered and complete metric space and* $A, B : X \to X$ *be two operators with closed graph. Suppose that:*

(i) *A and B are graphic contractions;*
(ii) *A is increasing;*

(iii) *$A(x) \preceq B(x)$, for every $x \in X$.*

Then, the following conclusions hold:

(a) *A and B are WPOs;*
(b) *for every $x, y \in X$ with $x \preceq y \Rightarrow A^\infty(x) \preceq B^\infty(y)$;*
(c) *if, additionally, the operator B is increasing, then for each $x \in X$ such that $x \succeq A(x) \Rightarrow x \succeq B^\infty(x)$;*

Fiber Contraction Theorem

The following result will be an important tool in our approach.

Theorem 8 (Fiber Contraction Principle) *Let (X, d) and (Y, ρ) be two metric spaces, such that ρ is a complete metric on Y. Let $A : X \times Y \to X \times Y$ given by*

$$A(x, y) := (B(x), C(x, y))$$

be a triangular operator, i.e., $B : X \to X$ and $C : X \times Y \to Y$. Suppose that:

(i) *B is a WPO;*
(ii) *there exists $\alpha \in [0, 1[$ such that the operator $C(x, \cdot) : Y \to Y$ is an α-contraction;*
(iii) *if $(x^*, y^*) \in F_A$, then the operator $C(\cdot, y^*) : Y \to Y$ is continuous in x^*.*

Then, the following conclusions hold:

(a) *A is a WPO in the L-space $(X \times Y, \to)$, where $\to$ denotes the termwise convergence;*
(b) *if, additionally, B is a PO, then A is a PO too.*

3 The Cauchy Problem for Halanay Equation

The Cauchy problem for equation (2) is equivalent to the following functional integral equation with maxima:

$$x(t) = \begin{cases} \varphi(t), \ t \in [t_0 - h, t_0], \\ e^{-\alpha(t-t_0)}\varphi(t_0) + \beta \displaystyle\int_{t_0}^{t} e^{\alpha(s-t)} \max_{\theta \in [s-h,s]} x(\theta) ds, \ t \geq t_0, \end{cases} \tag{3}$$

in the space $C([t_0 - h, +\infty[)$.

Now we consider the operators

$$A_\varphi : C([t_0 - h, +\infty[) \to C([t_0 - h, +\infty[),$$

defined by

$$A_\varphi(x)(t) := \text{ the right hand side of (3)},$$

and for each $T > t_0$, the operator

$$A_{\varphi,T} : C[t_0 - h, T] \to C[t_0 - h, T],$$

defined by

$$A_{\varphi,T}(x)(t) := \text{ the right hand side of (3)},$$

for $t \in [t_0 - h, T]$.

In what follows, we shall prove that the operator $A_{\varphi,T}$ has a unique fixed point, for each $T > t_0$, i.e., the operator A_φ has a unique fixed point, i.e., the Cauchy problem for the Halanay equation has a unique solution. For this, we use the Burton method of progressive contractions ([6]; see also forward step method in [22] and step by step contraction principle in [25]), in terms of max-norm. In the case of equations with $\max\limits_I$ operator, we cannot use the Bielecki norm technique.

Theorem 9 *The Cauchy problem for Halanay equation has in $C([t_0 - h, +\infty[) \cap C^1([t_0, +\infty[)$ a unique solution.*

Proof Let $T > t_0$. Let $m \in \mathbb{N}^*$ be such that

$$l := \frac{\beta(T - t_0)}{m} < 1 \quad \text{and} \quad h \geq \frac{T - t_0}{m}.$$

We denote

$$t_1 := t_0 + \frac{T - t_0}{m}, \ldots, t_k := t_0 + k\frac{T - t_0}{m}, \ldots, t_m := T.$$

We remark that the operator A_{φ,t_1} is an l-contraction. Let x_1^* its unique fixed point. For

$$C_{x_1^*}[t_0 - h, t_2] := \{x \in C[t_0 - h, t_2] \mid x\big|_{[t_0 - h, t_1]} = x_1^*\},$$

we consider the operator

$$A_{\varphi,t_2} : C_{x_1^*}[t_0 - h, t_2] \to C_{x_1^*}[t_0 - h, t_2].$$

This operator is an l-contraction. Let x_2^* be its unique fixed point. We remark that

$$x_2^*\Big|_{[t_0-h,t_1]} = x_1^*.$$

By induction, we consider the operator

$$A_{\varphi,t_{k+1}} : C_{x_k^*}[t_0-h,t_{k+1}] \to C_{x_k^*}[t_0-h,t_{k+1}],\ k=\overline{1,m-1},$$

which is an l-contraction with x_{k+1}^* its unique fixed point and

$$x_{k+1}^*\Big|_{[t_0-h,t_k]} = x_k^*.$$

So, x_m^* is the unique fixed point of $A_{\varphi,T}$.

Remark 1 By the fiber contraction theorem, it follows that A_φ is PO with respect to the uniform convergence on each compact subinterval of $[t_0-h,+\infty[$. For to prove this, it is sufficient to prove that for each $T > t_0$ the operator $A_{\varphi,T}$ is PO with respect to uniform convergence on $[t_0-h,T]$. For to do this, we shall use fiber contractions principle.

First, let $U := C[t_0-h,t_1]$ and $V := C[t_1,t_2]$ and (3) writing in the following form:

$$x(t) = \begin{cases} \varphi(t),\ t\in[t_0-h,t_0], \\ e^{-\alpha(t-t_0)}\varphi(t_0) + \beta\displaystyle\int_{t_0}^{t} e^{\alpha(s-t)} \max_{\theta\in[s-h,s]} x(\theta)ds,\ t\in[t_0,t_1], \\ e^{-\alpha(t-t_0)}\varphi(t_0) + \beta\displaystyle\int_{t_0}^{t_1} e^{\alpha(s-t)} \max_{\theta\in[s-h,s]} x(\theta)ds \\ \qquad\qquad + \beta\displaystyle\int_{t_1}^{t} e^{\alpha(s-t)} \max_{\theta\in[s-h,s]} x(\theta)ds,\ t\in[t_1,t_2). \end{cases} \tag{4}$$

If we denote $u := x\Big|_{[t_0-h,t_1]}$ and $v := x\Big|_{[t_1,t_2]}$, then we can write (4) in the following form:

$$u(t) = \begin{cases} \varphi(t),\ t\in[t_0-h,t_0], \\ e^{-\alpha(t-t_0)}\varphi(t_0) + \beta\displaystyle\int_{t_0}^{t} e^{\alpha(s-t)} \max_{\theta\in[s-h,s]} u(\theta)ds,\ t\in[t_0,t_1], \end{cases}$$

$$v(t) = e^{-\alpha(t-t_0)}\varphi(t_0) + \beta\int_{t_0}^{t_1} e^{\alpha(s-t)} \max_{\theta\in[s-h,s]} u(\theta)ds$$
$$+\beta\int_{t_1}^{t} e^{\alpha(s-t)} \max\left(\max_{\theta\in[s-h,t_1]} u(\theta),\ \max_{t\in[t_1,s]} v(\theta)\right) ds,\ t\in[t_1,t_2]$$

or

$$\begin{cases} u = A_1(u) \\ v = A_2(u, v), \end{cases}$$

where $A_1 := A_{\varphi,t_1}$ and $A_2 : U \times V \to U$ defined by

$$A_2(u, v)(t) := e^{-\alpha(t-t_0)}\varphi(t_0) + \beta \int_{t_0}^{t_1} \max u(\theta) ds$$
$$+ \beta \int_{t_1}^{t} e^{\alpha(s-t)} \max \left(\max_{\theta \in [s-t_0, t_1]} u(\theta), \max_{\theta \in [t_1, s]} v(\theta) \right) ds.$$

From the fiber contraction principle, we have that the operator

$$A : U \times V \to U \times V,\ A(u, v) = (A_1(u), A_2(u, v))$$

is PO.

From the definition of A, we have that the operator A_{φ,t_2} is PO. By a similar way, we prove that A_{φ,t_3} is PO, choosing

$$U := C[t_0 - h, t_2],\ V := C[t_2, t_3],\ A_1 := A_{\varphi,t_2},$$

and A_2 suitably defined. By induction, we prove that $A_{\varphi,T}$ is a PO.

Remark 2 Since A_φ is PO and is increasing, from Abstract Gronwall lemma we have that

$$x \in C([t_0 - h, +\infty[),\ x \le A_\varphi(x) \Rightarrow x \le x^*,$$

and

$$x \in C([t_0 - h, +\infty[),\ x \ge A_\varphi(x) \Rightarrow x \ge x^*.$$

Remark 3 Let us consider the operator

$$E : C([t_0 - h, +\infty[) \to C([t_0 - h, +\infty[)$$

defined by

$$E(x)(t) := \begin{cases} x(t),\ t \in [t_0 - h, t_0], \\ e^{-\alpha(t-t_0)} x(t_0) + \displaystyle\int_{t_0}^{t} e^{\alpha(s-t)} \max_{\theta \in [s-t_1, s]} x(\theta) ds,\ t \in [t_0, +\infty[. \end{cases}$$

If we denote for each $\varphi \in C[t_0 - h, t_0]$,

$$C_\varphi([t_0 - h, +\infty[) := \{x \in C([t_0 - h, +\infty[) \big| \, x|_{[t_0-h,t_0]} = \varphi\}.$$

It is clear that we have the following:

- $C([t_0 - h, +\infty[) = \bigcup\limits_{\varphi \in C[t_0-h,t_0]} ([t_0 - h, +\infty[),$
- $E(C_\varphi([t_0, +\infty[)) \subset C_\varphi([t_0, +\infty[),$
- $E\Big|_{C_\varphi([t_0,+\infty[)} = A_\varphi.$

From this, we have that the operator E is WPO.

For $\varphi \in C[t_0 - h, t_0]$, we denote by $\widetilde{\varphi}$, the function $\widetilde{\varphi} : C([t_0, +\infty[)$ defined by

$$\widetilde{\varphi} = \begin{cases} \varphi(t), & t \in [t_0 - h, t_0], \\ \varphi(t_0), & t \in [t_0, +\infty[. \end{cases}$$

We remark that for fixed point x^*_φ of A_φ, we have that $x^*_\varphi = E^\infty(\widetilde{\varphi})$.

Since E is WPO and E is increasing, we have Abstract Gronwall lemma for E, i.e.,

$$x \le E(x) \Rightarrow x \le E^\infty(x)$$
$$x \ge E(x) \Rightarrow x \ge E^\infty(x).$$

From this, we have that if $X^*_{\varphi_i}$ is the unique fixed point of A_{φ_i}, $i = \overline{1, 2}$, and if $\varphi_1 \le \varphi_2$, then $x^*_{\varphi_1} \le x^*_{\varphi_2}$. Indeed, it follows from the increasing of E^∞, and from

$$x^*_{\varphi_1} = E^\infty(\widetilde{\varphi}_1), \; x^*_{\varphi_2} = E^\infty(\widetilde{\varphi}_2).$$

If $x^* \in E_t$ and x is such that $x \le E(x)$ and $x\Big|_{[t_0-h,t_0]} \le x^*\Big|_{[t_0-h,t_0]}$, then $x \le x^*$.

4 Halanay Functional Differential Inequation: Halanay Lemma

For $\alpha > \beta > 0$ and $t_0 \in \mathbb{R}$, we consider the Halanay inequation

$$x'(t) \le -\alpha x(t) + \beta \max_{\theta \in [t-h,t]} x(\theta), \; t \in [t_0 - h, +\infty[. \tag{5}$$

Let $x \in C(t_0, +\infty[) \cap C^1([t_0 - h, +\infty[)$ be a solution of this equation. Then we have that $x \le E(x)$, where $E(x)$ was defined in Remark 3.

Let $\gamma > 0$ be such that $e^{-\gamma(t-t_0)}$, $t \in [t_0 - h, +\infty[$ is a solution of Halanay equation (2). Notice that there exists $k > 0$ such that

$$x(t) \leq ke^{-\gamma(t-t_0)}, \ \forall\, t \in [t_0 - h, t_0].$$

Since $e^{-\gamma(t-t_0)}$, $t \in [t_0 - h, +\infty[$ is a solution of (2), then $ke^{-\gamma(t-t_0)}$, $t \in [t_0 - h, +\infty[$ is also a solution of (2), i.e., it is a fixed point of the operator E. From Remark 3, we have that

$$x(t) \leq ke^{-\gamma(t-t_0)}, \ \forall\, t \in [t_0 - h, +\infty[.$$

So we have

Halanay Lemma. *If x is a solution of* (5)*, then there exists $k > 0$, $\gamma > 0$, such that*

$$x(t) \leq ke^{-\gamma(t-t_0)}, \ \forall\, t \in [t_0 - h, +\infty[.$$

5 Halanay-Type Results

Halanay's lemma generated several papers concerning functional differential equations and functional integral equations.

All these results are, in fact, Chaplygin-type results and Gronwall-type results. In our opinion, it is more appropriate to consider Halanay-type results, concrete Chaplygin-type results, and concrete Gronwall-type results for equations with maxima.

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An Inequality Related to Möbius Transformations

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Abstract The open unit ball $\mathbb{B} = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < 1\}$ is endowed with Möbius addition $\oplus_M$ defined by

$$\mathbf{u} \oplus_M \mathbf{v} = \frac{(1 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2},$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$. In this article, we prove the inequality

$$\frac{\|\mathbf{u}\| - \|\mathbf{v}\|}{1 + \|\mathbf{u}\|\|\mathbf{v}\|} \leq \|\mathbf{u} \oplus_M \mathbf{v}\| \leq \frac{\|\mathbf{u}\| + \|\mathbf{v}\|}{1 - \|\mathbf{u}\|\|\mathbf{v}\|}$$

in $\mathbb{B}$. This leads to a new metric on $\mathbb{B}$ defined by

$$d_T(\mathbf{u}, \mathbf{v}) = \tan^{-1} \| -\mathbf{u} \oplus_M \mathbf{v}\|,$$

which turns out to be an invariant of Möbius transformations on $\mathbb{R}^n$ carrying $\mathbb{B}$ onto itself. We also compute the isometry group of $(\mathbb{B}, d_T)$ and give a parametrization of the isometry group by vectors and rotations.

1 The Unit Ball of n-Dimensional Euclidean Space $\mathbb{R}^n$

Let $\mathbb{B}$ denote the open unit ball of n-dimensional Euclidean space $\mathbb{R}^n$, that is,

$$\mathbb{B} = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < 1\}, \tag{1}$$

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where $\|\cdot\|$ denotes the usual Euclidean norm on $\mathbb{R}^n$. It is known in the literature that $\mathbb{B}$ forms a bounded symmetric domain, naturally associated with the Poincaré and Beltrami–Klein models of n-dimensional hyperbolic geometry. In fact, the Poincaré metric d_P corresponding to a curvature of -1 is given by

$$d_P(\mathbf{x}, \mathbf{y}) = \cosh^{-1}\left(1 + \frac{2\|\mathbf{x} - \mathbf{y}\|^2}{(1 - \|\mathbf{x}\|^2)(1 - \|\mathbf{y}\|^2)}\right), \tag{2}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{B}$ [4, p. 1232]. Furthermore, the Cayley–Klein metric associated with the Beltrami–Klein model is defined via cross-ratios; see, for instance, [4, p. 1233].

From an algebraic point of view, the unit ball has a group-like structure when it is endowed with *Möbius addition* $\oplus_M$ defined by

$$\mathbf{u} \oplus_M \mathbf{v} = \frac{(1 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2}. \tag{3}$$

Möbius addition governs the unit ball in the same way that ordinary vector addition governs the Euclidean space; see, for instance, [3, 6, 11]. Furthermore, Möbius addition induces the well-known Möbius transformation of $\mathbb{B}$ of the form

$$L_{\mathbf{u}}(\mathbf{v}) = \mathbf{u} \oplus_M \mathbf{v} = \frac{(1 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2}, \tag{4}$$

called the *hyperbolic translation* by $\mathbf{u}$, for all $\mathbf{u} \in \mathbb{B}$ [6, p. 124]. A remarkable result of Kim and Lawson shows strong connections between the geometric and algebraic structures of the unit ball. In fact, they relate the Poincaré metric with Möbius addition:

$$d_P(\mathbf{x}, \mathbf{y}) = 2\tanh^{-1}\|-\mathbf{x} \oplus_M \mathbf{y}\| \tag{5}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{B}$; see Theorem 3.7 of [4]. Equation (5) includes what Ungar refers to as a *gyrometric* [10, Definition 6.8]. More precisely, the (Möbius) gyrometric and the rapidity metric of $(\mathbb{B}, \oplus_M)$ are defined by

$$\rho_M(\mathbf{x}, \mathbf{y}) = \|-\mathbf{x} \oplus_M \mathbf{y}\| \tag{6}$$

and by

$$d_M(\mathbf{x}, \mathbf{y}) = \tanh^{-1}(\rho_M(\mathbf{x}, \mathbf{y})) \tag{7}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{B}$, respectively.

A Nonassociative Structure of the Unit Ball

The space $(\mathbb{B}, \oplus_M)$ shares many properties with abelian groups, called by some a *gyrocommutative gyrogroup* and by others a *Bruck loop* or a *K-loop*. Henceforth, $(\mathbb{B}, \oplus_M)$ is referred to as the *Möbius gyrogroup*.

The group-like axioms satisfied by the Möbius gyrogroup are as follows.

(I) (IDENTITY) The zero vector $\mathbf{0}$ satisfies $\mathbf{0} \oplus_M \mathbf{v} = \mathbf{v} = \mathbf{v} \oplus_M \mathbf{0}$ for all $\mathbf{v} \in \mathbb{B}$.

(II) (INVERSE) For each $\mathbf{v} \in \mathbb{B}$, the negative vector $-\mathbf{v}$ belongs to $\mathbb{B}$ and satisfies

$$(-\mathbf{v}) \oplus_M \mathbf{v} = \mathbf{0} = \mathbf{v} \oplus_M (-\mathbf{v}).$$

(III) (THE GYROASSOCIATIVE LAW) For all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$, there are automorphisms $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ and $\mathrm{gyr}[\mathbf{v}, \mathbf{u}]$ in $\mathrm{Aut}\,(\mathbb{B}, \oplus_M)$, such that

$$\mathbf{u} \oplus_M (\mathbf{v} \oplus_M \mathbf{w}) = (\mathbf{u} \oplus_M \mathbf{v}) \oplus_M \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}$$

and

$$(\mathbf{u} \oplus_M \mathbf{v}) \oplus_M \mathbf{w} = \mathbf{u} \oplus_M (\mathbf{v} \oplus_M \mathrm{gyr}[\mathbf{v}, \mathbf{u}]\mathbf{w})$$

for all $\mathbf{w} \in \mathbb{B}$.

(IV) (THE LOOP PROPERTY) For all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$,

$$\mathrm{gyr}[\mathbf{u} \oplus_M \mathbf{v}, \mathbf{v}] = \mathrm{gyr}[\mathbf{u}, \mathbf{v}] \quad \text{and} \quad \mathrm{gyr}[\mathbf{u}, \mathbf{v} \oplus_M \mathbf{u}] = \mathrm{gyr}[\mathbf{u}, \mathbf{v}].$$

(V) (THE GYROCOMMUTATIVE LAW) For all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$,

$$\mathbf{u} \oplus_M \mathbf{v} = \mathrm{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{v} \oplus_M \mathbf{u}).$$

The automorphism $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ mentioned in Item (III) is called the *gyroautomorphism* generated by $\mathbf{u}$ and $\mathbf{v}$. It is uniquely determined by its generators via the *gyrator identity* described by the formula

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = -(\mathbf{u} \oplus_M \mathbf{v}) \oplus_M (\mathbf{u} \oplus_M (\mathbf{v} \oplus_M \mathbf{w})) \tag{8}$$

for all $\mathbf{w} \in \mathbb{B}$. Sometimes it is convenient to denote $-\mathbf{v}$ by $\ominus\mathbf{v}$, the (unique) inverse of $\mathbf{v}$ with respect to Möbius addition. Some elementary properties of the Möbius gyrogroup are collected in Table 1.

Table 1 Properties of the Möbius gyrogroup (cf. [7, 10])

Gyrogroup identity	Name/reference
$L_{\ominus\mathbf{u}} = L_{\mathbf{u}}^{-1}$	Inverse of gyrotranslation
$\ominus\mathbf{u} \oplus_M (\mathbf{u} \oplus_M \mathbf{v}) = \mathbf{v}$	Left cancellation law
$\ominus(\mathbf{u} \oplus_M \mathbf{v}) = \mathrm{gyr}[\mathbf{u}, \mathbf{v}](\ominus\mathbf{v} \oplus_M \ominus\mathbf{u})$	cf. $(gh)^{-1} = h^{-1}g^{-1}$
$(\ominus\mathbf{u} \oplus_M \mathbf{v}) \oplus_M \mathrm{gyr}[\ominus\mathbf{u}, \mathbf{v}](\ominus\mathbf{v} \oplus_M \mathbf{w}) = \ominus\mathbf{u} \oplus_M \mathbf{w}$	cf. $(g^{-1}h)(h^{-1}k) = g^{-1}k$
$\mathrm{gyr}[\ominus\mathbf{u}, \ominus\mathbf{v}] = \mathrm{gyr}[\mathbf{u}, \mathbf{v}]$	Even property
$\mathrm{gyr}[\mathbf{v}, \mathbf{u}] = \mathrm{gyr}^{-1}[\mathbf{u}, \mathbf{v}]$, the inverse of $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$	Inversive symmetry

Isometries of the Unit Ball

It is known in the literature that the transformation $L_{\mathbf{u}}\colon \mathbf{v} \mapsto \mathbf{u} \oplus_M \mathbf{v}$ preserves the gyrometric ρ_M; see, for instance, [4, Lemma 3.2 (v)]. Thus, $L_{\mathbf{u}}$ preserves the rapidity metric d_M. In fact, every isometry of $(\mathbb{B}, d_M)$ must be of the form $L_{\mathbf{u}} \circ \tau$, where τ is the restriction of an orthogonal transformation on $\mathbb{R}^n$ to the unit ball, due to the fact that any Möbius transformation that fixes $\mathbf{0}$ is orthogonal. The following theorem shows that the metric geometry of $\mathbb{B}$ with respect to d_M is homogeneous.

Theorem 1 (Homogeneity) *For each pair of points* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{B}$*, there is an isometry* T *of* $(\mathbb{B}, d_M)$ *such that* $T(\mathbf{x}) = \mathbf{y}$*. In particular,* $\mathbb{B}$ *is homogeneous.*

Proof Let $\mathbf{x}, \mathbf{y} \in \mathbb{B}$. Define $T = L_{\mathbf{y}} \circ L_{\ominus\mathbf{x}}$. Then T is an isometry of $\mathbb{B}$, being the composite of isometries of $\mathbb{B}$. Furthermore, $T(\mathbf{x}) = \mathbf{y} \oplus_M (\ominus\mathbf{x} \oplus_M \mathbf{x}) = \mathbf{y}$. □

By using the gyrogroup formalism, a *point-reflection* symmetry of $\mathbb{B}$ is easy to construct, as shown in the following theorem.

Theorem 2 (Symmetry) *For each point* $\mathbf{x} \in \mathbb{B}$*, there is a symmetry* $S_{\mathbf{x}}$ *of* $\mathbb{B}$*; that is,* $S_{\mathbf{x}}$ *is an isometry of* $(\mathbb{B}, d_M)$ *such that* $S_{\mathbf{x}}^2$ *is the identity transformation* I *of* $\mathbb{B}$ *and* $\mathbf{x}$ *is the unique fixed point of* $S_{\mathbf{x}}$.

Proof Let ι be the inversion map of $\mathbb{B}$, that is, $\iota(\mathbf{v}) = \ominus\mathbf{v}$ for all $\mathbf{v} \in \mathbb{B}$. Since $\ominus\mathbf{v} = -\mathbf{v}$ for all $\mathbf{v} \in \mathbb{B}$, ι is simply the negative map: $\mathbf{v} \mapsto -\mathbf{v}$. Note that ι is an isometry of $(\mathbb{B}, d_M)$ for ι is linear and preserves the Euclidean norm. Furthermore, $\iota(\mathbf{v}) = \mathbf{v}$ if and only if $\mathbf{v} = \mathbf{0}$.

Given $\mathbf{x} \in \mathbb{B}$, define $S_{\mathbf{x}} = L_{\mathbf{x}} \circ \iota \circ L_{\ominus\mathbf{x}}$. Then $S_{\mathbf{x}} = L_{\mathbf{x}} \circ \iota \circ L_{\mathbf{x}}^{-1}$, and so

$$S_{\mathbf{x}}^2 = (L_{\mathbf{x}} \circ \iota \circ L_{\mathbf{x}}^{-1}) \circ (L_{\mathbf{x}} \circ \iota \circ L_{\mathbf{x}}^{-1}) = L_{\mathbf{x}} \circ \iota^2 \circ L_{\mathbf{x}}^{-1} = L_{\mathbf{x}} \circ L_{\mathbf{x}}^{-1} = I.$$

Note that $S_{\mathbf{x}} \neq I$; otherwise, we would have $L_{\mathbf{x}} \circ \iota \circ L_{\mathbf{x}}^{-1} = I$ and would have $\iota = I$, a contradiction. It is clear that $S_{\mathbf{x}}$ is an isometry of $\mathbb{B}$. By construction, $\mathbf{x}$ is a fixed point of $S_{\mathbf{x}}$. Suppose that $\mathbf{y}$ is a fixed point of $S_{\mathbf{x}}$, that is, $S_{\mathbf{x}}(\mathbf{y}) = \mathbf{y}$. It follows that $\mathbf{x} \oplus_M \iota(\ominus\mathbf{x} \oplus_M \mathbf{y}) = \mathbf{y}$, and hence, $\iota(\ominus\mathbf{x} \oplus_M \mathbf{y}) = \ominus\mathbf{x} \oplus_M \mathbf{y}$. As mentioned previously, $\mathbf{0}$ is the unique fixed point of ι and so $\ominus\mathbf{x} \oplus_M \mathbf{y} = \mathbf{0}$. This implies that $\mathbf{x} = \mathbf{y}$. □

We close this section with the following theorem whose proof is straightforward (and so is omitted).

Theorem 3 *If* $\tau \in \operatorname{Aut}(\mathbb{B}, \oplus_M)$ *and* $\|\tau(\mathbf{v})\| = \|\mathbf{v}\|$ *for all* $\mathbf{v} \in \mathbb{B}$*, then* τ *is an isometry of* $\mathbb{B}$ *with respect to* d_M*. In particular, the gyroautomorphisms of* $(\mathbb{B}, \oplus_M)$ *are isometries.*

2 The Negative Euclidean Space and Its Clifford Algebra

It seems that the formalism of Clifford algebras is a suitable tool for the study of the Möbius gyrogroup [2, 5]. Let us begin with the definition of an underlying vector space that will be used to built a unital associative algebra in which Möbius addition has a compact formula. The *negative* Euclidean space has $\mathbb{R}^n$ as the underlying vector space, but its inner product is a variant of the Euclidean inner product defined by

$$B(\mathbf{u}, \mathbf{v}) = -\langle \mathbf{u}, \mathbf{v} \rangle, \qquad \mathbf{u}, \mathbf{v} \in \mathbb{R}^n. \tag{9}$$

Note that (9) defines a nondegenerate symmetric bilinear form on $\mathbb{R}^n$. Also, the associated quadratic form is given by $Q(\mathbf{v}) = -\|\mathbf{v}\|^2$ for all $\mathbf{v} \in \mathbb{R}^n$.

The negative Euclidean space induces a real unital associative algebra, which is unique up to isomorphism, called the *Clifford algebra* of $(\mathbb{R}^n, B)$ denoted by $C\ell_n$ [5]. To describe the structure of $C\ell_n$, let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$. Then $C\ell_n$ has a basis of the form

$$\{e_I : I = \emptyset \text{ or } I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\}\}, \tag{10}$$

where $e_I = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_k}$ for $I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$ and $e_\emptyset = 1$, the multiplicative identity of $C\ell_n$. Hence, a typical element of $C\ell_n$ is of the form $\sum_I \lambda_I e_I$ with λ_I in $\mathbb{R}$. The binary operations of vector addition and scalar multiplication in $C\ell_n$ are defined pointwise. The product of two elements in $C\ell_n$ is obtained by using the distributive law (but not assuming that algebra multiplication is commutative) subject to the defining relations

$$\mathbf{e}_i^2 = -1 \quad \text{and} \quad \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \tag{11}$$

for all $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$. The base field $\mathbb{R}$ is embedded into $C\ell_n$ by the map $\lambda \mapsto \lambda 1$, and the original space $\mathbb{R}^n$ is embedded into $C\ell_n$ by the inclusion map [7, Section 3].

There is a unique involutive algebra antiautomorphism of $C\ell_n$ that extends the identity automorphism I of $\mathbb{R}^n$, called the *reversion*, denoted by $a \mapsto \tilde{a}$. Furthermore, the *grade involution* denoted by $a \mapsto \hat{a}$ is a unique involutive automorphism of $C\ell_n$ that extends $-I$, whereas the (Clifford) *conjugation* denoted by $a \mapsto \bar{a}$ is a unique involutive antiautomorphism of $C\ell_n$ that extends $-I$. The grade involution is used to define a *Clifford group* (also called a *Lipschitz group*), which is a group under multiplication of $C\ell_n$ defined by

$$\Gamma_n = \{g \in C\ell_n : g \text{ is invertible and } \hat{g}\mathbf{v}g^{-1} \in \mathbb{R}^n \text{ for all } \mathbf{v} \in \mathbb{R}^n\}. \tag{12}$$

The conjugation of $C\ell_n$ gives rise to a group homomorphism of Γ_n. In fact, define a map η by

$$\eta(a) = a\bar{a}, \qquad a \in C\ell_n. \tag{13}$$

Then the restriction of η to Γ_n is a homomorphism from Γ_n to the multiplicative group of nonzero numbers, denoted by $\mathbb{R}^\times$ [8, Proposition 2]. If an element a in $C\ell_n$ has the property that $\eta(a) \in \mathbb{R}$ and $\eta(a) \geq 0$, we define $|a| = \sqrt{\eta(a)}$. It is not difficult to see that $|\mathbf{v}| = \|\mathbf{v}\|$ for all $\mathbf{v} \in \mathbb{R}^n$.

The following theorem summarizes basic properties of $C\ell_n$ that will be used in Section 3, especially the proof of Theorem 7.

Theorem 4 (Proposition 5, [8]) *The following properties hold in the Clifford algebra* $C\ell_n$.

1. $\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u} = -2\langle \mathbf{u}, \mathbf{v}\rangle$ *for all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.
2. $\mathbf{v}^2 = -\|\mathbf{v}\|^2$ *for all* $\mathbf{v} \in \mathbb{R}^n$.
3. $1 - \mathbf{u}\mathbf{v} \in \Gamma_n$ *and* $(1 - \mathbf{u}\mathbf{v})^{-1} = \dfrac{1 - \mathbf{v}\mathbf{u}}{\eta(1 - \mathbf{u}\mathbf{v})}$ *for all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *with* $\|\mathbf{u}\|\|\mathbf{v}\| \neq 1$.
4. $\eta(\mathbf{w}(1 - \mathbf{u}\mathbf{v})^{-1}) = \dfrac{\eta(\mathbf{w})}{\eta(1 - \mathbf{u}\mathbf{v})}$ *for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *with* $\|\mathbf{u}\|\|\mathbf{v}\| \neq 1$.

In view of Theorem 4 (2), if $\mathbf{v} \neq \mathbf{0}$, then $\mathbf{v}$ is invertible with respect to multiplication of $C\ell_n$ and $\mathbf{v}^{-1} = -\dfrac{1}{\|\mathbf{v}\|^2}\mathbf{v}$. Furthermore, by Lemma 1 of [8],

$$\hat{\mathbf{v}}\mathbf{w}\mathbf{v}^{-1} = \frac{1}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{w}\mathbf{v}$$

belongs to $\mathbb{R}^n$ for all nonzero vectors $\mathbf{v} \in \mathbb{R}^n$ and all $\mathbf{w} \in \mathbb{R}^n$. This implies that $\mathbb{R}^n \setminus \{\mathbf{0}\} \subseteq \Gamma_n$, and we obtain the following theorem.

Theorem 5 *Every transformation of the form* $\mathbf{w} \mapsto q\mathbf{w}q^{-1}$, *where* $\mathbf{w} \in \mathbb{R}^n$ *and* $q \in \Gamma_n$, *defines an orthogonal transformation on* $\mathbb{R}^n$.

Proof Let $\mathbf{w} \in \mathbb{R}^n$ and let $q \in \Gamma_n$. Clearly, $\|q\mathbf{0}q^{-1}\| = 0 = \|\mathbf{0}\|$. Therefore, we may assume that $\mathbf{w} \neq \mathbf{0}$ and hence $\mathbf{w} \in \Gamma_n$. Since η is a homomorphism from Γ_n to $\mathbb{R}^\times$, it follows that $\eta(q\mathbf{w}q^{-1}) = \eta(q)\eta(\mathbf{w})\eta(q)^{-1} = \eta(\mathbf{w})$ and so

$$\|q\mathbf{w}q^{-1}\| = \sqrt{\eta(q\mathbf{w}q^{-1})} = \sqrt{\eta(\mathbf{w})} = \|\mathbf{w}\|.$$

It is clear that the map $\mathbf{w} \mapsto q\mathbf{w}q^{-1}$ is linear and bijective for $\mathbf{w} \mapsto q^{-1}\mathbf{w}q$ defines its inverse with respect to composition of maps. □

Using the Clifford algebra formalism, one gains a compact formula for Möbius addition, as shown in the following theorem.

Theorem 6 (Theorem 5.2, [5]) *In* $C\ell_n$*, Möbius addition can be expressed as*

$$\mathbf{u} \oplus_M \mathbf{v} = (\mathbf{u} + \mathbf{v})(1 - \mathbf{u}\mathbf{v})^{-1} \tag{14}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$. *The gyroautomorphisms are given by* $\operatorname{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = q\mathbf{w}q^{-1}$, *where*

$$q = \frac{1 - \mathbf{u}\mathbf{v}}{|1 - \mathbf{u}\mathbf{v}|},$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{B}$.

3 Metrics on the Möbius Gyrogroup and Their Isometry Groups

In this section, we prove a useful inequality involving Möbius addition and the Euclidean norm as an application of the Cauchy–Schwarz inequality, using the Clifford algebra formalism. This enables us to define a variant of norm metric on the Möbius gyrogroup. This metric turns out to be a characteristic property of Möbius transformations on $\hat{\mathbb{R}}^n$ carrying $\mathbb{B}$ onto itself, where $\hat{\mathbb{R}}^n$ is the one-point compactification of $\mathbb{R}^n$. We then give a complete description of the corresponding isometry group via a gyrogroup approach.

Theorem 7 *The inequality*

$$\frac{\|\mathbf{u}\| - \|\mathbf{v}\|}{1 + \|\mathbf{u}\|\|\mathbf{v}\|} \leq \|\mathbf{u} \oplus_M \mathbf{v}\| \leq \frac{\|\mathbf{u}\| + \|\mathbf{v}\|}{1 - \|\mathbf{u}\|\|\mathbf{v}\|} \tag{15}$$

holds in the Möbius gyrogroup.

Proof Using the Cauchy–Schwarz inequality, we have

$$-\|\mathbf{u}\|\|\mathbf{v}\| \leq \langle \mathbf{u}, \mathbf{v} \rangle \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. This implies that

$$\eta(\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 - (\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u}) + \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 \leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$$

and that $\eta(\mathbf{u} + \mathbf{v}) \geq (\|\mathbf{u}\| - \|\mathbf{v}\|)^2$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{B}$. As in the proof of Proposition 5 (4) of [8], we have $\eta(1 - \mathbf{u}\mathbf{v}) \geq (1 - \|\mathbf{u}\|\|\mathbf{v}\|)^2$ and

$$\eta(1 - \mathbf{u}\mathbf{v}) = 1 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \le (1 + \|\mathbf{u}\|\|\mathbf{v}\|)^2.$$

Hence, by Theorem 4 (4),

$$\|\mathbf{u} \oplus_M \mathbf{v}\| = \sqrt{\frac{\eta(\mathbf{u}+\mathbf{v})}{\eta(1-\mathbf{u}\mathbf{v})}} \le \sqrt{\frac{(\|\mathbf{u}\| + \|\mathbf{v}\|)^2}{(1 - \|\mathbf{u}\|\|\mathbf{v}\|)^2}} = \frac{\|\mathbf{u}\| + \|\mathbf{v}\|}{1 - \|\mathbf{u}\|\|\mathbf{v}\|},$$

and similarly

$$\|\mathbf{u} \oplus_M \mathbf{v}\| = \sqrt{\frac{\eta(\mathbf{u}+\mathbf{v})}{\eta(1-\mathbf{u}\mathbf{v})}} \ge \sqrt{\frac{(\|\mathbf{u}\| - \|\mathbf{v}\|)^2}{(1 + \|\mathbf{u}\|\|\mathbf{v}\|)^2}} \ge \frac{\|\mathbf{u}\| - \|\mathbf{v}\|}{1 + \|\mathbf{u}\|\mathbf{v}\|\|},$$

as required. □

In view of (15) and the well-known trigonometric identity, the tangent function is needed in order to obtain a bounded metric on the unit ball of $\mathbb{R}^n$. In fact, define a function $\|\cdot\|_T$ by

$$\|\mathbf{v}\|_T = \tan^{-1}\|\mathbf{v}\| \tag{16}$$

for all $\mathbf{v} \in \mathbb{B}$. Here, T stands for "$\tan^{-1}$."

Theorem 8 *$\|\cdot\|_T$ satisfies the following properties:*

1. $\|\mathbf{x}\|_T \ge 0$ *and* $\|\mathbf{x}\|_T = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$*;*
2. $\|\ominus\mathbf{x}\|_T = \|\mathbf{x}\|_T$*;*
3. $\|\mathbf{x}\|_T - \|\mathbf{y}\|_T \le \|\mathbf{x} \oplus_M \mathbf{y}\|_T \le \|\mathbf{x}\|_T + \|\mathbf{y}\|_T$*;*
4. $\|\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{x}\|_T = \|\mathbf{x}\|_T$

for all $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y} \in \mathbb{B}$.

Proof Item (1) follows from the fact that $\tan^{-1}$ is a strictly increasing injective function on $(-\infty, \infty)$. Item (2) follows from the fact that $\|-\mathbf{x}\| = \|\mathbf{x}\|$.

To prove (3), set $x = \tan^{-1}\|\mathbf{x}\|$ and $y = \tan^{-1}\|\mathbf{y}\|$. By Theorem 7,

$$\frac{\|\mathbf{x}\| - \|\mathbf{y}\|}{1 + \|\mathbf{x}\|\|\mathbf{y}\|} \le \|\mathbf{x} \oplus_M \mathbf{y}\| \le \frac{\|\mathbf{x}\| + \|\mathbf{y}\|}{1 - \|\mathbf{x}\|\|\mathbf{y}\|},$$

and so $\tan(x - y) \le \|\mathbf{x} \oplus_M \mathbf{y}\| \le \tan(x + y)$. Since $\tan^{-1}$ is an increasing function, it follows that $x - y \le \tan^{-1}\|\mathbf{x} \oplus_M \mathbf{y}\| \le x + y$, as claimed. By Theorem 6, there is an element $q \in \Gamma_n$ for which $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{x} = q\mathbf{x}q^{-1}$. It follows from Theorem 5 that

$$\|\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{x}\|_T = \tan^{-1}\|q\mathbf{x}q^{-1}\| = \tan^{-1}\|\mathbf{x}\| = \|\mathbf{x}\|_T,$$

which proves (4). □

As a consequence of Theorem 8, we obtain a new metric on the Möbius gyrogroup. Unlike the Poincaré metric, this metric is bounded as shown in the following theorem.

Theorem 9 *Define d_T by*

$$d_T(\mathbf{x}, \mathbf{y}) = \| \ominus \mathbf{x} \oplus_M \mathbf{y}\|_T \tag{17}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{B}$*. Then d_T is a bounded metric on* $\mathbb{B}$*.*

Proof By Theorem 8 (1), $d_T(\mathbf{x}, \mathbf{y}) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{B}$ and $d_T(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{B}$. Using appropriate properties of the Möbius gyrogroup in Table 1, together with Theorem 8, we obtain

$$\| \ominus \mathbf{y} \oplus_M \mathbf{x}\|_T = \| \ominus (\ominus\mathbf{y} \oplus_M \mathbf{x})\|_T = \|\mathrm{gyr}[\ominus\mathbf{y}, \mathbf{x}](\ominus\mathbf{x} \oplus_M \mathbf{y})\|_T = \| \ominus \mathbf{x} \oplus_M \mathbf{y}\|_T,$$

and so $d_T(\mathbf{y}, \mathbf{x}) = d_T(\mathbf{x}, \mathbf{y})$. Furthermore, we obtain

$$\begin{aligned} d_T(\mathbf{x}, \mathbf{z}) &= \| \ominus \mathbf{x} \oplus_M \mathbf{z}\|_T \\ &= \|(\ominus\mathbf{x} \oplus_M \mathbf{y}) \oplus_M \mathrm{gyr}[\ominus\mathbf{x}, \mathbf{y}](\ominus\mathbf{y} \oplus_M \mathbf{z})\|_T \\ &\leq \| \ominus \mathbf{x} \oplus_M \mathbf{y}\|_T + \|\mathrm{gyr}[\ominus\mathbf{x}, \mathbf{y}](\ominus\mathbf{y} \oplus_M \mathbf{z})\|_T \\ &= \| \ominus \mathbf{x} \oplus_M \mathbf{y}\|_T + \| \ominus \mathbf{y} \oplus_M \mathbf{z}\|_T \\ &= d_T(\mathbf{x}, \mathbf{y}) + d_T(\mathbf{y}, \mathbf{z}). \end{aligned}$$

This proves that d_T satisfies the defining properties of a metric.

Note that $d_T(\mathbf{0}, \mathbf{v}) = \|\mathbf{v}\|_T = \tan^{-1} \|\mathbf{v}\| < \tan^{-1} 1 = \dfrac{\pi}{4}$ for all $\mathbf{v} \in \mathbb{B}$. Hence,

$$d_T(\mathbf{x}, \mathbf{y}) \leq d_T(\mathbf{x}, \mathbf{0}) + d_T(\mathbf{0}, \mathbf{y}) < \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{B}$. □

Although d_T is quite different from the Poincaré metric, both generate the same topology on the unit ball. It is clear that the Poincaré metric and the rapidity metric of the Möbius gyrogroup generate the same topology since the former is twice the latter.

Theorem 10 *The topologies induced by d_T and d_M are equivalent.*

Proof Note that $d_T(\mathbf{u}, \mathbf{v}) \leq d_M(\mathbf{u}, \mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$ since

$$f(x) = \tanh^{-1} x - \tan^{-1} x$$

defines a strictly increasing function on the open interval (0, 1). This implies that the topology generated by d_M is finer than the topology generated by d_T. Next, we prove that the topology generated by d_T is finer than the topology generated by d_M. Let $\mathbf{u} \in \mathbb{B}$ and let $\epsilon > 0$. Choose $\delta = \tan^{-1}(\tanh \epsilon)$. Let $\mathbf{v} \in B_{d_T}(\mathbf{u}, \delta)$. Then $d_T(\mathbf{u}, \mathbf{v}) < \delta$, that is, $\| \ominus \mathbf{u} \oplus_M \mathbf{v}\|_T < \tan^{-1}(\tanh \epsilon)$. It follows that

$$d_M(\mathbf{u}, \mathbf{v}) = \tanh^{-1} \| \ominus \mathbf{u} \oplus_M \mathbf{v}\| < \epsilon$$

for tan and $\tanh^{-1}$ are strictly increasing functions. Hence, $\mathbf{v} \in B_{d_M}(\mathbf{u}, \epsilon)$. This proves $B_{d_T}(\mathbf{u}, \delta) \subseteq B_{d_M}(\mathbf{u}, \epsilon)$. □

Let $\mathrm{O}(\mathbb{R}^n)$ be the orthogonal group of $\mathbb{R}^n$, that is,

$$\mathrm{O}(\mathbb{R}^n) = \{\tau : \tau \text{ is a bijective orthogonal transformation on } \mathbb{R}^n\}. \tag{18}$$

Set

$$\mathrm{O}(\mathbb{B}) = \{\tau|_{\mathbb{B}} : \tau \in \mathrm{O}(\mathbb{R}^n)\}, \tag{19}$$

where $\tau|_{\mathbb{B}}$ is the restriction of τ to $\mathbb{B}$. It is clear that $\mathrm{O}(\mathbb{B})$ forms a group under composition of maps since $\mathbb{B}$ is preserved under orthogonal transformations on $\mathbb{R}^n$. Given $\mathbf{u}, \mathbf{v} \in \mathbb{B}$, note that $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ satisfies the following properties:

1. $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{0} = \mathbf{0}$;
2. $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is an automorphism of $(\mathbb{B}, \oplus_M)$;
3. $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ preserves the Möbius gyrometric.

Hence, by Theorem 3.2 of [1], there is a bijective orthogonal transformation on $\mathbb{R}^n$, denoted by $\mathrm{Gyr}[\mathbf{u}, \mathbf{v}]$, for which $\mathrm{Gyr}[\mathbf{u}, \mathbf{v}]|_{\mathbb{B}} = \mathrm{gyr}[\mathbf{u}, \mathbf{v}]$. This proves the following inclusion:

$$\{\mathrm{gyr}[\mathbf{u}, \mathbf{v}] : \mathbf{u}, \mathbf{v} \in \mathbb{B}\} \subseteq \mathrm{O}(\mathbb{B}).$$

Next, we compute the isometry group of $(\mathbb{B}, d_T)$.

Lemma 1 *The left gyrotranslation $L_{\mathbf{u}} : \mathbf{v} \mapsto \mathbf{u} \oplus_M \mathbf{v}$ defines an isometry of $(\mathbb{B}, d_T)$ for all $\mathbf{u} \in \mathbb{B}$.*

Proof By Theorem 10 (1) of [9], $L_{\mathbf{u}}$ is a bijective self-map of $\mathbb{B}$. Using appropriate properties of the Möbius gyrogroup in Table 1, we obtain

$$\begin{aligned} \| \ominus (\mathbf{u} \oplus_M \mathbf{x}) \oplus_M (\mathbf{u} \oplus_M \mathbf{y})\| &= \|\mathrm{gyr}[\mathbf{u}, \mathbf{x}](\ominus\mathbf{x} \ominus \mathbf{u}) \oplus_M (\mathbf{u} \oplus_M \mathbf{y})\| \\ &= \|(\ominus\mathbf{x} \ominus \mathbf{u}) \oplus_M \mathrm{gyr}[\mathbf{x}, \mathbf{u}](\mathbf{u} \oplus_M \mathbf{y})\| \\ &= \|(\ominus\mathbf{x} \ominus \mathbf{u}) \oplus_M \mathrm{gyr}[\ominus\mathbf{x}, \ominus\mathbf{u}](\mathbf{u} \oplus_M \mathbf{y})\| \\ &= \| \ominus \mathbf{x} \oplus_M \mathbf{y}\|. \end{aligned}$$

It follows that

$$d_T(L_{\mathbf{u}}(\mathbf{x}), L_{\mathbf{u}}(\mathbf{y})) = \| \ominus L_{\mathbf{u}}(\mathbf{x}) \oplus_M L_{\mathbf{u}}(\mathbf{y})\|_T = \| \ominus \mathbf{x} \oplus_M \mathbf{y}\|_T = d_T(\mathbf{x}, \mathbf{y}). \quad \square$$

Theorem 11 *The isometry group of* $(\mathbb{B}, d_T)$ *is given by*

$$\operatorname{Iso}(\mathbb{B}, d_T) = \{L_{\mathbf{u}} \circ \tau : \mathbf{u} \in \mathbb{B}, \tau \in \mathrm{O}(\mathbb{B})\}. \tag{20}$$

Proof For convenience, if $\rho \in \mathrm{O}(\mathbb{R}^n)$, then the restriction of ρ to $\mathbb{B}$ is simply denoted by ρ. By Lemma 1, $L_{\mathbf{u}}$ is an isometry of $\mathbb{B}$ with respect to d_T. Let $\rho \in \mathrm{O}(\mathbb{R}^n)$. Using (3), we have $\rho(\mathbf{x}) \oplus_M \rho(\mathbf{y}) = \rho(\mathbf{x} \oplus_M \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{B}$ since ρ is linear and preserves the Euclidean inner product. Hence, the restriction of ρ to $\mathbb{B}$ is indeed an automorphism of $(\mathbb{B}, \oplus_M)$ since $\rho(\mathbb{B}) \subseteq \mathbb{B}$ and $\rho^{-1} \in \mathrm{O}(\mathbb{R}^n)$. It follows that

$$d_T(\rho(\mathbf{x}), \rho(\mathbf{y})) = \|\rho(\ominus\mathbf{x} \oplus_M \mathbf{y})\|_T = \| \ominus \mathbf{x} \oplus_M \mathbf{y}\|_T = d_T(\mathbf{x}, \mathbf{y}).$$

Thus, ρ is an isometry of $\mathbb{B}$ and so $\{L_{\mathbf{u}} \circ \tau : \mathbf{u} \in \mathbb{B}, \tau \in \mathrm{O}(\mathbb{B})\} \subseteq \operatorname{Iso}(\mathbb{B}, d_T)$.

Let $T \in \operatorname{Iso}(\mathbb{B}, d_T)$. By definition, T is a bijective self-map of $\mathbb{B}$. By Theorem 11 of [9], $T = L_{T(\mathbf{0})} \circ \rho$, where ρ is a bijective self-map of $\mathbb{B}$ fixing $\mathbf{0}$. As in the proof of Theorem 18 (2) of [7], $L^{-1}_{T(\mathbf{0})} = L_{\ominus T(\mathbf{0})}$ and so $\rho = L_{\ominus T(\mathbf{0})} \circ T$. Therefore, ρ is an isometry of $(\mathbb{B}, d_T)$. Since $d_T(\rho(\mathbf{x}), \rho(\mathbf{y})) = d_T(\mathbf{x}, \mathbf{y})$, and $\tan^{-1}$ is injective, it follows that

$$\| \ominus \rho(\mathbf{x}) \oplus_M \rho(\mathbf{y})\| = \| \ominus \mathbf{x} \oplus_M \mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{B}$. Thus, ρ preserves the Möbius gyrometric. By Theorem 3.2 of [1], $\rho = \tau|_{\mathbb{B}}$, where τ is a bijective orthogonal transformation on $\mathbb{R}^n$. This proves that

$$\operatorname{Iso}(\mathbb{B}, d_T) \subseteq \{L_{\mathbf{u}} \circ \tau : \mathbf{u} \in \mathbb{B}, \tau \in \mathrm{O}(\mathbb{B})\}. \quad \square$$

By Theorem 11, every isometry of $\mathbb{B}$ with respect to d_T can be expressed as the composite of a left gyrotranslation with an orthogonal transformation restricted to $\mathbb{B}$. This expression is unique in the sense that if $L_{\mathbf{u}} \circ \alpha = L_{\mathbf{v}} \circ \beta$ with $\mathbf{u}, \mathbf{v}$ in $\mathbb{B}$ and α, β in $\mathrm{O}(\mathbb{B})$, then $\mathbf{u} = \mathbf{v}$ and $\alpha = \beta$. Furthermore, we have the following composition law of isometries of $(\mathbb{B}, d_T)$:

$$(L_{\mathbf{u}} \circ \alpha) \circ (L_{\mathbf{v}} \circ \beta) = L_{\mathbf{u} \oplus_M \alpha(\mathbf{v})} \circ (\operatorname{gyr}[\mathbf{u}, \alpha(\mathbf{v})] \circ \alpha \circ \beta) \tag{21}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$ and $\alpha, \beta \in \mathrm{O}(\mathbb{B})$, a formula comparable to the composition law of Euclidean isometries.

Since $\mathbf{v} \mapsto L_{\mathbf{v}}$ defines a one-to-one correspondence from $\mathbb{B}$ to the set of left gyrotranslations of $\mathbb{B}$, we have

$$\operatorname{Iso}(\mathbb{B}, d_T) \cong \mathbb{B} \rtimes_{\text{gyr}} \mathrm{O}(\mathbb{B}). \tag{22}$$

Here, $\mathbb{B} \rtimes_{\text{gyr}} \mathrm{O}(\mathbb{B})$ is the semidirect-product-like group whose underlying set is

$$\mathbb{B} \rtimes_{\text{gyr}} \mathrm{O}(\mathbb{B}) = \{(\mathbf{v}, \tau) : \mathbf{v} \in \mathbb{B}, \tau \in \mathrm{O}(\mathbb{B})\} \tag{23}$$

with group law

$$(\mathbf{u}, \alpha)(\mathbf{v}, \beta) = (\mathbf{u} \oplus_M \alpha(\mathbf{v}), \text{gyr}[\mathbf{u}, \alpha(\mathbf{v})] \circ \alpha \circ \beta) \tag{24}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}$ and $\alpha, \beta \in \mathrm{O}(\mathbb{B})$. This is a result analogous to the fact that the isometry group of the Euclidean space is the semidirect product of $\mathbb{R}^n$ and $\mathrm{O}(\mathbb{R}^n)$:

$$\mathbb{R}^n \rtimes \mathrm{O}(\mathbb{R}^n) = \{(\mathbf{v}, \tau) : \mathbf{v} \in \mathbb{R}^n, \tau \in \mathrm{O}(\mathbb{R}^n)\},$$

where the group law is given by

$$(\mathbf{u}, \alpha)(\mathbf{v}, \beta) = (\mathbf{u} + \alpha(\mathbf{v}), \alpha \circ \beta)$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathrm{O}(\mathbb{R}^n)$. The group $\mathbb{B} \rtimes_{\text{gyr}} \mathrm{O}(\mathbb{B})$ is known as the *gyrosemidirect* of $\mathbb{B}$ and $\mathrm{O}(\mathbb{B})$ [10, Section 2.6].

Theorem 12 *Let T be a self-map of $\mathbb{B}$. The following are equivalent:*

1. *T preserves the Poincaré metric d_P;*
2. *T preserves the rapidity metric d_M;*
3. *T preserves the Möbius gyrometric ρ_M;*
4. *T preserves the metric d_T generated by $\|\cdot\|_T$.*

Proof The theorem follows directly from the fact that $d_P(\mathbf{x}, \mathbf{y}) = 2d_M(\mathbf{x}, \mathbf{y})$ and that $\tanh^{-1}$ and $\tan^{-1}$ are injective. □

Corollary 1 $\mathrm{Iso}(\mathbb{B}, d_P) = \mathrm{Iso}(\mathbb{B}, d_M) = \mathrm{Iso}(\mathbb{B}, \rho_M) = \mathrm{Iso}(\mathbb{B}, d_T)$.

Recall that a Möbius transformation of $\hat{\mathbb{R}}^n$ that leaves $\mathbb{B}$ invariant is called a Möbius transformation of $\mathbb{B}$ [6, p. 120]. It is known that the isometry group of the Poincaré ball model $(\mathbb{B}, d_P)$, also called the *conformal ball model*, can be identified with the group of Möbius transformations of $\mathbb{B}$; see, for instance, [6, Corollary 1 on p. 125]. By Corollary 1, Equation (24) provides a parametric realization of the Möbius transformation group of $\mathbb{B}$ in terms of vectors and rotations. Furthermore, d_T is an invariant of Möbius transformations of $\mathbb{B}$ in the sense of the following theorem.

Theorem 13 *Every Möbius transformation of $\mathbb{B}$ restricts to an isometry of $(\mathbb{B}, d_T)$, and every isometry of $(\mathbb{B}, d_T)$ extends to a unique Möbius transformation of $\mathbb{B}$.*

Proof Let ϕ be a Möbius transformation of $\mathbb{B}$. By Theorem 4.5.2 of [6], ϕ restricts to an isometry of $(\mathbb{B}, d_P)$. By Corollary 1, $\phi|_{\mathbb{B}}$ is an isometry of $(\mathbb{B}, d_T)$. Let σ be an isometry of $(\mathbb{B}, d_T)$. By the same corollary, σ is an isometry of $(\mathbb{B}, d_P)$ and, hence, extends to a unique Möbius transformation of $\mathbb{B}$ by the same theorem. □

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On a Half-Discrete Hilbert-Type Inequality in the Whole Plane with the Hyperbolic Tangent Function and Parameters

Michael Th. Rassias, Bicheng Yang, and Andrei Raigorodskii

Abstract In this paper, introducing multi-parameters and using properties of series, we prove a half-discrete Hilbert-type inequality in the whole plane with kernel in terms of the hyperbolic tangent function. The constant factor related to the Riemann zeta function and the gamma function is proved to be the best possible. In the form of applications, we also present equivalent forms, a few particular inequalities, operator expressions and reverses.

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1 Introduction

Assuming that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n > 0$,

$$0 < \sum_{m=1}^{\infty} a_m^p < \infty, \ \ 0 < \sum_{n=1}^{\infty} b_n^q < \infty,$$

we have the following well-known Hardy–Hilbert inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [1]):

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}. \tag{1}$$

If $f(x), g(y) \geq 0$,

$$0 < \int_0^{\infty} f^p(x)dx < \infty \ \ and \ \ 0 < \int_0^{\infty} g^q(y)dy < \infty,$$

then the following Hardy–Hilbert integral inequality holds true (cf. [2]):

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^{\infty} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(y)dy \right)^{\frac{1}{q}}, \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

In 2011, the following half-discrete Hardy–Hilbert inequality with the same best possible constant factor was proved (cf. [3]):

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{b_n f(x)}{x+n} dx < \frac{\pi}{\sin(\pi/p)} \left(\int_0^{\infty} f^p(x)dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}. \tag{3}$$

Inequalities of the form (1), (2) and (3) are essential for various applications in mathematical analysis (cf. [2, 4–6]).

A survey of the work conducted in the area of Hilbert-type inequalities with homogeneous kernels of negative degree was presented in 2009 in [7]. Some new inequalities with homogenous kernels of degree 0 as well as with non-homogenous kernels were investigated in [8–10]. The inequalities in all of these works are constructed in the quarter plane of the first quadrant. Other kinds of Hilbert-type inequalities were also established in [11–32].

In 2007, a Hilbert-type integral inequality in the whole plane was proved by Yang in [30]. Additionally, the following Hilbert-type integral inequality in the whole plane was established in [31]:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{|1+xy|^{\lambda}}f(x)g(y)dxdy$$
$$< k_{\lambda}\left[\int_{-\infty}^{\infty}|x|^{p(1-\frac{\lambda}{2})-1}f^{p}(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\frac{\lambda}{2})-1}g^{q}(y)dy\right]^{\frac{1}{q}}, \tag{4}$$

where the constant factor

$$k_{\lambda}=B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)+2B\left(1-\lambda,\frac{\lambda}{2}\right)\ (0<\lambda<1)$$

is the best possible. He et al. have also proved some new Hilbert-type inequalities in the whole plane with the best possible constant factors (cf. [33–40]).

In this paper, introducing multi-parameters and using properties of series, we prove the half-discrete Hilbert-type inequality (5) in the whole plane with kernel in terms of the hyperbolic tangent function:

$$1-\tanh(u)=1-\frac{\sinh(u)}{\cosh(u)}=\frac{2e^{-u}}{e^{u}+e^{-u}}\ \ (u\geq 0)$$

and a best possible constant factor:

$$\sum_{|n|=1}^{\infty}\int_{-\infty}^{\infty}\left(1-\tanh\left(\rho\left(\frac{|n|}{|x|}\right)^{\gamma}\right)\right)f(x)b_{n}dx$$
$$<\frac{4(2^{\sigma/\gamma}-2)}{\gamma(4\rho)^{\sigma/\gamma}}\Gamma\left(\frac{\sigma}{\gamma}\right)\zeta\left(\frac{\sigma}{\gamma}\right)$$
$$\times\left[\int_{-\infty}^{\infty}|x|^{p(1+\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}|n|^{q(1-\sigma)-1}b_{n}^{q}\right]^{\frac{1}{q}}, \tag{5}$$

where $\rho>0$, $0<\gamma<\sigma\leq 1$,

$$\zeta(s):=\sum_{k=1}^{\infty}\frac{1}{k^{s}}\ \ (Re\ s>1)$$

is the Riemann zeta function and

$$\Gamma(s):=\int_{0}^{\infty}e^{-v}v^{s-1}dv\ \ (Re\ s>0)$$

is the gamma function (cf. [41]).

Moreover, we also obtain an extension of (5) with multi-parameters. In the form of applications, we additionally present equivalent forms, a few particular inequalities, operator expressions and reverses.

2 Weight Functions and Some Lemmas

In what follows, we assume that $p \in \mathbf{R}\backslash\{0, 1\}$,

$$\frac{1}{p} + \frac{1}{q} = 1, \delta \in \{-1, 1\},\ a, b \in (-1, 1), \rho > 0,\ \ 0 < \gamma < \sigma \leq 1.$$

We set

$$\begin{aligned} g(x, y) &:= 1 - \tanh\left(\rho\left[\frac{|y| + by}{(|x| + ax)^{\delta}}\right]^{\gamma}\right) \\ &= \frac{2e^{-\rho[\frac{|y|+by}{(|x|+ax)^{\delta}}]^{\gamma}}}{e^{\rho[\frac{|y|+by}{(|x|+ax)^{\delta}}]^{\gamma}} + e^{-\rho[\frac{|y|+by}{(|x|+ax)^{\delta}}]^{\gamma}}}\ (x \neq 0, y \neq 0), \end{aligned} \tag{6}$$

wherefrom

$$g(x, y) = 1 - \tanh\left(\rho[\frac{y(1 + b)}{(|x| + ax)^{\delta}}]^{\gamma}\right)\ (y > 0),$$

$$g(x, y) = 1 - \tanh\left(\rho\{\frac{|y| + by}{[x(1 + a)]^{\delta}}\}^{\gamma}\right)\ (x > 0),$$

$$g(-x, y) = 1 - \tanh\left(\rho\{\frac{|y| + by}{[x(1 - a)]^{\delta}}\}^{\gamma}\right)\ (x > 0),$$

$$g(x, -y) = 1 - \tanh\left(\rho[\frac{y(1 - b)}{(|x| + ax)^{\delta}}]^{\gamma}\right)\ (y > 0).$$

Lemma 1 *We define two weight functions $\omega(\sigma, n)$ and $\varpi(\sigma, x)$ as follows:*

$$\omega(\sigma, n) : = \int_{-\infty}^{\infty} g(x, n)\frac{(|n| + bn)^{\sigma}}{(|x| + ax)^{1+\delta\sigma}}dx\ \ (|n| \in \mathbf{N}), \tag{7}$$

$$\varpi(\sigma, x) : = \sum_{|n|=1}^{\infty} g(x, n)\frac{(|x| + ax)^{-\delta\sigma}}{(|n| + bn)^{1-\sigma}}\ \ (x \in \mathbf{R}\backslash\{0\}). \tag{8}$$

Then,

(i) *we have*

$$\omega(\sigma, n) = k_a(\sigma) := \frac{4(2^{\sigma/\gamma} - 2)\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}{\gamma(4\rho)^{\sigma/\gamma}(1-a^2)} \in \mathbf{R}_+ \ (|n| \in \mathbf{N}); \tag{9}$$

(ii) we also have

$$k_b(\sigma)(1-\theta(\sigma, x)) < \varpi(\sigma, x) < k_b(\sigma) \ (x \in \mathbf{R}\backslash\{0\}), \tag{10}$$

where

$$\begin{aligned}\theta(\sigma, x) &:= \frac{2^{\sigma/\gamma}}{2(2^{\sigma/\gamma} - 2)\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})} \\ &\quad \times \int_0^{\rho\left[\frac{1+b}{(|x|+ax)^{\delta}}\right]^{\gamma}} u^{\frac{\sigma}{\gamma}-1}(1-\tanh(u))du \\ &= O\left(\frac{1}{(|x|+ax)^{\delta\sigma}}\right) \in (0,1).\end{aligned} \tag{11}$$

Proof We obtain

$$\begin{aligned}\omega(\sigma, n) &= \int_{-\infty}^{0} g(x, n)\frac{(|n|+bn)^{\sigma}dx}{[x(a-1)]^{1+\delta\sigma}} + \int_0^{\infty} g(x, n)\frac{(|n|+bn)^{\sigma}dx}{[x(a+1)]^{1+\delta\sigma}} \\ &= \int_0^{\infty} g(-x, n)\frac{(|n|+bn)^{\sigma}dx}{[x(1-a)]^{1+\delta\sigma}} + \int_0^{\infty} g(x, n)\frac{(|n|+bn)^{\sigma}dx}{[x(1+a)]^{1+\delta\sigma}}.\end{aligned}$$

Setting

$$u = \rho\left\{\frac{|n|+bn}{[x(1-a)]^{\delta}}\right\}^{\gamma} \ \left(\text{resp. } u = \rho\left\{\frac{|n|+bn}{[x(1+a)]^{\delta}}\right\}^{\gamma}\right)$$

in the first (resp. second) integral above, by the Lebesgue term-by-term integration theorem (cf. [42]), we deduce that

$$\begin{aligned}\omega(\sigma, n) &= \left(\frac{1}{1-a} + \frac{1}{1+a}\right)\frac{1}{\gamma\rho^{\sigma/\gamma}}\int_0^{\infty} u^{\frac{\sigma}{\gamma}-1}(1-\tanh(u))du \\ &= \frac{4}{\gamma\rho^{\sigma/\gamma}(1-a^2)}\int_0^{\infty}\frac{e^{-2u}u^{\frac{\sigma}{\gamma}-1}}{1+e^{-2u}}du \\ &= \frac{4}{\gamma\rho^{\sigma/\gamma}(1-a^2)}\int_0^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k u^{\frac{\sigma}{\gamma}-1}}{e^{(2k+2)u}}du\end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{\gamma\rho^{\sigma/\gamma}(1-a^2)}\int_0^\infty \sum_{k=0}^{\infty}[e^{-(4k+2)u}-e^{-(4k+4)u}]u^{\frac{\sigma}{\gamma}-1}du\\
&= \frac{4}{\gamma\rho^{\sigma/\gamma}(1-a^2)}\sum_{k=0}^{\infty}\int_0^\infty [e^{-(4k+2)u}-e^{-(4k+4)u}]u^{\frac{\sigma}{\gamma}-1}du\\
&= \frac{4}{\gamma\rho^{\sigma/\gamma}(1-a^2)}\sum_{k=0}^{\infty}\int_0^\infty (-1)^k e^{-(2k+2)u}u^{\frac{\sigma}{\gamma}-1}du \quad (v=(2k+2)u)\\
&= \frac{4}{\gamma(2\rho)^{\sigma/\gamma}(1-a^2)}\int_0^\infty e^{-v}v^{\frac{\sigma}{\gamma}-1}dv\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^{\sigma/\gamma}}\\
&= \frac{4\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}(1-a^2)}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{\sigma/\gamma}}.
\end{aligned}
$$

Since $\frac{\sigma}{\gamma}>1$, we obtain that

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{\sigma/\gamma}} &= \sum_{k=1}^{\infty}\frac{1}{k^{\sigma/\gamma}}-2\sum_{k=1}^{\infty}\frac{1}{(2k)^{\sigma/\gamma}}\\
&= \frac{1}{2^{\sigma/\gamma}}\left(2^{\sigma/\gamma}-2\right)\zeta(\frac{\sigma}{\gamma}).
\end{aligned}
$$

We then deduce (9).

We have

$$
\begin{aligned}
\varpi(\sigma,x) &= \sum_{n=-1}^{-\infty} g(x,n)\frac{(|x|+ax)^{-\delta\sigma}}{(|n|+bn)^{1-\sigma}}+\sum_{n=1}^{\infty} g(x,n)\frac{(|x|+ax)^{-\delta\sigma}}{(|n|+bn)^{1-\sigma}}\\
&= \frac{(|x|+ax)^{-\delta\sigma}}{(1-b)^{1-\sigma}}\sum_{n=1}^{\infty}\frac{g(x,-n)}{n^{1-\sigma}}+\frac{(|x|+ax)^{-\delta\sigma}}{(1+b)^{1-\sigma}}\sum_{n=1}^{\infty}\frac{g(x,n)}{n^{1-\sigma}}. \quad (12)
\end{aligned}
$$

Since for $\gamma>0$ we have

$$
\frac{d}{du}(1-\tanh(u^\gamma)) = \frac{-2\gamma u^{\gamma-1}(1-e^{-2u^\gamma})}{(e^{u^\gamma}+e^{-u^\gamma})^2}-\frac{2\gamma u^{\gamma-1}e^{-u^\gamma}}{e^{u^\gamma}+e^{-u^\gamma}}<0\,,
$$

it follows that for $0<\sigma\le 1$ both

$$
\frac{g(x,-y)}{y^{1-\sigma}} \;\; and \;\; \frac{g(x,y)}{y^{1-\sigma}}
$$

are strictly decreasing in $y \in (0, \infty)$. By (12) and the decreasing property of series, we have

$$\varpi(\sigma, x) < \frac{(|x| + ax)^{-\delta\sigma}}{(1-b)^{1-\sigma}} \int_0^\infty \frac{g(x, -y)}{y^{1-\sigma}} dy + \frac{(|x| + ax)^{-\delta\sigma}}{(1+b)^{1-\sigma}} \int_0^\infty \frac{g(x, y)}{y^{1-\sigma}} dy.$$

Setting

$$u = \rho\left[\frac{y(1-b)}{(|x| + ax)^{\delta}}\right]^{\gamma} \left(\text{resp. } u = \rho\left[\frac{y(1+b)}{(|x| + ax)^{\delta}}\right]^{\gamma}\right)$$

in the first (resp. second) integral above, by simplifications, we obtain

$$\varpi(\sigma, x) < \frac{4(2^{\sigma/\gamma} - 2)\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}{\gamma(4\rho)^{\sigma/\gamma}(1-b^2)} = k_b(\sigma).$$

By (12) and the decreasing property of series, we also obtain that

$$\varpi(\sigma, x) > \frac{(|x| + ax)^{-\delta\sigma}}{(1-b)^{1-\sigma}} \int_1^\infty \frac{g(x, -y)}{y^{1-\sigma}} dy + \frac{(|x| + ax)^{-\delta\sigma}}{(1+b)^{1-\sigma}} \int_1^\infty \frac{g(x, y)}{y^{1-\sigma}} dy.$$

Setting again

$$u = \rho\left[\frac{y(1-b)}{(|x| + ax)^{\delta}}\right]^{\gamma} \left(\text{resp. } u = \rho\left[\frac{y(1+b)}{(|x| + ax)^{\delta}}\right]^{\gamma}\right)$$

in the first (resp. second) integral above, by simplifications, we have

$$\begin{aligned}
\varpi(\sigma, x) &> \frac{1}{\gamma\rho^{\sigma/\gamma}(1-b)} \int_{\rho\left[\frac{1-b}{(|x|+ax)^{\delta}}\right]^{\gamma}}^{\infty} u^{\frac{\sigma}{\gamma}-1}(1 - \tanh(u))du \\
&\quad + \frac{1}{\gamma\rho^{\sigma/\gamma}(1+b)} \int_{\rho\left[\frac{1+b}{(|x|+ax)^{\delta}}\right]^{\gamma}}^{\infty} u^{\frac{\sigma}{\gamma}-1}(1 - \tanh(u))du \\
&\geq \frac{2}{\gamma\rho^{\sigma/\gamma}(1-b^2)} \int_{\rho\left[\frac{1+b}{(|x|+ax)^{\delta}}\right]^{\gamma}}^{\infty} u^{\frac{\sigma}{\gamma}-1}(1 - \tanh(u))du \\
&= k_b(\sigma)(1 - \theta(\sigma, x)) > 0.
\end{aligned}$$

We obtain that

$$\lim_{u\to 0}(1-\tanh(u)) = 1,\ \lim_{u\to\infty}(1-\tanh(u)) = 0$$

and thus $0 < 1 - \tanh(u) \le 1$ ($u \in (0,\infty)$). Hence, we have

$$0 < \theta(\sigma, x) = \frac{2^{\sigma/\gamma}}{2(2^{\sigma/\gamma}-2)\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}$$

$$\times \int_0^{\rho\left[\frac{1+b}{(|x|+ax)^\delta}\right]^\gamma} u^{\frac{\sigma}{\gamma}-1}(1-\tanh(u))du$$

$$\le \frac{2^{\sigma/\gamma}}{2(2^{\sigma/\gamma}-2)\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}\int_0^{\rho\left[\frac{1+b}{(|x|+ax)^\delta}\right]^\gamma} u^{\frac{\sigma}{\gamma}-1}du$$

$$= \frac{\gamma(2\rho)^{\sigma/\gamma}}{2\sigma(2^{\sigma/\gamma}-2)\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}\left[\frac{1+b}{(|x|+ax)^\delta}\right]^\sigma,$$

namely, (10) and (11) follow.
This completes the proof of the lemma.

Lemma 2 *If* $\varepsilon > 0$, *and*

$$H_\varepsilon(b) := \sum_{|n|=1}^{\infty}\frac{1}{(|n|+bn)^{1+\varepsilon}},$$

then it holds

$$H_\varepsilon(b) = \frac{1}{\varepsilon}\left(\frac{2}{1-b^2}+o_1(1)\right)(1+o_2(1))\ (\varepsilon\to 0^+). \tag{13}$$

Proof We have

$$H_\varepsilon(b) = \sum_{n=-1}^{-\infty}\frac{1}{[n(b-1)]^{1+\varepsilon}} + \sum_{n=1}^{\infty}\frac{1}{[n(b+1)]^{1+\varepsilon}}$$

$$= \left[\frac{1}{(1-b)^{1+\varepsilon}}+\frac{1}{(1+b)^{1+\varepsilon}}\right]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}. \tag{14}$$

By (14) and the decreasing property of series, we derive that

$$
\begin{aligned}
H_\varepsilon(b) &= \left[\frac{1}{(1+b)^{1+\varepsilon}} + \frac{1}{(1-b)^{1+\varepsilon}}\right]\left(1 + \sum_{n=2}^{\infty} \frac{1}{n^{1+\varepsilon}}\right) \\
&< \left[\frac{1}{(1+b)^{1+\varepsilon}} + \frac{1}{(1-b)^{1+\varepsilon}}\right]\left(1 + \int_1^{\infty} \frac{dy}{y^{1+\varepsilon}}\right) \\
&= \frac{1}{\varepsilon}(\frac{2}{1-b^2} + o_1(1))(1+\varepsilon), \\
H_\varepsilon(b) &> \left[\frac{1}{(1+b)^{1+\varepsilon}} + \frac{1}{(1-b)^{1+\varepsilon}}\right]\int_1^{\infty} \frac{dy}{y^{1+\varepsilon}} \\
&= \frac{1}{\varepsilon}\left(\frac{2}{1-b^2} + o_1(1)\right).
\end{aligned}
$$

Hence, we obtain (13).
This completes the proof of the lemma.

Lemma 3 *For* $\varepsilon > 0$, *setting*

$$
E_\delta := \left\{x \in \mathbf{R}\backslash\{0\}; \frac{1}{(|x| + ax)^\delta} \geq 1\right\},
$$

we have

$$
H_\delta := \int_{E_\delta} \frac{1}{(|x| + ax)^{1+\delta\varepsilon}} dx = \frac{1}{\varepsilon} \frac{2}{1-a^2}. \tag{15}
$$

Proof Setting

$$
\begin{aligned}
E_\delta^+ &: = \left\{x > 0; \frac{1}{[x(1+a)]^\delta} \geq 1\right\}, \\
E_\delta^- &: = \left\{x < 0; \frac{1}{[(-x)(1-a)]^\delta} \geq 1\right\},
\end{aligned}
$$

it follows that $E_\delta = E_\delta^+ \cup E_\delta^-$ and

$$
H_\delta = \frac{1}{(1+a)^{1+\delta\varepsilon}} \int_{E_\delta^+} \frac{dx}{x^{1+\delta\varepsilon}} + \frac{1}{(1-a)^{1+\delta\varepsilon}} \int_{E_\delta^-} \frac{dx}{(-x)^{1+\delta\varepsilon}}.
$$

Setting $u = [x(1+a)]^\delta$ (resp. $u = [(-x)(1-a)]^\delta$) in the first (resp. second) integral above, we obtain

$$H_\delta = \left(\frac{1}{1+a} + \frac{1}{1-a}\right) \int_1^\infty \frac{du}{u^{1+\varepsilon}} = \frac{1}{\varepsilon} \frac{2}{1-a^2}.$$

Therefore, (15) follows.
This completes the proof of the lemma.

3 Main Results

Theorem 1 *Suppose that* $p > 1$, *and*

$$K_{a,b}(\sigma) := k_a^{\frac{1}{q}}(\sigma) k_b^{\frac{1}{p}}(\sigma) = \frac{4(2^{\sigma/\gamma} - 2)\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}{\gamma(4\rho)^{\sigma/\gamma}(1-a^2)^{1/q}(1-b^2)^{1/p}}. \tag{16}$$

If $f(x), b_n \geq 0$, *such that*

$$0 < \int_{-\infty}^{\infty} (|x| + ax)^{p(1+\delta\sigma)-1} f^p(x) dx < \infty, \text{ and}$$

$$0 < \sum_{|n|=1}^{\infty} (|n| + bn)^{q(1-\sigma)-1} b_n^q < \infty,$$

then we have the following equivalent inequalities:

$$I := \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} \left(1 - \tanh\left(\frac{\rho(|n| + bn)^\gamma}{(|x| + ax)^{\delta\gamma}}\right)\right) f(x) b_n dx < K_{a,b}(\sigma)$$

$$\times \left[\int_{-\infty}^{\infty} (|x| + ax)^{p(1+\delta\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}} \left[\sum_{|n|=1}^{\infty} (|n|+bn)^{q(1-\sigma)-1} b_n^q\right]^{\frac{1}{q}} \tag{17}$$

$$J_1 := \left\{\sum_{|n|=1}^{\infty} (|n| + bn)^{p\sigma-1} \left[\int_{-\infty}^{\infty} \left(1 - \tanh\left(\frac{\rho(|n| + bn)^\gamma}{(|x| + ax)^{\delta\gamma}}\right)\right) f(x) dx\right]^p\right\}^{\frac{1}{p}}$$

$$< K_{a,b}(\sigma) \left[\int_{-\infty}^{\infty} (|x| + ax)^{p(1+\delta\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}}, \tag{18}$$

$$J_2 := \left\{ \int_{-\infty}^{\infty} (|x| + ax)^{-q\delta\sigma - 1} \left[\sum_{|n|=1}^{\infty} \left(1 - \tanh\left(\frac{\rho(|n| + bn)^{\gamma}}{(|x| + ax)^{\delta\gamma}} \right) \right) b_n \right]^q dx \right\}^{\frac{1}{q}}$$

$$< K_{a,b}(\sigma) \left[\sum_{|n|=1}^{\infty} (|n| + bn)^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}}. \quad (19)$$

In particular, for $a = b = 0$, we deduce the following equivalent inequalities:

$$\sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} \left(1 - \tanh\left(\rho \left(\frac{|n|}{|x|^{\delta}} \right)^{\gamma} \right) \right) f(x) b_n dx$$

$$< \frac{4(2^{\sigma/\gamma} - 2)}{\gamma (4\rho)^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma} \right) \zeta\left(\frac{\sigma}{\gamma} \right)$$

$$\times \left(\int_{-\infty}^{\infty} |x|^{p(1+\delta\sigma)-1} f^p(x) dx \right)^{\frac{1}{p}} \left(\sum_{|n|=1}^{\infty} |n|^{q(1-\sigma)-1} b_n^q \right)^{\frac{1}{q}}, \quad (20)$$

$$\left\{ \sum_{|n|=1}^{\infty} |n|^{p\sigma-1} \left[\int_{-\infty}^{\infty} \left(1 - \tanh\left(\rho \left(\frac{|n|}{|x|^{\delta}} \right)^{\gamma} \right) \right) f(x) dx \right]^p \right\}^{\frac{1}{p}}$$

$$< \frac{4(2^{\sigma/\gamma} - 2)}{\gamma(4\rho)^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma} \right) \zeta\left(\frac{\sigma}{\gamma} \right) \left[\int_{-\infty}^{\infty} |x|^{p(1+\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}, \quad (21)$$

$$\left\{ \int_{-\infty}^{\infty} |x|^{-q\delta\sigma-1} \left[\sum_{|n|=1}^{\infty} \left(1 - \tanh\left(\rho \left(\frac{|n|}{|x|^{\delta}} \right)^{\gamma} \right) \right) b_n \right]^q dx \right\}^{\frac{1}{q}}$$

$$< \frac{4(2^{\sigma/\gamma} - 2)}{\gamma(4\rho)^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma} \right) \zeta\left(\frac{\sigma}{\gamma} \right) \left[\sum_{|n|=1}^{\infty} |n|^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}}. \quad (22)$$

Proof By Hölder's inequality with weight (cf. [43]) and (7), we obtain

$$\left[\int_{-\infty}^{\infty} g(x, n) f(x) dx \right]^p$$

$$= \left[\int_{-\infty}^{\infty} g(x, n) \frac{(|x| + ax)^{(1+\delta\sigma)/q} f(x)}{(|n| + bn)^{(1-\sigma)/p}} \frac{(|n| + bn)^{(1-\sigma)/p}}{(|x| + ax)^{(1+\delta\sigma)/q}} dx \right]^p$$

$$\leq \int_{-\infty}^{\infty} g(x,n)\frac{(|x|+ax)^{(1+\delta\sigma)(p-1)}}{(|n|+bn)^{1-\sigma}} f^{p}(x)dx$$
$$\times \left[\int_{-\infty}^{\infty} g(x,n)\frac{(|n|+bn)^{(1-\sigma)(q-1)}}{(|x|+ax)^{1+\delta\sigma}} dx\right]^{p-1}$$
$$= \frac{\omega^{p-1}(\sigma,n)}{(|n|+bn)^{p\sigma-1}} \int_{-\infty}^{\infty} g(x,n)\frac{(|x|+ax)^{(1+\delta\sigma)(p-1)}}{(|n|+bn)^{1-\sigma}} f^{p}(x)dx.$$

Then by (9) and the Lebesgue term-by-term integration theorem (cf. [42]), in view of (8), we deduce that

$$\begin{aligned} J_1 &\leq k_a^{\frac{1}{q}}(\sigma)\left[\sum_{|n|=1}^{\infty}\int_{-\infty}^{\infty} g(x,n)\frac{(|x|+ax)^{(1+\delta\sigma)(p-1)}}{(|n|+bn)^{1-\sigma}} f^{p}(x)dx\right]^{\frac{1}{p}} \\ &= k_a^{\frac{1}{q}}(\sigma)\left[\int_{-\infty}^{\infty}\sum_{|n|=1}^{\infty} g(x,n)\frac{(|x|+ax)^{(1+\delta\sigma)(p-1)}}{(|n|+bn)^{1-\sigma}} f^{p}(x)dx\right]^{\frac{1}{p}} \\ &= k_a^{\frac{1}{q}}(\sigma)\left[\int_{-\infty}^{\infty} \varpi(\sigma,x)(|x|+ax)^{p(1+\delta\sigma)-1} f^{p}(x)dx\right]^{\frac{1}{p}}. \end{aligned} \tag{23}$$

Hence, by (10), since

$$K_{a,b}(\sigma) = k_a^{\frac{1}{q}}(\sigma)k_b^{\frac{1}{p}}(\sigma),$$

we deduce (18).
By Hölder's inequality (cf. [43]), we have

$$\begin{aligned} I &= \sum_{|n|=1}^{\infty}\left[(|n|+bn)^{\frac{-1}{p}+\sigma}\int_{-\infty}^{\infty} g(x,n)f(x)dx\right][(|n|+bn)^{\frac{1}{p}-\sigma}b_n] \\ &\leq J_1\left[\sum_{|n|=1}^{\infty}(|n|+bn)^{q(1-\sigma)-1}b_n^{q}\right]^{\frac{1}{q}}. \end{aligned} \tag{24}$$

Then by (18), we deduce (17). On the other hand, assuming that (17) is valid, we set

$$b_n := (|n|+bn)^{p\sigma-1}\left[\int_{-\infty}^{\infty} g(x,n)f(x)dx\right]^{p-1} \quad (|n| \in \mathbf{N}).$$

Then, we obtain that

$$J_1 = \left[\sum_{|n|=1}^{\infty} (|n| + bn)^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{p}}.$$

In view of (23), it follows that $J_1 < \infty$. If $J_1 = 0$, then (18) is trivially valid; if $J_1 > 0$, then by (17), we have

$$\sum_{|n|=1}^{\infty} (|n| + bn)^{q(1-\sigma)-1} b_n^q$$

$$= J_1^p = I < K_{a,b}(\sigma) \left[\int_{-\infty}^{\infty} (|x| + ax)^{p(1+\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}$$

$$\times \left[\sum_{|n|=1}^{\infty} (|n| + bn)^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}},$$

$$J_1 = \left[\sum_{|n|=1}^{\infty} (|n| + bn)^{q(1-\sigma)-1} b_n^q \right]^{1-\frac{1}{q}}$$

$$< K_{a,b}(\sigma) \left[\int_{-\infty}^{\infty} (|x| + ax)^{p(1+\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}.$$

That is, (18) follows, which is equivalent to (17).
By Hölder's inequality with weight, we also obtain that

$$\left[\sum_{|n|=1}^{\infty} g(x,n) b_n \right]^q$$

$$= \left[\sum_{|n|=1}^{\infty} g(x,n) \frac{(|x| + ax)^{(1+\delta\sigma)/q}}{(|n| + bn)^{(1-\sigma)/p}} \frac{(|n| + bn)^{(1-\sigma)/p}}{(|x| + ax)^{(1+\delta\sigma)/q}} b_n \right]^q$$

$$\leq \left[\sum_{|n|=1}^{\infty} g(x,n) \frac{(|x| + ax)^{(1+\delta\sigma)(p-1)}}{(|n| + bn)^{1-\sigma}} \right]^{q-1}$$

$$\times \sum_{|n|=1}^{\infty} g(x,n) \frac{(|n| + bn)^{(1-\sigma)(q-1)}}{(|x| + ax)^{1+\delta\sigma}} b_n^q$$

$$= \frac{(\varpi(\sigma,x))^{q-1}}{(|x| + ax)^{-q\delta\sigma-1}} \sum_{|n|=1}^{\infty} g(x,n) \frac{(|n| + bn)^{(1-\sigma)(q-1)}}{(|x| + ax)^{1+\delta\sigma}} b_n^q.$$

By (10) and the Lebesgue term-by-term theorem, we obtain that

$$J_2 < k_a^{\frac{1}{p}}(\sigma)\left[\int_{-\infty}^{\infty}\sum_{|n|=1}^{\infty} g(x,n)\frac{(|n|+bn)^{(1-\sigma)(q-1)}}{(|x|+ax)^{1+\delta\sigma}}b_n^q dx\right]^{\frac{1}{q}}$$

$$= k_a^{\frac{1}{p}}(\sigma)\left[\sum_{|n|=1}^{\infty}\omega(\sigma,n)(|n|+bn)^{q(1-\sigma)-1}b_n^q\right]^{\frac{1}{q}}. \tag{25}$$

Hence, by (9), we deduce (19).
We have proved that (17) is true. Setting

$$f(x) := (|x|+ax)^{-q\delta\sigma-1}\left[\sum_{|n|=1}^{\infty} g(x,n)b_n\right]^{q-1} \quad (x \in \mathbf{R}\backslash\{0\}),$$

it follows that

$$J_2 = \left[\int_{-\infty}^{\infty}(|x|+ax)^{p(1+\delta\sigma)-1}f^p(x)dx\right]^{\frac{1}{q}},$$

and in view of (25), we derive that $J_2 < \infty$. If $J_2 = 0$, then (19) is trivially true; if $J_2 > 0$, then by (17), we have

$$\int_{-\infty}^{\infty}(|x|+ax)^{p(1+\delta\sigma)-1}f^p(x)dx$$

$$= J_2^q = I < K_{a,b}(\sigma)\left[\int_{-\infty}^{\infty}(|x|+ax)^{p(1+\delta\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}$$

$$\times\left[\sum_{|n|=1}^{\infty}(|n|+bn)^{q(1-\sigma)-1}b_n^q\right]^{\frac{1}{q}},$$

$$J_2 = \left[\int_{-\infty}^{\infty}(|x|+ax)^{p(1+\delta\sigma)-1}f^p(x)dx\right]^{1-\frac{1}{p}}$$

$$< K_{a,b}(\sigma)\left[\sum_{|n|=1}^{\infty}(|n|+bn)^{q(1-\sigma)-1}b_n^q\right]^{\frac{1}{q}},$$

namely, (19) follows.

On the other hand, assuming that (19) is true, by Hölder's inequality and similarly to as we proved (24), we have

$$I \le \left[\int_{-\infty}^{\infty} (|x| + ax)^{p(1+\delta\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} J_2. \tag{26}$$

Then by (19), we obtain (17), which is equivalent to (19).
Therefore, the inequalities (17), (18) and (19) are equivalent.
This completes the proof of the theorem.

Theorem 2 *With regard to the assumptions of Theorem 1, the constant factor $K_{a,b}(\sigma)$ in (17), (18) and (19) is the best possible.*

Proof For $0 < \varepsilon < q\sigma$, we set $\widetilde{\sigma} = \sigma - \frac{\varepsilon}{q}$ $(\in (0, 1))$,

$$\widetilde{f}(x) := \begin{cases} \frac{1}{(|x|+ax)^{\delta(\sigma+\frac{\varepsilon}{p})+1}}, x \in E_\delta, \\ 0, x \in \mathbf{R}\backslash E_\delta, \end{cases}$$

and

$$\widetilde{b}_n := (|n| + bn)^{(\sigma-\frac{\varepsilon}{q})-1} \quad (|n| \in \mathbf{N}).$$

By (13) and (15), we have

$$\begin{aligned} \widetilde{I}_1 &:= \left[\int_{-\infty}^{\infty} (|x| + ax)^{p(1+\delta\sigma)-1} \widetilde{f}^p(x)dx\right]^{\frac{1}{p}} \\ &\times \left[\sum_{|n|=1}^{\infty} (|n| + bn)^{q(1-\sigma)-1} \widetilde{b}_n^q\right]^{\frac{1}{q}} \\ &= \left[\int_{-\infty}^{\infty} \frac{dx}{(|x| + ax)^{\delta\varepsilon+1}}\right]^{\frac{1}{p}} \left[\sum_{|n|=1}^{\infty} \frac{1}{(|n| + bn)^{\varepsilon+1}}\right]^{\frac{1}{q}} \\ &\le \frac{1}{\varepsilon}\left(\frac{2}{1-a^2}\right)^{\frac{1}{p}} \left[(\frac{2}{1-b^2} + o_1(1))(1 + o_2(1))\right]^{\frac{1}{q}}. \end{aligned}$$

By (10), we also have that

$$\begin{aligned} \widetilde{I} \ : &= \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} g(x, n)\widetilde{f}(x)\widetilde{b}_n dx \\ &= \int_{E_\delta} \sum_{|n|=1}^{\infty} g(x, n) \frac{(|x| + ax)^{-\delta(\widetilde{\sigma}+\varepsilon)-1}}{(|n| + bn)^{1-\widetilde{\sigma}}} dx \end{aligned}$$

$$= \int_{E_\delta} \frac{\varpi(\widetilde{\sigma}, x)}{(|x| + ax)^{\delta\varepsilon+1}} dx \geq k_b(\widetilde{\sigma}) \int_{E_\delta} \frac{1 - \theta(\widetilde{\sigma}, x)}{(|x| + ax)^{\delta\varepsilon+1}} dx$$

$$= k_b(\widetilde{\sigma}) \left[\int_{E_\delta} \frac{dx}{(|x| + ax)^{\delta\varepsilon+1}} - \int_{E_\delta} \frac{dx}{O((|x| + ax)^{\delta(\sigma + \frac{\varepsilon}{p})+1})} \right]$$

$$= \frac{1}{\varepsilon} k_b(\sigma - \frac{\varepsilon}{q})(\frac{2}{1 - a^2} - \varepsilon O(1)).$$

If the constant factor $K_{a,b}(\sigma)$ in (17) is not the best possible, then there exists a positive number k, with $K_{a,b}(\sigma) > k$, such that (17) is satisfied when replacing $K_{a,b}(\sigma)$ by k. Then, in particular, we have $\varepsilon\tilde{I} < \varepsilon k \widetilde{I}_1$, namely,

$$k_b \left(\sigma - \frac{\varepsilon}{q} \right) \left(\frac{2}{1 - a^2} - \varepsilon O(1) \right)$$

$$< k \cdot \left(\frac{2}{1 - a^2} \right)^{\frac{1}{p}} \left[\left(\frac{2}{1 - b^2} + o_1(1) \right) (1 + o_2(1)) \right]^{\frac{1}{q}}.$$

It follows that

$$k_b(\sigma) \frac{2}{1 - a^2} \leq k \left(\frac{2}{1 - a^2} \right)^{\frac{1}{p}} \left(\frac{2}{1 - b^2} \right)^{\frac{1}{q}} \quad (\varepsilon \to 0^+),$$

namely,

$$K_{a,b}(\sigma) = \frac{4(2^{\sigma/\gamma} - 2)\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}{\gamma(4\rho)^{\sigma/\gamma}(1 - a^2)^{1/q}(1 - b^2)^{1/p}} \leq k.$$

This is a contradiction. Hence, the constant factor $K_{a,b}(\sigma)$ in (17) is the best possible.

The constant factor $K_{a,b}(\sigma)$ in (18) (resp. (19)) is still the best possible. Otherwise, we would reach the contradiction by (24) (resp. (26)) that the constant factor $K_{a,b}(\sigma)$ in (17) is not the best possible.

This completes the proof of the theorem.

4 Operator Expressions

Suppose that $p > 1$. We set the following functions:

$$\Phi(x) := (|x| + ax)^{p(1+\delta\sigma)-1} \;\; and \;\; \Psi(n) := (|n| + bn)^{q(1-\sigma)-1},$$

wherefrom

$$\Phi^{1-q}(x) = (|x| + ax)^{-q\delta\sigma-1},$$

and

$$\Psi^{1-p}(n) = (|n| + bn)^{p\sigma-1} \ (x \in \mathbf{R}\backslash\{0\}, |n| \in \mathbf{N}).$$

Define the following real weight normed linear spaces:

$$L_{p,\Phi}(\mathbf{R}) := \left\{ f = f(x); ||f||_{p,\Phi} := \left(\int_{-\infty}^{\infty} \Phi(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$L_{q,\Phi^{1-q}}(\mathbf{R}) := \left\{ h = h(x); ||h||_{q,\Phi^{1-q}} := \left(\int_{-\infty}^{\infty} \Phi^{1-q}(x)|h(x)|^q dx \right)^{\frac{1}{q}} < \infty \right\},$$

$$l_{q,\Psi} := \left\{ b = \{b_n\}_{|n|=1}^{\infty}; ||b||_{q,\Psi} := \left(\sum_{|n|=1}^{\infty} \Psi(n)|b_n|^q \right)^{\frac{1}{q}} < \infty \right\}$$

$$l_{p,\Psi^{1-p}} := \left\{ c = \{c_n\}_{|n|=1}^{\infty}; ||c||_{p,\Psi^{1-p}} := \left(\sum_{|n|=1}^{\infty} \Psi^{1-p}(n)|c_n|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

(a) In view of Theorem 1, for $f \in L_{p,\Phi}(\mathbf{R})$, setting

$$H^{(1)}(n) := \int_{-\infty}^{\infty} g(x,n)|f(x)|dx \ (|n| \in \mathbf{N}),$$

by (18), we have

$$||H^{(1)}||_{p,\Psi^{1-p}} = \left[\sum_{|n|=1}^{\infty} \Psi^{1-p}(n)(H^{(1)}(n))^p \right]^{\frac{1}{p}} < K_{\alpha,\beta}(\sigma)||f||_{p,\Phi} < \infty, \tag{27}$$

namely, $H^{(1)} \in l_{p,\Psi^{1-p}}$.

Definition 1 Define a Hilbert-type operator

$$T^{(1)} : L_{p,\Phi}(\mathbf{R}) \to l_{p,\Psi^{1-p}}$$

in the whole plane as follows: For any $f \in L_{p,\Phi}(\mathbf{R})$, there exists a unique representation $T^{(1)}f = H^{(1)} \in l_{p,\Psi^{1-p}}$, satisfying

$$(T^{(1)}f)(n) = H^{(1)}(n),$$

for any $|n| \in \mathbf{N}$.

In view of (27), it follows that

$$||T^{(1)}f||_{p,\Psi^{1-p}} = ||H^{(1)}||_{p,\Psi^{1-p}} \leq K_{a,b}||f||_{p,\Phi},$$

and then the operator $T^{(1)}$ is bounded, satisfying

$$||T^{(1)}|| = \sup_{f(\neq\theta)\in L_{p,\Phi}(\mathbf{R})} \frac{||T^{(1)}f||_{p,\Psi^{1-p}}}{||f||_{p,\Phi}} \leq K_{a,b}(\sigma).$$

In virtue of the fact that the constant factor $K_{a,b}(\sigma)$ in (27) is the best possible, we have

$$||T^{(1)}|| = K_{a,b}(\sigma) = \frac{4(2^{\sigma/\gamma} - 2)\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}{\gamma(4\rho)^{\sigma/\gamma}(1-a^2)^{1/q}(1-b^2)^{1/p}}. \tag{28}$$

If we define the formal inner product of $T^{(1)}f$ and b $(\in l_{q,\Psi})$ as follows:

$$(T^{(1)}f, b) := \sum_{|n|=1}^{\infty} (\int_{-\infty}^{\infty} g(x,n)f(x)dx)b_n,$$

then we can rewrite the equivalent forms (17) and (18) in the following manner:

$$(T^{(1)}f, b) < ||T^{(1)}||\cdot||f||_{p,\Psi}||b||_{q,\Phi}, ||T^{(1)}f||_{p,\Psi^{1-p}} < ||T^{(1)}||\cdot||f||_{p,\Phi}. \tag{29}$$

(b) In view of Theorem 1, for $b \in l_{q,\Psi}$, setting

$$H^{(2)}(x) := \sum_{|n|=1}^{\infty} g(x,n)b_n \ (x \in \mathbf{R}\backslash\{0\}),$$

we obtain, by (19), that

$$||H^{(2)}||_{q,\Phi^{1-q}} = \left[\int_{-\infty}^{\infty} \Phi^{1-q}(x)(H^{(2)}(x))^q dx\right]^{\frac{1}{q}} < K_{a,b}(\sigma)||b||_{q,\Psi} < \infty, \tag{30}$$

namely, $H^{(2)} \in L_{q,\Psi^{1-q}}(\mathbf{R})$.

Definition 2 Define a Hilbert-type operator

$$T^{(2)} : l_{q,\Psi} \to L_{q,\Psi^{1-q}}(\mathbf{R})$$

in the whole plane as follows: For any $b \in l_{q,\Psi}$, there exists a unique representation

$$T^{(2)}b = H^{(2)} \in L_{q,\Psi^{1-q}}(\mathbf{R}),$$

satisfying

$$(T^{(2)}b)(x) = H^{(2)}(x),$$

for any $x \in \mathbf{R}$.

In view of (30), we have

$$||T^{(2)}b||_{q,\Phi^{1-q}} = ||H^{(2)}||_{q,\Phi^{1-q}} \leq K_{a,b}(\sigma)||b||_{q,\Psi},$$

and then the operator $T^{(2)}$ is bounded, satisfying

$$||T^{(2)}|| = \sup_{b(\neq\theta)\in l_{q,\Psi}} \frac{||T^{(2)}b||_{q,\Phi^{1-q}}}{||b||_{q,\Psi}} \leq K_{a,b}(\sigma).$$

As the constant factor $K_{a,b}(\sigma)$ in (30) is the best possible, we have

$$||T^{(2)}|| = K_{a,b}(\sigma) = ||T^{(1)}||. \tag{31}$$

If we define the formal inner product of $T^{(2)}b$ and $f \ (\in L_{p,\Phi}(\mathbf{R}))$ as follows:

$$(T^{(2)}b, f) := \int_{-\infty}^{\infty} \sum_{|n|=1}^{\infty} g(x,n) b_n f(x) dx,$$

then we can rewrite the equivalent forms (17) and (19) in the following manner:

$$(T^{(2)}b, f) < ||T^{(2)}|| \cdot ||f||_{p,\Psi}||b||_{q,\Phi}, ||T^{(2)}b||_{q,\Phi^{1-q}} < ||T^{(2)}|| \cdot ||b||_{q,\Psi}. \tag{32}$$

Remark 1

(i) For $\delta = 1$, (20) reduces to (5). If $f(-x) = f(x) \ (x > 0)$, $b_{-n} = b_n \ \ (n \in \mathbf{N})$, then (5) reduces to the following half-discrete Hilbert-type inequality (cf. [6]):

$$\sum_{n=1}^{\infty} \int_0^{\infty} \left(1 - \tanh\left(\rho\left(\frac{n}{x}\right)^{\gamma}\right)\right) f(x) b_n dx$$
$$< \frac{2(2^{\sigma/\gamma} - 2)}{\gamma(4\rho)^{\sigma/\gamma}} \Gamma\left(\frac{\sigma}{\gamma}\right) \zeta\left(\frac{\sigma}{\gamma}\right)$$

$$\times \left[\int_0^\infty x^{p(1+\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}}. \tag{33}$$

(ii) For $\delta = 1$, (17) reduces to the following particular inequality with homogeneous kernel of degree 0:

$$\sum_{|n|=1}^\infty \int_{-\infty}^\infty \left(1 - \tanh\left(\rho\left(\frac{|n| + bn}{|x| + ax}\right)^\gamma\right)\right) f(x) b_n dx$$

$$< K_{a,b}(\sigma) \left[\int_{-\infty}^\infty (|x| + ax)^{p(1+\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}$$

$$\times \left[\sum_{|n|=1}^\infty (|n| + bn)^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}}. \tag{34}$$

(iii) For $\delta = -1$, (17) reduces to the following particular inequality with non-homogeneous kernel:

$$\sum_{|n|=1}^\infty \int_{-\infty}^\infty (1 - \tanh\left(\rho[(|x| + ax)(|n| + bn)]^\gamma\right)) f(x) b_n dx$$

$$< K_{a,b}(\sigma) \left[\int_{-\infty}^\infty (|x| + ax)^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}$$

$$\times \left[\sum_{|n|=1}^\infty (|n| + bn)^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}}. \tag{35}$$

The constant factors in the above inequalities are the best possible.

5 Two Kinds of Equivalent Reverse Inequalities

In the following, for the cases in $0 < p < 1$ and $p < 0$, we still use $||b||_{q,\Phi}$ and $||f||_{p,\Psi}$ as the formal symbol.

Theorem 3 *Suppose that* $0 < p < 1$. *If* $f(x), b_n \geq 0$, *satisfying*

$$0 < ||f||_{p,\Psi} < \infty, \; 0 < ||b||_{q,\Phi} < \infty,$$

then we have the following equivalent inequalities:

$$I = \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} \left(1 - \tanh\left(\frac{\rho(|n| + bn)^{\gamma}}{(|x| + ax)^{\delta\gamma}}\right)\right) f(x) b_n dx$$

$$> K_{a,b}(\sigma) \left[\int_{-\infty}^{\infty} (1-\theta(\sigma, x))(|x|+ax)^{p(1+\delta\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}} ||b||_{q,\Phi}, \quad (36)$$

$$J_1 = \left\{\sum_{|n|=1}^{\infty} (|n| + bn)^{p\sigma-1} \left[\int_{-\infty}^{\infty} \left(1 - \tanh\left(\frac{\rho(|n| + bn)^{\gamma}}{(|x| + ax)^{\delta\gamma}}\right)\right) f(x) dx\right]^p\right\}^{\frac{1}{p}}$$

$$> K_{a,b}(\sigma) \left[\int_{-\infty}^{\infty} (1 - \theta(\sigma, x))(|x| + ax)^{p(1+\delta\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}}, \quad (37)$$

$$\widetilde{J}_2 := \left\{\int_{-\infty}^{\infty} \frac{(1 - \theta(\sigma, x))^{1-q}}{(|x| + ax)^{q\delta\sigma+1}} \left[\sum_{|n|=1}^{\infty} \left(1 - \tanh\left(\frac{\rho(|n| + bn)^{\gamma}}{(|x| + ax)^{\delta\gamma}}\right)\right) b_n\right]^q dx\right\}^{\frac{1}{q}}$$

$$> K_{a,b}(\sigma)||b||_{q,\Phi}, \quad (38)$$

where the constant factor $K_{a,b}(\sigma)$ is the best possible.

Proof Similarly, by the reverse Hölder inequality (cf. [43]) and (7), we obtain that

$$\left[\int_{-\infty}^{\infty} g(x, n) f(x) dx\right]^p$$
$$\geq \frac{\omega^{p-1}(\sigma, n)}{(|n| + bn)^{p\sigma-1}} \int_{-\infty}^{\infty} g(x, n) \frac{(|x| + ax)^{(1+\delta\sigma)(p-1)}}{(|n| + bn)^{1-\sigma}} f^p(x) dx.$$

In view of (9), the Lebesgue term-by-term integration theorem (cf. [42]) and (8), we deduce that

$$J_1 \geq k_a^{\frac{1}{q}}(\sigma) \left[\int_{-\infty}^{\infty} \varpi(\sigma, x)(|x| + ax)^{p(1+\delta\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}}. \quad (39)$$

Hence, by (10), we obtain (37).

By the reverse Hölder inequality (cf. [43]), we also have that

$$I \geq J_1 \left[\sum_{|n|=1}^{\infty} (|n| + bn)^{q(1-\sigma)-1} b_n^q\right]^{\frac{1}{q}}. \quad (40)$$

In view of (37), we obtain (36).

On the other hand, assuming that (36) is valid, we set

$$b_n := (|n| + bn)^{p\sigma-1} \left[\int_{-\infty}^{\infty} g(x,n) f(x) dx \right]^{p-1} \quad (|n| \in \mathbf{N}).$$

Then, we obtain that

$$J_1 = \left[\sum_{|n|=1}^{\infty} (|n| + bn)^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{p}}.$$

In view of (39), it follows that $J_1 > 0$. If $J_1 = \infty$, then (37) is trivially valid; if $J_1 < \infty$, then by (36), we have

$$\begin{aligned} ||b||_{q,\Phi}^q &= J_1^p = I \\ &> K_{a,b}(\sigma) \left[\int_{-\infty}^{\infty} (1 - \theta(\sigma,x))(|x| + ax)^{p(1+\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} ||b||_{q,\Phi}, \end{aligned}$$

$$||b||_{q,\Phi}^{q-1} = J_1 > K_{a,b}(\sigma) \left[\int_{-\infty}^{\infty} (1 - \theta(\sigma,x))(|x| + ax)^{p(1+\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}},$$

namely, (37) holds, which is equivalent to (36).

Similarly to as we obtained (39), we have

$$\widetilde{J}_2 > k_a^{\frac{1}{p}}(\sigma) \left[\sum_{|n|=1}^{\infty} \omega(\sigma,n)(|n| + bn)^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}}. \tag{41}$$

Hence, by (9), we deduce (38). We have proved that (36) is valid. Setting

$$f(x) := \frac{(1 - \theta(\sigma,x))^{1-q}}{(|x| + ax)^{q\delta\sigma+1}} \left[\sum_{|n|=1}^{\infty} g(x,n) b_n \right]^{q-1} \quad (x \in \mathbf{R} \backslash \{0\}),$$

it then follows that

$$\widetilde{J}_2 = \left[\int_{-\infty}^{\infty} (1 - \theta(\sigma,x))(|x| + ax)^{p(1+\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{q}},$$

and in view of (41), we obtain that $\widetilde{J}_2 > 0$. If $\widetilde{J}_2 = \infty$, then (38) is trivially valid; if $\widetilde{J}_2 < \infty$, then by (36), we have

$$\int_{-\infty}^{\infty}(1-\theta(\sigma,x))(|x|+ax)^{p(1+\delta\sigma)-1}f^p(x)dx=\widetilde{J}_2^q=I$$

$$>K_{a,b}(\sigma)\left[\int_{-\infty}^{\infty}(1-\theta(\sigma,x))(|x|+ax)^{p(1+\delta\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}||b||_{q,\Phi},$$

$$\widetilde{J}_2=\left[\int_{-\infty}^{\infty}(1-\theta(\sigma,x))(|x|+ax)^{p(1+\delta\sigma)-1}f^p(x)dx\right]^{1-\frac{1}{p}}>K_{a,b}(\sigma)||b||_{q,\Phi},$$

namely, (38) follows. On the other hand, assuming that (38) is valid, by the reverse Hölder inequality (cf. [43]), we obtain

$$I\geq\left[\int_{-\infty}^{\infty}(1-\theta(\sigma,x))(|x|+ax)^{p(1+\delta\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\widetilde{J}_2. \tag{42}$$

Then by (38), we get (16), which is equivalent to (38).

Therefore, inequalities (36), (37) and (38) are equivalent.

For $\varepsilon>0$, we set $\widetilde{\sigma}=\sigma+\frac{\varepsilon}{p}$ $(>\gamma)$,

$$\widetilde{f}(x):=\begin{cases}\frac{1}{(|x|+ax)^{\delta(\sigma+\frac{\varepsilon}{p})+1}},\ x\in E_\delta,\\ 0,\ x\in\mathbf{R}\backslash E_\delta,\end{cases}$$

and

$$\widetilde{b}_n:=(|n|+bn)^{(\sigma-\frac{\varepsilon}{q})-1}\quad(|n|\in\mathbf{N}).$$

Then by (13) and (15), we obtain that

$$\begin{aligned}\widetilde{I}_1\ :&=\left[\int_{-\infty}^{\infty}(1-\theta(\sigma,x))(|x|+ax)^{p(1+\delta\sigma)-1}\widetilde{f}^p(x)dx\right]^{\frac{1}{p}}\\ &\times\left[\sum_{|n|=1}^{\infty}(|n|+bn)^{q(1-\sigma)-1}\widetilde{b}_n^q\right]^{\frac{1}{q}}\\ &=\left[\int_{-\infty}^{\infty}\frac{(1-\theta(\sigma,x))dx}{(|x|+ax)^{\delta\varepsilon+1}}\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}\frac{1}{(|n|+bn)^{\varepsilon+1}}\right]^{\frac{1}{q}}\\ &=\left[\frac{2}{\varepsilon}\frac{1}{1-a^2}-\int_{-\infty}^{\infty}\frac{dx}{O((|x|+ax)^{\delta(\sigma+\varepsilon)+1})}\right]^{\frac{1}{p}}\end{aligned}$$

$$
\times \left[\sum_{|n|=1}^{\infty} \frac{1}{(|n| + bn)^{\varepsilon+1}} \right]^{\frac{1}{q}}
$$

$$
= \frac{1}{\varepsilon} \left(\frac{2}{1-a^2} - \varepsilon O(1) \right)^{\frac{1}{p}} \left[\left(\frac{2}{1-b^2} + o_1(1) \right) (1 + o_2(1)) \right]^{\frac{1}{q}}.
$$

By (10), we also have that

$$
\begin{aligned}
\widetilde{I} &:= \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} g(x,n) \widetilde{f}(x) \widetilde{b}_n dx \\
&= \sum_{|n|=1}^{\infty} \int_{E_\delta} g(x,n) \frac{(|n| + bn)^{(\sigma - \frac{\varepsilon}{q})-1}}{(|x| + ax)^{\delta(\sigma + \frac{\varepsilon}{p})+1}} dx \\
&\le \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} g(x,n) \frac{(|n| + bn)^{(\widetilde{\sigma} - \varepsilon)-1}}{(|x| + ax)^{\delta\widetilde{\sigma}+1}} dx \\
&= \sum_{|n|=1}^{\infty} \frac{\omega(\widetilde{\sigma}, n)}{(|n| + bn)^{\varepsilon+1}} = k_a(\widetilde{\sigma}) \sum_{|n|=1}^{\infty} \frac{1}{(|n| + bn)^{\varepsilon+1}} \\
&= \frac{1}{\varepsilon} k_a \left(\sigma + \frac{\varepsilon}{p} \right) \left(\frac{2}{1-b^2} + o_1(1) \right) (1 + o_2(1)).
\end{aligned}
$$

If the constant factor $K_{a,b}(\sigma)$ in (37) is not the best possible, then there exists a positive number k, with $K_{a,b}(\sigma) < k$, such that (37) is valid when replacing $K_{a,b}(\sigma)$ by k. Then, in particular, we have $\varepsilon \tilde{I} > \varepsilon k \widetilde{I}_1$, namely,

$$
\begin{aligned}
& k_a \left(\sigma + \frac{\varepsilon}{p} \right) \left(\frac{2}{1-b^2} + o_1(1) \right) (1 + o_2(1)) \\
> k \cdot & \left(\frac{2}{1-a^2} - \varepsilon O(1) \right)^{\frac{1}{p}} \left[\left(\frac{2}{1-b^2} + o_1(1) \right) (1 + o_2(1)) \right]^{\frac{1}{q}}.
\end{aligned}
$$

It follows that

$$
k_a(\sigma) \frac{2}{1-b^2} \ge k \cdot \left(\frac{2}{1-a^2} \right)^{2/p} \left(\frac{2}{1-b^2} \right)^{2/q} \quad (\varepsilon \to 0^+),
$$

namely,

$$
K_{a,b}(\sigma) = \frac{4(2^{\sigma/\gamma} - 2)\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}{\gamma(4\rho)^{\sigma/\gamma}(1-a^2)^{1/q}(1-b^2)^{1/p}} \ge k.
$$

This is a contradiction. Hence, the constant factor $K_{a,b}(\sigma)$ in (36) is the best possible.

The constant factor $K_{a,b}(\sigma)$ in (37) ((38)) is still the best possible. Otherwise, we would reach a contradiction by (40) ((42)) that the constant factor $K_{a,b}(\sigma)$ in (36) is not the best possible.

Theorem 4 *Suppose that* $p < 0$. *If* $f(x), b_n \geq 0$, *satisfying* $0<||f||_{p,\Psi}, ||b||_{q,\Phi} < \infty$, *then we have the following equivalent inequalities:*

$$I = \sum_{|n|=1}^{\infty} \int_{-\infty}^{\infty} \left(1 - \tanh\left(\frac{\rho(|n| + bn)^{\gamma}}{(|x| + ax)^{\delta\gamma}}\right)\right) f(x) b_n dx$$
$$> K_{a,b}(\sigma)||f||_{p,\Psi}||b||_{q,\Phi}, \tag{43}$$

$$J_1 = \left\{\sum_{|n|=1}^{\infty} (|n| + bn)^{p\sigma-1} \left[\int_{-\infty}^{\infty} \left(1 - \tanh\left(\frac{\rho(|n| + bn)^{\gamma}}{(|x| + ax)^{\delta\gamma}}\right)\right) f(x) dx\right]^p\right\}^{\frac{1}{p}}$$
$$> K_{a,b}(\sigma)||f||_{p,\Psi}, \tag{44}$$

$$J_2 = \left\{\int_{-\infty}^{\infty} \frac{1}{(|x| + ax)^{q\delta\sigma+1}} \left[\sum_{|n|=1}^{\infty} \left(1 - \tanh\left(\frac{\rho(|n| + bn)^{\gamma}}{(|x| + ax)^{\delta\gamma}}\right)\right) b_n\right]^q dx\right\}^{\frac{1}{q}}$$
$$> K_{a,b}(\sigma)||b||_{q,\Phi}, \tag{45}$$

where the constant factor $K_{a,b}(\sigma)$ *is the best possible.*

Proof For $p < 0$, by the reverse Hölder inequality (cf. [43]) and (7), we find

$$\left[\int_{-\infty}^{\infty} g(x, n) f(x) dx\right]^p$$
$$\leq \frac{\omega^{p-1}(\sigma, n)}{(|n| + bn)^{p\sigma-1}} \int_{-\infty}^{\infty} g(x, n) \frac{(|x| + ax)^{(1+\delta\sigma)(p-1)}}{(|n| + bn)^{1-\sigma}} f^p(x) dx.$$

Then by (9), the Lebesgue term-by-term integration theorem (cf. [42]) and (8), we deduce that

$$J_1 \geq k_a^{\frac{1}{q}}(\sigma) \left[\int_{-\infty}^{\infty} \varpi(\sigma, x)(|x| + ax)^{p(1+\delta\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}}. \tag{46}$$

Hence, by (10), we obtain (44).

By the reverse Hölder inequality (cf. [43]), we have

$$I \geq J_1 \left[\sum_{|n|=1}^{\infty} (|n| + bn)^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}}. \tag{47}$$

Then by (44), we deduce (43).

On the other hand, assuming that (43) is valid, we set

$$b_n := (|n| + bn)^{p\sigma-1} \left(\int_{-\infty}^{\infty} g(x, n) f(x) dx \right)^{p-1} \quad (|n| \in \mathbf{N})$$

and find $J_1 = ||b||_{q,\Phi}^{\frac{q}{p}}$. In view of (46), it follows that $J_1 > 0$. If $J_1 = \infty$, then (44) is trivially valid; if $J_1 < \infty$, then by (43), we have

$$||b||_{q,\Phi}^q = J_1^p = I > K_{a,b}(\sigma)||f||_{p,\Psi}||b||_{q,\Phi},$$

$$J_1 = ||b||_{q,\Phi}^{q-1} > K_{a,b}(\sigma)||f||_{p,\Psi},$$

namely, (44) holds, which is equivalent to (43).

Similarly, we obtain that

$$J_2 > k_b^{\frac{1}{p}}(\sigma) \left[\sum_{|n|=1}^{\infty} \omega(\sigma, n)(|n| + bn)^{q(1-\sigma)-1} b_n^q \right]^{\frac{1}{q}}. \tag{48}$$

Hence, by (9), we deduce (38). We have proved that (43) is valid. Setting

$$f(x) := \frac{1}{(|x| + ax)^{q\delta\sigma+1}} \left(\sum_{|n|=1}^{\infty} g(x, n) b_n \right)^{q-1} \quad (x \in \mathbf{R} \backslash \{0\}),$$

it follows that $J_2 = ||f||_{p,\Psi}^{\frac{p}{q}}$ and in view of (48), we get $J_2 > 0$. If $J_2 = \infty$, then (45) is trivially valid; if $J_2 < \infty$, then by (43), we have

$$||f||_{p,\Psi}^p = J_2^q = I > K_{a,b}(\sigma)||f||_{p,\Psi}||b||_{q,\Phi},$$

$$J_2 = ||f||_{p,\Psi}^{p-1} > K_{a,b}(\sigma)||b||_{q,\Phi},$$

namely, (45) follows.

On the other hand, assuming that (45) is valid, by the reverse Hölder inequality (cf. [43]), we obtain

$$I \geq \left[\int_{-\infty}^{\infty}(|x|+ax)^{p(1+\delta\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}} J_{2}. \tag{49}$$

Then by (45), we get (23), which is equivalent to (45).

Therefore, inequalities (43), (44) and (45) are equivalent.

For $0<\varepsilon<|p|(\sigma-\gamma)$, we set $\widetilde{\sigma}=\sigma+\frac{\varepsilon}{p}\ (>\gamma)$,

$$\widetilde{f}(x) := \begin{cases} \frac{1}{(|x|+ax)^{\delta(\sigma+\frac{\varepsilon}{p})+1}}, x\in E_{\delta}, \\ 0, x\in \mathbf{R}\backslash E_{\delta}, \end{cases}$$

and

$$\widetilde{b}_{n} := (|n|+bn)^{(\sigma-\frac{\varepsilon}{q})-1}(|n|\in \mathbf{N}).$$

Then by (13) and (15), we obtain that

$$\begin{aligned}
\widetilde{I}_{1} &:= \left[\int_{-\infty}^{\infty}(|x|+ax)^{p(1+\delta\sigma)-1}\widetilde{f}^{p}(x)dx\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}(|n|+bn)^{q(1-\sigma)-1}\widetilde{b}_{n}^{q}\right]^{\frac{1}{q}} \\
&= \left[\int_{-\infty}^{\infty}\frac{dx}{(|x|+ax)^{\delta\varepsilon+1}}\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}\frac{1}{(|n|+bn)^{\varepsilon+1}}\right]^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left(\frac{2}{1-a^{2}}\right)^{\frac{1}{p}}\left[(\frac{2}{1-b^{2}}+o_{1}(1))(1+o_{2}(1))\right]^{\frac{1}{q}}.
\end{aligned}$$

By (10), we still have

$$\begin{aligned}
\widetilde{I} &:= \sum_{|n|=1}^{\infty}\int_{-\infty}^{\infty}g(x,n)\widetilde{f}(x)\widetilde{b}_{n}dx \\
&= \sum_{|n|=1}^{\infty}\int_{E_{\delta}}g(x,n)\frac{(|n|+bn)^{(\sigma-\frac{\varepsilon}{q})-1}}{(|x|+ax)^{\delta(\sigma+\frac{\varepsilon}{p})+1}}dx \\
&\leq \sum_{|n|=1}^{\infty}\int_{-\infty}^{\infty}g(x,n)\frac{(|n|+bn)^{(\widetilde{\sigma}-\varepsilon)-1}}{(|x|+ax)^{\delta\widetilde{\sigma}+1}}dx \\
&= \sum_{|n|=1}^{\infty}\frac{\omega(\widetilde{\sigma},n)}{(|n|+bn)^{\varepsilon+1}} = k_{a}(\widetilde{\sigma})\sum_{|n|=1}^{\infty}\frac{1}{(|n|+bn)^{\varepsilon+1}} \\
&= \frac{1}{\varepsilon}k_{a}(\sigma+\frac{\varepsilon}{p})(\frac{2}{1-b^{2}}+o_{1}(1))(1+o_{2}(1)).
\end{aligned}$$

If the constant factor $K_{a,b}(\sigma)$ in (43) is not the best possible, then there exists a positive number k, with $K_{a,b}(\sigma) < k$, such that (43) is valid when replacing $K_{a,b}(\sigma)$ by k. Then, in particular, we have $\varepsilon \tilde{I} > \varepsilon k \widetilde{I_1}$, namely,

$$
\begin{aligned}
& k_a\left(\sigma + \frac{\varepsilon}{p}\right)\left(\frac{2}{1-b^2} + o_1(1)\right)(1 + o_2(1)) \\
& > k \cdot \left(\frac{2}{1-a^2}\right)^{\frac{1}{p}}\left[\left(\frac{2}{1-b^2} + o_1(1)\right)(1 + o_2(1))\right]^{\frac{1}{q}}.
\end{aligned}
$$

It follows that

$$
k_a(\sigma)\frac{2}{1-b^2} \geq k\left(\frac{2}{1-a^2}\right)^{\frac{1}{p}}\left(\frac{2}{1-b^2}\right)^{\frac{1}{q}} \quad (\varepsilon \to 0^+),
$$

namely,

$$
K_{a,b}(\sigma) = \frac{4(2^{\sigma/\gamma} - 2)\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}{\gamma(4\rho)^{\sigma/\gamma}(1-a^2)^{1/q}(1-b^2)^{1/p}} \geq k.
$$

This is a contradiction. Hence, the constant factor $K_{a,b}(\sigma)$ in (43) is the best possible.

The constant factor $K_{a,b}(\sigma)$ in (44) ((45)) is still the best possible. Otherwise, we would reach a contradiction by (47) ((49)) that the constant factor $K_{a,b}(\sigma)$ in (43) is not the best possible.

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Analysis of Apostol-Type Numbers and Polynomials with Their Approximations and Asymptotic Behavior

Yilmaz Simsek

Abstract In this chapter, using the methods and techniques of approximation of some classical polynomials and numbers including the Apostol–Bernoulli numbers and polynomials, we survey and investigate various properties of the Boole type combinatorial numbers and polynomials. By applying the p-adic q-integrals including the bosonic and fermionic p-adic integrals on p-adic integers, we study on generating functions for the generalized Boole type combinatorial numbers and polynomials attached to the Dirichlet character. These numbers and polynomials are related to the generalized Apostol–Bernoulli numbers and polynomials, the generalized Apostol–Euler numbers and polynomials, generalized Apostol–Daehee numbers and polynomials, and also generalized Apostol–Changhee numbers and polynomials. With the help of these generating functions, PDEs and their functional equation, many formulas, identities and relations involving the generalized Apostol–Daehee and Apostol–Changhee numbers and polynomials, the Stirling numbers, the Bernoulli numbers of the second kind, the generalized Bernoulli numbers and the generalized Euler numbers, and the Frobenius–Euler polynomials are given. Finally, by using asymptotic estimates for the Apostol–Bernoulli polynomials, asymptotic estimates for Boole type combinatorial numbers and polynomials are given.

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1 Introduction, Definitions and Notations

Special numbers and polynomials have played an important role in theory of the approximation and analytic inequalities. These numbers and polynomials have been used in almost all areas of mathematics, in physics, and in engineering problems.

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In this chapter, we use the following standard notations and definitions:

$$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}.$$

$\mathbb{Z}$ denotes the set of integers, $\mathbb{Q}$ denotes the set of rational numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_p$ denotes the set of p-adic integers.

We assume that $\ln z$ denotes the principal branch of the multi-valued function $\ln z$ with the imaginary part $\Im(\ln z)$ constrained by the interval $(-\pi, \pi]$. For example, for $z \in \mathbb{C}$, we have

$$\ln z = \ln |z| + i \arg z$$

with $-\pi < i \arg z \leq \pi$.

For $z \in \mathbb{C}$, setting

$$\exp(z) = e^z.$$

For $x \in \mathbb{R}$, $[x]$ denotes the integral part of x.

In addition to the above standard notations, we also give the following notations:

$$0^n = \begin{cases} 1, & (n = 0) \\ 0, & (n \in \mathbb{N}), \end{cases}$$

and the Pochhammer's symbol for the rising factorial is given by the following notation:

$$(\lambda)^{\overline{v}} = \frac{\Gamma(\lambda + v)}{\Gamma(\lambda)} = \lambda(\lambda + 1) \cdots (\lambda + v - 1),$$

and

$$(\lambda)^{\overline{0}} = 1$$

for $\lambda \neq 1$, where $v \in \mathbb{N}$, $\lambda \in \mathbb{C}$, and $\Gamma(\lambda)$ denotes the gamma function, which is an important special function in mathematics.

$$\binom{z}{v} = \frac{z(z-1)\cdots(z-v+1)}{v!} = \frac{(z)_v}{v!} \ (v \in \mathbb{N}, z \in \mathbb{C})$$

and

$$\binom{z}{0} = 1.$$

Observe that

$$(-\lambda)^{\overline{v}} = (-1)^v (\lambda)_v$$

(*cf.* [8, 83, 86]).

Bernoulli Type Polynomials and Numbers and Euler Type Polynomials and Numbers

Here, the generating functions of Bernoulli type numbers and polynomials and Euler type numbers and polynomials are given. With the help of approximation theory, asymptotic estimates for these polynomials are also given.

The Apostol–Bernoulli polynomials, $\mathcal{B}_n(x;\lambda)$, are defined by means of the following generating function:

$$F_A(t,x;\lambda) = \frac{t}{\lambda e^t - 1} e^{tx} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda) \frac{t^n}{n!}, \tag{1.1}$$

where $\lambda \in \mathbb{C}$, and the following set denotes the poles of the function $F_A(t,x;\lambda)$.

If

$$P = \{2\pi i n - \ln \lambda : n \in \mathbb{Z}\},$$

when $\lambda \neq 1$, and

$$P = \{2\pi i n : n \in \mathbb{Z}\}$$

when $\lambda = 1$; under this condition, 0 is a removable singularity. Setting $\lambda \to 1$, the set P has been reflected in various discontinuities. That is, when $\lambda = 1$ and $\lambda \neq 1$, the radius of convergence of the series in (1.1) is 2π and $|\ln \lambda|$.

Substituting $x = 0$ into (1.1), we have

$$\mathcal{B}_n(\lambda) = \mathcal{B}_n(0;\lambda).$$

Here, $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol–Bernoulli numbers (*cf.* [16, 31, 53, 60, 84, 87]; see also the references cited in each of these earlier works).

By using (1.1), few values the Apostol–Bernoulli numbers and polynomials are given as follows:

$$\mathcal{B}_0(\lambda) = 0,$$

$$\mathcal{B}_1(\lambda) = \frac{1}{\lambda - 1},$$

$$\mathcal{B}_2(\lambda) = \frac{-2\lambda}{(\lambda-1)^2},$$
$$\mathcal{B}_3(\lambda) = \frac{3\lambda(\lambda+1)}{(\lambda-1)^3}, \ldots$$

and

$$\mathcal{B}_0(x;\lambda) = 0,$$
$$\mathcal{B}_1(x;\lambda) = \frac{1}{\lambda-1},$$
$$\mathcal{B}_2(x;\lambda) = \frac{1}{\lambda-1}x - \frac{2\lambda}{(\lambda-1)^2},$$
$$\mathcal{B}_3(x;\lambda) = \frac{3}{\lambda-1}x^2 - \frac{6\lambda}{(\lambda-1)^2}x + \frac{3\lambda(\lambda+1)}{(\lambda-1)^3}, \ldots$$

and so on (*cf.* [1–43, 46–48, 50–86]; see also the references cited in each of these earlier works).

By using the following Lipschitz summation formula, the following Fourier series of the polynomials $\mathcal{B}_n(x;\lambda)$, for any $(\lambda \in \mathbb{C}, (\lambda \neq 0))$, was given by Luo [52]:

$$\mathcal{B}_n(x;\lambda) = -\delta_n(x;\lambda) - \frac{n!}{\lambda^x} \sum_{v \in \mathbb{Z}\setminus\{0\}}^{\infty} \frac{\exp(2\pi i v x)}{(2\pi i v - \log\lambda)^n}, \tag{1.2}$$

where

$$\delta_n(x;\lambda) = \begin{cases} 0, & \lambda = 1 \\ \frac{(-1)^n n!}{\lambda^x (\log\lambda)^n}, & \lambda \neq 1 \end{cases}$$

(*cf.* see also for detail et al. [56]).

In the work of Navas et al. [56], using the appropriate approximating sums over the sets $F \sqsubseteq P$, substituting $x = 0$ into the Fourier series of the polynomials $\mathcal{B}_n(x;\lambda)$ in equation (1.2), Navas et al. gave an asymptotic expansion for the Apostol–Bernoulli numbers $\mathcal{B}_n(\lambda)$, by the following theorem:

Theorem 1 ([56]) *Let $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. Let F be a finite subset of the set of P satisfying*

$$\max\{|w| : w \in F\} < \min\{|w| : w \in P\setminus F\} = \beta.$$

For all integers n (≥ 2), then we have

$$(\mathcal{B}_n(\lambda))/(n!)) = -\sum_{w \in F} \frac{1}{w^n} + O\left(\frac{1}{\beta^n}\right),$$

where the constant implicit in the order term depends only on λ *and* F.

It is easy to see the following relation:

$$B_n(x) = \lim_{\lambda \to 1} \mathcal{B}_n(x; \lambda),$$

where $B_n(x)$ denotes the Bernoulli polynomials (of the first kind) (*cf.* [3–43, 46–48, 50–78, 80–90]; see also the references cited in each of these earlier works).

Some properties of the Bernoulli polynomials are given as follows:

$$\frac{d}{dx}\{B_n(x)\} = nB_{n-1}(x),$$

$$B_{2n}\left(\frac{1}{2}\right) = \left(2^{1-2n} - 1\right) B_{2n}, \textit{and}$$

$$B_{2n-1}\left(\frac{1}{2}\right) = 0 \quad (n \geq 1),$$

and

$$B_n = B_n(0)$$

denotes the Bernoulli numbers (of the first kind). Observe that

$$B_n\left(x + \frac{1}{2}\right) = \sum_{j=0}^{n} \binom{n}{2j} \left(2^{1-2j} - 1\right) B_{2j} x^{n-2j}.$$

The well-known Euler formula including the Riemann zeta function $\zeta(z)$ and the Bernoulli numbers is given as follows:

$$B_{2n} = (-1)^{n-1} 2(2n)!(2\pi)^{-2n} \zeta(2n),$$

where $n \in \mathbb{N}$ (*cf.* [3–43, 46–48, 50–78, 80–90]).

The following well-known inequality was given by Kouba [42]:

$$1 < \zeta(2n) < 2, \tag{1.3}$$

where $n \in \mathbb{N}$.

Remark 1 We observe that with the help of the above inequality, the lower and upper bounds of Bernoulli numbers B_{2n} can be easily found. Consequently, it is known that the finite arithmetic sums, which contain the Dedekind sums and the Hardy sums, are closely related to the Bernoulli numbers and polynomials. Perhaps with the help of (1.3), the lower and upper bounds of these sums may be easily found.

Let $z \in \mathbb{C}$. The cosine and sine functions are defined, respectively, by

$$T_{2k}(z) = \sum_{j=0}^{k} (-1)^j \frac{z^{2j}}{(2j)!},$$

and

$$T_{2k+1}(z) = \sum_{j=0}^{k} (-1)^j \frac{z^{2j+1}}{(2j+1)!}$$

(*cf.* [10]).

It is well known that the series of the function $T_n(2\pi z)$ is uniformly convergent on a compact subset to *cos* $(2\pi z)$ if n is even, and to $\sin(2\pi z)$ if n is odd (*cf.* [10, 42, 56]). Therefore, we have the following well-known approximation result for Bernoulli polynomials $B_n(z)$ and the function $T_n(2\pi z)$:

Theorem 2 *For all $z \in \mathbb{C}$, $n \geq 2$, we have (with $k = \left[\frac{n}{2}\right]$)*

$$\left|(-1)^k \frac{(2\pi)^n}{2n!} B_n\left(z + \frac{1}{2}\right) - T_n(2\pi z)\right| < 2^{-n} e^{(4\pi|z|)}$$

(cf. [10]*).*

Corollary 1 *The following sequences converge uniformly on compact subsets of $\mathbb{C}$:*

$$(-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}(z) \to \cos(2\pi z)$$

and

$$(-1)^{k-1} \frac{(2\pi)^{2k+1}}{2(2k+1)!} B_{2k+1}(z) \to \sin(2\pi z)$$

(cf. [10]*).*

For $\lambda = 1$, $n > 1$, and $0 < x < 1$, equation (1.2) reduces to the following well-known Fourier series for the Bernoulli polynomials:

$$B_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{v \in \mathbb{Z} \setminus \{0\}}^{\infty} \frac{\exp(2\pi i v x)}{v^n}$$

(*cf.* [2, 8, 52, 86]).

The Apostol-Euler numbers of the first kind, $\mathcal{E}_n(x;\lambda)$, are defined by means of the following generating function:

$$F_{P1}(t,x;\lambda) = \frac{2}{\lambda e^t + 1} e^{tx} = \sum_{n=0}^{\infty} \mathcal{E}_n(x;\lambda) \frac{t^n}{n!} \tag{1.4}$$

($\lambda \in \mathbb{C}$; $|t| < \pi$ when $\lambda = 1$ and $|t| < |\ln(-\lambda)|$ when $\lambda \neq 1$).

By using the following Lipschitz summation formula, Luo [52] gave the following Fourier series of the polynomials $\mathcal{E}_n(x;\lambda)$, for any ($\lambda \in \mathbb{C}$, ($\lambda \neq 0$)):

$$\mathcal{E}_n(x;\lambda) = \frac{2n!}{\lambda^x} \sum_{v \in \mathbb{Z}}^{\infty} \frac{\exp((2v-1)\pi i x)}{((2v-1)\pi i - \log \lambda)^{n+1}}. \tag{1.5}$$

Substituting $x = 0$ into (1.4), we have the first kind Apostol–Euler numbers:

$$\mathcal{E}_n(\lambda) = \mathcal{E}_n(0;\lambda). \tag{1.6}$$

When $\lambda \to 1$ into (1.4) and (1.6), we have the first kind Euler polynomials and the first kind Euler numbers, respectively:

$$E_n(x) = \lim_{\lambda \to 1} \mathcal{E}_n(x;\lambda) \tag{1.7}$$

and

$$E_n = \lim_{\lambda \to 1} \mathcal{E}_n(\lambda) \tag{1.8}$$

(*cf.* [3–43, 46–48, 50–78, 80–88]; see also the references cited in each of these earlier works).

For $\lambda = 1$, $n > 1$, and $0 < x < 1$, equation (1.5) reduces to the following well-known Fourier series for the Euler polynomials:

$$E_n(x) = \frac{2n!}{(\pi i)^{n+1}} \sum_{v \in \mathbb{Z}}^{\infty} \frac{\exp((2v-1)\pi i v x)}{(2v-1)^{n+1}}$$

(*cf.* [2, 8, 52, 86]).

For $x = 0$, by using (1.4), few values of the Apostol–Euler numbers of the first kind are given as follows:

$$\mathcal{E}_0(\lambda) = \frac{2}{\lambda + 1},$$

$$\mathcal{E}_1(\lambda) = -\frac{2\lambda}{(\lambda + 1)^2},$$

$$\mathcal{E}_2(\lambda) = \frac{2\lambda(\lambda-1)}{(\lambda+1)^3},$$

$$\mathcal{E}_3(\lambda) = -\frac{2\lambda\left(\lambda^2-4\lambda+1\right)}{(\lambda+1)^4}, \ldots$$

(*cf.* [9–43, 46–48, 50–78, 80–86]; see also the references cited in each of these earlier works).

The second kind Apostol–Euler polynomials, $\mathcal{E}_n^*(x,\lambda)$, are defined by means of the following generating function:

$$F_{P2}(t,x;\lambda) = \frac{2}{\lambda e^t + \lambda^{-1}e^{-t}}e^{tx} = \sum_{n=0}^{\infty}\mathcal{E}_n^*(x;\lambda)\frac{t^n}{n!} \tag{1.9}$$

(*cf.* [75, 77, 86]).

Combining (1.4) with (1.9), we have the following well-known relation:

$$\mathcal{E}_n^*(x;\lambda) = \lambda 2^n \mathcal{E}_n\left(\frac{x+1}{2},\lambda^2\right). \tag{1.10}$$

By combining (1.5) with (1.10), we arrive at Fourier series of the polynomials $\mathcal{E}_n^*(x;\lambda)$ by the following theorem:

Theorem 3 *Let* $\lambda \in \mathbb{C}(\lambda \neq 0)$. *Then we have*

$$\mathcal{E}_n^*(x;\lambda) = \frac{2^{n+1}n!}{\lambda^{2x-1}}\sum_{v\in\mathbb{Z}}^{\infty}\frac{\exp\left(\left(\frac{2v-1}{2}\right)\pi i(x+1)\right)}{\left((2v-1)\pi i - \log\lambda^2\right)^{n+1}}. \tag{1.11}$$

Substituting $\lambda = 1$ and $x = 0$ into the above relation, a relation between the first and second kind Euler numbers is given as follows:

$$E_n^* = 2^n E_n\left(\frac{1}{2}\right)$$

(*cf.* [36, 69, 75, 77, 86]; see also the references cited in each of these earlier works).

Substituting $\lambda = 1$ into (1.11), we have

$$\mathcal{E}_n^*(x) = \frac{2^{n+1}n!}{(\pi i)^{n+1}}\sum_{v\in\mathbb{Z}}^{\infty}\frac{\exp\left(\left(\frac{2v-1}{2}\right)\pi i(x+1)\right)}{(2v-1)^{n+1}}.$$

Combining (1.1) with (1.4), we have the following well-known relation:

$$\mathcal{B}_n(x;\lambda) = -\frac{n}{2}\mathcal{E}_{n-1}(x;-\lambda) \tag{1.12}$$

(*cf.* [75, 77, 86]; see also the references cited in each of these earlier works).

The λ-Bernoulli polynomials (Apostol-type Bernoulli polynomials), $\mathfrak{B}_n(x;\lambda)$, are defined by means of the following generating function (see [39]):

$$F_B(t,x;\lambda) = \frac{\log\lambda + t}{\lambda e^t - 1} e^{tx} = \sum_{n=0}^{\infty} \mathfrak{B}_n(x;\lambda) \frac{t^n}{n!} \tag{1.13}$$

($|t| < 2\pi$ when $\lambda = 1$ and $|t| < |\log\lambda|$ when $\lambda \neq 1$) with

$$\mathfrak{B}_n(\lambda) = \mathfrak{B}_n(0;\lambda)$$

denotes the λ-Bernoulli numbers (Apostol-type Bernoulli numbers) (*cf.* [18, 39, 68, 74, 87]).

By using (1.13), a few values of the λ-Bernoulli numbers are given by

$$\mathfrak{B}_0(\lambda) = \frac{\log\lambda}{\lambda - 1}$$

and

$$\mathfrak{B}_1(\lambda) = \frac{\lambda - 1 - \lambda\log\lambda}{(\lambda-1)^2}.$$

If $n > 1$, then we have

$$\mathfrak{B}_n(\lambda) = \lambda \sum_{j=0}^{n} \binom{n}{j} \mathfrak{B}_j(\lambda).$$

Therefore,

$$\mathfrak{B}_2(\lambda) = \frac{\lambda\log\lambda - \lambda^2}{(\lambda-1)^3}.$$

A relation between the λ-Bernoulli numbers and the Frobenius–Euler numbers is given as follows (see [39, Theorem 1, p. 439]):

$$\mathfrak{B}_0(\lambda) = \frac{\log\lambda}{\lambda - 1} H_0\left(\frac{1}{\lambda}\right)$$

and

$$\mathfrak{B}_n(\lambda) = \frac{\log\lambda}{\lambda - 1} H_n\left(\frac{1}{\lambda}\right) + \frac{n}{\lambda - 1} H_{n-1}\left(\frac{1}{\lambda}\right),$$

where $H_n\left(\frac{1}{\lambda}\right)$ denote the Frobenius–Euler numbers, which are defined by means of the following generating function:

$$F_f(t,u) = \frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u)\frac{t^n}{n!}, \tag{1.14}$$

where $u \in \mathbb{C}$ with $u \neq 1$ (*cf.* [27], [39, Theorem 1, p. 439], [67, 87]; see also the references cited in each of these earlier works).

We also note that $e^{tx}F_f(t,u)$ gives us well-known generating function for the Frobenius–Euler polynomials $H_n(x;u)$.

In [72], we gave the following functional equation:

$$F_B(t,0;\lambda) = \frac{\log\lambda}{\lambda-1}F_f\left(t,\frac{1}{\lambda}\right) + F_A(t,0;\lambda).$$

By using the above functional equation, we have

$$H_n\left(\frac{1}{\lambda}\right) = (\lambda-1)\,\frac{\mathfrak{B}_n(\lambda) - \mathcal{B}_n(\lambda)}{\log\lambda},$$

where $\log\lambda \neq 0$.

The Humbert polynomials $\Pi_{n,m}^{(\lambda)}(x)$ defined by Humbert in [15] with the following generating function:

$$\left(1 - mxt + t^m\right)^{-\lambda} = \sum_{n=0}^{\infty} \Pi_{n,m}^{(\lambda)}(x)\, t^n$$

(*cf.* [15], [57, 83, p. 86, Eq-(26)]), and the recurrence relation for these polynomials is given as follows:

$$(n+1)\,\Pi_{n+1,m}^{(\lambda)}(x) - mx\,(n+\lambda)\,\Pi_{n,m}^{(\lambda)}(x) - (n+m\lambda-m+1)\,\Pi_{n-m+1,m}^{(\lambda)}(x) = 0$$

(*cf.* [9, 55]; see also the references cited in each of these earlier works).

The generalized Humbert polynomials $P_n(m,x,y,p,C)$ are defined by the following generating function:

$$\left(C - mxt + yt^m\right)^p = \sum_{n=0}^{\infty} P_n(m,x,y,p,C)t^n,$$

and it is clear that

$$P_n(m,x,1,-\lambda,1) = \Pi_{n,m}^{(\lambda)}(x)$$

(*cf.* [9, 13, 55, 57]).

Apostol–Bernoulli Polynomials and Numbers and Apostol–Euler Polynomials and Numbers Attached to Dirichlet Character

Here, the generating functions of generalized Bernoulli type numbers and polynomials attached to Dirichlet character and generalized Euler type numbers and polynomials attached to Dirichlet character are given.

Let $d \in \mathbb{N}$ and $(\mathbb{Z}/d\mathbb{Z})^*$ denotes the unit group of reduced residue class modulo d. Throughout this paper, χ is a Dirichlet character with modulo d, which is a group homomorphism, i.e.,

$$\chi : (\mathbb{Z}/d\mathbb{Z})^* \to \mathbb{C}\setminus\{0\}$$

(*cf.* [2]).

Let χ be a non-trivial Dirichlet character with conductor d. Let λ be a complex number. The generalized Apostol–Bernoulli numbers attached to Dirichlet character, $\mathcal{B}_{n,\chi}(\lambda)$, are defined by means of the following generating function:

$$\sum_{j=0}^{d-1} \frac{\lambda^j e^{tj} t\chi(j)}{\lambda^d e^{td} - 1} = \sum_{n=0}^{\infty} \mathcal{B}_{n,\chi}(\lambda)\frac{t^n}{n!} \tag{1.15}$$

(*cf.* [1, 28, 32, 34, 65, 87]; see also the references cited in each of these earlier works).

By combining (1.15) with (1.1), we have

$$\mathcal{B}_{n,\chi}(\lambda) = d^{n-1}\sum_{j=0}^{d-1} \lambda^j \chi(j)\mathcal{B}_n\left(\frac{j}{d}; \lambda^p\right),$$

and for the trivial character $\chi \equiv 1$, we have

$$\mathcal{B}_n(\lambda) = \mathcal{B}_{n,1}(\lambda)$$

(*cf.* [1, 28, 32, 34, 65, 87]).

Let χ be a non-trivial Dirichlet character with conductor d. Let λ be a complex number. The generalized Apostol–Euler numbers attached to Dirichlet character, $\mathcal{E}_{n,\chi}(\lambda)$ are defined by means of the following generating function:

$$2\sum_{j=0}^{d-1} \frac{(-\lambda)^j e^{tj}\chi(j)}{\lambda^d e^{td} + 1} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\chi}(\lambda)\frac{t^n}{n!} \tag{1.16}$$

(*cf.* [32, 34, 87]; see also the references cited in each of these earlier works).

By combining (1.16) with (1.4), we have

$$\mathcal{E}_{n,\chi}(\lambda) = d^n \sum_{j=0}^{d-1} (-\lambda)^j \chi(j)\mathcal{E}_n\left(\frac{j}{d}; \lambda^p\right).$$

For the trivial character $\chi \equiv 1$, we have

$$\mathcal{E}_n(\lambda) = \mathcal{E}_{n,1}(\lambda)$$

(*cf.* [32, 34, 87]).

Combinatorial Type Numbers and Polynomials

Here, the generating functions of combinatorial type numbers and polynomials, including the Stirling numbers, the Bernoulli numbers and polynomials of the second kind, and the combinatorial numbers and polynomials, are given.

The Stirling numbers of the first kind, $S_1(n, k)$ are defined by means of the following generating function:

$$F_{S1}(t, k) = \frac{(\log(1 + t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n, k)\frac{t^n}{n!}. \tag{1.17}$$

These numbers have the following properties:

$$S_1(0, 0) = 1.$$

If $k > 0$, then

$$S_1(0, k) = 0.$$

If $n > 0$, then

$$S_1(n, 0) = 0.$$

If $k > n$, then

$$S_1(n, k) = 0.$$

By using (1.17), one easily has the following recurrence equation for the number $S_1(n, k)$:

$$S_1(n + 1, k) = -nS_1(n, k) + S_1(n, k - 1)$$

(*cf.* [3, 6, 7, 64, 70, 71]; see also the references cited in each of these earlier works).

The Bernoulli polynomials of the second kind, $b_n(x)$, are defined by means of the following generating function:

$$F_{b2}(t,x) = \frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} \tag{1.18}$$

(*cf.* [64, pp. 113–117]; see also the references cited in each of these earlier works).

The Bernoulli numbers of the second kind, $b_n(0)$ are defined by means of the following generating function:

$$F_{b2}(t) = \frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n(0)\frac{t^n}{n!}. \tag{1.19}$$

These numbers are computed by the following formula:

$$\sum_{k=0}^{n-1}(-1)^k \binom{n}{k} b_k(0) = n!\delta_{n,1},$$

where $\delta_{n,1}$ denotes the Kronecker delta (*cf.* [64, p. 116]). The Bernoulli polynomials of the second kind are defined by

$$b_n(x) = \int_x^{x+1} (u)_n du.$$

Substituting $x = 0$ into the above equation, one has

$$b_n(0) = \int_0^1 (u)_n du.$$

The numbers $b_n(0)$ are also the so-called the *Cauchy numbers* (*cf.* [64, p. 116], [23, 41, 62, 64, 72]; see also the references cited in each of these earlier works). In [38], Kim et al. gave a computation method for the Bernoulli polynomials of the second kind that is defined as follows:

$$b_n(x) = \sum_{l=0}^{n} \frac{S_1(n,l)}{l+1}\left((x+1)^{l+1} - x^{l+1}\right),$$

and also Roman [64, p.115] gave

$$b_n(x) = b_n(0) + \sum_{l=1}^{n} \frac{nS_1(n-l, l-1)}{l}x^l.$$

By using the above formula for the Bernoulli polynomials and numbers of the second kind, few of these numbers are computed as follows, respectively:

$$\begin{aligned}
b_0(x) &= 1, \\
b_1(x) &= \frac{1}{2}(2x+1), \\
b_2(x) &= \frac{1}{6}(6x^2-1), \\
b_3(x) &= (\frac{1}{4})(4x^3-6x^2+1), \\
b_4(x) &= \frac{1}{30}(30x^4-120x^3+120x^2-19), \ldots
\end{aligned}$$

and

$$b_0(0)=1, b_1(0)=\frac{1}{2}, b_2(0)=-\frac{1}{6}, b_3(0)=\frac{1}{4}, b_4(0)=-\frac{19}{30}, \cdots$$

Here we note that when the ordinary generating function is taken instead of the exponential generating function, each of these numbers must be multiplied by $1/n!$, where n denotes the index of each bernoulli numbers of the second kind, that is one takes the following numbers: $\frac{b_n(0)}{n!}$.

The λ-array polynomials $S_v^n(x;\lambda)$ by the following generating function (see [70]):

$$F_A(t,x,v;\lambda) = \frac{\left(\lambda e^t - 1\right)^v}{v!} e^{tx} = \sum_{n=0}^{\infty} S_v^n(x;\lambda) \frac{t^n}{n!}, \tag{1.20}$$

where $v \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [3, 6, 70, 71]; see also the references cited in each of these earlier works).

The λ-Stirling numbers, $S_2(n,v;\lambda)$, are defined by means of the following generating function:

$$F_S(t,v;\lambda) = \frac{\left(\lambda e^t - 1\right)^v}{v!} = \sum_{n=0}^{\infty} S_2(n,v;\lambda) \frac{t^n}{n!}, \tag{1.21}$$

where $v \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [53, 70, 84]; see also the references cited in each of these earlier works). By using (1.21), one easily compute the following values for $S_2(n,v;\lambda)$:

$$S_2(0,0;\lambda)=1,$$
$$S_2(1,0;\lambda)=0,$$
$$S_2(1,1;\lambda)=\lambda,$$
$$S_2(2,0;\lambda)=0,$$
$$S_2(2,1;\lambda)=\lambda,\dots$$

and

$$S_2(0,v;\lambda)=\frac{(\lambda-1)^v}{v!}.$$

Substituting $\lambda=1$ into (1.21), then one easily arrives at the Stirling numbers of the second kind:

$$S_2(n,v)=S_2(n,v;1)$$

(*cf.* [7–43, 46–48, 50–78, 80–88]; see also the references cited in each of these earlier works).

The Daehee polynomials are defined by means of the following generating functions:

$$F_D(z,t)=\frac{\log(1+t)}{t}(1+t)^z=\sum_{n=0}^{\infty}D_n(z)\frac{t^n}{n!},$$

so that, obviously,

$$D_n=D_n(0)$$

denotes the Daehee numbers (*cf.* [25, 62, 72]).

The Peters polynomials $s_k(x;\lambda,\mu)$, which are Sheffer polynomials, are defined by means of the following generating functions:

$$\frac{1}{\left(1+(1+t)^{\lambda}\right)^{\mu}}(1+t)^x=\sum_{n=0}^{\infty}s_k(x;\lambda,\mu)\frac{t^n}{n!}$$

(*cf.* [21, 64]). If $\mu=1$, then the polynomials $s_k(x;\lambda,\mu)$ are reduced to the Boole polynomials. If $\lambda=1$ and $\mu=1$, then these polynomials are also reduced to the Changhee polynomials, which are defined by means of the following generating functions:

$$\frac{2}{t+2}(1+t)^x=\sum_{n=0}^{\infty}Ch_n(x)\frac{t^n}{n!}.$$

By using the above equation, we have

$$Ch_n = Ch_n(0),$$

where Ch_n denote the Changhee numbers (*cf.* [24, 35], and also see [18, 24–26, 30]).

In [76], we defined Boole type polynomials and numbers. That is, the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$ are defined by the following generating functions, respectively:

$$F(t,x,\lambda) = \frac{2(1+\lambda t)^x}{\lambda(1+\lambda t)-1} = \sum_{n=0}^{\infty} Y_n(x;\lambda)\frac{t^n}{n!} \tag{1.22}$$

and

$$F(t,\lambda) = \frac{2}{\lambda(1+\lambda t)-1} = \sum_{n=0}^{\infty} Y_n(\lambda)\frac{t^n}{n!}. \tag{1.23}$$

Note that

$$Y_n(\lambda) = Y_n(0;\lambda).$$

Recently, some generalizations of these numbers $Y_n(\lambda)$ and polynomials $Y_n(x;\lambda)$ have been studied (*cf.* [22, 43–45, 73, 76, 81, 82, 89]).

From (1.22) and (1.23), we have a few values of the polynomials $Y_n(x;\lambda)$ and the numbers $Y_n(\lambda)$ as follows:

$$\begin{aligned}
Y_0(x;\lambda) &= \frac{2}{\lambda-1},\\
Y_1(x;\lambda) &= \frac{2\lambda}{\lambda-1}x - \frac{2\lambda^2}{(\lambda-1)^2},\\
Y_2(x;\lambda) &= \frac{2\lambda^2}{\lambda-1}x^2 - \frac{6\lambda^3-2\lambda^2}{(\lambda-1)^2}x + \frac{4\lambda^4}{(\lambda-1)^3},\\
Y_3(x;\lambda) &= \frac{2\lambda^3}{\lambda-1}x^3 - \frac{12\lambda^4-6\lambda^3}{(\lambda-1)^2}x^2 + \frac{22\lambda^5-14\lambda^4+4\lambda^3}{(\lambda-1)^3}x - \frac{12\lambda^6}{(\lambda-1)^4}, \ldots
\end{aligned}$$

and

$$\begin{aligned}
Y_0(\lambda) &= \frac{2}{\lambda-1},\\
Y_1(\lambda) &= -\frac{2\lambda^2}{(\lambda-1)^2},
\end{aligned}$$

$$Y_2(\lambda) = \frac{4\lambda^4}{(\lambda-1)^3},$$

$$Y_3(\lambda) = -\frac{12\lambda^6}{(\lambda-1)^4},$$

$$Y_4(\lambda) = \frac{48\lambda^8}{(\lambda-1)^5}, \ldots$$

(*cf.* [76, 89]).

By using the above generating function, we get the following recurrence relation for the numbers $Y_n(\lambda)$:

Theorem 4 *Let*

$$Y_0(\lambda) = \frac{2}{\lambda-1}.$$

If $n > 1$, *we have*

$$Y_n(\lambda) = \frac{n\lambda^2}{\lambda-1} Y_{n-1}(\lambda) \tag{1.24}$$

(cf. [73, 76]*).*

In [48], Kucukoglu and Simsek defined the following combinatorial numbers and polynomials, respectively:

$$\mathcal{F}_d(t;\lambda,q) = \frac{\log(1+\lambda t)}{(\lambda q)^d(1+\lambda t)^d - 1} = \sum_{n=0}^{\infty} I_{n,d}(\lambda,q)\frac{t^n}{n!},$$

and

$$\mathcal{G}_d(t,x;\lambda,q) = (1+\lambda t)^x \mathcal{F}_d(t;\lambda,q) = \sum_{n=0}^{\infty} I_{n,d}(x;\lambda,q)\frac{t^n}{n!}. \tag{1.25}$$

In [47], Kucukoglu defined higher order of the polynomials $I_{n,d}(x;\lambda,q)$ and the numbers $I_{n,d}(\lambda,q)$. She gave various properties of these numbers and polynomials (*cf.* for detail, see [43, 46–49]).

Simsek and So [81] defined the following special polynomials $y_{7,n}(x;\lambda,q,d)$:

$$K_d(t,x;\lambda,q) = \frac{(1+q)(1+\lambda t)^x}{(\lambda q)^d(1+\lambda t)^d + 1} = \sum_{n=0}^{\infty} y_{7,n}(x;\lambda,q,d)\frac{t^n}{n!}. \tag{1.26}$$

Substituting $x = 0$ into (1.26), we have the special combinatorial numbers:

$$y_{7,n}(\lambda, q, d) = y_{7,n}(0; \lambda, q, d),$$

and also substituting $d = 1$ into (1.26), we also have the special combinatorial polynomials:

$$y_{7,n}(x; \lambda, q) = y_{7,n}(x; \lambda, q, 1).$$

(*cf*. [80])

By combining (1.25) with (1.26), we get the following functional equation:

$$\mathcal{G}_d(t, x; \lambda, q)K_d(t, y; \lambda, q) = (1+q)\mathcal{G}_{2d}(t, x+y; \lambda, q).$$

By using the above functional equation, we derive

$$\sum_{n=0}^{\infty} I_{n,d}(x; \lambda, q)\frac{t^n}{n!} \sum_{n=0}^{\infty} y_{7,n}(y; \lambda, q, d)\frac{t^n}{n!} = [2] \sum_{n=0}^{\infty} I_{n,2d}(x+y; \lambda, q)\frac{t^n}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} I_{m,d}(x; \lambda, q) y_{7,n-m}(y; \lambda, q, d)\frac{t^n}{n!} = [2] \sum_{n=0}^{\infty} I_{n,2d}(x+y; \lambda, q)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 5

$$I_{n,2d}(x+y; \lambda, q) = \frac{1}{[2]} \sum_{m=0}^{n} \binom{n}{m} I_{m,d}(x; \lambda, q) y_{7,n-m}(y; \lambda, q, d).$$

In [82], they gave generalization of the numbers $y_{7,n}(\lambda, q, d)$ that are defined by the following generating function:

$$\mathcal{F}_v(t; \lambda, q, d) = \left(\frac{1+q}{(\lambda q)^d (1+\lambda t)^d + 1}\right)^v = \sum_{n=0}^{\infty} y_{7,n}^{(v)}(\lambda, q, d)\frac{t^n}{n!}. \tag{1.27}$$

They also defined generalization of the polynomials $y_{7,n}(x; \lambda, q, d)$ as follows:

$$\mathcal{G}_v(t, x; \lambda, q, d) = (1+\lambda t)^x \mathcal{F}_v(t; \lambda, q, d) = \sum_{n=0}^{\infty} y_{7,n}^{(v)}(x; \lambda, q, d)\frac{t^n}{n!}. \tag{1.28}$$

By (1.27) and (1.28), we have

$$y_{7,n}^{(v)}(\lambda, q, d) = y_{7,n}^{(v)}(0; \lambda, q, d),$$

and

$$y_{7,n}(x; \lambda, q, d) = y_{7,n}^{(1)}(x; \lambda, q, d).$$

Few values of the numbers $y_{7,n}^{(2)}(\lambda, q, d)$ are given as follows:

$$y_{7,0}^{(2)}(\lambda, q, d) = \left(\frac{1+q}{(\lambda q)^d + 1}\right)^2,$$

$$y_{7,1}^{(2)}(\lambda, q, d) = -\frac{2d\lambda(\lambda q)^d (1+q)^2}{((\lambda q)^d + 1)^3},$$

and

$$y_{7,2}^{(2)}(\lambda, q, d) = \frac{8(d\lambda)^2(\lambda q)^{2d}(1+q)^2}{((\lambda q)^d + 1)^4} - \frac{\lambda^2(\lambda q)^d((d)_2 + (2d)_2(\lambda q)^d)(1+q)^2}{((\lambda q)^d + 1)^4}$$

(*cf.* [81, 82]).

We have recently defined various kind Peters and Boole type combinatorial numbers and polynomials. Thus, we inserted some notations for these numbers and polynomials. For instance, in order to distinguish them from each other, these polynomials are labeled by the following symbols:

$$y_{j,n}(x; \lambda, q),$$

$j = 1, 2, \ldots, 7$, and also $Y_n(x; \lambda)$. Therefore, the number 7 is only used for index representation for these polynomials (*cf.* [82]).

2 *p*-Adic *q*-Integrals Equations

Here, we survey some fundamental properties of p-adic q-integrals equations. We give some examples for these integrals. By using p-adic q-integrals on $\mathbb{Z}_p$, generating functions for the generalized Apostol-type numbers attached to Dirichlet character are given in [76]. Using these generating functions with their functional equations, relations between these numbers, the λ-Bernoulli numbers, and the Stirling numbers are given.

We now give standard notations for p-adic q-integrals on $\mathbb{Z}_p$. Let $\mathbb{Z}_p$ be a set of p-adic integers. Let $\mathbb{K}$ be a field with a complete valuation and $C^1(\mathbb{Z}_p \to \mathbb{K})$ be a set of continuous derivative functions. That is, $C^1(\mathbb{Z}_p \to \mathbb{K})$ is contained in

$$\left\{ f : \mathbb{X} \to \mathbb{K} : f(x) \text{ is differentiable and } \frac{d}{dx} f(x) \text{ is continuous} \right\}.$$

We assume that p is a fixed prime in the next section.

The distribution on $\mathbb{Z}_p$ is defined by

$$\mu_q(x + p^N \mathbb{Z}_p) = \frac{q^x}{[p^N]},$$

where $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and

$$[x] = [x:q] = \begin{cases} \frac{1-q^x}{1-q}, & q \neq 1 \\ x, & q = 1. \end{cases}$$

Let $f \in C^1(\mathbb{Z}_p \to \mathbb{K})$. Therefore, the p-adic q-integrals of the function f are defined by

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]} \sum_{x=0}^{p^N-1} f(x) q^x \tag{2.1}$$

(*cf.* [29]).

Let $f \in C^1(\mathbb{Z}_p \to \mathbb{K})$ and

$$E^d \{f(x)\} = f(x+d).$$

A p-adic q-integral equation of (2.1) is defined on $\mathbb{Z}_p$ as follows:

$$q^n \int_{\mathbb{Z}_p} E^n \{f(x)\} d\mu_q(x) - \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \frac{q-1}{\log q} \left(\sum_{j=0}^{n-1} q^j f'(j) + \log q \sum_{j=0}^{n-1} q^j f(j) \right), \tag{2.2}$$

where n is a positive integer (*cf.* [29, 34]).

Example 1 Substituting $f(x) = e^{tx}$ into (2.2), we obtain

$$\int_{\mathbb{Z}_p} e^{tx} d\mu_q(x) = \frac{1}{q^n e^t - 1} \frac{q-1}{\log q} \left(t \sum_{j=0}^{n-1} q^j e^{tj} + \log q \sum_{j=0}^{n-1} q^j e^{tj} \right).$$

From the above equation, we get

$$\int_{\mathbb{Z}_p} e^{tx} d\mu_q(x) = \frac{1}{q^n e^{nt} - 1} \frac{q-1}{\log q} \left(t \frac{q^n e^t - 1}{q e^t - 1} + \frac{q^n e^t - 1}{q e^t - 1} \log q \right).$$

Thus

$$\int_{\mathbb{Z}_p} e^{tx} d\mu_q(x) = \left(\frac{q-1}{\log q} \right) \frac{t + \log q}{q e^t - 1}.$$

Combining the above equation with (1.13), we have

$$\sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} x^n d\mu_q(x) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{q-1}{\log q} \mathfrak{B}_n(x; \lambda) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get

$$\int_{\mathbb{Z}_p} x^n d\mu_q(x) = \frac{q-1}{\log q} \mathfrak{B}_n(x; \lambda).$$

In [34, Theorem 3], Kim gave *the following formula for p-adic integral*:

$$q^n \int_{\mathbb{Z}_p} E^n f(x) d\mu_{-q}(x) - (-1)^n \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = [2] \sum_{j=0}^{n-1} (-1)^{n-l-1} q^j f(j). \tag{2.3}$$

If d is an odd positive integer, then (2.3) reduces to

$$q^d \int_{\mathbb{Z}_p} E^d f(x) d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = [2] \sum_{j=0}^{d-1} (-1)^l q^j f(j), \tag{2.4}$$

and if d is an even positive integer, then (2.3) reduces to

$$q^{d}\int\limits_{\mathbb{Z}_p} E^{d} f(x)\, d\mu_{-q}(x) - \int\limits_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = [2]\sum_{j=0}^{d-1}(-1)^{l} q^{j} f(j). \tag{2.5}$$

The Volkenborn (p-Adic Bosonic) Integral

When $q \to 1$, (2.1) reduces to the Volkenborn (p-adic Bosonic) integral, which is defined as follows:

Let $f \in C^{1}(\mathbb{Z}_p \to \mathbb{K})$. Then we have the following Volkenborn (p-adic bosonic) integral on $\mathbb{Z}_p$:

$$\int\limits_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x), \tag{2.6}$$

where

$$\mu_1(x) = \mu_1\left(x + p^N\mathbb{Z}_p\right)$$

denotes the Haar distribution, which is defined by

$$\mu_1\left(x + p^N\mathbb{Z}_p\right) = \frac{1}{p^N}$$

(*cf.* [66]; see also the references cited in each of these earlier works).

In the work of Kim [29], the Volkenborn integral is also the so-called *bosonic* p-adic integral or the Volkenborn integral on $\mathbb{Z}_p$. The Volkenborn integral on $\mathbb{Z}_p$ is used to construct generating functions including Bernoulli type numbers and polynomials and the other special numbers and polynomials.

Some basic properties of this integral are given as follows.

The Volkenborn integral in terms of the Mahler coefficients is given by the following formula:

$$\int\limits_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}a_n,$$

where

$$f(x) = \sum_{n=0}^{\infty} a_n\binom{x}{j} \in C^{1}(\mathbb{Z}_p \to \mathbb{K}),$$

and

$$\binom{x}{j} = \frac{x(x-1)(x-2)\cdots(x-j+1)}{j!}$$

denotes the Mahler coefficients (*cf.* [66, Proposition 55.3, p. 168]).

Let $f : \mathbb{Z}_p \to \mathbb{K}$ be an analytic function and $x \in \mathbb{Z}_p$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

The Volkenborn integral of this analytic function is given by

$$\int_{\mathbb{Z}_p} \left(\sum_{n=0}^{\infty} a_n x^n \right) d\mu_1(x) = \sum_{n=0}^{\infty} a_n \int_{\mathbb{Z}_p} x^n d\mu_1(x)$$

(*cf.* [66, Proposition 55.4, p. 168]).

Some nice and interesting results of p-adic integral or Volkenborn integral are given as follows.

The following property is very important in order to construct generating functions for special numbers and polynomials:

$$\int_{\mathbb{Z}_p} f(x+m) d\mu_1(x) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) + \sum_{j=0}^{m-1} f'(j), \tag{2.7}$$

where

$$f'(j) = \frac{d}{dx} f(x)\,\Big|_{x=j}$$

(*cf.* [29, 31, 66, 91]; see also the references cited in each of these earlier works).

The p-adic integral representations of the Bernoulli numbers and polynomials are given as follows:

$$\int_{\mathbb{Z}_p} x^n d\mu_1(x) = B_n \tag{2.8}$$

and

$$\int_{\mathbb{Z}_p} (z+x)^n d\mu_1(x) = B_n(z) \tag{2.9}$$

(*cf.* [29, 31, 66]; see also the references cited in each of these earlier works). Formulas in (2.8) and (2.9) are known as Witt's type formulas for the Bernoulli numbers and Bernoulli polynomials, respectively.

Theorem 6

$$\int_{\mathbb{Z}_p} \binom{x}{j} d\mu_1(x) = \frac{(-1)^j}{j+1} \tag{2.10}$$

(cf. [66]*).*

The Fermionic p-Adic Integral

The fermionic p-adic integral on $\mathbb{Z}_p$ is used to construct generating functions for Euler type numbers and polynomials and also other special numbers and polynomials.

The *fermionic* p-adic integral on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x), \tag{2.11}$$

where

$$\mu_{-1}(z + p^N\mathbb{Z}_p) = (-1)^x$$

(*cf.* [31]).

Let $f \in C^1(\mathbb{Z}_p \to \mathbb{K})$. When $q \to -1$ in (2.3), Kim [32] gave the following integral equation:

$$\int_{\mathbb{Z}_p} E^d f(x)\, d\mu_{-1}(x) - (-1)^d \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = 2\sum_{j=0}^{d-1} (-1)^{d-1-j} f(j), \tag{2.12}$$

where d is a positive integer. When $d = 1$, equation (2.12) is reduced to the following well-known integral equation:

$$\int_{\mathbb{Z}_p} f(x+1)\, d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = 2f(0)$$

(*cf.* [32]).

By using (2.11), the Witt's formulas for the Euler numbers and polynomials are given as follows, respectively,

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n \tag{2.13}$$

and

$$\int_{\mathbb{Z}_p} (z+x)^n \, d\mu_{-1}(x) = E_n(z) \tag{2.14}$$

(*cf.* [16, 31]; see also the references cited in each of these earlier works).

Theorem 7

$$\int_{\mathbb{Z}_p} \binom{x}{j} d\mu_{-1}(x) = \frac{(-1)^j}{2^j} \tag{2.15}$$

(cf. [24]*).*

3 Generalized Apostol-Type Numbers Attached to Dirichlet Character χ

Let χ be a non-trivial Dirichlet character with conductor d. Let λ be a p-adic integer. We set

$$f(x,t;\lambda) = \lambda^x (1+\lambda t)^x \chi(x) \tag{3.1}$$

(*cf.* [73, 76]). Substituting (3.1) into (2.2), we have

$$\begin{aligned}\int_{\mathbb{Z}_p} \lambda^x (1+\lambda t)^x \chi(x) d\mu_q(x) &= \frac{q-1}{\left((\lambda q)^d (1+\lambda t)^d - 1\right)\log q} \sum_{j=0}^{d-1} (\lambda q)^j (1+\lambda t)^j \chi(j) \log\left(\lambda + \lambda^2 t\right) \\ &+ \frac{q-1}{(\lambda q)^d (1+\lambda t)^d - 1} \sum_{j=0}^{d-1} (\lambda q)^j (1+\lambda t)^j \chi(j). \end{aligned} \tag{3.2}$$

From the above integral equation, we constructed the following generating function for the generalized Apostol–Daehee numbers attached to Dirichlet character χ with conductor d as follows:

$$F_{\mathfrak{D}}(t;q,\lambda,\chi)=\frac{(q-1)\log\left(\lambda+\lambda^{2}t\right)+(q-1)\log q}{\log q}\sum_{j=0}^{d-1}\frac{(\lambda q)^{j}\,(1+\lambda t)^{j}\,\chi(j)}{(\lambda q(1+\lambda t))^{d}-1},$$

where

$$F_{\mathfrak{D}}(t;q,\lambda,\chi)=\sum_{n=0}^{\infty}\mathfrak{D}_{n,\chi}(\lambda,q)\frac{t^{n}}{n!} \tag{3.3}$$

(*cf.* [73, 76]).

where $H_{m-1}(j/d;1/\lambda d)$ denotes the Frobenius-Euler polinomials (*cf.* [73, 76]).

The generalized Apostol–Daehee numbers attached to Dirichlet character χ with conductor d are related to the Bernoulli numbers of the second kind and the polynomials $I_{n,d}(x;\lambda,q)$. This relation is given below.

By combining (1.25) with (1.19), we get the following functional equation:

$$\lambda tF_{\mathfrak{D}}(t;q,\lambda,\chi){=}\frac{q-1}{\log q}\left(\log\left(\lambda q\right)F_{b2}(\lambda t){+}\lambda t\right)\sum_{j=0}^{d-1}(-1)^{j}\chi(j)\left(\lambda q\right)^{j}\mathcal{G}_{d}(t,j;\lambda,q).$$

By using the above functional equation, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\lambda n\mathfrak{D}_{n-1,\chi}(\lambda,q)\frac{t^{n}}{n!}\\&=\sum_{n=0}^{\infty}\frac{(q-1)\log\left(\lambda q\right)}{\log q}\sum_{j=0}^{d-1}(-1)^{j}\chi(j)\left(\lambda q\right)^{j}\sum_{m=0}^{n}\binom{n}{m}\lambda^{m}b_{m}(0)I_{n-m,d}(j;\lambda,q)\frac{t^{n}}{n!}\\&\quad+\lambda\frac{q-1}{\log q}\sum_{n=0}^{\infty}n\sum_{j=0}^{d-1}(-1)^{j}\chi(j)\left(\lambda q\right)^{j}I_{n-1,d}(j;\lambda,q)\frac{t^{n}}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 8 *Let $n\in\mathbb{N}$. Then we have*

$$\begin{aligned}&\mathfrak{D}_{n-1,\chi}(\lambda,q)\\&=\frac{(q-1)\log\left(\lambda q\right)}{n\log q}\sum_{j=0}^{d-1}(-1)^{j}\chi(j)\left(\lambda q\right)^{j}\sum_{m=0}^{n}\binom{n}{m}\lambda^{m-1}b_{m}(0)I_{n-m,d}(j;\lambda,q)\\&\quad+\frac{q-1}{\log q}\sum_{j=0}^{d-1}(-1)^{j}\chi(j)\left(\lambda q\right)^{j}I_{n-1,d}(j;\lambda,q).\end{aligned}$$

By (3.3), we get

$$F_{\mathfrak{D}}(t;q,\lambda,\chi)=\left(\frac{(q-1)\log\lambda}{\log q}+q-1\right)\frac{F_{b2}(\lambda t)}{d\lambda t}\sum_{j=0}^{d-1}(\lambda q)^{j}\,\chi(j)F_{A}\left(d\log(1+\lambda t)\,,\frac{j}{d};(\lambda q)^{d}\right)$$

$$+\frac{(q-1)}{d\log q}\sum_{j=0}^{d-1}(\lambda q)^{j}\,\chi(j)F_{A}\left(d\log(1+\lambda t)\,,\frac{j}{d};(\lambda q)^{d}\right).$$

Combining the above functional equation with (1.1) and (1.19), we get the following result:

Theorem 9 *Let $m\in\mathbb{N}$. Then we have*

$$\mathfrak{D}_{m-1,\chi}(\lambda,q)=\left(\frac{(q-1)\log\lambda}{\log q^{m}}+\frac{q-1}{m}\right)\sum_{j=0}^{d-1}q^{j}\chi(j)\sum_{l=0}^{m}\binom{m}{l}\lambda^{m+j-l-1}b_{m-l}(0)$$
$$\times\sum_{n=0}^{l}d^{n-1}\mathcal{B}_{n}\left(\frac{j}{d};(\lambda q)^{d}\right)S_{1}(l,n) \tag{3.4}$$
$$+\frac{q-1}{\log q}\sum_{j=0}^{d-1}(\lambda q)^{j}\,\chi(j)\sum_{n=0}^{m-1}d^{n-1}\mathcal{B}_{n}\left(\frac{j}{d};(\lambda q)^{d}\right)S_{1}(m-1,n)$$

(cf. [73, 76]*).*

If $q\to 1$ in (3.4), we get the following corollary:

Corollary 2 *Let $m\in\mathbb{N}$. Then we have*

$$\mathfrak{D}_{m-1,\chi}(\lambda)=\frac{\log\lambda}{m}\sum_{j=0}^{d-1}\chi(j)\sum_{l=0}^{m}\binom{m}{l}\lambda^{m+j-l-1}b_{m-l}(0) \tag{3.5}$$
$$\times\sum_{n=0}^{l}d^{n-1}\mathcal{B}_{n}\left(\frac{j}{d};\lambda^{d}\right)S_{1}(l,n)+\sum_{j=0}^{d-1}\lambda^{j}\chi(j)$$
$$\times\sum_{n=0}^{m-1}d^{n-1}\mathcal{B}_{n}\left(\frac{j}{d};\lambda^{d}\right)S_{1}(m-1,n).$$

Substituting $\lambda=1$ into (3.5), for $m\in\mathbb{N}_{0}$, we get

$$\mathfrak{D}_{m,\chi}=\sum_{j=0}^{d-1}\chi(j)\sum_{n=0}^{m}d^{n-1}B_{n}\left(\frac{j}{d}\right)S_{1}(m,n) \tag{3.6}$$

(*cf.* [73, 76]).

By combining (3.6) with the following well-known identity

$$B_{n,\chi} = d^{n-1} \sum_{j=0}^{d-1} \chi(j) B_n \left(\frac{j}{d} \right),$$

we arrive at the following result.

Corollary 3 *Let $m \in \mathbb{N}$. Then we have*

$$\mathfrak{D}_{m,\chi} = \sum_{n=0}^{m} B_{n,\chi} S_1(m,n) \tag{3.7}$$

(cf. [73, 76]*).*

Substituting $\lambda t = e^t - 1$ into (3.3), we get:

Theorem 10 *Let $m \in \mathbb{N}$. Then we have*

$$\sum_{n=0}^{m-1} \frac{\mathfrak{D}_{n,\chi}(\lambda, q) S_2(m-1,n)}{\lambda^n} = \frac{d^{m-1}(q-1)\log(\lambda q)}{\log q} \sum_{j=0}^{d-1} (\lambda q)^j \chi(j) \mathcal{B}_{m-1}\left(\frac{j}{d}; (\lambda q)^d \right)$$
$$+ \frac{(q-1)\, d^{m-1}}{m \log q} \sum_{j=0}^{d-1} (\lambda q)^j \chi(j) \mathcal{B}_m \left(\frac{j}{d}; (\lambda q)^d \right) \tag{3.8}$$

(cf. [73, 76]*).*

When $\lambda = 1$ and $q \to 1$, (3.8) reduces to the following corollary:

Corollary 4 *Let $m \in \mathbb{N}$. Then we have*

$$B_{m,\chi} = \sum_{n=0}^{m} \mathfrak{D}_{n,\chi} S_2(m,n)$$

and

$$\sum_{n=0}^{m} \frac{\mathfrak{D}_{n,\chi}(\lambda, 1) S_2(m,n)}{\lambda^n} = m d^{m-1} \sum_{j=0}^{d-1} \lambda^j \chi(j) H_{m-1}\left(\frac{j}{d}; \frac{1}{\lambda^d} \right),$$

where $H_{m-1}(j/d; 1/\lambda^d)$ denotes the Frobenius-Euler polinomials (cf. [73, 76]*).*

The generalized Apostol–Daehee polynomials attached to the Dirichlet character χ, with conductor d, are defined by means of the following generating function:

$$F_{\mathfrak{D}}(z,t;q,\lambda,\chi) = F_{\mathfrak{D}}(t;q,\lambda,\chi)(1+\lambda t)^z = \sum_{n=0}^{\infty} \mathfrak{D}_{n,\chi}(z;\lambda,q) \frac{t^n}{n!} \tag{3.9}$$

(*cf.* [73, 76]).

Substituting $z=\sum_{j=1}^{v} x_j$ (3.9), we have

$$F_{\mathfrak{D}}\left(\sum_{j=1}^{v} x_j, t; q, \lambda, \chi\right) = F_{\mathfrak{D}}(t; q, \lambda, \chi)(1+\lambda t)^{\sum_{j=1}^{v} x_j}.$$

By using (3.3) and (3.9), and also we assume that $|\lambda t| < 1$, then we have

$$\mathfrak{D}_{n,\chi}\left(\sum_{j=1}^{v} x_j; q, \lambda\right) = \sum_{j=0}^{n} \binom{n}{j} \mathfrak{D}_{n-j,\chi}(\lambda, q)\lambda^j \left(x_1+x_2+\cdots+x_v\right)_j.$$

Since

$$(x+y)_n = \sum_{j=0}^{n} \binom{n}{j} (x)_j (y)_{n-j},$$

we obtain the following theorem:

Theorem 11 *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathfrak{D}_{n,\chi}\left(\sum_{j=1}^{v} x_j; q, \lambda\right) = \sum_{j=0}^{n} \binom{n}{j} \lambda^j \mathfrak{D}_{n-j,\chi}(\lambda, q)$$
$$\times \sum_{j_1+\cdots+j_v=v} M(j_1,\ldots,j_v) \prod_{j(j_1,\ldots,j_v)=1}^{v} (x)_{j(j_1,\ldots,j_v)},$$

where

$$\sum_{j_1+\cdots+j_v=j} M(j_1,\ldots,j_v) \prod_{j(j_1,\ldots,j_v)=1}^{v} (x)_{j(j_1,\ldots,j_v)}$$
$$= \sum_{j_1=0}^{j} \sum_{j_2=0}^{j-j_1} \cdots \sum_{j_v=0}^{j-j_1-j_2-\cdots-j_{v-1}} \binom{j}{j_1}\binom{j-j_1}{j_2}\cdots\binom{j-j_1-j_2-\cdots-j_{v-1}}{j_v}$$
$$\times (x_1)_{j_1} (x_2)_{j_2} \cdots (x_v)_{j-j_1-j_2-\cdots-j_{v-1}}.$$

Remark 2 Substituting $v=2$ into Theorem 11, we have

$$\mathfrak{D}_{n,\chi}(x_1+x_2; \lambda, q) = \sum_{j=0}^{n} \sum_{j_1=0}^{j} \binom{n}{j}\binom{j}{j_1} \mathfrak{D}_{n-j,\chi}(\lambda, q)\lambda^j (x_1)_j (x_2)_{j-j_1}.$$

Substituting $v = 1$ into Theorem 11, we have

$$\mathfrak{D}_{n,\chi}(x_1; \lambda, q) = \sum_{j=0}^{n} \binom{n}{j} \mathfrak{D}_{n-j,\chi}(\lambda, q) \lambda^j (x_1)_j$$

(*cf.* [73, 76]).

Remark 3 If $q \to 1$ and $\lambda \to 1$ and $\chi \equiv 1$, then $\mathfrak{D}_{n,\chi}(z; \lambda, q)$ reduces to the polynomials $D_n(z)$ (*cf.* [25, 26, 30, 68, 72]).

4 Generalized Apostol–Changhee Numbers Attached to the Dirichlet Character with Odd Conductor

Substituting (3.1) into (2.4), we get

$$\int_{\mathbb{Z}_p} \lambda^x (1+\lambda t)^x \chi(x) d\mu_{-q}(x) = \frac{[2]}{(\lambda q)^d (1+\lambda t)^d + 1} \sum_{j=0}^{d-1} (-1)^j \chi(j) (\lambda q)^j (1+\lambda t)^j. \tag{4.1}$$

Therefore, the above equation gives us generating functions for the generalized Apostol–Changhee numbers and polynomials by means of the following generating functions, respectively:

$$F_{\mathfrak{C}}(t; \lambda, q, \chi) = \sum_{j=0}^{d-1} (-1)^j \frac{[2]\, \chi(j) (\lambda q)^j (1+\lambda t)^j}{(\lambda q)^d (1+\lambda t)^d + 1} = \sum_{n=0}^{\infty} \mathfrak{Ch}_{n,\chi}(\lambda, q) \frac{t^n}{n!}, \tag{4.2}$$

where d is an odd positive integer (*cf.* [73, 76]).

By using (4.2), the following functional equation is given by

$$F_{\mathfrak{C}}(t, x; \lambda, q, \chi) = \frac{1+q}{2} \sum_{j=0}^{d-1} (-1)^j \chi(j) (\lambda q)^j F_{P1}\left(d \log(1+\lambda t), \frac{j}{d}; (\lambda q)^d\right)$$

(*cf.* [73, 76]).

Combining the above equation with (1.4) and (4.2), and using $S_1(m, n) = 0$, $m < n$, we have

$$\mathfrak{Ch}_{m,\chi}(\lambda, q) = \sum_{j=0}^{d-1} (-q)^j \chi(j) \sum_{n=0}^{m} \lambda^{j+m} d^n \mathcal{E}_n\left(\frac{j}{d}; (\lambda q)^d\right) S_1(m, n), \tag{4.3}$$

and

$$\mathfrak{Ch}_{m,\chi}(\lambda, q) = \sum_{n=0}^{m} \mathcal{E}_{n,\chi}(q\lambda) S_1(m, n) \tag{4.4}$$

(*cf.* [73, 76]).

Substituting $\lambda t = e^u - 1$ into (4.2) yields

$$\frac{1+q}{(\lambda q)^d e^{du} + 1} \sum_{j=0}^{d-1} (-1)^j \chi(j) (\lambda q)^j e^{ju} = \sum_{n=0}^{\infty} \frac{\mathfrak{Ch}_{n,\chi}(\lambda, q)}{\lambda^n} \frac{(e^u - 1)^n}{n!}. \tag{4.5}$$

Thus, by (1.4), we get the following formulas:

$$\frac{1+q}{2} \sum_{j=0}^{d-1} (-1)^j \chi(j) (\lambda q)^j d^m \mathcal{E}_m \left(\frac{j}{d}, (\lambda q)^d \right) = \sum_{n=0}^{m} \frac{\mathfrak{Ch}_{n,\chi}(\lambda, q) S_2(m, n)}{\lambda^n}$$

and

$$\frac{1+q}{2} \sum_{n=0}^{m} \sum_{j=0}^{d-1} \sum_{l=0}^{m-n} (-1)^j \binom{m}{n} \chi(j) (\lambda q)^j d^n \mathcal{E}_n \left((\lambda q)^d \right) S_2(m-n, l) (j)_l$$
$$= \sum_{n=0}^{m} \frac{\mathfrak{Ch}_{n,\chi}(\lambda, q) S_2(m, n)}{\lambda^n}.$$

Combining (1.16) with (4.5), we have the following formula:

Theorem 12 *Let $m \in \mathbb{N}_0$. Then we have*

$$\mathcal{E}_{m,\chi}(\lambda) = \frac{2}{1+q} \sum_{n=0}^{m} \frac{\mathfrak{Ch}_{n,\chi}(\lambda, q) S_2(m, n)}{\lambda^n}$$

(cf. [73, 76]*).*

By using (4.5), we get

$$\frac{[2]}{(\lambda q)^d e^{du} + 1} \sum_{j=0}^{d-1} (-1)^j \chi(j) \sum_{v=0}^{j} \binom{j}{v} \left(\lambda q e^u - 1 \right)^v = \sum_{n=0}^{\infty} \frac{\mathfrak{Ch}_{n,\chi}(\lambda, q)}{\lambda^n} \frac{(e^u - 1)^n}{n!}.$$

Combining the above equation with (1.21), we have

$$\sum_{j=0}^{d-1}(-1)^j\chi(j)\sum_{v=0}^{j}\binom{j}{v}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{m}{n}d^n\mathcal{E}_n\left(\frac{j}{d},(\lambda q)^d\right)v!S_2(m-n,v;\lambda q)\frac{u^m}{m!}$$
$$=\frac{2}{[2]}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{\mathfrak{Ch}_{n,\chi}(\lambda,q)S_2(m,n)}{\lambda^n}\frac{u^m}{m!}.$$

Comparing the coefficients of $\frac{u^m}{m!}$ on both sides of the above equation, we get the following theorem:

Theorem 13

$$\sum_{j=0}^{d-1}(-1)^j\chi(j)\sum_{v=0}^{j}\binom{j}{v}\sum_{n=0}^{m}\binom{m}{n}d^n\mathcal{E}_n\left(\frac{j}{d},(\lambda q)^d\right)v!S_2(m-n,v;\lambda q)$$
$$=\frac{2}{[2]}\sum_{n=0}^{m}\frac{\mathfrak{Ch}_{n,\chi}(\lambda,q)S_2(m,n)}{\lambda^n}.$$

The polynomials $\mathfrak{Ch}_{n,\chi}(z;\lambda,q)$ are defined by means of the following generating function:

$$F_{\mathfrak{C}}(t,z;\lambda,q,\chi)=F_{\mathfrak{C}}(t;\lambda,q,\chi)(1+\lambda t)^z \tag{4.6}$$
$$=\sum_{n=0}^{\infty}\mathfrak{Ch}_{n,\chi}(z;\lambda,q)\frac{t^n}{n!}$$

(*cf*. [73, 76]).

Substituting $z=\sum_{j=1}^{v}x_j$ (4.6), we have

$$F_{\mathfrak{C}}\left(t,\sum_{j=1}^{v}x_j;\lambda,q,\chi\right)=F_{\mathfrak{C}}(t;\lambda,q,\chi)(1+\lambda t)^{\sum_{j=1}^{v}x_j}.$$

Combining the above equation with (4.2), we have

$$\sum_{n=0}^{\infty}\mathfrak{Ch}_{n,\chi}\left(\sum_{j=1}^{v}x_j;\lambda,q\right)\frac{t^n}{n!}=(1+\lambda t)^{\sum_{j=1}^{v}x_j}\sum_{n=0}^{\infty}\mathfrak{Ch}_{n,\chi}(\lambda,q)\frac{t^n}{n!}.$$

We assume that $|\lambda t|<1$. Then we have

$$\sum_{n=0}^{\infty} \mathfrak{Ch}_{n,\chi}\left(\sum_{j=1}^{v} x_j;\lambda,q\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\mathfrak{Ch}_{n-j,\chi}(\lambda,q)\lambda^j(x_1+x_2+\cdots+x_v)_j\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get

$$\mathfrak{Ch}_{n,\chi}\left(\sum_{j=1}^{v} x_j;q,\lambda\right)=\sum_{j=0}^{n}\binom{n}{j}\mathfrak{Ch}_{n-j,\chi}(\lambda,q)\lambda^j\,(x_1+x_2+\cdots+x_v)_j\,.$$

Since

$$(x+y)_n=\sum_{j=0}^{n}\binom{n}{j}(x)_j(y)_{n-j},$$

we obtain the following theorem:

Theorem 14 *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathfrak{Ch}_{n,\chi}\left(\sum_{j=1}^{v} x_j;q,\lambda\right)=\sum_{j=0}^{n}\binom{n}{j}\mathfrak{Ch}_{n-j,\chi}(\lambda,q)\lambda^j$$
$$\times\sum_{j_1+\cdots+j_v=v} M(j_1,\ldots,j_v)\prod_{j(j_1,\ldots,j_v)=1}^{v}(x)_{j(j_1,\ldots,j_v)}\,,$$

where

$$\sum_{j_1+\cdots+j_v=v} M(j_1,\ldots,j_v)\prod_{j(j_1,\ldots,j_v)=1}^{v}(x)_{j(j_1,\ldots,j_v)}$$

is given by Theorem 11.

Remark 4 Substituting $v=2$ into Theorem 14, we have

$$\mathfrak{Ch}_{n,\chi}(x_1+x_2;\lambda,q)=\sum_{j=0}^{n}\sum_{j_1=0}^{j}\binom{n}{j}\binom{j}{j_1}(x_1)_j(x_2)_{j-j_1}\lambda^j\mathfrak{Ch}_{n-j,\chi}(\lambda,q).$$

Substituting $v=1$ into Theorem 14, we have

$$\mathfrak{Ch}_{n,\chi}(x_1;\lambda,q)=\sum_{j=0}^{n}\binom{n}{j}\mathfrak{Ch}_{n-j,\chi}(\lambda,q)\lambda^j(x_1)_j$$

(*cf.* [73, 76]).

Remark 5 If $q \to 1$ and $\lambda \to 1$ and $\chi \equiv 1$, then $\mathfrak{Ch}_{n,\chi}(z;\lambda,q)$ reduces to the Changhee polynomials $Ch_n(z)$ (*cf.* [24, 35, 68]).

Combining (4.6) with (1.26), we have the following functional equation:

$$F_{\mathfrak{C}}(t,z;\lambda,q,\chi) = \sum_{j=0}^{d-1}(-1)^j \chi(j)\,(\lambda q)^j\, K_d(t,z+j;\lambda,q).$$

By using the above functional equation, we obtain

$$\sum_{n=0}^{\infty}\mathfrak{Ch}_{n,\chi}(z;\lambda,q)\frac{t^n}{n!} = \sum_{j=0}^{d-1}(-1)^j \chi(j)\,(\lambda q)^j \sum_{n=0}^{\infty} y_{7,n}(z+j;\lambda,q,d)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

$$\mathfrak{Ch}_{n,\chi}(z;\lambda,q) = \sum_{j=0}^{d-1}(-1)^j \chi(j)\,(\lambda q)^j\, y_{7,n}(z+j;\lambda,q,d).$$

Here we note that some properties of the polynomials $y_{7,n}(z+j;\lambda,q,d)$ and the numbers $y_{7,n}(\lambda,q,d)$ and also computation formulas are given in detail by Simsek and So [81, 82].

5 Generalized Apostol-Type Numbers Attached to the Dirichlet Character with Even Conductor

Substituting (3.1) into (2.5), we also constructed the following generating function for a new family of the generalized Apostol-type numbers attached to the Dirichlet character with even conductor:

$$H(t;\lambda,q) = [2]\sum_{j=0}^{d-1}(-1)^{j+1}\frac{\chi(j)\,(\lambda q)^j\,(1+\lambda t)^j}{(\lambda q)^d\,(1+\lambda t)^d - 1} = \sum_{n=0}^{\infty} Y_{n,\chi}(\lambda,q)\frac{t^n}{n!}, \qquad (5.1)$$

where d is an even positive integer and $\lambda \neq 1$ (*cf.* [73, 76]).

If $q \to 1$ in (5.1), then we have

$$\frac{2}{\lambda^d(1+\lambda t)^d - 1}\sum_{j=0}^{d-1}(-1)^{j+1}\chi(j)\lambda^j(1+\lambda t)^j = \sum_{n=0}^{\infty} Y_{n,\chi}(\lambda,1)\frac{t^n}{n!}.$$

A new family of the generalized Apostol-type polynomials attached to the Dirichlet character with even conductor, $Y_{n,\chi}(z;\lambda,q)$ are defined by means of the following generating function:

$$H(t,z;\lambda,q)=(1+\lambda t)^{z}H(t;\lambda,q)=\sum_{n=0}^{\infty}Y_{n,\chi}(z;\lambda,q)\frac{t^{n}}{n!}. \tag{5.2}$$

Remark 6 When $\chi\equiv 1$ and $q\to 1$, equations (5.2) and (5.1) reduce to (1.22) and (1.23), respectively. Therefore, the polynomials $Y_{n,\chi}(z;\lambda,q)$ and the numbers $Y_{n,\chi}(\lambda,q)$ are generalized of the Boole type polynomials and numbers.

By using equation (5.2), we get

$$\sum_{n=0}^{\infty}Y_{n,\chi}(z;\lambda,q)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}(z)_{n}\lambda^{n}\frac{t^{n}}{n!}\sum_{n=0}^{\infty}Y_{n,\chi}(\lambda,q)\frac{t^{n}}{n!}.$$

By using the Cauchy rule of product series in the above equation, we obtain

$$\sum_{n=0}^{\infty}Y_{n,\chi}(z;\lambda,q)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}(z)_{n-j}Y_{j,\chi}(\lambda,q)\right)\frac{t^{n}}{n!}.$$

Comparing the coefficients of $\frac{t^{n}}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 15 *Let $n\in\mathbb{N}_{0}$. Then we have*

$$Y_{n,\chi}(z;\lambda,q)=\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}(z)_{n-j}Y_{j,\chi}(\lambda,q).$$

By using (5.2), we get

$$H(t,x+y;\lambda,q)=(1+\lambda t)^{x}H(t,y;\lambda,q).$$

By using the above equation, we derive

$$\sum_{n=0}^{\infty}Y_{n,\chi}(x+y;\lambda,q)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}(x)_{n}\lambda^{n}\frac{t^{n}}{n!}\sum_{n=0}^{\infty}Y_{n,\chi}(y;\lambda,q)\frac{t^{n}}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty}Y_{n,\chi}(x+y;\lambda,q)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(x)_{j}\lambda^{j}Y_{n-j,\chi}(y;\lambda,q)\frac{t^{n}}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 16 *Let $n \in \mathbb{N}_0$. Then we have*

$$Y_{n,\chi}(x+y;\lambda,q)=\sum_{j=0}^{n}\binom{n}{j}(x)_j\lambda^j Y_{n-j,\chi}(y;\lambda,q).$$

Combining (5.1) with (1.1), we obtain the following functional equation:

$$\left(d\ln(\lambda q+\lambda^2 qt)\right)H(t;\lambda,q)=[2]\sum_{j=0}^{d-1}(-1)^{j+1}\chi(j)F_A\left(d\ln(\lambda q+\lambda^2 qt),\frac{j}{d};1\right).$$

By using the above equation, we get

$$\left(d\ln(\lambda q+\lambda^2 qt)\right)\sum_{n=0}^{\infty}Y_{n,\chi}(\lambda,q)\frac{t^n}{n!}$$
$$=[2]\sum_{j=0}^{d-1}(-1)^{j+1}\chi(j)\sum_{n=0}^{\infty}B_n\left(\frac{j}{d}\right)\frac{d^n\ln(\lambda q+\lambda^2 qt)^n}{n!}.$$

Thus, we have

$$d\ln(\lambda q)\sum_{n=0}^{\infty}Y_{n,\chi}(\lambda,q)\frac{t^n}{n!}+(d\ln(1+\lambda t))\sum_{n=0}^{\infty}Y_{n,\chi}(\lambda,q)\frac{t^n}{n!}$$
$$=[2]\sum_{j=0}^{d-1}(-1)^{j+1}\chi(j)\sum_{n=0}^{\infty}B_n\left(\frac{j}{d}\right)\frac{d^n}{n!}\sum_{v=0}^{n}v!\binom{n}{v}(\ln(\lambda q))^{n-v}F_{S1}(\lambda t,v).$$

Combining the above functional equation with (1.17), we obtain

$$d\ln(\lambda q)\sum_{m=0}^{\infty}mY_{m-1,\chi}(\lambda,q)\frac{t^{m+1}}{m!}+d\sum_{m=1}^{\infty}\sum_{c=1}^{m}(-1)^{c-1}\frac{\lambda^c Y_{m-1-c,\chi}(\lambda,q)}{c(m-c)!}t^m$$
$$=[2]\sum_{j=0}^{d-1}(-1)^{j+1}\chi(j)\sum_{n=0}^{\infty}B_n\left(\frac{j}{d}\right)\frac{d^n}{n!}\sum_{v=0}^{n}v!\binom{n}{v}(\ln(\lambda q))^{n-v}$$
$$\times\sum_{m=0}^{\infty}\lambda^m S_1(m,v)\frac{t^{m+1}}{m!}.$$

Since $S_1(m, v) = 0$ if $m < v$, the above equation reduces to the following relation:

$$\sum_{m=0}^{\infty}\left(mY_{m-1,\chi}(\lambda, q) + \frac{m!}{\ln(\lambda q)}\sum_{c=1}^{m}(-1)^c\frac{\lambda^c Y_{m-1-c,\chi}(\lambda, q)}{c(m-c)!}\right)\frac{t^m}{m!}$$
$$= \sum_{m=0}^{\infty}\sum_{j=0}^{d-1}\sum_{n=0}^{m-1}\sum_{v=0}^{n}(-1)^j\binom{n}{v}B_n\left(\frac{j}{d}\right)\frac{[2]\,md^{n-1}\chi(j)v!\lambda^m}{n!}(\ln(\lambda q))^{n-v-1}S_1(m-1, v)\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 17 *Let $m \in \mathbb{N}$. Then we have*

$$Y_{m-1,\chi}(\lambda, q) + \frac{(m-1)!}{\ln(\lambda q)}\sum_{c=1}^{m}(-1)^{c-1}\frac{\lambda^c Y_{m-1-c,\chi}(\lambda, q)}{c(m-c)!}$$
$$= \sum_{j=0}^{d-1}\sum_{n=0}^{m-1}\sum_{v=0}^{n}(-1)^j\binom{n}{v}\frac{[2]\,d^{n-1}\chi(j)v!\lambda^m}{n!}(\ln(\lambda q))^{n-v-1}S_1(m-1, v)B_n\left(\frac{j}{d}\right).$$

Integrals of the Polynomials $Y_{n,\chi}(z; \lambda, q)$

Riemann integral of the polynomials $Y_{n,\chi}(z; \lambda, q)$:

$$\int_0^1 Y_{n,\chi}(z; \lambda, q)\,dz = \sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}Y_{j,\chi}(\lambda, q)b_{n-j}(0),$$

(*cf.* [76]).

The p-adic integrals of the polynomials $Y_{n,\chi}(z; \lambda, q)$:

$$\int_{\mathbb{Z}_p} Y_{n,\chi}(z; \lambda, q)\,d\mu_1(z) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{(n-j)!\lambda^{n-j}}{n+1-j}Y_{j,\chi}(\lambda, q)$$

and

$$\int_{\mathbb{Z}_p} Y_{n,\chi}(z; \lambda, q)\,d\mu_{-1}(z) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{(n-j)!\lambda^{n-j}}{2^{n-j}}Y_{j,\chi}(\lambda, q)$$

(*cf.* [76]).

6 Partial Derivatives of the Functions $F(t,x,\lambda)$

Differentiating both side of (1.22) with respect to t, we have the following PDE (*cf.* [89]):

$$\frac{\partial F(t,x,\lambda)}{\partial t}=F(t,x,\lambda)\left(\frac{\lambda x}{1+\lambda t}-\frac{\lambda^2}{2}F(t,\lambda)\right). \tag{6.1}$$

Combining (1.22) and (1.23) with (6.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}Y_{n+1}(x;\lambda)\frac{t^n}{n!}&=\lambda x\sum_{n=0}^{\infty}(-\lambda t)^n\sum_{n=0}^{\infty}Y_n(x;\lambda)\frac{t^n}{n!}\\&\quad-\frac{\lambda^2}{2}\sum_{n=0}^{\infty}Y_n(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}Y_n(x;\lambda)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the following theorem:

Theorem 18 ([89]) *Let $n\in\mathbb{N}_0$. Then we have*

$$\begin{aligned}Y_{n+1}(x;\lambda)&=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}Y_k(x;\lambda)\left(2(-1)^{n-k}\lambda^{n-k+1}x(n-k)!-\lambda^2Y_{n-k}(\lambda)\right)\\&=\sum_{k=0}^{n}\binom{n}{k}Y_k(x;\lambda)\left((-1)^{n-k}\lambda^{n-k+1}x(n-k)!-2^{n-k}\left(\frac{\lambda^2}{\lambda-1}\right)^{n-k+1}Ch_{n-k}\right).\end{aligned}$$

Differentiating both side of (1.22) v times with respect to t, we have

$$\begin{aligned}\frac{\partial^{(v)}F(t,x,\lambda)}{\partial t^v}&=\left[\sum_{j=0}^{v}(-1)^j(v)_j(x)_{v-j}\lambda^{v+j}(1+\lambda t)^{j-v}\right.\\&\qquad\left.\times\left(\lambda^2t+\lambda-1\right)^{-j}\right]F(t,x,\lambda).\end{aligned} \tag{6.2}$$

By using (1.22) and (6.2) and the following well-known binomial series

$$\frac{1}{(1+\lambda t)^{v-j}}=\sum_{k=0}^{\infty}(-1)^k\binom{v-j+k-1}{k}\lambda^kt^k, \tag{6.3}$$

$$\left(\lambda^2t+\lambda-1\right)^{-j}=\frac{1}{(\lambda-1)^j}\sum_{k=0}^{\infty}(-1)^k\binom{j+k-1}{k}\frac{\lambda^{2k}}{(\lambda-1)^k}t^k, \tag{6.4}$$

and

$$\frac{1}{(1+\lambda t)^{v-j}}\frac{1}{\left(\lambda^2 t+\lambda-1\right)^j}=\frac{1}{(\lambda-1)^j}\sum_{k=0}^{\infty}C_k\left(j,v,\lambda\right)\frac{t^k}{k!}, \tag{6.5}$$

where

$$C_k\left(j,v,\lambda\right)=\sum_{m=0}^{k}(-1)^k\binom{k}{m}(v-j+k-1)_m\,(j+k-1)_{k-m}\,\frac{\lambda^{2k-m}}{(\lambda-1)^{k-m}},$$

we have

$$\frac{\partial^{(v)}F\left(t,x,\lambda\right)}{\partial t^v}=\sum_{n=0}^{\infty}Y_{n+v}\left(x;\lambda\right)\frac{t^n}{n!}. \tag{6.6}$$

Setting (6.6), (6.3), (6.4), and (6.5) in (6.2), we have

$$\sum_{n=0}^{\infty}Y_{n+v}\left(x;\lambda\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left[\sum_{j=0}^{v}(-1)^j\,(v)_j\,(x)_{v-j}\,\lambda^v\left(\frac{\lambda}{\lambda-1}\right)^j\right.$$
$$\left.\times\sum_{k=0}^{n}\binom{n}{k}C_k\left(j,v,\lambda\right)Y_{n-k}\left(x;\lambda\right)\right]\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the following theorem:

Theorem 19 ([89]) *Let* $n,v\in\mathbb{N}_0$. *Then*

$$Y_{n+v}\left(x;\lambda\right)=\sum_{j=0}^{v}(v)_j\,(x)_{v-j}\,\lambda^v\left(-\frac{\lambda}{\lambda-1}\right)^j\sum_{k=0}^{n}\binom{n}{k}C_k\left(j,v,\lambda\right)Y_{n-k}\left(x;\lambda\right), \tag{6.7}$$

where

$$C_k\left(j,v,\lambda\right)=\sum_{m=0}^{k}(-1)^k\binom{k}{m}(v-j+k-1)_m\,(j+k-1)_{k-m}\,\frac{\lambda^{2k-m}}{(\lambda-1)^{k-m}}.$$

Differentiating both side of (1.22) with respect to x, we have the following PDE (*cf.* [89]):

$$\frac{\partial F\left(t,x,\lambda\right)}{\partial x}=F\left(t,x,\lambda\right)\log\left(\lambda t+1\right). \tag{6.8}$$

By using the above equation, we have the following theorem:

Theorem 20 ([89]) *Let $n \in \mathbb{N}$. Then we have*

$$Y_{n-1}(x;\lambda) = \frac{1}{\lambda n}\sum_{k=0}^{n}\binom{n}{k}\frac{\partial}{\partial x}\{Y_k(x;\lambda)\}\lambda^{n-k}b_{n-k}(0).$$

7 Identities for the Polynomials $Y_n(x;\lambda)$

By using (1.22), we derive

$$\sum_{n=0}^{\infty} Y_n(x;\lambda)\frac{t^n}{n!} = \frac{2e^{x\log(1+\lambda t)}}{\lambda e^{\log(1+\lambda t)} - 1}.$$

Combining the above equation with (1.1) and (1.19), we obtain

$$\sum_{n=0}^{\infty} Y_n(x;\lambda)\frac{t^n}{n!} = \frac{2}{\log(1+\lambda t)}\sum_{n=0}^{\infty}\mathcal{B}_n(x;\lambda)\frac{(\log(1+\lambda t))^n}{n!}.$$

Therefore,

$$\sum_{k=0}^{\infty} kY_{k-1}(x;\lambda)\frac{t^k}{k!} = \frac{2}{\lambda}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}b_{k-j}(0)\,\lambda^{k-j}\sum_{n=0}^{j}\mathcal{B}_n(x;\lambda)S_1(j,n)\lambda^j\frac{t^k}{k!}.$$

Comparing the coefficients of $\frac{t^k}{k!}$ on both sides of the above equation, we obtain the following theorem:

Theorem 21 *Let $k \in \mathbb{N}_0$. Then we have*

$$Y_k(x;\lambda) = \frac{2}{k+1}\sum_{j=0}^{k+1}\sum_{n=0}^{j}\binom{k+1}{j}\lambda^k b_{k+1-j}(0)\,\mathcal{B}_n(x;\lambda)S_1(j,n). \tag{7.1}$$

Lemma 1 ([89])

$$Y_n(x;-1) = (-1)^{n+1}Ch_n(x). \tag{7.2}$$

Proof Substituting $\lambda = -1$ into (1.22), we have

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n Y_n(x;-1)\frac{t^n}{n!} &= \frac{2(1-t)^x}{t-2} \\ &= -\sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Lemma 2 ([89]) *Let* $n \in \mathbb{N}_0$. *Then we have*

$$Y_n(-1) = (-1)^{n+1} Ch_n. \tag{7.3}$$

Corollary 5 ([89]) *Let* $n \in \mathbb{N}_0$. *Then we have*

$$\begin{aligned}\frac{d}{dx}\{Ch_{n+1}(x)\} = x\sum_{l=0}^{n-1}\sum_{k=0}^{l}(-1)^{n+l+1}\binom{l}{k}\frac{n!D_k Ch_{l-k}(x)}{l!}\\ -\frac{1}{2}\sum_{l=0}^{n-1}\sum_{k=0}^{l}\binom{n-1}{l}\binom{l}{k} nD_{n-1-l}Ch_k Ch_{l-k}(x)\\ +\sum_{k=0}^{n}\frac{(-1)^k n!Ch_{n-k}(x)}{(n-k)!}.\end{aligned}$$

Theorem 22 ([89]) *Let* $n \in \mathbb{N}$. *Then we have*

$$\begin{aligned}&\lambda\frac{\partial}{\partial x}\{Y_{n+1}(x;\lambda)\}+\frac{1}{n+1}\frac{\partial}{\partial x}\{Y_{n+2}(x;\lambda)\}\\ &= x\sum_{k=0}^{n}\frac{(-1)^k\lambda^{k+2}n!Y_{n-k}(x;\lambda)}{(k+1)(n-k)!}\\ &-\frac{1}{2}\sum_{l=0}^{n-1}\sum_{k=0}^{l}\frac{(-1)^{n-1-l}\lambda^{n+3-l}}{n-l}\binom{l}{k}\frac{n!Y_k(\lambda)Y_{l-k}(x;\lambda)}{l!}\\ &-\frac{1}{2}\sum_{l=0}^{n}\frac{(-1)^{n-l}\lambda^{n+3-l}}{n-l+1}\sum_{k=0}^{l}\binom{l}{k}\frac{n!Y_k(\lambda)Y_{l-k}(x;\lambda)}{l!}\\ &+\frac{\lambda}{n+1}Y_{n+1}(x;\lambda).\end{aligned}$$

Theorem 23 ([89]) *Let* $m \in \mathbb{N}_0$. *Then we have*

$$Y_m(-\lambda) = (-1)^{m+1}\lambda^m\sum_{n=0}^{m}\mathcal{E}_n(\lambda)S_1(m,n) \tag{7.4}$$

and

$$Y_m(\lambda) = 2\lambda^m\sum_{n=0}^{m}\frac{S_1(m,n)B_{n+1}(\lambda)}{n+1}.$$

Remark 7 ([89]) Let $\lambda = 1$. Equation (7.4) reduces to the following relation:

$$Y_m(-1) = (-1)^{m+1} \sum_{n=0}^{m} \mathcal{E}_n(1)\, S_1(m,n).$$

Combining (7.3) and (1.8) with the above equation, we have the following well-known identity:

$$Ch_m = \sum_{n=0}^{m} E_n S_1(m,n),$$

which was proven by Kim et al. [24, Theorem 2.7].

Theorem 24 ([89]) *Let $m \in \mathbb{N}_0$. Then we have*

$$Y_m(x;-\lambda) = (-1)^{m+1} \lambda^m \sum_{n=0}^{m} \mathcal{E}_n(x;\lambda)\, S_1(m,n). \tag{7.5}$$

Remark 8 ([89]) When $\lambda = 1$. Equation (7.5) reduces to the following identity:

$$Y_m(x;-1) = (-1)^{m+1} \sum_{n=0}^{m} \mathcal{E}_n(x;1)\, S_1(m,n).$$

Combining (7.2) and (1.7) with the above equation, we have the following well-known identity:

$$Ch_m(x) = \sum_{n=0}^{m} E_n(x)\, S_1(m,n),$$

which was proven by Kim et al. [24, Theorem 2.5].

Relations Between the Numbers $Y_n(\lambda)$, the Polynomials $Y_n(x;\lambda)$, and Hypergeometric Function

Generalized hypergeometric function ${}_pF_p$ is defined by

$${}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix};z\right] = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)^{\overline{m}}}{\prod_{j=1}^{q} (\beta_j)^{\overline{m}}} \frac{z^m}{m!} \tag{7.6}$$

(*cf.* [40, 83, 84]).

We give some comments for series converges in (7.6):

for all z if $p < q+1$, and also for $|z| < 1$ if $p = q+1$ (*cf.* [40, 83, 84]).

For the series in (7.6), we assume that all parameters have general values, real or complex, except for the β_j, $j = 1, 2, \ldots, q$ none of which is equal to zero or to a negative integer.

Substituting $p = q = 0$ into (7.6), we have

$$ {}_0F_0\left(z\right) = e^z. $$

Substituting $p = 2$ and $q = 1$ into (7.6), we have

$$ {}_2F_1\left(\alpha_1; \alpha_2; \beta_1; z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)^{\overline{k}}\,(\alpha_2)^{\overline{k}}}{(\beta_1)^{\overline{k}}} \frac{z^k}{k!}. $$

Substituting $p = 1$ and $q = 0$ into (7.6), we have

$$ {}_1F_0\left[\begin{matrix} b \\ - \end{matrix}; x\right] = \frac{1}{(1-x)^b} $$

(*cf.* [40, 83, 84]).

Relations between hypergeometric function and integral of the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$ are given as follows:

Theorem 25 ([89])

$$ \int_0^u Y_n(\lambda)d\lambda = \frac{-2n!u^{2n+1}}{2n+1}\, {}_2F_1\left(-n-1, -2n-1; -2n-2; -u\right), $$

where ${}_2F_1$ denotes the Gauss hypergeometric functions.

Theorem 26 ([89])

$$ \begin{aligned} \int_0^u Y_n(x;\lambda)d\lambda = {} & -2n!u^{2n+1} \sum_{k=0}^{n} \frac{\binom{x}{k} u^{-k}}{(2n-k+1)} \\ & \times {}_2F_1\left(k-n-1, k-2n-1; k-2n-2; -u\right). \end{aligned} $$

Relations Between Infinite Series and the Numbers $Y_n(\lambda)$, the Humbert Polynomial, the Changhee Numbers, the Daehee Numbers, and the Lucas Numbers

Relations between infinite series and the numbers $Y_n(\lambda)$, the Humbert polynomial, the Changhee numbers, the Daehee numbers, and the Lucas numbers are given as follows.

The numbers $Y_n(\lambda)$ have the following infinite series:

$$\sum_{n=0}^{\infty} \frac{1}{Y_n(\lambda)} = \frac{\lambda-1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda-1}{\lambda^2} \right)^n .$$

By using the above equation, we have the following interesting series:

Theorem 27 ([89])

$$\sum_{n=0}^{\infty} \frac{1}{Y_n(\lambda)} = \frac{\lambda-1}{2} e^{\frac{1-\lambda}{\lambda^2}} .$$

Theorem 28 ([89]) *Let* $\left| \frac{\lambda^2}{\lambda-1} \right| < 1$. *Then*

$$\sum_{n=0}^{\infty} \frac{Y_n(\lambda)}{D_n} = \frac{2\lambda^2}{\left(1-\lambda+\lambda^2\right)^2} - \frac{2}{1-\lambda+\lambda^2}. \tag{7.7}$$

Theorem 29 ([89])

$$\sum_{n=0}^{\infty} \frac{D_n}{Y_n(\lambda)} = -\frac{\lambda^2}{2} \log\left(1+\frac{1-\lambda}{\lambda^2}\right).$$

By using the following series

$$\sum_{n=0}^{\infty} \frac{D_n}{Y_n(-1)} = -\frac{\log(3)}{2}$$

(*cf.* [89]) and

$$\sum_{n=1}^{\infty} \frac{L_n}{n2^n} = 2\log(2),$$

where L_n denotes the Lucas numbers (*cf.* [54, p. 7]), we have

$$\sum_{n=1}^{\infty} \left(\frac{D_n}{Y_n(-1)} + \frac{L_n}{n2^n} \right) = \log\left(\frac{4e}{\sqrt{3}}\right),$$

where $\log e = 1$.

Combining the above series with the following series

$$\sum_{n=0}^{\infty} \frac{Ch_n}{D_n} = 4$$

(*cf.* [89]), we have the following interesting series:

$$\sum_{n=1}^{\infty} \left(\frac{D_n}{Y_n(-1)} + \frac{Ch_n}{D_n} + \frac{L_n}{n2^n} \right) = 4 + \log \left(\frac{4e}{\sqrt{3}} \right)$$

$$(log2 - log3)/2 + 1$$

Theorem 30 ([89]) *Let* $\left| \frac{\lambda^2}{\lambda-1} \right| < \frac{1}{2}$. *Then*

$$\sum_{n=0}^{\infty} \frac{Y_n(\lambda)}{Ch_n} = \frac{2}{\lambda - 1 - 2\lambda^2}. \tag{7.8}$$

Theorem 31 ([89]) *Let* $\left| \frac{\lambda-1}{2\lambda^2} \right| < 1$. *Then*

$$\sum_{n=0}^{\infty} \frac{Ch_n}{Y_n(\lambda)} = \frac{\lambda^2(\lambda-1)}{2\lambda^2 - \lambda + 1}. \tag{7.9}$$

Remark 9 ([89]*)*

$$\sum_{n=0}^{\infty} \frac{Y_n(\lambda)}{D_n} = 2\lambda^2 \sum_{n=0}^{\infty} \Pi_{n,2}^{(2)} \left(\frac{1}{2} \right) \lambda^n - 2 \sum_{n=0}^{\infty} \Pi_{n,2}^{(1)} \left(\frac{1}{2} \right) \lambda^n.$$

By using (7.8) and (7.9), we have

Remark 10 ([89]*)*

$$\sum_{n=0}^{\infty} \frac{Y_n(\lambda)}{Ch_n} = -2 \sum_{n=0}^{\infty} P_n(2, \frac{1}{2}, 2, -1, 1)\lambda^n.$$

Remark 11 ([89]*)*

$$\sum_{n=0}^{\infty} \frac{Ch_n}{Y_n(\lambda)} = \lambda^2(\lambda-1) \sum_{n=0}^{\infty} P_n(2, \frac{1}{2}, 2, -1, 1)\lambda^n.$$

8 The Lerch Transcendent Function and Apostol Type Numbers and Polynomials: Approximation to the Polynomials $Y_n(x; \lambda)$

In this section, we give not only Fourier series, but also asymptotic estimates for Boole type combinatorial numbers and polynomials with the help of equation (1.2), Theorem 1, and (1.5).

The Lerch transcendent function is defined by

$$\Phi(\lambda, s, a) = \sum_{k=0}^{\infty} \frac{\lambda^k}{(k+a)^s},$$

where $a \notin \mathbb{Z}^- \cup \{0\}$, and either $|\lambda| < 1$, $s \in \mathbb{C}$ or $\lambda = 1$, Re $s > 1$ guarantees convergence (we use λ as a variable in order to maintain a unified notation) (*cf.* [2, 9, 63, 83, 86, 87]). By using analytic continuation, we have

$$\mathcal{B}_n(a; \lambda) = -n\Phi(\lambda, 1-n, a) \tag{8.1}$$

(*cf.* [2, 9, 56, 63, 83, 86, 87]).

By combining (8.1) with (7.1), we arrive at the following theorem:

Theorem 32 *Let $n \in \mathbb{N}_0$. Then we have*

$$Y_k(x; \lambda) = \frac{-2}{k+1} \sum_{j=0}^{k+1} \sum_{n=0}^{j} \binom{k+1}{j} \lambda^k b_{k+1-j}(0) S_1(j, n) n\Phi(\lambda, 1-n, x). \tag{8.2}$$

By combining (1.2) with (7.1), we arrive at the following theorem:

Theorem 33 *Let $\lambda \in \mathbb{C}$, $(\lambda \neq 0)$. Then we have*

$$\begin{aligned} Y_k(x; \lambda) = &-\frac{2}{k+1} \sum_{j=0}^{k+1} \sum_{n=0}^{j} \binom{k+1}{j} \lambda^k b_{k+1-j}(0) S_1(j, n) \delta_n(x; \lambda) \\ &-\frac{2}{k+1} \sum_{j=0}^{k+1} \sum_{n=0}^{j} \binom{k+1}{j} \lambda^{k-x} b_{k+1-j}(0) S_1(j, n) n! \\ &\times \sum_{v \in \mathbb{Z} \setminus \{0\}}^{\infty} \frac{\exp(2\pi i v x)}{(2\pi i v - \log \lambda)^n}, \end{aligned}$$

where

$$\delta_n(x;\lambda) = \begin{cases} 0, & \lambda = 1 \\ \frac{(-1)^n n!}{\lambda^x (\log\lambda)^n}, & \lambda \neq 1. \end{cases}$$

Since

$$\int_0^1 \lambda^x \mathcal{B}_n(x;\lambda) \exp(-2\pi i m x)\, dx = -\frac{n!}{(2\pi i m - \log\lambda)^n}, \tag{8.3}$$

where $\lambda \in \mathbb{C}$, $(\lambda \neq 0, 1)$, $m \in \mathbb{Z}$, and $n \in \mathbb{N}$ (*cf.* [56]), we deduce that

$$\int_0^1 \lambda^x Y_k(x;\lambda) e^{-2\pi i l x} dx = -\frac{2}{k+1} \sum_{j=0}^{k+1} \sum_{n=0}^{j} \binom{k+1}{j} \frac{n! \lambda^k b_{k+1-j}(0)\, S_1(j,n)}{(2\pi i l - \log\lambda)^n}$$

is a Fourier coefficient (or Laplace transform) of the following function:

$$\lambda^x Y_k(x;\lambda).$$

Combining (7.1) with the well-known identity given by (1.12), we arrive at the following theorem:

Theorem 34

$$Y_k(x;\lambda) = \frac{-1}{k+1} \sum_{j=0}^{k+1} \sum_{n=0}^{j} \binom{k+1}{j} n \lambda^k b_{k+1-j}(0)\, S_1(j,n) \mathcal{E}_{n-1}(x;-\lambda). \tag{8.4}$$

In [76], we gave the following novel identity:

$$\mathcal{B}_m(\lambda) = \frac{m}{2} \sum_{n=0}^{m-1} \lambda^{-n} Y_n(\lambda) S_2(m-1,n). \tag{8.5}$$

Combining (8.5) with (1.12), we arrive at the following theorem:

Theorem 35

$$\mathcal{E}_m(-\lambda) = \sum_{n=0}^{m} (-1)^n \lambda^{-n} Y_n(-\lambda) S_2(m,n). \tag{8.6}$$

Combining the above well-known formula

$$\mathcal{E}_n(x;-\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{E}_m(-\lambda) x^{n-m}$$

(*cf.* [86]), with (8.6), we get the following corollary:

Corollary 6

$$\mathcal{E}_n(x;-\lambda)=\sum_{m=0}^{n}\sum_{k=0}^{m}(-1)^k\binom{n}{m}\lambda^{-k}Y_k(-\lambda)S_2(m,k)x^{n-m}.$$

By combining (1.5) with (8.4), we get the following theorem:

Theorem 36 *Let* $\lambda\in\mathbb{C}(\lambda\neq 0)$. *Then we have*

$$Y_k(x;-\lambda)=\frac{2}{k+1}\sum_{j=0}^{k+1}\sum_{n=0}^{j}(-1)^{k+1}\binom{k+1}{j}nn!\lambda^{k-x}b_{k+1-j}\,(0)\,S_1(j,n)$$
$$\times\sum_{v\in\mathbb{Z}}^{\infty}\frac{\exp\left((2v-1)\pi ix\right)}{((2v-1)\pi i-\log\lambda)^n}.$$

By using (8.2), and under the conditions of Theorem 1, we arrive at the following theorem, which gives us an asymptotic expansion for the numbers $Y_k(\lambda)$:

Theorem 37

$$Y_k(\lambda)=\frac{2}{k+1}\sum_{j=0}^{k+1}\sum_{n=0}^{j}\binom{k+1}{j}\lambda^k b_{k+1-j}\,(0)\,S_1(j,n)$$
$$\times\sum_{v\in\mathbb{Z}}^{\infty}\frac{\exp\left((2v-1)\,\pi ix\right)}{((2v-1)\,\pi i-\log\lambda)^n}.$$

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A General Lower Bound for the Asymptotic Convergence Factor

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Abstract We provide a rather general and very simple to compute lower bound for the asymptotic convergence factor of compact subsets of $\mathbb{C}$ with connected complement and finitely many connected components.

MSC (2010): 41A17, 41A29, 65F10

1 Introduction

The subject of this work has many connections with the theory of approximation [4], the problem of solving large-scale matrix problems by Krylov subspace iterations and digital filtering. We denote $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of natural, real and complex numbers, respectively.

Our point of view is a classical problem of approximation.

Let us see how this problem arises.

Suppose that L is a non-connected compact subset of $\mathbb{C}$ with connected complement, of the form. $L := \bigcup\limits_{i=0}^{m} K_i$, for some $m \in \mathbb{N}$, $m \geq 1$, where K_i, $i = 0, 1, \ldots, m$, be of the form connected components of L. We also assume that each component K_i does not reduce to a point. Let p_i, $i = 0, 1, \ldots, m$, be $m + 1$ different complex polynomials; that is, $p_i \neq p_j$ for every $i, j \in \{0, 1, \ldots, m\}$, $i \neq j$.

Consider the function $F : L \to \mathbb{C}$, which is defined by the formula:

$$F(z) = p_j(z) \text{ if } z \in K_j, \quad \text{for every } \; j \in \{0, 1, \ldots, m\}.$$

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Let $A \subseteq \mathbb{C}$, $F : A \to \mathbb{C}$ be a complex function. As usual we denote $\|F\|_A = \sup\{x \in \mathbb{R} \mid \exists\, a \in A : x = |F(a)|\} = \sup_{x \in A} |F(x) \in [0, +\infty]$.

We fix some positive number δ. The general problem is to find explicitly an arbitrary polynomial p such that $\|F - p\|_L < \delta$ and also to find the relation between p and δ, if possible. It turns out that the notion of asymptotic convergence factor for a compact set is extremely useful in studying the above problem. Based on the previous notation, we proceed with the relevant definition.

For every $n = 1, 2, \ldots$, let V_n be the set of complex polynomials on L with degree at most n; that is, $V_n = \{p : L \to \mathbb{C},\ p \text{ is polynomial, } degp \leq n\}$

$$dist(V_n, F) := \min\{\|F - p\|_L,\ p \in V_n\} \ \text{ for } \ n = 1, 2, \ldots .$$

Of course, for every $n \in \mathbb{N}$, there exists some $p \in V_n$ such that $dist(V_n, F) = \|F - p\|_L$, and the polynomial p is unique [12] for every $n \geq 1$. Even, if the formulation of the problem of finding the above best polynomial p that minimizes the quantity $\|F - p\|_L$ is simple, and this is usually unknown and difficult to compute (see [7], page 11).

However, if the compact set L has a simple structure or good regularity properties, the previous approximation problem can be solved. Despite this, the computation of the best polynomial is difficult even in simple cases, for instance, the union of two disjoint closed discs, and in most of the cases, this is done with complicated numerical methods [7].

A classical theorem in this area (see [7, 12]) is the following:

Theorem 1 *The number* $\rho_L := \limsup_{n \to +\infty} (dist(V_n, F))^{\frac{1}{n}}$ *is a positive constant such that* $\rho_L \in (0, 1)$*, it is independent from the function* F *and it is dependent only on the compact set* L*.*

The number ρ_L is called **the asymptotic convergence factor of** $\boldsymbol{L}$ and is a characteristic for the compact set L. Of course, the knowledge of the above number ρ_L for L is a crucial point for the solution of the initial approximation problem.

However, the number ρ_L is very difficult to be computed in general, [6, 7, 11]. So, it is desirable for a simple compact set L to obtain "good" estimates from above and below of the number ρ_L.

In this paper, we give an easily computed lower bound for the number ρ_L, which is best possible in a certain sense. Our main result is the following:

Theorem 2 *Under the above assumptions and notation, we have*

$$\rho_L \geq \max_{j=0,\ldots,m} \sup_{z \in K_j^0} \frac{dist(z, K_j^c)}{dist(z, L \backslash K_j)}.$$

In order to prove this theorem, we introduce another characteristic number θ_L for a compact set L, which is defined in a bit complicated way in the next section.

However, this number gives us the necessary potential theoretic tools in order to establish that $\rho_L = \theta_L$; that is, θ_L is nothing but the asymptotic convergence factor. The potential theoretic view assigns to the number ρ_L new important properties. We also provide a variety of simple examples of certain compact sets where the number θ_L can be computed by a very simple algebraic formula and not with numerical methods.

Results concerning the computation of ρ_L for certain compact set L can be found in [1, 6, 7, 11].

Remark We note that Proposition 4 of this paper is used in a substantial way in order to prove the main result of Theorem 1 of [12]. The theme of [12] is related to universal Taylor series. On the other hand, according to Remark 3.3 of [12], half of the main result of this paper, namely Theorem 3, can be deduced by a completely different method based entirely on the results of universality for Taylor series [12]. This means that there exists a close relation between the results of this paper and that of [12].

2 The Number θ_L and Its Lower Bound

We begin with the necessary terminology. For the topological concepts of this paper, we refer to the classical book of Burckel [3].

More specifically, for the definitions of a curve, or a loop, or an arc, or a simple curve, or a smooth curve, see Definition 1.11 [3].

For the definition of a simply connected subset A of $\mathbb{C}$, see Definition 1.36 [3]. With a Jordan curve, we mean a homeomorphism in $\mathbb{C}$ of a circle. It is trivial to see that a Jordan curve is also a loop. If $\gamma : [\alpha, \beta] \to \mathbb{C}$ is a curve, where $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, we denote $\gamma^* = \gamma([\alpha, \beta])$. If γ is a loop and $w \in \mathbb{C} \setminus \gamma^*$, we denote $\mathrm{Ind}_\gamma(w)$ the index of γ with respect to w; that is, the number $\mathrm{Ind}_\gamma(w) := \dfrac{1}{2\pi i} \cdot \displaystyle\int_\gamma \frac{1}{z-w} dz$ (see Definition 4.2, page 84 of [3]).

For a compact subset K of $\mathbb{C}$ and a Jordan curve γ, we write

$$\mathrm{Ind}_\gamma(K) := \{\mathrm{Ind}_\gamma(w),\ w \in K\},$$

when $\gamma^* \cap K = \emptyset$.

The definition of interior, $\mathrm{Int}(\gamma)$, and Exterior ,$Ex(\gamma)$, of a Jordan curve γ is given in Definition 4.45 (i), page 10–4 of [3]. For results about potential theory, we refer to the classical books [2] and [9].

Below we prove a series of useful topological lemmas and the main result of this paper, which is a simple estimation of the lower bound of the number ρ_L, for many cases of compact sets. The first lemma is a variation of Exercise 10.10, page 347 of [3].

Lemma 1 *Let $V \subseteq \mathbb{C}$, $V \neq \mathbb{C}$, be a simply connected domain, $K \subseteq V$, K compact, set.*

Then, there exists a smooth Jordan curve $\gamma \subset V$ such that $Ind_\gamma(K) = \{1\}$.

Proof Let D be the open unit disc. By the Riemann mapping theorem, there exists a conformal mapping $f : D \to V$, that is, $1-1$ and onto. We set $L := f^{-1}(K)$. Of course, the set L is a compact subset of D. Let $r_0 \in (0, 1)$ such that $L \subset D(0, r_0)$, where

$$D(0, r_0) := \{z \in \mathbb{C} \mid |z| < r_0\}.$$

We set

$$\Gamma := C(0, r_0) := \{z \in \mathbb{C} \mid |z| = r_0\}.$$

We consider the circle $\gamma_0 : [0, 1] \to \mathbb{C}$, $\gamma_0(t) = r_0 e^{2\pi i t}$, $t \in [0, 1]$, and we set $\gamma := f \circ \gamma_0$. We set $\gamma^* := \gamma([0, 1])$, and we write simply $\gamma^* = \gamma$ without confusion, for simplicity.

It is easy to show that the curve γ is a smooth Jordan curve such that $K \cap \gamma = \emptyset$. So the number $\mathrm{Ind}_\gamma(w)$ has sense for every $w \in K$.

We fix some $w_0 \in K$. We compute the number $\mathrm{Ind}_\gamma(w_0)$.

We have

$$\begin{aligned} \mathrm{Ind}_\gamma(w_0) :&= \frac{1}{2\pi i} \int_\gamma \frac{1}{z - w_0} dz \\ &= \frac{1}{2\pi i} \int_0^1 \frac{1}{\gamma(t) - w_0} \cdot \gamma'(t) dt \\ &= \frac{1}{2\pi i} \int_0^1 \frac{f'(\gamma_0(t)) \cdot \gamma_0'(t)}{f(\gamma_0(t)) - w_0} dt. \end{aligned} \tag{1}$$

We consider the function $g : D \setminus \{z_0\} \to \mathbb{C}$ defined by the formula:

$$g(z) := \frac{f'(z)}{f(z) - w_0}, \quad \text{where } z_0 := f^{-1}(w_0), \quad z \in D \setminus \{z_0\}.$$

Obviously, the function g is well defined and holomorphic in $D \setminus \{z_0\}$ and has a singularity in z_0. Obviously z_0 is pole of g. It holds $z_0 \notin \gamma_0$, so the integral $\int_{\gamma_0} g(z) dz$ is well defined.

Now, we have

$$\begin{aligned} \int_{\gamma_0} g(z) dz &= \int_0^1 g(\gamma_0(t)) \cdot \gamma_0'(t) dt \\ &= \int_0^1 \frac{f'(\gamma_0(t)) \cdot \gamma_0'(t)}{f(\gamma_0(t)) - w_0}. \end{aligned} \tag{2}$$

By (1) and (2), we take

$$\mathrm{Ind}_\gamma(w_0) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{f'(z)}{f(z)-w_0}dz. \tag{3}$$

Because $L \subset D(0, r_0)$, by definition we have that $z_0 \in \mathrm{Int}(\gamma_0)$.

We compute easily that $\lim\limits_{z\to z_0}(z-z_0)g(z) = 1$. This gives that z_0 is a simple pole for g and $Res(g, z_0) = 1$. So by the calculus of residues we take

$$\int_{\gamma_0} g(z)dz = 2\pi i\ Res(g, z_0)\cdot \mathrm{Ind}\gamma_0(z_0) = 2\pi i. \tag{4}$$

By (3) and (4), we take $\mathrm{Ind}\gamma(w_0) = 1$ and the result follows. ■

Using Lemma 1, we prove now the following lemma, for compact subsets of $\mathbb{C}$, with finite many connected components only.

Lemma 2 *Let* $K = \bigcup\limits_{i=1}^{m} K_i$ *be a compact set with connected complement, where* K_i*,* $i = 1, 2, \ldots, m$*, be the connected components of* K*,* $m > 1$*.*

Then, there exist smooth Jordan curves δ_i*,* $i = 1, 2, \ldots, m$*, pairwise disjoint such that* $Ind_{\delta_i}(K_i) = \{1\}$ *and* $K_i \subset Int(\delta_i)$*, for* $i = 1, 2, \ldots, m$ *and every one of them has all the others in its exterior.*

Proof First of all, we can choose bounded open subsets of $\mathbb{C}$, G_i, $i = 1, 2, \ldots, m$, pairwise disjoint such that $K_i \subset G_i$, for $i = 1, 2, \ldots, m$.

By Proposition (iv), page 99 of [3] we have that K_i^c is connected for $i = 1, 2, \ldots, m$. Now, by Costakis and Grosse-Erdmann lemma [5, 8], we take that there exist open simply connected sets V_i, $i = 1, 2, \ldots, m$, such that $K_i \subset V_i \subset G_i$, for $i = 1, 2, \ldots, m$. By Corollary 4.66, page 114 of [3], we have that if we write $V_i = V_i^j$, $j \in J$ for every $i = 1, 2, \ldots, m$, where V_i^j, $j \in J$, be the connected components of V_i, and J is a set of indices, then $(V_i^j)^c$, $j \in J$, are connected sets. It is easy to see that there exists unique $j_i \in J$ such that $K_i \subset V_i^{j_i}$ for every $i = 1, 2, \ldots, m$. So, we have that for every $i = 1, 2, \ldots, m$ there exists a bounded and simply connected domain $V_i^{j_i}$ such that $K_i \subset V_i^{j_i} \subset G_i$.

To avoid complicated symbolism, we write simply V_i instead of $V_i^{j_i}$. So, we have that for every $i = 1, 2, \ldots, m$ there exists a bounded and simply connected domain V_i, such that

$$K_i \subset V_i \subset G_i, \qquad i = 1, 2, \ldots, m.$$

Now, we apply Lemma 1 and we take that for every $i = 1, 2, \ldots, m$ there exists a smooth Jordan curve $\delta_i \subset V_i$ such that $\mathrm{Ind}_{\delta_i}(K_i) = \{1\}$, where it is supposed, of course, that $\delta_i \cap K_i = \emptyset$ for $i = 1, 2, \ldots, m$. Because $G_i \cap G_j = \emptyset$ for $i, j \in$

$\{1,2,\ldots,m\}$, $i \neq j$, we have that the curves δ_i, $i = 1,2,\ldots,m$, are pairwise disjoint.

Now, let $w \in K_i$, for some $i \in \{1,2,\ldots,m\}$. Because $K_i \cap \delta_i = \emptyset$, we have that $w_i \in \mathrm{Int}(\delta_i)$ or $w_i \in Ex(\delta_i)$. If $w_i \in Ex(\delta_i)$, then $\mathrm{Ind}_{\delta_i}(w) = 0$, by Theorem 10.10 of [10]. So we have $K_i \subset \mathrm{Int}(\delta_i)$ for every $i = 1,2,\ldots,m$.

Now, let $i, j \in \{1,2,\ldots,m\}$, $i \neq j$. We show now that $\delta_j \subset Ex(\delta_i)$. We have $\delta_i \subset V_i$, so $V_i^c \subset \delta_i^c$. But $\delta_i^c = Ex(\delta_i) \cup \mathrm{Int}(\delta_i)$. Because V_i^c is connected, we have that $V_i^c \subset \mathrm{Int}(\delta_i)$ or $V_i^c \subset Ex(\delta_i)$. But V_i^c is unbounded, because V_i is bounded and $\mathrm{Int}(\delta_i)$ is bounded, so

$$V_i^c \subset Ex(\delta_i). \tag{5}$$

Now we have

$$V_i \subset G_i \Rightarrow G_i^c \subset V_i^c. \tag{6}$$

We have also

$$G_i \cap G_j = \emptyset \Rightarrow G_j \subset G_i^c. \tag{7}$$

Also we have

$$\delta_j \subset V_j \subset G_j. \tag{8}$$

By (5), (6), (7) and (8), we have $\delta_j \subset Ex(\delta_j)$. This gives, of course, that every one from the smooth Jordan curves δ_i, $i = 1,2,\ldots,m$, has all the others in its exterior, and the proof of Lemma 2 is complete. ∎

Now, we fix a compact subset L of $\mathbb{C}$, with connected complement such that

$$L := \bigcup_{i=0}^{m_0} K_i, \quad m_0 \in \mathbb{N}, \quad m_0 \geq 1, \quad \text{where} \;\; K_i, \quad i = 0,1,\ldots,m_0,$$

be the connected components of L.

We consider the set $\mathfrak{D}_L$ that is a subset of the set of all finite unions of pairwise disjoint Jordan curves, where every one of them has all the remaining curves in its exterior; that is, $\mathfrak{D}_L := \{\Delta \in \mathscr{P}(\mathbb{C}) \mid$ and there exist $m_0 + 1$ smooth Jordan curves δ_i, for $i = 0,1,\ldots,m_0$ such that $\Delta = \bigcup\limits_{i=0}^{m_0} \delta_i$, $K_i \subset \mathrm{Int}(\delta_i)$ and $\mathrm{Ind}_{\delta_i}(K_i) = \{1\}$ for every $i = 0,1,\ldots,m_0$ and $\bigcup\limits_{\substack{i=0\\ i\neq j}}^{m_0} \delta_i \subset Ex(\delta_j)$ for every $j = 0,1,\ldots,m_0$.

By Lemma 2, we have $\mathfrak{D}_L \neq \emptyset$, and by the proof of Lemma 2, we can show easily that the set $\mathfrak{D}_L$ is uncountable.

The set $\mathfrak{D}_L$ is of course well defined and non-empty without any other restriction.

From now on, we suppose that $\overset{\circ}{K_0} \neq \emptyset$.

Let $\Omega := (\mathbb{C} \setminus L) \cup \{\infty\}$. Then Ω is a proper subdomain of $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$, and by the fact that $\overset{\circ}{K_0} \neq \emptyset$, we take easily that $\partial\Omega$ is non-polar. This gives that there exists the unique Green's function g_Ω for Ω, with pole at infinity (Definition 4.4.1 and Theorem 4.4.2, [9]). Let $\Delta \in \mathfrak{D}_L$. We write

$$\theta_{L,\Delta} := \max_{z\in\Delta} e^{-g_\Omega(z,\infty)} := \max\{x \in \mathbb{R} \mid \exists\, z \in \Delta : x = e^{-g_\Omega(z,\infty)}\}.$$

It is obvious that the number $\theta_{L,\Delta}$ is a well-defined positive number in $(0, 1)$, because Δ is a compact set and the green's function g_Ω is continuous in Ω and $\Delta \subset \Omega \setminus \{\infty\}$.

We define

$$\theta_L := \inf\{x \in \mathbb{R} \mid \exists\, \Delta \in \mathfrak{D}_L : x = \theta_{L,\Delta}\} = \inf_{\Delta\in\mathfrak{D}_L} \{\theta_{L,\Delta}\}.$$

By the above the number θ_L is a well-defined number in $[0, 1)$ and corresponds uniquely to the compact set L by its definition.

We fix $z_0 \in \overset{\circ}{K_0}$.

We write $L_1 := L \setminus K_0$, $r_0 := dist(z_0, K_0^c)$ and $h_0 := dist(z_0, L_1)$.

We have the following lemma.

Lemma 3 *By the above definitions, we have easily that*

(1) $r_0 > 0$.
(2) $r_0 < h_0$.
(3) $h_0 < +\infty$.

Let some $\Delta = \bigcup_{i=0}^{m_0} \delta_i \in \mathfrak{D}_L$.

We fix some $w_0 \in K_0^c$ *such that* $h_0 = |z_0 - w_0|$. *Obviously there exists the unique* $i_0 \in \{1, 2, \dots, m_0\}$ *such that* $w_0 \in K_{i_0}$. *We denote*

$$I := [z_0, w_0] := \{z \in \mathbb{C} \mid \exists\, t \in [0, 1] : z = (1 - t)z_0 + tw_0\}.$$

Then we have $\delta_{i_0} \cap I \neq \emptyset$.

Proof We suppose that $\delta_0 \cap I = \emptyset$ to take a contradiction. Then

$$I \subset \delta_0^c. \tag{9}$$

Because δ_0 is a Jordan curve, we have

$$\delta_0^c = \mathrm{Int}(\delta_0) \cup Ex(\delta_0). \tag{10}$$

Because the segment I is connected, we take by (9) and (10) that

$$I \subset \mathrm{Int}(\delta_0) \quad \text{or} \tag{11}$$

$$I \subset Ex(\delta_0). \tag{12}$$

We suppose that relation (12) holds. Then

$$z_0 \in I \subset Ex(\delta_0) \Rightarrow z_0 \in Ex(\delta_0). \tag{13}$$

We have

$$z_0 \in \overset{\circ}{K}_0 \subset K_0 \subset \mathrm{Int}(\delta_0), \tag{14}$$

by Lemma 2.

By (13) and (14), we have $\mathrm{Int}(\delta_0) \cap Ex(\delta_0) \neq \emptyset$, that is, false. So (11) holds. Thus

$$w_0 \in \mathrm{Int}(\delta_0).$$

Because $w_0 \in K_{i_0} \Rightarrow w_0 \in K_{i_0} \subset \mathrm{Int}(\delta_{i_0})$ by Lemma 2,

$$w_0 \in \mathrm{Int}(\delta_{i_0}).$$

By the above, we have

$$\mathrm{Int}(\delta_0) \cap \mathrm{Int}(\delta_{i_0}) \neq \emptyset. \tag{15}$$

We have $\delta_{i_0} \subset Ex(\delta_0)$. So

$$\delta_{i_0} \cap \mathrm{Int}(\delta_0) = \emptyset \Rightarrow \mathrm{Int}(\delta_0) \subset \delta_{i_0}^c.$$

But

$$\delta_{i_0}^c = \mathrm{Int}(\delta_{i_0}) \cup Ex(\delta_{i_0}).$$

Thus

$$\mathrm{Int}(\delta_0) \subset \mathrm{Int}(\delta_{i_0}) \;\; \text{or} \;\; \mathrm{Int}(\delta_0) \subset Ex(\delta_{i_0}). \tag{16}$$

Because of (15), relation (16) gives

$$\mathrm{Int}(\delta_0) \subset \mathrm{Int}(\delta_{i_0}). \tag{17}$$

Similarly exactly with (17), we take

$$\mathrm{Int}(\delta_{i_0}) \subset \mathrm{Int}(\delta_0). \tag{18}$$

By (17) and (18), we take

$$\mathrm{Int}(\delta_0) = \mathrm{Int}(\delta_{i_0}). \tag{19}$$

By (19) and Theorem 4.41 of [3], we take

$$\delta_{i_0} = \partial\mathrm{Int}(\delta_{i_0}) = \partial\mathrm{Int}(\delta_0) = \delta_0,$$

which is false because $\delta_0 \cap \delta_{i_0} = \emptyset$. So, we have a contradiction that gives us that $\delta_0 \cap I \neq \emptyset$. ■

In addition to the above, we suppose now that every connected component K_i, $i = 1, 2, \ldots, m_0$, of L contains more than one point. We prove now the main result of this paper.

Theorem 3 *With the above notations, we have*

$$\theta_L \geq \frac{r_0}{h_0}.$$

Proof We take some $w_0 \in L_1$ such that $h_0 = |z_0 - w_0|$. Let $i_0 \in \{1, 2, \ldots, m_0\}$ be the unique natural number such that $w_0 \in K_{i_0}$. We set $D(z_0, h_0) := \{z \in \mathbb{C} \mid |z - z_0| < h_0\}$ and $V := D(z_0, h_0) \setminus K_0 = D(z_0, h_0) \cap K_0^c$.

The set V is open, and it is easy to see that $\emptyset \neq V \subset \Omega$.

We consider the function $\widetilde{g} : \overline{\Omega} \to \mathbb{R}$ that is defined by the formula:

$$\widetilde{g}(z) := \left.\begin{array}{cl} 0 & \text{if } z \in \partial\Omega \\ g_\Omega(z, \infty) & \text{if } z \in \Omega. \end{array}\right\}$$

The set $\overline{V}$ is a compact subset of $\overline{\Omega}$, and the function $\widetilde{g}$ is continuous, so it takes its maximum value on $\overline{V}$ in a point (say) $w_1 \in \overline{V}$. That is, we have

$$\widetilde{g}(w_1) = \max_{z \in \overline{V}} \widetilde{g}(z).$$

Using Identity Principle (Theorem 1.1.7, page 6, [9]), maximum principle (Theorem 1.1.8, page 6, [9]) and geometrical properties of Ω, we get that $w_1 \in C(z_0, h_0) \cap \Omega$, where

$$C(z_0, h_0) := \{z \in \mathbb{C} \mid |z - z_0| = h_0\}, \quad \text{and}$$

$$g_\Omega(w_1, \infty) = \max_{z \in \overline{V}} \widetilde{g}(z) > 0.$$

Set

$$C_1 := \{z \in \mathbb{C} \mid |z - z_0| \leq r_0\} \text{ and } \Omega_1 := C_1^c \cup \{\infty\}.$$

By Corollary 4.4.5, page 108, [9], we have

$$g_\Omega(z, \infty) \leq g_{\Omega_1}(z, \infty), \quad \text{for every } z \in \Omega.$$

So we have

$$g_\Omega(w_1, \infty) \leq g_{\Omega_1}(w_1, \infty) = \log\left(\frac{h_0}{z_0}\right). \tag{20}$$

We define

$$I := [z_0, w_0] := \{z \in \mathbb{C} \mid \exists\, t \in [0, 1] : z = (1 - t)z_0 + t w_0\}.$$

In this point exactly, we apply Lemma 3.

We take a $\Delta \in \mathfrak{D}_L$, where $\Delta = \bigcup_{i=0}^{m_0} \delta_i$, $K_i \subset \text{Int}(\delta_i)$ and $\text{Ind}_{\delta_i}(K_i) = \{1\}$ for every $i = 0, 1, \ldots, m_0$ and

$$\bigcup_{\substack{i=0\\ i\neq j}}^{m_0} \delta_i \subset Ex(\delta_j) \text{ for every } j = 0, 1, \ldots, m_0.$$

We apply Lemma 3, and we get $\emptyset \neq \delta_0 \cap I \subset V$.

Let some $z_1 \in \Delta_0 \cap I$. Then we have

$$g_\Omega(z_1, \infty) \leq g_\Omega(w_1, \infty). \tag{21}$$

By (20) and (21), we get

$$g_\Omega(z_1, \infty) \leq \log\left(\frac{h_0}{r_0}\right) \Rightarrow \frac{z_0}{h_0} \leq e^{-g_\Omega(z_1,\infty)} \leq \theta_{L,\Delta}.$$

This implies that $\dfrac{r_0}{h_0} \leq \theta_L$, and the proof of this theorem is complete. ■

Theorem 3 gives us a simple lower bound for the number θ_L.

We will prove that in some cases this lower bound is optimal in some sense.

More specifically, let D be the open unit disc, and we denote by $K_0 := \overline{D} = \{z \in \mathbb{C} \mid \mid z \mid\leq 1\}$ the closed unit disc, for the sequel.

We fix some positive number, $h_0 > 1$. We set

$$C_{h_0} := \{L \subseteq \mathbb{C} \mid L$$

is compact with connected complement, $L = \bigcup_{i=0}^{m} K_i$, $m \geq 1$, $K_0 := \overline{D}$, where K_i, $i = 0, 1, \ldots, m$, be the connected components of L and K_i, $i = 1, \ldots, m$, contains more than one point and $dist(\{0\}, L \setminus K_0) = h_0, h_0 \in L\}$.

Of course, by Theorem 3, we have

$$\theta_L \geq \frac{1}{h_0} \text{ for every } L \in C_{h_0}.$$

We prove the following proposition.

Proposition 1 *It holds* $\inf\{\theta_L,\ L \in C_{h_0}\} = \dfrac{1}{h_0}$.

Proof We set $I := \inf\{\theta_L, L \in C_{h_0}\}$. By Theorem 3, we have $I \geq \dfrac{1}{h_0}$. It holds $1 - \dfrac{1}{h_0} > 0$.

We prove that for every $\delta \in \left(0, 1 - \dfrac{1}{h}\right)$ there exists some $L' \in C_{h_0}$ such that $\theta_{L'} < \delta + \dfrac{1}{h_0}$ that implies the desired result. We fix some

$$\delta_0 \in \left(0, 1 - \frac{1}{h_0}\right).$$

Then we fix some

$$\ell_0 \in \left(\frac{h_0}{\delta_0 h_0 + 1}, h_0\right).$$

We get

$$\frac{1}{\ell_0} < \delta_0 + \frac{1}{h_0} \text{ and } \ell_0 > 1.$$

Then we fix some

$$r_0 \in (0, h_0 - \ell_0).$$

Finally, we fix some positive number ε_0 such that

$$\varepsilon_0 < \frac{1}{4} \text{ and} \tag{22}$$

$$\varepsilon_0 < \frac{r_0}{2}. \tag{23}$$

That is, we have defined the positive numbers h_0, δ_0, ℓ_0, r_0 and ε_0 that satisfy the above inequalities.

We set

$$K_1 := \overline{D}(h_0+\varepsilon_0, \varepsilon_0) := \{z \in \mathbb{C} \mid |z-(h_0+\varepsilon_0)| \leq \varepsilon_0\}$$

and $L' := K_0 \cup K_1$. Of course, we have $L' \in C_{h_0}$. We prove that for the compact set L' we have $\theta_{L'} < \delta_0 + \dfrac{1}{h_0}$.

First, we consider the circles $\delta_1 := \gamma_1([0,1])$, where $\gamma_1 : [0,1] \to \mathbb{C}$, such that $\gamma_1(t) = \ell_0 \cdot e^{2\pi i t}$ for every $t \in [0,1]$, $\delta_2 := \gamma_2([0,1])$, where $\gamma_2 : [0,1] \to \mathbb{C}$, such that $\gamma_2(t) := (h_0+\varepsilon_0) + r_0 e^{2\pi i t}$, for every $t \in [0,1]$. We set $\Delta := \delta_1 \cup \delta_2$. We get $\Delta \in \mathfrak{D}_{L'}$.

Now, we fix some natural number $N_0 \in \mathbb{N}$, such that

$$(h_0 - r_0)^{N_0} > 2, \tag{24}$$

$$\left(\frac{h_0 - r_0}{\ell_0}\right)^{N_0} > \frac{2(h_0-\ell_0)}{z_0}, \tag{25}$$

$$\ell_0^{N_0} > 2, \tag{26}$$

and

$$\left(\frac{8h_0}{h_0-\ell_0}\right)^{\frac{1}{N_0}} \cdot \frac{1}{\ell_0^{\frac{N_0}{N_0+1}}} < \delta_0 + \frac{1}{h_0}. \tag{27}$$

We fix also some natural number $n_0 > 2$.

Let j_k, $k = 0, 1, \ldots, n_0$, be the n_0-roots of unity; that is, $j_k := e^{\frac{2k\pi i}{n_0}}$, $k = 0, 1, \ldots, n_0 - 1$. We set $w_k := (h_0 + \varepsilon_0) + \varepsilon_0 j_k$, $k = 0, 1, \ldots, n_0 - 1$. We consider the polynomial:

$$p(z) := (z^{n_0 N_0} - 1) \cdot \prod_{k=0}^{n_0-1} (z - w_k).$$

Using (22) and the fact that $h_0 > 1$, we get

$$\|p\|_{L'} \leq 2 \cdot (h_0 + 2\varepsilon_0 + 1)^{n_0}. \tag{28}$$

Using inequalities (23), (24), (25) and (26), we take

$$\min_{z \in \Delta} |p(z)| \geq (\ell_0^{n_0 N_0} - 1) \cdot (h_0 - \ell_0)^{n_0}. \tag{29}$$

By inequalities (28) and (29), we get

$$\frac{\|p\|_{L'}}{\min_{z\in\Delta}|p(z)|} \le \frac{2\cdot(h_0+2\varepsilon_0+1)^{n_0}}{(\ell_0^{n_0N_0}-1)\cdot(h_0-\ell_0)^{n_0}}. \tag{30}$$

By inequalities (23), (26), (27) and (30), we have

$$\left(\frac{\|p\|_{L'}}{\min_{z\in\Delta}|p(z)|}\right)^{\frac{1}{n_0N_0+n_0}} < \delta_0 + \frac{1}{n_0}. \tag{31}$$

We denote $\Omega_{L'} := (\mathbb{C}\setminus L') \cup \{\infty\}$.

Applying Bernstein's lemma (5.5.7), (a), page 156 of [9], for the polynomial p of degree $n_0N_0+n_0$, we take

$$\theta_{L',\Delta} \le \left(\frac{\|p\|_{L'}}{\min_{z\in\Delta}|p(z)|}\right)^{\frac{1}{n_0N_0+n_0}}. \tag{32}$$

By (31) and (32), we have $\theta_{L'} < \delta_0 + \dfrac{1}{h}$, and the proof of this Proposition 1 is complete. ∎

We set

$$L_0 := K_0 \cup \{h_0\} \in C_{h_0}.$$

We prove now the following proposition.

Proposition 2 *We have*

$$\theta_{L_0} = \frac{1}{h_0}.$$

Proof We consider a compact set $L_1 \in C_{h_0}$. Obviously, we have $L_0 \subset L_1$. We set $\Omega_0 := (\mathbb{C}\setminus L_0)\cup\{\infty\}$ and $\Omega_1 := (\mathbb{C}\setminus L_1)\cup\{\infty\}$. Of course, $\Omega_1 \subset \Omega_0$. By Corollary 4.4.5, page 108, [9], we have

$$g_{\Omega_1}(z,\infty) \le g_{\Omega_0}(z,\infty), \quad z\in\Omega_1 \tag{33}$$

where by g_{Ω_1} and g_{Ω_0} we denote the Green's functions on Ω_1 and Ω_0, respectively. Relation (33) gives

$$\theta_{L_0,\Delta} \le \theta_{L_1,\Delta} \ \text{ for every } \ \Delta \in \mathfrak{D}_{L_1}. \tag{34}$$

Of course, we have $\mathfrak{D}_{L_1} \subset \mathfrak{D}_{L_0}$. This gives

$$\{\theta_{L_0,\Delta} \mid \Delta \in \mathfrak{D}_{L_1}\} \subseteq \{\theta_{L_0,\Delta} \mid \Delta \in \mathfrak{D}_{L_0}\},$$

thus

$$\inf\{\theta_{L_0,\Delta} \mid \Delta \in \mathfrak{D}_{L_0}\} \leq \inf\{\theta_{L_0,\Delta} \mid \Delta \in \mathfrak{D}_{L_1}\}.$$

The last inequality gives us

$$\theta_{L_0} \leq \inf\{\theta_{L_0,\Delta} \mid \Delta \in \mathfrak{D}_{L_1}\}. \tag{35}$$

By (34), we get

$$\inf\{\theta_{L_0,\Delta} \mid \Delta \in \mathfrak{D}_{L_1}\} \leq \inf\{\theta_{L_1}, \Delta \mid \Delta \in \mathfrak{D}_{L_1}\} = \theta_{L_1}. \tag{36}$$

By (35) and (36), we obtain

$$\theta_{L_0} \leq \theta_{L_1}. \tag{37}$$

By (37) and Proposition 1, we have

$$\theta_{L_0} \leq \frac{1}{h_0}. \tag{38}$$

By the proof of Proposition 1, it is easy to see that we can construct a strictly decreasing sequence of compact sets $L_n \in C_{h_0}$, for $n > \dfrac{1}{1-\dfrac{1}{n_0}}$, that is, $L_{n+1} \subset L_n$, for $n > \dfrac{1}{1-\dfrac{1}{n_0}}$ such that $\theta_{L_n} < \dfrac{1}{n} + \dfrac{1}{h_0}$ for every $n \in \mathbb{N}$, $n > \dfrac{1}{1-\dfrac{1}{h_0}}$. Of course, $\cap L_n = L_0$. We set $a := \dfrac{1}{1-\dfrac{1}{h_0}}$.

We set $\Omega_n := (\mathbb{C} \setminus L_n) \cup \{\infty\}$, $n > a$. We have $\Omega_0 = \bigcup\limits_{n>a} \Omega_n$. We fix some $\Delta_0 \in \mathfrak{D}_{L_0}$. It is easy to see that there exists some $m_0 > a$ such that $\Delta_0 \in \mathfrak{D}_{L_n}$ for every $n \geq m_0$. Of course, we have $\Omega_0 = \bigcup\limits_{n \geq m_0} \Omega_n$.

By Theorem 4.4.6, page 108, [9], we have

$$\lim_{n\to+\infty} g_{\Omega_n}(z,\infty) = g_{\Omega_0}(z,\infty) \ \text{ for } \ z \in \Omega_0,$$

so

$$\lim_{n\to+\infty} g_{\Omega_n}(z,\infty) = g_{\Omega_0}(z,\infty), \quad \text{for every } \ z \in \Delta_0,$$

where we denote by g_{Ω_n}, $n > a$ the Green's function on Ω_n.

Of course, $\Omega_n \subset \Omega_{n+1}$, $n \geq n_0$, so

$$g_{\Omega_n}(z,\infty) \leq g_{\Omega_{n+1}}(z,\infty), \quad n \geq m_0, \tag{39}$$

by the Corollary 4.4.5, page 108, [9] .

By this inequality (39), we have that the sequence of functions $(-g_{\Omega_n})$, $n \geq m_0$, is a decreasing sequence of continuous functions on the compact set Δ_0, so by Dini's theorem we take $g_{\Omega_n} \to g_{\Omega_0}$ uniformly on Δ_0.

We fix some positive number ε_0. Then there exists $m_1 \geq m_0$ such that

$$|g_{\Omega_n}(z,\infty) - g_{\Omega_0}(z,\infty)\,| < \varepsilon_0 \ \text{ for every } \ z \in \Delta_0, \quad n \geq m_1.$$

This gives that $e^{-\varepsilon_0}\dfrac{1}{h_0} < \theta_{L_0,\Delta}$ for the arbitrary positive number ε_0. So $\dfrac{1}{h_0} \leq \theta_{L_0,\Delta}$, for the arbitrary contour Δ_0, thus

$$\frac{1}{h_0} \leq \theta_{L_0}. \tag{40}$$

By (38) and (40), we get $\theta_{L_0} = \dfrac{1}{h_0}$, and the proof of this proposition is complete. ■

3 Final Step of the Proof of Theorem 2

So, by the above Theorem 3, we have proved that the number θ_L is positive and we have found an easy-computed (in all simple cases) lower bound of θ_L.

For the sequel, we refer to [9] for the respective terminology.

More specifically, for the definition of Harnack distance, see Definition 1.3.4. We note that the Harnack distance is a continuous function. For the definition of logarithmic capacity, see Definition 5.1.1. For the definition of a Fekete n-tuple and the n-th diameter $\delta_n(K)$ of a compact set $K \subseteq \mathbb{C}$, see Definition 5.5.1.

For the definition of a Fekete polynomial of degree $n \geq 2$ of a compact set K, see Definition 5.5.3

We remind here (Bernstein's lemma) Theorem 5.5.7 of [9].

Let L be a non-polar compact subset of $\mathbb{C}$, and let Ω be the component of $(\mathbb{C} \cup \{\infty\}) \setminus L$ containing ∞. If q_n is a Fekete polynomial of degree $n \geq 2$ of L, then

$$\left(\frac{|q_n(z)|}{\|q_n\|_L}\right)^{1/n} \geq e^{g_\Omega(z,\infty)}\left(\frac{c(L)}{\delta_n(L)}\right)^{T_\Omega(z,\infty)} \quad \text{for every } \ z \in \Omega \setminus \{\infty\}. \tag{$*$}$$

We consider now the fixed set L of our work, where $L = \bigcup_{i=0}^{m_0} K_i$, $m_0 > 1$, $\overset{\circ}{K}_0 \neq \emptyset$ and K_i, $i = 1, \ldots, m_0$, contains more than one point, K_i, $i = 0, 1, \ldots, m_0$, are the connected components of L and L^c is connected.

We choose some Fekete polynomial q_m, for every $m = 2, 3, \ldots$, for L, and we fix them for the sequel.

We set

$$\inf_{z\in\Delta} |q_m(z)| := \inf\{x \in \mathbb{R} \mid \exists\, z \in \Delta : x = |q_m(z)|\} \text{ for every } m = 2, 3, \ldots, \quad \Delta \in \mathfrak{D}_L.$$

By the above terminology, we get the following lemma using inequality $(*)$.

Lemma 4 *For every positive constant $c_L \in (\theta_L, 1)$ that depends on L, there exists some $\Delta \in \mathfrak{D}_L$ that depends on L, c_L and some natural number m_Δ that depends on Δ such that*

$$\frac{\|q_n\|_L}{\inf\limits_{z\in\Delta} |q_n(z)|} < c_L^n, \quad \textit{for every } n \geq m_\Delta.$$

Proof Take arbitrary $c_L \in (\theta_L, 1)$. By the definition of the number θ_L, we can take some $\Delta_{L,c_L} \in \mathfrak{D}_L$, such that $\theta_L < \theta_{L,\Delta} < c_L$, where Δ_{L,c_L} depends on L, c_L. We write $\Delta = \Delta_{L,c_L}$ for simplicity.

We fix some $m_0 \geq 2$. By $(*)$, we get

$$\left(\frac{\|q_{m_0}\|_L}{|q_{m_0}(z)|}\right)^{1/m_0} \leq \frac{1}{e^{g_\Omega(z,\infty)}} \left(\frac{\delta_{m_0}(L)}{c(L)}\right)^{T_\Omega(z,\infty)} \quad \text{for every } z \in \Delta.$$

So we have

$$\left(\frac{\|q_{m_0}\|_L}{|q_{m_0}(z)|}\right)^{1/m_0} \leq \theta_{L,\Delta} \cdot \left(\frac{\delta_{m_0}(L)}{c(L)}\right)^{\|T_\Omega\|_\Delta} \quad \text{for every } z \in \Delta. \tag{41}$$

By (41), we have

$$\left(\frac{\|q_{m_0}\|_L}{\inf\limits_{z\in\Delta} |q_{m_0}(z)|}\right)^{1/m_0} \leq \theta_{L,\Delta} \cdot \left(\frac{\delta_{m_0}(L)}{c(L)}\right)^{\|T_\Omega\|_\Delta} \quad \text{for every } m_0 \geq 2. \tag{42}$$

By Fekete–Szegö theorem (Theorem 5.5.2), we have that

$$\delta_m(L) \to c(L) \text{ as } m \to +\infty. \tag{43}$$

The number $\theta_{L,\Delta}$ depends on Δ.

Thus, because

$$\theta_{L,\Delta} < c_L, \tag{44}$$

there exists some natural number m_Δ that depends on Δ, such that

$$\frac{\|q_m\|_L}{\inf\limits_{z\in\Delta} |q_m(z)|} < c_L^m \ \text{ for every } \ m \in \mathbb{N}, \quad m \geq m_\Delta,$$

by (42), (43) and (44).

This completes the proof. ■

We will need also a proposition, which is a variation of the well-known Bernstein–Walsh theorem.

Proposition 3 *Let some compact set $L = \bigcup\limits_{i=0}^{m_0} K_i$, $m_0 \in \mathbb{N}$, as above, where K_i, $i = 0, 1, \ldots, m_0$, be the connected components of L. Let some complex polynomials p_j, $j = 0, 1, \ldots, m_0$. We consider the function $F : L \to \mathbb{C}$ that is defined by the following formula:*

$$F(z) = p_j(z) \ \textit{ if } \ z \in K_j \ \textit{ for every } \ j = 0, 1, \ldots, m_0.$$

Then, for every positive number $c_L \in (\theta_L, 1)$, there exists $\Delta \in \mathfrak{D}_L$ that depends on L, c_L, some natural number $m = m_\Delta$ that depends on Δ, some positive constant $A = A_{\Delta,F}$ that depends on Δ, F and some sequence of polynomials (r_j) that depends on Δ, F such that the following inequality holds:

$$\|F - r_m\|_L < A \cdot c_L^m \ \textit{ for every } \ m \in \mathbb{N}, \quad m \geq m_\Delta, \quad \deg(r_m) \leq m - 1.$$

Proof We consider the compact set $L = \bigcup\limits_{i=0}^{m_0} K_i$, the polynomials p_j, $j = 0, 1, \ldots, m_0$, and the complex function F as in the suppositions of this proposition. We fix some positive number $c_L \in (\theta_L, 1)$. Then we fix some $\Delta = \Delta_{L,c_L} \in \mathfrak{D}_L$ that depends on L, c_L such that $\theta_{L,\Delta} \in (\theta_L, c_L)$.

Afterwards we apply Lemma 4 and we get that there exists some natural number $m_\Delta > 2$ that depends on Δ such that

$$\frac{\|q_n\|_L}{\inf\limits_{z\in\Delta} |q_n(z)|} < c_L^n, \quad \text{for every } \ n \in \mathbb{N} \ \ n \geq m_\Delta > 2. \tag{45}$$

Let $\Delta = \bigcup\limits_{i=0}^{m_0} \delta_i$, where δ_i, $i = 0, 1, \ldots, m_0$, be the connected components of Δ. There exist V_i, $i = 0, 1, \ldots, m_0$, bounded simply connected domains pairwise disjoint such that $\delta_i \subset V_i$ for every $i = 0, 1, \ldots, m_0$ by the proof of Lemma 2.

We set $V := \bigcup_{i=0}^{m_0} V_i$.

We consider the function $\Phi : V \to \mathbb{C}$ by the following formula:

$$\Phi(z) = p_j(z) \;\text{ for every }\; z \in V_j \;\text{ for every }\; j = 0, 1, \ldots, m_0.$$

Of course, we have $\Phi \restriction_L = F$, and Φ is holomorphic, for the restriction $\Phi \restriction L$ of Φ on L.

We consider some arbitrary fixed sequence (q_m), $m \geq 2$, of Fekete polynomials for L of degree m, $m \geq 2$. We define now the functions $r_m : L \to \mathbb{C}$ with the formula:

$$r_m(w) := \sum_{j=0}^{m_0} \frac{1}{2\pi i} \int_{\delta_j} \frac{\Phi(z)}{q_m(z)} \cdot \frac{q_m(w) - q_m(z)}{w - z} dz \tag{46}$$

for every $m \in \mathbb{N}$, $m \geq 2$, $w \in L$. For every $m \geq 2$, the functions r_m are polynomials of degree at most $m - 1$, as we can see easily.

It is obvious that $\sum_{j=0}^{m_0} \mathrm{Ind}_{\delta_j}(a) = 0$ for every $a \in \mathbb{C} \setminus V$.

We fix some $n_0 \geq 2$.

We apply now the global Cauchy's integral formula for the function Φ and the smooth Jordan curves δ_j, $j = 0, 1, \ldots, m_0$, and we take

$$\sum_{j=0}^{m_0} \mathrm{Ind}_{\delta_j}(w) \cdot \Phi(w) = \sum_{j=0}^{m_0} \frac{1}{2\pi i} \int_{\delta_j} \frac{\Phi(z)}{z - w} dz \;\text{ for every }\; w \in V \setminus \Delta. \tag{47}$$

By (46) and (47), we get

$$\Phi(w) - r_{n_0}(w) = \sum_{j=0}^{m_0} \frac{1}{2\pi i} \int_{\delta_j} \frac{\Phi(z) q_{n_0}(w)}{(z - w) q_{n_0}(z)} dz \;\text{ for every }\; w \in L. \tag{48}$$

It is time to use the supposition that every one from the curves δ_j, $j = 0, 1, \ldots, m_0$ is smooth. This gives that the curves δ_j, $j = 0, 1, \ldots, m_0$, have length (see Definition 2.8, page 44 of [3]). We set $\ell_j = \mathrm{length}(\delta_j)$ for every $j = 0, 1, \ldots, m_0$, $\lambda_0 := \sum_{j=0}^{m_0} \ell_j$,

$$dist(\Delta, L) := \min\{x \in \mathbb{R} \mid \exists z_1 \in \Delta, z_2 \in L : x = |z_1 - z_2|\}.$$

Then we define the number:

$$A := \frac{\lambda_0 \cdot \|\Phi\|_L}{2\pi \cdot dist(\Delta, L)}. \tag{49}$$

Of course, the above number A depends on F, Δ. By (48) and (49), we take easily that

$$\|F - r_{n_0}\|_L \le A \cdot \frac{\|q_{n_0}\|_L}{\inf\limits_{z \in \Delta} |q_{n_0}(z)|}. \tag{50}$$

Of course, the positive number A is independent from the natural number n_0.

So by (50), we get

$$\|F - r_n\|_L \le A \cdot \frac{\|q_n\|_L}{\inf\limits_{z \in \Delta} |q_n(z)|} \quad \text{for every } \ n \in \mathbb{N}, \quad n \ge 2. \tag{51}$$

By (45) and (51), we get that

$$\|F - r_n\|_L < A \cdot c_L^n \ \text{ for every } \ n \in \mathbb{N}, \quad n \ge m_\Delta, \tag{52}$$

where the natural number m_Δ depends on Δ, the set Δ depends on L, c_L, the positive number c_L depends on L, the constant A depends on F, Δ and the polynomials $r_n, n \ge 2$, depend on F, Δ, and the sequence $q_n, n \in \mathbb{N}, n \ge 2$.

The above inequality (52) completes the proof of this proposition. ∎

The above Proposition 3 gives some important role to the number θ_L. It shows that the number θ_L plays a crucial role in the problem of approximation by polynomials.

We have the following very important information about the number ρ_L below.

Proposition 4 *By the previous notations, we have that $\rho_L = \theta_L$.*

Proof We take some $\Delta \in \mathfrak{D}_L$.

Let $\Delta = \bigcup\limits_{i=0}^{m_0} \delta_i$, where $\delta_i, i = 0, 1, \ldots, m_0$, be the connected components of Δ.

Let G_i, $i = 0, 1, \ldots, m_0$, be bounded simply connected domains, pairwise disjoint such that $\delta_i \subset G_i$ for $i = 0, 1, \ldots, m_0$ (using the proof of Lemma 2).

We consider $m_0 + 1$ polynomials $p_i, i = 0, 1, \ldots, m_0$, where $p_i \neq p_j$ for every $i, j \in \{0, 1, \ldots, m_0\}, i \neq j$. We define the holomorphic function $F : G \to \mathbb{C}$, where $G := \bigcup\limits_{i=0}^{m_0} G_i$ with the following formula:

$$F(z) = p_i(z) \ \text{ for every } \ z \in G_i, \quad i = 0, 1, \ldots, m_0.$$

We apply Proposition 3, and for the above function F, there exists a sequence of polynomials r_n, $n \geq 2$, some positive number A and some natural number n_0 such that

$$\|F - r_n\|_L < A \cdot c_L^n \ \text{ for every } \ n \geq n_0$$

for some positive constant $c_L \in (\theta_{L,\Delta}, 1)$ (see the proof of Lemma 4 also).

This gives that

$$\limsup_{n \to +\infty} \|F - r_n\|_L^{1/n} \leq c_L. \tag{53}$$

Let S_n, $n = 2, 3, \ldots$, be the unique polynomial of degree at most n (that there exists see [13]) that minimizes the quantity $\|F - S_n\|_L$. We write $t_{F,n} := \|F - S_n\|_L$ for simplicity. It is known [7, 13] that

$$\limsup_{n \to +\infty} t_{F,n}^{1/n} = \rho_L. \tag{54}$$

By the definition of the number $t_{F,n}$, we have, of course,

$$t_{F,n} \leq \|F - r_n\|_L \ \text{ for } \ n \geq 2. \tag{55}$$

By (53), (54) and (55), we get $\rho_L \leq c_L$. But the number c_L is some arbitrary positive number such that $\theta_{L,\Delta} < c_L < 1$. This gives that $\rho_L \leq \theta_{L,\Delta}$. Because this holds for every $\Delta \in \mathfrak{D}_L$, we get

$$\rho_L \leq \theta_L. \tag{56}$$

Now $\rho_L = \exp(-g_c)$, where g_c is the critical potential (see [7]), and $\gamma := \{z \in \mathbb{C} : g_\Omega(z) = g_c\}$ is the critical level curve, where $\Omega := (\mathbb{C} \cup \{\infty\}) \setminus L$ and g_Ω is the Green's function for L. It is simple to see, by the continuity of g_Ω, that there exists a sequence of curves Δ_n, $n = 1, 2, \ldots$, where $\Delta_n \in \mathfrak{D}_L$ for $n = 1, 2, \ldots$ such that

$$\theta_{L,\Delta_n} \to \rho_L \ \text{ as } \ n \to +\infty. \tag{57}$$

By (56) and (57), we obtain that $\rho_L = \theta_L$, and the proof of this proposition is complete. ∎

Propositions 1 and 4 can afford us quantitative examples of compact sets such that the following remark holds:

Remark 3 For every $\rho \in (0, 1)$ and $\delta \in (0, 1 - \rho)$, we can construct a compact set L that is a union of two disjoint closed discs, one of them be the closed unit disc such that

$$|\rho_L - \rho| < \delta.$$

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Orlicz Version of Mixed Mean Dual Affifine Quermassintegrals

C.-J. Zhao and W.-S. Cheung

Abstract In this paper, our main aim is to generalize the mixed mean dual affine quermassintegrals to the Orlicz space. Under the framework of Orlicz dual Brunn–Minkowski theory, we introduce a new geometric operator by calculating the first Orlicz variation of the mixed mean dual affine quermassintegrals and call it the Orlicz mixed mean dual affine quermassintegrals. The fundamental notions and conclusions of the mixed mean dual affine quermassintegrals, and the Minkowski and Brunn–Minkowski inequalities for the mixed mean dual affine quermassintegrals are extended to an Orlicz setting, and the related concepts and inequalities of Orlicz dual quermassintegrals are also included in our conclusions. The new Orlicz isoperimetric inequalities in special case yield the Orlicz dual Minkowski inequality and Orlicz dual Brunn–Minkowski inequality, which also imply the L_p-dual Minkowski inequality and Brunn–Minkowski inequality for the mixed mean dual affine quermassintegrals.

1 Introduction

The radial addition $K\widetilde{+}L$ of star sets (compact sets that are star-shaped at o and contain o) K and L can be defined by

$$K\widetilde{+}L = \{x\widetilde{+}y : x \in K, y \in L\},$$

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where $x\widetilde{+}y = x + y$ if x, y, and o are collinear, $x\widetilde{+}y = o$, otherwise, or by

$$\rho(K\widetilde{+}L, \cdot) = \rho(K, \cdot) + \rho(L, \cdot),$$

where $\rho(K, \cdot)$ denotes the radial function of star set K, which is defined by $\rho(K, u) = \max\{c \geq 0 : cu \in K\}$, for $u \in S^{n-1}$, where S^{n-1} is the surface of the unit sphere. Hints as to the origins of the radial addition can be found in [8, p. 235]. If $\rho(K, \cdot)$ is positive and continuous, K will be called a star body. Let $\mathscr{S}^n$ denote the set of star bodies about the origin in $\mathbb{R}^n$. When combined with volume, radial addition gives rise to another substantial appendage to the classical theory, called the dual Brunn–Minkowski theory. Radial addition is the basis for the dual Brunn–Minkowski theory (see, e.g., [2, 3, 7, 14–17, 21–23, 27, 28, 32, 34, 37] for recent important contributions). The original is originated from Lutwak [24]. He introduced the concept of dual mixed volume laid the foundation of the dual Brunn–Minkowski theory. In particular, the dual Brunn–Minkowski theory can count among its successes the solution of the Busemann–Petty problem in [7, 14, 23, 33, 38]. More generally, for any $p < 0$ (or $p > 0$), the p-radial addition $K\widetilde{+}_pL$ is defined by

$$\rho(K\widetilde{+}_pL, x)^p = \rho(K, x)^p + \rho(L, x)^p,$$

for $x \in \mathbb{R}^n$ and $K, L \in \mathscr{S}^n$ (see [12]). In 1996, L_p-harmonic radial combination for star bodies was defined by Lutwak [22]: If K and L are star bodies, for $p \geq 1$, the L_p-harmonic radial addition is defined by

$$\rho(K\widehat{+}_pL, u)^{-p} = \rho(K, u)^{-p} + \rho(L, u)^{-p}, \quad u \in \mathscr{S}^{n-1}. \tag{1.1}$$

For convex bodies, the L_p-harmonic addition was first investigated by Firey [6].

If K is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then

$$h(K, x) = \max\{x \cdot y : y \in K\},$$

for $x \in \mathbb{R}^n$, defined the support function $h(K, x)$ of K. A nonempty closed convex set is uniquely determined by its support function. L_p-addition and inequalities are the fundamental and core content in the L_p-Brunn–Minkowski theory. In recent years, a new extension of L_p-Brunn–Minkowski theory is to Orlicz–Brunn–Minkowski theory, initiated by Lutwak, Yang, and Zhang [25, 26]. Gardner, Hug, and Weil [11] constructed a general framework for the Orlicz–Brunn–Minkowski theory and made clear for the first time the relation to Orlicz spaces and norms. The Orlicz centroid inequality for star bodies was introduced in [44]. The Orlicz addition of convex bodies was introduced, and it extended the L_p-Brunn–Minkowski inequality to the Orlicz–Brunn–Minkowski inequality (see [35]). Other articles that have advanced the theory can be found in the literature [4, 18, 20, 21, 30, 42, 43].

Just as L_p-Brunn–Minkowski theory is extended to Orlicz–Brunn–Minkowski theory, it has recently turned to a study extending from L_p-dual Brunn–Minkowski theory to Orlicz dual Brunn–Minkowski theory. The dual Orlicz–Brunn–Minkowski theory has also attracted people's attention [12, 13, 36, 41]. In particular, in 2014, Zhu, Zhou, and Xu [40] introduced the Orlicz harmonic radial sum $K\widehat{+}_{\phi}L$ of two star bodies K and L, defined by for $u \in S^{n-1}$

$$\rho(K\widehat{+}_{\phi}L, u) = \sup\left\{\lambda > 0 : \phi\left(\frac{\rho(K,u)}{\lambda}\right) + \phi\left(\frac{\rho(L,u)}{\lambda}\right) \leq \phi(1)\right\}, \tag{1.2}$$

where $\phi : (0, \infty) \to (0, \infty)$ is a convex and strictly decreasing function such that $\phi(0) = \infty$, $\lim_{t\to\infty} \phi(t) = 0$, and $\lim_{t\to 0} \phi(t) = \infty$. Let $\mathscr{C}$ denote the class of the convex and strictly decreasing functions ϕ. When $p \geq 1$ and $\phi(t) = t^{-p}$, the Orlicz harmonic addition $\widehat{+}_{\phi}$ becomes the L_p-harmonic radial addition $\widehat{+}_p$. The Orlicz dual mixed volume with respect to Orlicz harmonic radial addition, denoted by $\widetilde{V}_{\phi}(K, L)$, is defined by

$$\begin{aligned}\widetilde{V}_{\phi}(K, L) =: \frac{\phi'_{+}(1)}{n} \lim_{\varepsilon\to 0^{+}} \frac{\widetilde{V}(K\widehat{+}_{\phi}\varepsilon \cdot L) - V(K)}{\varepsilon} \\ = \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^{n} dS(u),\end{aligned} \tag{1.3}$$

where $K\widehat{+}_{\phi}\varepsilon \cdot L$ is the Orlicz linear combination of K and L (see Sect. 2), and $\phi'_{+}(1)$ denotes the value of the right derivative of convex function ϕ at point 1.

The dual affine quermassintegral was proposed by Lutwak (orally, in the 1980s), for $K \in \mathscr{S}^n$, by letting $\widetilde{\Phi}_0(K) := V(K)$, $\widetilde{\Phi}_n(K) := \omega_n$, and $0 < j < n$, defined by

$$\widetilde{\Phi}_{n-j}(K) := \omega_n \left[\int_{G_{n,j}} \left(\frac{\mathrm{vol}_j(K\cap\xi)}{\omega_j}\right)^{n} d\mu_j(\xi)\right]^{1/n}, \tag{1.4}$$

where $G_{n,j}$ denotes the Grassman manifold of j-dimensional subspaces in $\mathbb{R}^n$, μ_j denotes the gauge Haar measure on $G_{n,j}$, $\mathrm{vol}_j(K\cap\xi)$ denotes the j-dimensional volume of intersection of K on j-dimensional subspace $\xi \subset \mathbb{R}^n$, and ω_j denotes the volume of j-dimensional unit ball. Gardner [9] showed the Brunn–Minkowski inequality for the dual affine quermassintegrals. If $K, L \in S^{n-1}$ and $0 \leq i \leq n-1$, then

$$\widetilde{\Phi}_j(K\widetilde{+}L)^{1/(n-j)} \leq \widetilde{\Phi}_j(K)^{1/(n-j)} + \widetilde{\Phi}_j(L)^{1/(n-j)}, \tag{1.5}$$

with equality if and only if K is a dilatate of L, modulo a set of measure zero. In analogy to (1.4), one may also define mixed mean dual affine quermassintegrals by

$$\overline{\Phi}_{n-j,i}(K) := \omega_n \left[\int_{A_{n,j}} \left(\frac{w_i^{(j)}(K\cap\xi)}{\omega_j}\right)^{n-i+1} d\nu_j(\xi)\right]^{1/(n-i+1)}, \tag{1.6}$$

for $0 \le i < j \le n$ and $K \in \mathscr{S}^n$ and by letting $\overline{\Phi}_{0,i}(K) := \widetilde{W}_i(K)$. Here, $A_{n,j}$ denotes the space of the j-dimensional affine subspace in $\mathbb{R}^n$, $w_i^{(j)}(K \cap \xi)$ denotes the dual quermassintegrals of intersection of K on j-dimensional subspace $\xi \subset \mathbb{R}^n$, and ν_j denotes the gauge Haar measure on $A_{n,j}$ (see Sect. 5).

In this paper, our main aim is to generalize the mixed mean dual affine quermassintegrals to the Orlicz space. Under the framework of Orlicz dual Brunn–Minkowski theory, we introduce a new affine geometric quantity such as Orlicz mixed mean dual affine quermassintegrals. The fundamental notions and conclusions of the mixed mean dual affine quermassintegrals and the Minkowski and Brunn–Minkowski inequalities for the mixed mean dual affine quermassintegrals are extended to an Orlicz setting. The new Orlicz–Minkowski and Brunn–Minkowski inequalities for the Orlicz mean dual affine quermassintegrals in special case yield the L_p-dual Minkowski inequality and Brunn–Minkowski inequalities for the mixed mean dual affine quermassintegrals, which also imply the Orlicz dual Minkowski inequality and Brunn–Minkowski inequalities for dual quermassintegrals.

Comply with the basic spirit of Aleksandrov [1], Fenchel and Jensen's [5] introduction of mixed quermassintegrals, and introduction of Lutwak's L_p-mixed quermassintegrals (see [22, 29]), we are based on the study of the first-order Orlicz variation of the mixed mean dual affine quermassintegrals. In Sect. 3, we prove that the first Orlicz variation of the mixed mean dual affine quermassintegrals can be expressed as: For $\phi \in \mathscr{C}$, $\varepsilon > 0$, $0 \le i, j \le n$, and $K, L \in \mathscr{S}^n$,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \overline{\Phi}_{n-j,i}(K \widehat{+}_\phi \varepsilon \cdot L) = \frac{j-i}{\phi'_+(1)} \overline{\Phi}_{n-j,i}(K)^{i-n} \overline{\Phi}_{\phi,n-j,i}(K,L)^{n-i+1}. \tag{1.7}$$

In this first variational equation (1.7), we find a new geometric quantity. Based on this, we extract the required geometric quantity, denote by $\overline{\Phi}_{\phi,n-j,i}(K,L)$, and call as Orlicz mean dual affine quermassintegrals, defined by

$$\overline{\Phi}_{\phi,n-j,i}(K,L) := \left(\frac{\phi'_+(1)}{(j-i) \cdot \overline{\Phi}_{n-j,i}(K)^{i-n}} \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \overline{\Phi}_{n-j,i}(K \widehat{+}_\phi \varepsilon \cdot L) \right)^{1/(n-i+1)}, \tag{1.8}$$

where $\phi \in \mathscr{C}$, $0 \le i < j \le n$, and $K, L \in \mathscr{S}^n$. We also prove the new affine geometric quantity $\overline{\Phi}_{\phi,n-j,i}(K,L)$ has an integral representation:

$$\overline{\Phi}_{\phi,n-j,i}(K,L) = \omega_n \left[\int_{A_{n,j}} \frac{\widetilde{W}_{\phi,i}^{(j)}(K \cap \xi, L \cap \xi)}{w_i^{(j)}(K \cap \xi)} \left(\frac{w_i^{(j)}(K \cap \xi)}{\omega_j} \right)^{n-i+1} d\nu_j(\xi) \right]^{1/(n-i+1)}, \tag{1.9}$$

where $\widetilde{W}^{(j)}_{\phi,i}(K\cap\xi, L\cap\xi)$ denotes the Orlicz dual quermassintegral (see Sect. 2) of j-dimensional star bodies $K\cap\xi$ and $L\cap\xi$ in j-dimensional subspace ξ.

Because the Orlicz mixed mean dual affine quermassintegrals is an extension of the mean dual affine quermassintegrals, a very natural question is raised: is there a Minkowski type isoperimetric inequality for the Orlicz mean dual affine quermassintegrals? In Sect. 4, we give a positive answer to this question and establish the Orlicz dual Minkowski inequality for the affine geometric quantity. We prove the Orlicz Minkowski inequality for Orlicz mixed mean dual affine quermassintegrals.

Theorem 1 *If $\phi\in\mathscr{C}$, $0\le i<j\le n$, and $K, L\in\mathscr{S}^n$, then*

$$\left(\frac{\overline{\Phi}_{\phi,n-j,i}(K,L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{n-i+1}\ge\phi\left(\left(\frac{\overline{\Phi}_{n-j,i}(L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{1/(j-i)}\right). \tag{1.10}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Putting $j=n$ in (1.10), (1.10) becomes the following Orlicz dual Minkowski inequality for dual mixed quermassintegrals established in [41]. If $K, L\in\mathscr{S}^n$, $0\le i<n$, and $\phi\in\mathscr{C}$, then

$$\widetilde{W}_{\phi,i}(K,L)\ge\widetilde{W}_i(K)\phi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\right)^{1/(n-i)}\right).$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

In Sect. 5, on the basis of the dual Minkowski inequality for the Orlicz mean dual affine quermassintegrals, we establish a dual Brunn–Minkowski inequality for the Orlicz mixed mean dual affine quermassintegrals.

Theorem 2 *If $K, L\in\mathscr{S}^n$, $0\le i<j\le n$, and $\phi\in\mathscr{C}$, then for nay $\varepsilon>0$*

$$\begin{aligned}\phi(1)\ge\phi&\left(\left(\frac{\overline{\Phi}_{n-j,i}(K)}{\overline{\Phi}_{n-j,i}(K\widehat{+}_\phi\varepsilon\cdot L)}\right)^{1/(j-i)}\right)\\&+\varepsilon\cdot\phi\left(\left(\frac{\overline{\Phi}_{n-j,i}(L)}{\overline{\Phi}_{n-j,i}(K\widehat{+}_\phi\varepsilon\cdot L)}\right)^{1/(j-i)}\right).\end{aligned} \tag{1.11}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Putting $j=n$ and $\varepsilon=1$ in (1.11), (1.11) becomes the following Orlicz dual Brunn–Minkowski inequality established in [41]. If $K, L\in\mathscr{S}^n$, $0\le i<n$, and $\phi\in\mathscr{C}$, then

$$\phi(1) \geq \phi\left(\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K \widehat{+}_\phi L)}\right)^{1/(n-i)}\right) + \phi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K \widehat{+}_\phi L)}\right)^{1/(n-i)}\right).$$

If ϕ is strictly convex, equality holds with if and only if K and L are dilates. Moreover, putting $\varepsilon = 1$, $\phi(t) = t^{-p}$, $1 \leq p < \infty$ in (1.11), (1.11) becomes the L_p-dual Brunn–Minkowski inequality for the mean dual affine quermassintegrals. If $K, L \in \mathscr{S}^n$, $\varepsilon > 0$, $0 \leq i < j \leq n$, $1 \leq p$, and $\phi \in \mathscr{C}$, then

$$\overline{\Phi}_{n-j,i}(K \widehat{+}_p \varepsilon \cdot L)^{-p/(j-i)} \geq \overline{\Phi}_{n-j,i}(K)^{-p/(j-i)} + \overline{\Phi}_{n-j,i}(L)^{-p/(j-i)}, \qquad (1.12)$$

with equality if and only if K and L are dilates. When $j = n$ and $i = 0$, (1.12) becomes Lutwak's dual Brunn–Minkowski inequality (2.12) for the volumes.

2 Preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. A body in $\mathbb{R}^n$ is a compact set equal to the closure of its interior. For a compact set $K \subset \mathbb{R}^n$, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the volume of K. Let $\mathscr{K}^n$ denote the class of nonempty compact convex subsets containing the origin in their interiors in $\mathbb{R}^n$. Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin and contains the origin, its radial function is $\rho(K, \cdot) : S^{n-1} \to [0, \infty)$, defined by $\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}$. Note that the class (star sets) is closed under unions, intersection, and intersection with subspace. The radial function is homogeneous of degree -1, that is, $\rho(K, ru) = r^{-1}\rho(K, u)$, for all $x \in \mathbb{R}^n$ and $r > 0$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows, if $K, L \in \mathscr{S}^n$, then (see e.g. [31])

$$\tilde{\delta}(K, L) = |\rho(K, u) - \rho(L, u)|_\infty.$$

From the definition of the radial function, it follows immediately that for $g \in GL(n)$ the radial function of the image $gK = \{gy : y \in K\}$ of K is given by

$$\rho(gK, x) = \rho(K, g^{-1}x), \qquad (2.1)$$

for all $x \in \mathbb{R}^n$.

Dual Mixed Volumes and L_p-Dual Mixed Volumes

If $K_1, \ldots, K_n \in \mathscr{S}^n$, the dual mixed volume $\widetilde{V}(K_1, \ldots, K_n)$ is defined by (see [24])

$$\widetilde{V}(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) dS(u).$$

If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the dual mixed volume $\widetilde{V}(K_1, \dots, K_n)$ is written as $\widetilde{V}_i(K, L)$. If $L = B$, the dual mixed $\widetilde{V}_i(K, L)$ is written as $\widetilde{W}_i(K)$, and we call it dual quermassintegral. Obviously, for $K \in \mathscr{S}^n$, we have

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} dS(u), \tag{2.2}$$

and (see [24])

$$\widetilde{V}_1(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K \widetilde{+} \varepsilon \cdot L) - V(K)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-1} \rho(L, u) dS(u). \tag{2.3}$$

The fundamental inequality for dual mixed volumes stated that: If $K, L \in \mathscr{S}^n$, then

$$\widetilde{V}_1(K, L)^n \le V(K)^{n-1} V(L), \tag{2.4}$$

with equality if and only if K and L are dilates. The Brunn–Minkowski inequality for the radial addition is the following: If $K, L \in \mathscr{S}^n$, then

$$V(K \widetilde{+} L)^{1/n} \le V(K)^{1/n} + V(L)^{1/n}, \tag{2.5}$$

with equality if and only if K and L are dilates.

The following result follows immediately from the definition of L_p-radial addition, with $p < 0$ (or $p > 0$).

$$\frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K \widetilde{+}_p \varepsilon \cdot L) - V(L)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} \rho(K.u)^{n-i-p} \rho(L.u)^p dS(u).$$

Let $K, L \in \mathscr{S}^n$ and $p < 0$, define L_p-dual mixed volume of star K and L, $\widetilde{V}_p(K, L)$, by

$$\widetilde{V}_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K.u)^{n-p} \rho(L.u)^p dS(u). \tag{2.6}$$

This integral representation (2.6), together with the Hölder inequality, yields the p-dual Minkowski inequality (see [39]): If $K, L \in \mathscr{S}^n$ and $p < 0$, then

$$\widetilde{V}_p(K, L)^n \ge V(K)^{n-p} V(L)^p, \tag{2.7}$$

with equality if and only if K and L are dilates. The definition of L_p-radial addition, together with (2.7), yields Gardner's Brunn–Minkowski inequality for p-radial addition (see [10]). If $K, L \in \mathscr{S}^n$ and $p < 0$, then

$$V(K \widetilde{+}_p L)^{p/n} \geq V(K)^{p/n} + V(L)^{p/n}, \tag{2.8}$$

with equality if and only if K and L are dilates.

L_p-Harmonic Mixed Volumes

The following result follows immediately from (1.1) with $p \geq 1$:

$$-\frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K \widehat{+}_p \varepsilon \cdot L) - V(L)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} \rho(K.u)^{n+p} \rho(L.u)^{-p} dS(u). \tag{2.9}$$

Let $K, L \in \mathscr{S}^n$ and $p \geq 1$. The L_p-harmonic mixed volume of star K and L, denoted by $\widetilde{V}_{-p}(K, L)$, is defined by (see [29])

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K.u)^{n+p} \rho(L.u)^{-p} dS(u). \tag{2.10}$$

This integral representation (2.10), together with the Hölder inequality, yields Lutwak's L_p-dual Minkowski inequality, as follows: If $K, L \in \mathscr{S}^n$ and $p \geq 1$, then

$$\widetilde{V}_{-p}(K, L)^n \geq V(K)^{n+p} V(L)^{-p}, \tag{2.11}$$

with equality if and only if K and L are dilates. This integral representation (2.10), together with the definition of p-harmonic addition, yields Lutwak's L_p-Brunn–Minkowski inequality for harmonic p-addition (see [29]). If $K, L \in \mathscr{S}^n$ and $p \geq 1$, then

$$V(K \widehat{+}_p L)^{-p/n} \geq V(K)^{-p/n} + V(L)^{-p/n}, \tag{2.12}$$

with equality if and only if K and L are dilates.

Orlicz Harmonic Linear Combination

Definition 1 Let $m \geq 2, \phi \in \mathscr{C}$, $K_j \in \mathscr{S}^n$, and $j = 1, \ldots, m$, define the Orlicz harmonic addition of $K_1, \ldots, K_m$, denoted by $K_1 \widehat{+}_\phi \cdots \widehat{+}_\phi K_m$, defined by

$$\rho(K_1\widehat{+}_\phi\cdots\widehat{+}_\phi K_m,u)=\sup\left\{\lambda>0:\sum_{j=1}^m\phi\left(\frac{\rho(K_j,x)}{\lambda}\right)\le\phi(1)\right\},\tag{2.13}$$

for all $x\in\mathbb{R}^n$.

Equivalently, the Orlicz harmonic addition $K_1\widehat{+}_\phi\cdots\widehat{+}_\phi K_m$ can be defined implicitly by

$$\phi\left(\frac{\rho(K_1,x)}{\rho(K_1\widehat{+}_\phi\cdots\widehat{+}_\phi K_m,x)}\right)+\cdots+\phi\left(\frac{\rho(K_m,x)}{\rho(K_1\widehat{+}_\phi\cdots\widehat{+}_\phi K_m,x)}\right)=\phi(1),\tag{2.14}$$

for all $x\in\mathbb{R}^n$.

The Orlicz harmonic linear combination on the case $m=2$ is defined.

Definition 2 Orlicz harmonic linear combination $\widehat{+}_\phi(K,L,\alpha,\beta)$ for $K,L\in\mathscr{S}^n$, $\phi\in\mathscr{C}$, and $\alpha,\beta\ge 0$ (not both zero), defined by

$$\alpha\cdot\phi\left(\frac{\rho(K,x)}{\rho(\widehat{+}_\phi(K,L,\alpha,\beta),x)}\right)+\beta\cdot\phi\left(\frac{\rho(L,x)}{\rho(\widehat{+}_\phi(K,L,\alpha,\beta),x)}\right)=\phi(1),\tag{2.15}$$

for all $x\in\mathbb{R}^n$.

When $\phi(t)=t^{-p}$ and $p\ge 1$, then Orlicz harmonic linear combination $\widehat{+}_\phi(K,L,\alpha,\beta)$ changes to the L_p-harmonic linear combination $\alpha\cdot K\widehat{+}_p\beta\cdot L$ (see [23]). Moreover, we shall write $K\widehat{+}_\phi\varepsilon\cdot L$ instead of $\widehat{+}_\phi(K,L,1,\varepsilon)$, for $\varepsilon\ge 0$, and assume throughout that this is defined by (2.15), where $\alpha=1$, $\beta=\varepsilon$, and $\phi\in\mathscr{C}$, and write $\widehat{+}_\phi(K,L,1,1)$ as $K\widehat{+}_\phi L$.

Orlicz Dual Quermassintegral

Lemma 1 *If $K,L\in\mathscr{S}^n$, $0\le i<n$, and $\phi\in\mathscr{C}$, then*

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}\rho(K\widehat{+}_\phi\varepsilon\cdot L,u)^{n-i}=\frac{n-i}{\phi'_+(1)}\cdot\phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^{n-i},\tag{2.16}$$

uniformly for $u\in S^{n-1}$.

Proof Form (2.15) and notice that ϕ and ϕ^{-1} are continuous functions, we obtain for $0\le i<n$

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}\rho(K\widehat{+}_\phi\varepsilon\cdot L,u)^{n-i}$$
$$=\lim_{\varepsilon\to 0^+}(n-i)\rho(K,u)^{n-i-1}\left(\rho(K,u)\phi\left(\frac{\rho(L,u)}{\rho(K\widehat{+}_\phi\varepsilon\cdot L,u)}\right)\right)\cdot\frac{d}{d\varepsilon}\bigg|_{y=1^+}\phi^{-1}(y)$$

$$= \frac{n-i}{\phi'_+(1)} \cdot \phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^{n-i},$$

where

$$y = \phi^{-1}\left(1 - \varepsilon\phi\left(\frac{\rho(L,u)}{\rho(K\widehat{+}_\phi\varepsilon \cdot L,u)}\right)\right),$$

and note that $y \to 1^+$ as $i \to 0^+$.

This lemma plays a central role in our deriving new concept of the Orlicz dual quermassintegrals.

Lemma 2 *If $\phi \in \mathscr{C}$, $0 \le i < n$, and $K, L \in \mathscr{S}^n$, then*

$$\frac{(\phi'_+)(1)}{n-i} \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \widetilde{W}_i(K\widehat{+}_\phi\varepsilon \cdot L) = \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^{n-i}dS(u). \tag{2.17}$$

Proof This follows immediately from (2.2) and Lemma 1.

Denoting by $\widetilde{W}_{\phi,i}(K,L)$, for any $\phi \in \mathscr{C}$ and $0 \le i < n$, the integral on the right-hand side of (2.17), we see that either side of the equation (2.17) is equal to $\widetilde{W}_{\phi,i}(K,L)$, and hence this new Orlicz dual mixed volume $\widetilde{W}_{\phi,i}(K,L)$ has been born.

Definition 3 For $\phi \in \mathscr{C}$ and $0 \le i < n$, Orlicz dual quermassintegral of star bodies K and L, $\widetilde{W}_{\phi,i}(K,L)$, is defined by

$$\widetilde{W}_{\phi,i}(K,L) := \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^{n-i}dS(u). \tag{2.18}$$

3 Orlicz Dual Mean Mixed Affine Quermassintegrals

In order to define Orlicz dual mixed mean affine Quermassintegrals, we need the following lemmas.

Lemma 3 *If $K, L \in \mathscr{S}^n$, $0 \le i < n$, and $\phi \in \mathscr{C}$, then for $\varepsilon > 0$*

$$\widetilde{W}_i(K\widehat{+}_\phi\varepsilon \cdot L) = \widetilde{W}_{\phi,i}(K\widehat{+}_\phi\varepsilon \cdot L, K) + \varepsilon \cdot \widetilde{W}_{\phi,i}(K\widehat{+}_\phi\varepsilon \cdot L, L). \tag{3.1}$$

Proof This follows immediately from (1.3) and (2.15).

Lemma 4 ([40]) *If $K, L \in \mathscr{S}^n$ and $\phi \in \mathscr{C}$, then for $\varepsilon > 0$*

$$K\widehat{+}_\phi\varepsilon \cdot L \to K, \tag{3.2}$$

in the radial Hausdorff metric as $\varepsilon \to 0^+$.

Lemma 5 ([43]) *If* $K, L \in \mathscr{S}^n$, $\varepsilon > 0$, *and* $\phi \in \mathscr{C}$, *then*

$$(K \widehat{+}_\phi \varepsilon \cdot L) \cap \xi = K \cap \xi \widehat{+}_\phi \varepsilon \cdot L \cap \xi. \tag{3.3}$$

In order to define the Orlicz mean dual affine querlmassintegrals, we need also calculate the first Orlicz variation of the mean dual affine querlmassintegrals.

Lemma 6 *If* $\phi \in \mathscr{C}$, $0 \le i \le n$, $0 < j \le n$, *and* $K, L \in \mathscr{S}^n$, *then for any* $\varepsilon > 0$

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \overline{\Phi}_{n-j,i}(K \widehat{+}_\phi \varepsilon \cdot L) = \frac{j-i}{\phi'_+(1)} \overline{\Phi}_{n-j,i}(K)^{i-n} \overline{\Phi}_{\phi,n-j,i}(K, L)^{n-i+1}. \tag{3.4}$$

Proof On the one hand, from (1.3), we have

$$\begin{aligned} &\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \int_{A_{n,j}} w_i^{(j)}((K \widehat{+}_\phi \varepsilon \cdot L) \cap \xi)^{n-i+1} d\nu_j(\xi) \\ &\quad = \frac{(n-i+1)(j-i)}{\phi'_+(1)} \int_{A_{n,j}} w_i^{(j)}(K \cap \xi)^{n-i} \widetilde{W}_{\phi,i}^{(j)}(K \cap \xi, L \cap \xi) d\nu_j(\xi), \end{aligned} \tag{3.5}$$

and on the other hand, from (1.6), (1.9), and (3.5), we obtain

$$\begin{aligned} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \overline{\Phi}_{n-j,i}(K \widehat{+}_\phi \varepsilon \cdot L) &= \frac{\omega_n}{(n-i+1)\omega_j} \left(\int_{A_{n,j}} w_i^{(j)}(K \cap \xi)^{n-i+1} d\nu_j(\xi) \right)^{(i-n)/(n-i+1)} \\ &\quad \times \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \int_{A_{n,j}} w_i^{(j)}((K \widehat{+}_\phi \varepsilon \cdot L) \cap \xi)^{n-i+1} d\nu_j(\xi) \\ &= \frac{j-i}{\phi'_+(1)} \frac{\omega_n}{\omega_j} \left(\int_{A_{n,j}} w_i^{(j)}(K \cap \xi)^{n-i+1} d\nu_j(\xi) \right)^{(i-n)/(n-i+1)} \\ &\quad \times \int_{A_{n,j}} \frac{\widetilde{W}_{\phi,i}^{(j)}(K \cap \xi, L \cap \xi)}{w_i^{(j)}(K \cap \xi)} w_i^{(j)}(K \cap \xi)^{n-i+1} d\nu_j(\xi) \\ &= \frac{j-i}{\phi'_+(1)} \overline{\Phi}_{n-j,i}(K)^{i-n} \overline{\Phi}_{\phi,n-j,i}(K, L)^{n-i+1}. \end{aligned}$$

Definition 4 If $\phi \in \mathscr{C}$, $0 \le i < j \le n$, and $K, L \in \mathscr{S}^n$, then Orlicz dual mixed mean affine querlmassintegral of K and L, denoted by $\overline{\Phi}_{\phi,n-j,i}(K, L)$, is defined by

$$\overline{\Phi}_{\phi,n-j,i}(K,L)$$
$$:=\omega_n\left[\int_{A_{n,j}}\frac{\widetilde{W}^{(j)}_{\phi,i}(K\cap\xi,L\cap\xi)}{w_i^{(j)}(K\cap\xi)}\left(\frac{w_i^{(j)}(K\cap\xi)}{\omega_j}\right)^{n-i+1}d\nu_j(\xi)\right]^{1/(n-i+1)}. \tag{3.6}$$

Specifically, we agreed

$$\overline{\Phi}_{\phi,0,i}(K,L)=\left(\frac{\widetilde{W}_{\phi,i}(K,L)}{\widetilde{W}_i(K)}\right)^{1/(n-i+1)}\widetilde{W}_i(K).$$

In order to prove our main results, we still need the following lemmas.

Lemma 7 *If $K,L\in\mathscr{S}^n$, $0\le i<n$, $0<j\le n$, and $\phi\in\mathscr{C}$, then*

$$\overline{\Phi}_{\phi,n-j,i}(K,K)=\phi(1)^{1/(n-i+1)}\overline{\Phi}_{n-j,i}(K). \tag{3.7}$$

Proof The definition of the Orlicz dual mixed mean affine quermassintegrals, together with (1.3) and (1.6), gives (3.7).

If $\phi(t)=t^{-p}$ and $p\ge 1$, then $\overline{\Phi}_{\phi,n-j,i}(K,L)$ denotes $\overline{\Phi}_{-p,n-j,i}(K,L)$ and calls the ith L_p-mean dual mixed affine quermassintegral of K and L, and

$$\overline{\Phi}_{-p,n-j,i}(K,L)$$
$$=\omega_n\left[\int_{A_{n,j}}\frac{\widetilde{W}^{(j)}_{-p,i}(K\cap\xi,L\cap\xi)}{w_i^{(j)}(K\cap\xi)}\left(\frac{w_i^{(j)}(K\cap\xi)}{\omega_j}\right)^{n-i+1}d\nu_j(\xi)\right]^{1/(n-i+1)},$$

where $\widetilde{W}^{(j)}_{-p,i}(K\cap\xi,L\cap\xi)$ denotes the ith L_p-dual mixed volume of j-dimensional star bodies $K\cap\xi$ and $L\cap\xi$ in j-dimensional subspace ξ.

Lemma 8 ([40]) *If $K,L\in\mathscr{S}^n$, $\phi\in\mathscr{C}$ and any $g\in SL(n)$, then for $\varepsilon>0$*

$$g(K\widehat{+}_\phi\varepsilon\cdot L)=(gK)\widehat{+}_\phi\varepsilon\cdot(gL). \tag{3.8}$$

In the following, we will prove that Orlicz dual mixed mean affine querlmass-integral $\overline{\Phi}_{\phi,n-j,i}(K,L)$ is invariant under simultaneous unimodular centro-affine transformation.

Lemma 9 *If $K,L\in\mathscr{S}^n$, $0\le i<j\le n$, $\phi\in\mathscr{C}$, and any $g\in SL(n)$, then*

$$\overline{\Phi}_{\phi,n-j,i}(gK,gL)=\overline{\Phi}_{\phi,n-j,i}(K,L). \tag{3.9}$$

Proof This follows immediately from Lemmas 6 and 8.

We also need the following lemma to prove our main results.

Lemma 10 (Jensen's inequality) *Let μ be a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. If $\phi : I \to \mathbb{R}$ is a convex function, then*

$$\int_X \phi(g(x))d\mu(x) \geq \phi\left(\int_X g(x)d\mu(x)\right). \tag{3.10}$$

If ϕ is strictly convex, equality holds if and only if $g(x)$ is constant for μ-almost all $x \in X$ (see [19, p.165]).

4 Minkowski Inequality for Orlicz Dual Mean Mixed Affine Quermassintegrals

Theorem 3 *If $K, L \in \mathscr{S}^n$, $\phi \in \mathscr{C}$, and $0 \leq i < j \leq n$, then*

$$\left(\frac{\overline{\Phi}_{\phi,n-j,i}(K,L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{n-i+1} \geq \phi\left(\left(\frac{\overline{\Phi}_{n-j,i}(L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{1/(j-i)}\right). \tag{4.1}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof When $j = n$, (4.1) becomes the well-known Orlicz dual Minkowski inequality for dual quermassintegrals; hence, we assume $0 \leq i < j < n$. Since

$$\int_{A_{n,j}} d\nu(\xi) = 1,$$

so the above equation defines a Borel probability measure ν on $A_{n,j}$, namely:

$$d\nu(\xi) = \frac{w_i^{(j)}(K \cap \xi)^{n-i+1}}{\int_{A_{n,j}} w_i^{(j)}(K \cap \xi)^{n-i+1} d\nu_j(\xi)} d\nu_j(\xi). \tag{4.2}$$

From (1.6), (3.7), and (4.2), and using Orlicz dual Minkowski inequality, Jensen inequality, and Hölder inequality, we obtain

$$\begin{aligned}\left(\frac{\overline{\Phi}_{\phi,n-j,i}(K,L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{n-i+1} &= \int_{A_{n,j}} \frac{\widetilde{W}_{\phi,i}^{(j)}(K \cap \xi, L \cap \xi)}{w_i^{(j)}(K \cap \xi)} d\nu \\ &\geq \int_{A_{n,j}} \phi\left(\left(\frac{w_i^{(j)}(L \cap \xi)}{w_i^{(j)}(K \cap \xi)}\right)^{1/(j-i)}\right) d\nu\end{aligned}$$

$$\geq \phi\left(\int_{A_{n,j}}\left(\frac{w_i^{(j)}(L\cap\xi)}{w_i^{(j)}(K\cap\xi)}\right)^{1/(j-i)}d\nu\right)$$

$$\geq \phi\left(\left(\frac{\overline{\Phi}_{n-j,i}(L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{1/(j-i)}\right).$$

Next, we discuss the equal condition of (4.1). If ϕ is strictly convex, suppose that K and L are dilates, i.e., there exists $\lambda > 0$ such that $L = \lambda K$. Hence,

$$\begin{aligned}\left(\frac{\overline{\Phi}_{\phi,n-j,i}(K,L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{n-i+1} &= \left(\frac{\phi(\lambda)^{1/(n-i+1)}\overline{\Phi}_{n-j,i}(K)}{\overline{\Phi}_{n-j,i}(K)}\right)^{n-i+1}\\ &= \phi(\lambda)\\ &= \phi\left(\left(\frac{\overline{\Phi}_{n-j,i}(L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{1/(j-i)}\right).\end{aligned}$$

This implies the equality in (4.1) holds.

On the other hand, suppose the equality holds in (4.1), then these three inequalities in the above proof must satisfy the equal sign. Since the first inequality in the above proof is the Orlicz dual Minkowski inequality. Form the equality condition of Orlicz dual Minkowski inequality, if the equality holds, then $K\cap\xi$ and $L\cap\xi$ must be dilates. The second inequality in the above proof is Jensen inequality. From the equality condition of Jensen inequality, if ϕ is strictly convex and the equality holds, then $\frac{w_i^{(j)}(L\cap\xi)}{w_i^{(j)}(K\cap\xi)}$ must be a constant, and this yields that $K\cap\xi$ and $L\cap\xi$ must be dilates. In this proof, the third inequality is obtained by applying the Hölder inequality. From the equality condition of Hölder inequality, this yields that equality holds $w_i^{(j)}(K\cap\xi)$ and $w_i^{(j)}(L\cap\xi)$ must be proportional, namely, $K\cap\xi$ and $L\cap\xi$ be dilates. Combinations of these equal conditions, it follows that equality in (4.1) holds, if ϕ is strictly convex, and equality holds if and only if K and L are dilates.

Corollary 1 *If $K, L \in \mathscr{S}^n$, $p \geq 1$, $0 \leq i < j \leq n$, then*

$$\left(\frac{\overline{\Phi}_{-p,n-j,i}(K,L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{n-i+1} \geq \left(\frac{\overline{\Phi}_{n-j,i}(L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{-p/(j-i)}, \tag{4.3}$$

with equality if and only if K and L are dilates.

Proof This follows immediately from (4.1) and with $\phi(t) = t^{-p}$ and $1 < p < \infty$.

Corollary 2 *If $K, L \in \mathscr{S}^n$, $0 \leq i < n$, and $\phi \in \mathscr{C}$, then*

$$\widetilde{W}_{\phi,i}(K,L) \geq \widetilde{W}_i(K)\phi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\right)^{1/(n-i)}\right). \tag{4.4}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof This follows immediately from (4.1) and with $j = n$.

The following uniqueness is a direct consequence of the Minkowski inequality for the Orlicz mean dual affine quermassintegrals.

Theorem 4 *If $\phi \in \mathscr{C}$ and is strictly convex, $0 \leq i < n$, $0 < j \leq n$, and $\mathscr{M} \subset \mathscr{S}^n$ such that $K, L \in \mathscr{M}$. If*

$$\overline{\Phi}_{\phi,n-j,i}(M,K) = \overline{\Phi}_{\phi,n-j,i}(M,L), \quad for\ all\ M \in \mathscr{M} \tag{4.5}$$

or

$$\frac{\overline{\Phi}_{\phi,n-j,i}(K,M)}{\overline{\Phi}_{n-j,i}(K)} = \frac{\overline{\Phi}_{\phi,n-j,i}(L,M)}{\overline{\Phi}_{n-j,i}(L)}, \quad for\ all\ M \in \mathscr{M}, \tag{4.6}$$

then $K = L$.

Proof Suppose (4.5) holds. Taking K for M, then from Lemma 7 and (4.1), we obtain

$$\phi(1)\overline{\Phi}_{n-j,i}(K)^{n-i+1} \geq \overline{\Phi}_{n-j,i}(K)^{n-i+1}\phi\left(\left(\frac{\overline{\Phi}_{n-j,i}(L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{1/j}\right),$$

with equality if and only if K and L are dilates. Hence,

$$\overline{\Phi}_{n-j,i}(K) \leq \overline{\Phi}_{n-j,i}(L),$$

with equality if and only if K and L are dilates. On the other hand, if taking L for M, we similarly get $\overline{\Phi}_{n-j,i}(K) \geq \overline{\Phi}_{n-j,i}(L)$, with equality if and only if K and L are dilates. Hence, $\overline{\Phi}_{n-j,i}(K) = \overline{\Phi}_{n-j,i}(L)$, and K and L are dilates, and it follows that K and L must be equal.

Suppose (4.6) holds. Taking L for M, then from Lemma 7 and (4.1), we obtain

$$\phi(1) = \frac{\overline{\Phi}_{\phi,n-j,i}(K,L)^{n-i+1}}{\overline{\Phi}_{n-j,i}(K)^{n-i+1}} \geq \phi\left(\left(\frac{\overline{\Phi}_{n-j,i}(L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{1/j}\right),$$

with equality if and only if K and L are dilates. Hence,

$$\overline{\Phi}_{n-j,i}(K) \leq \overline{\Phi}_{n-j,i}(L),$$

with equality if and only if K and L are dilates. On the other hand, if taking L for M, we similarly get $\overline{\Phi}_{n-j,i}(K) \geq \overline{\Phi}_{n-j,i}(L)$, with equality if and only if K and L are dilates. Hence, $\overline{\Phi}_{n-j,i}(K) = \overline{\Phi}_{n-j,i}(L)$, and K and L are dilates, and it follows that K and L must be equal.

Corollary 3 *If $\phi \in \mathscr{C}$ and is strictly convex, $0 \leq i < n$, $0 < j \leq n$, and $\mathscr{M} \subset \mathscr{S}^n$ such that $K, L \in \mathscr{M}$. If*

$$\widetilde{W}_{\phi,i}(M, K) = \widetilde{W}_{\phi,i}(M, L), \quad \text{for all } M \in \mathscr{M}$$

or

$$\frac{\widetilde{W}_{\phi,i}(K, M)}{\widetilde{W}_i(K)} = \frac{\widetilde{W}_{\phi,i}(L, M)}{\widetilde{W}_i(L)}, \quad \text{for all } M \in \mathscr{M},$$

then $K = L$.

Proof This follows immediately from Theorem 4 and with $j = n$.

5 Brunn–Minkowski Inequality for the Orlicz Dual Affine Quermassintegrals

Definition 5 (ith mean dual affine querlmassintegrals) The ith mean dual affine quermassintegral of star body K, $\overline{\Phi}_{n-j,i}(K)$, is defined by

$$\overline{\Phi}_{n-j,i}(K) := \omega_n \left[\int_{A_{n,j}} \left(\frac{w_i^{(j)}(K \cap \xi)}{\omega_j} \right)^{n-i+1} d\nu_j(\xi) \right]^{1/(n-i+1)}, \tag{5.1}$$

for $0 \leq i < j \leq n$ and $K \in \mathscr{S}^n$ and by letting $\overline{\Phi}_{0,i}(K) := \widetilde{W}_i(K)$. When $i = 0$, $w_i^{(j)}(K \cap \xi)$ denotes the well-known j-dimensional volume $\mathrm{vol}_j(K \cap \xi)$. Obviously, when $i = 0$, $\overline{\Phi}_{n-j,i}(K) = \overline{\Phi}_{n-j,0}(K) = \overline{\Phi}_{n-j}(K)$, when $i = 0$ and $j = n$, $\overline{\Phi}_{n-j,i}(K) = \overline{\Phi}_{0,0}(K) = V(K)$, and when $j = 0$ and $i = 0$, $\overline{\Phi}_{n,0}(K) = \omega_n$.

Lemma 11 *If $K, L \in \mathscr{S}^n$, $0 \leq i < j \leq n$, and $\phi \in \mathscr{C}$, then for any $\varepsilon > 0$*

$$\phi(1) = \left(\frac{\overline{\Phi}_{\phi,n-j,i}(K \widehat{+}_\phi \varepsilon \cdot L, K)}{\overline{\Phi}_{n-j,i}(K \widehat{+}_\phi \varepsilon \cdot L)} \right)^{n-i+1} + \varepsilon \cdot \left(\frac{\overline{\Phi}_{\phi,n-j,i}(K \widehat{+}_\phi \varepsilon \cdot L, L)}{\overline{\Phi}_{n-j,i}(K \widehat{+}_\phi \varepsilon \cdot L)} \right)^{n-i+1}. \tag{5.2}$$

Proof This follows immediately from Lemmas 3 and 5.

Lemma 12 ([43]) *Let $K, L \in \mathscr{S}^n$, $\varepsilon > 0$, and $\phi \in \mathscr{C}$.*

(1) *If K and L are dilates, then K and $K \widehat{+}_\phi \lambda \cdot L$ are dilates.*
(2) *If K and $K \widehat{+}_\phi \varepsilon \cdot L$ are dilates, then K and L are dilates.*

Theorem 5 *If $K, L \in \mathscr{S}^n$, $0 \leq i < j \leq n$, and $\phi \in \mathscr{C}$, then for $\varepsilon > 0$*

$$\begin{aligned}\phi(1) \geq \phi&\left(\left(\frac{\overline{\Phi}_{n-j,i}(K)}{\overline{\Phi}_{n-j,i}(K \widehat{+}_\phi \varepsilon \cdot L)}\right)^{1/(j-i)}\right)\\&+\varepsilon \cdot \phi\left(\left(\frac{\overline{\Phi}_{n-j,i}(L)}{\overline{\Phi}_{n-j,i}(K \widehat{+}_\phi \varepsilon \cdot L)}\right)^{1/(j-i)}\right).\end{aligned} \tag{5.3}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof This follows immediately from (4.1) and Lemma 11.

Corollary 4 *If $K, L \in \mathscr{S}^n$, $p \geq 1$, $0 \leq i < j \leq n$, then for $\varepsilon > 0$*

$$\overline{\Phi}_{n-j,i}(K \widehat{+}_p \varepsilon \cdot L)^{-p/(j-i)} \geq \overline{\Phi}_{n-j,i}(K)^{-p/(j-i)} + \varepsilon \cdot \overline{\Phi}_{n-j,i}(L)^{-p/(j-i)}, \tag{5.4}$$

with equality if and only if K and L are dilates.

Proof This follows immediately from (5.3) and with $\phi(t) = t^{-p}$, $1 < p < \infty$.

Putting $j = n$ and $\varepsilon = 1$ in (5.4), (5.4) becomes Lutwak's L_p-dual Brunn–Minkowski inequality (2.12).

Corollary 5 *If $K, L \in \mathscr{S}^n$, $0 \leq i < n$, and $\phi \in \mathscr{C}$, then*

$$1 \geq \phi\left(\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K \widehat{+}_\phi L)}\right)^{1/(n-i)}\right) + \phi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K \widehat{+}_\phi L)}\right)^{1/(n-i)}\right). \tag{5.5}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof This follows immediately from (5.3) and with $\varepsilon = 1$ and $j = n$.

Corollary 6 *If $K, L \in \mathscr{S}^n$, $\phi \in \mathscr{C}$, and $0 \leq i < j \leq n$, then*

$$\left(\frac{\overline{\Phi}_{\phi,n-j,i}(K, L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{n-i+1} \geq \phi\left(\left(\frac{\overline{\Phi}_{n-j,i}(L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{1/(j-i)}\right). \tag{5.6}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Proof Let

$$K_\varepsilon = K \widehat{+}_\phi \varepsilon \cdot L.$$

From Lemmas 4 and 6 and using the Orlicz–Brunn–Minkowski inequality (5.3), we obtain

$$\begin{aligned}
&\frac{j-i}{\phi'_-(1)}\overline{\Phi}_{n-j,i}(K)^{i-n}\overline{\Phi}_{\phi,n-j,i}(K,L)^{n-i+1} = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+}\overline{\Phi}_{n-j,i}(K_\varepsilon)\\
&= \lim_{\varepsilon\to 0^+}\frac{1-t}{\phi(1)-\phi\left(t^{1/(j-i)}\right)}\cdot\lim_{\varepsilon\to 0^+}\frac{\phi(1)-\phi\left(\left(\frac{\overline{\Phi}_{n-j,i}(K)}{\overline{\Phi}_{n-j}(K_\varepsilon)}\right)^{1/(j-i)}\right)}{\varepsilon}\cdot\lim_{\varepsilon\to 0^+}\overline{\Phi}_{n-j,i}(K_\varepsilon)\\
&\geq \frac{j-i}{\phi'_-(1)}\cdot\lim_{\varepsilon\to 0^+}\phi\left(\left(\frac{\overline{\Phi}_{n-j,i}(L)}{\overline{\Phi}_{n-j}(K_\varepsilon)}\right)^{1/(j-i)}\right)\cdot\lim_{\varepsilon\to 0^+}\overline{\Phi}_{n-j.i}(K_\varepsilon)\\
&= \frac{j-i}{\phi'_-(1)}\cdot\phi\left(\left(\frac{\overline{\Phi}_{n-j,i}(L)}{\overline{\Phi}_{n-j,i}(K)}\right)^{1/(j-i)}\right)\cdot\overline{\Phi}_{n-j,i}(K).
\end{aligned}\tag{5.7}$$

From (5.7), (5.6) easily follows.

This proof is complete.

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A Reduced-Basis Polynomial-Chaos Approach with a Multi-parametric Truncation Scheme for Problems with Uncertainties

Theodoros T. Zygiridis

Abstract Polynomial-chaos (PC) expansions constitute an invaluable tool for the investigation of uncertainty quantification problems, yet minimizing the consequences of the so-called curse of dimensionality requires methodologies that ensure reliable performance with a set of basis functions with reduced cardinality. In this work, we propose the construction of the PC basis set using a multi-parametric truncation scheme that generalizes standard ones and enables the derivation of anisotropic surrogates in a flexible fashion. The specification of the truncation rule's design parameters relies on a preliminary variance analysis, which entails only a fraction of the overall computational cost and enables a sufficient screening of the input variables. Despite its simplicity, the proposed approach is capable of deriving as credible results as the original PC method with fewer basis functions, due to the elimination of unnecessary terms, thus providing a more efficient framework for the study of demanding stochastic problems.

1 Introduction

There exists a multitude of problems, pertinent to engineering as well as other disciplines, where some or all of their aspects are of probabilistic nature. In such cases, the impact of these uncertainties on the quantities of interest (QoIs) needs to be assessed both reliably and efficiently, so that realistic conclusions are drawn and credible predictions can be made. In the context of developing stochastic models, generalized polynomial-chaos (PC) expansions [1] are widely used for the representation of random quantities with finite variance, whose general form depends on the distributions of the input variables, and conditionally display attractive convergence properties. Once the PC expansion of a QoI is available, its statistical moments and distribution can be easily computed. PC-based methods

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usually outperform standard Monte Carlo (MC) approaches [2], especially in problems with low or moderate numbers of random variables, as MC solutions are known for their slow convergence. The latter attribute translates into a necessity for considerable amounts of input samples from the random space, which needs to be avoided, especially in cases of computationally expensive deterministic solvers.

The computational cost involved in calculating the PC expansion of a QoI is directly related to the number of basis functions involved. In case of high-dimensional stochastic problems, typical choices regarding the selection of the basis functions result in large sets, which unavoidably imply considerable computational effort. This attribute is known as the "curse of dimensionality" and is practically related to the degradation of the efficiency of PC expansions, when several random variables need to be considered. Finding cost-effective solutions for high-dimensional uncertainty problems remains a very active research fields, and some of the remedies that have been proposed include integration on sparse grids [3, 4], alternative truncation schemes for constructing the basis index sets [5], optimal design of experiments [6], adaptive methods [7, 8], compressed sensing (CS) for representations with sparse coefficient vectors [9], etc.

In various cases, selecting a priori a reduced set of basis functions has proven to be a simple, and at the same time quite effective, approach for decreasing the computational cost of calculating the PC expansion coefficients. A predetermined basis set does not require complex algorithms for its implementation and does not affect the typical line of work followed in standard PC implementations. For instance, applying the so-called hyperbolic truncation scheme [8] to the set of multi-indices of the basis functions eliminates terms that describe complex interactions and are less likely to affect the QoI significantly. The main weakness of these solutions is their lack of flexibility (i.e., only a single parameter q that defines a q-norm needs to be selected), as well as the fact that some intuition is required, so that terms that may actually be influential are not rejected beforehand.

In this chapter, we propose a simple approach for constructing reliable PC approximations of stochastic QoIs that, compared to standard methodologies, feature a lower number of basis functions. The proposed line of work is realized in two steps. An approximate and cost-effective variance analysis is performed initially, utilizing only univariate polynomials. In this way, partial variances for all input factors are roughly estimated with a limited number of deterministic samples, enabling at the same time the ordering of the input variables, according to their impact on the output. The information gathered in the first stage of the procedure is exploited next, for the construction of the set of basis functions. To allow for additional flexibility, a modification of a standard truncation scheme is proposed, which enables the determination of anisotropic expansions in a more general way than before. The expansion coefficients are obtained using a typical sampling strategy and a least-squares approach; hence, no further or complex modifications are required, compared to ordinary PC techniques. Despite its simplicity, we provide numerical results that verify the improved performance accomplished by the proposed scheme, when compared to other methodologies, proving that a more consistent choice of basis functions is possible, without necessarily resorting to complicated implementations.

2 Existing Polynomial-Chaos Methods

Algorithms relying on PC approximations have been developed for the investigation of uncertainty problems in several areas (e.g., electrical, mechanical, and chemical engineering) of diverse nature, ranging from diffusion problems and control systems to flow simulations, mechanical vibrations, and transistors. The key concept of PC was originally introduced in [10], where second-order Gaussian random processes were represented via Hermite polynomials. This type of polynomial representation was much later combined with the finite element method in an intrusive fashion, resulting in an extension of the corresponding deterministic approach, in the context of modeling uncertainty problems in solid mechanics [11]. The aforementioned concepts can be generalized, so that other types of random variables can be represented via polynomial series, as described in [1], based on hypergeometric polynomials (the so-called Askey scheme). The considered polynomial families have the attractive property that, in several cases, their weighting functions coincide with frequently encountered probability distributions. In the general instance of arbitrary input distributions, the multi-element generalized PC approach [12] is a more suitable choice, which suggests the splitting of the random space and the construction of a family of orthogonal polynomials within each element. When the exact knowledge of the involved probability density functions (pdfs) is not available, the data-driven or arbitrary PC can be applied [13], where the necessary basis set is constructed according to the statistical moments of the input variables.

PC-based methodologies generally fall into two categories: intrusive and non-intrusive. Intrusive approaches [14–17] necessitate the modification of existing deterministic solvers, which can be a complex task in various cases. On the other hand, they perform the calculation of the expansion coefficients in the context of a single run via Galerkin projection, without requiring the selection of a sampling strategy. Furthermore, they are more likely to take advantage of the available modern computing capabilities [18]. Non-intrusive methods [19–21] are commonly easier to implement, as they rely solely on repetitive calls to deterministic solvers, similar to MC techniques. However, unlike the latter, the required amount of random samples is significantly smaller, especially when the number of stochastic inputs is small or moderate. In such cases, the necessary sampling of the random space can be performed in different ways (e.g., random sampling, Latin hypercube approaches, optimal design of experiments [6], etc.) and may influence the accuracy of the QoI estimations.

In practice, only the computation of truncated PC expansions is possible in realistic problems; hence, the set of basis functions that will be used needs to be specified. The number of retained elements depends on factors such as the dimensionality of the problem, the selected polynomial order (higher orders imply better approximations but are more costly to compute), and the chosen truncation scheme. A brief reference to the most common truncation rules is given in section "Definition of the Basis Index Set." Although the standard implementation of PC approaches is based on predetermined basis sets, various efforts have been devoted to the development of adaptive versions [7, 8, 22], where only the necessary bases are retained via an error-checking procedure, while the less influential ones are rejected. Usually, the implementation of adaptive solutions entails higher computational complexity, compared to methods with a predetermined basis set.

More recently, the potential of CS has been exploited for building reliable polynomial surrogates comprising only a few elements [8, 9, 23]. CS enables the recovery of a signal from a limited number of samples, as long as it is sparse (i.e., it involves only a few nonzero components). In many cases, the PC coefficients can be approximated by a sparse formula, as it is not uncommon for their majority to have negligible magnitudes, compared to the most dominant terms. The recovery of the sparse vectors is accomplished via the solution of specific constrained minimization problems, and the involved computational cost remains at reasonable levels, even when dealing with large numbers of random inputs.

3 Proposed Methodology

The technique developed in this work features a preliminary and computationally cheap stage, where information regarding the impact of each input variable on the output is estimated. The second step follows, where a new multi-parametric truncation rule is defined for the construction of the set of basis functions. We describe the latter stage first, so that the motivation necessitating the introduction of the first stage is clarified. Before that, a brief summary of the basic elements of the PC theory is given.

Polynomial-Chaos Expansions

Let us assume that the QoI y depends on d independent random variables ξ_i, $i = 1, \dots, d$ with $\rho_1, \dots, \rho_d$ pdfs, respectively. In case of second-order random variables, the PC expansion of y has the form

$$y(\boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^d} c_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}), \tag{1}$$

where $\boldsymbol{\xi} = [\xi_1, \ldots, \xi_d]^{\mathrm{T}}$ is the vector of input variables, $c_{\boldsymbol{\alpha}}$ are the expansion coefficients, $\Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$ are the polynomial basis functions, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ is a multi-index denoting the polynomial order at each dimension. The basis functions are constructed as products of univariate polynomials,

$$\Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \prod_{i=1}^{d} \psi_{\alpha_i}(\xi_i), \tag{2}$$

and have the orthogonality property. The proper selection of each univariate function ψ_i depends on the type of the corresponding random variable ξ_i [1] (e.g., we will consider variables with uniform distributions in the numerical tests, which are properly represented by Legendre polynomials). Considering the independence of the variables, the joint pdf is computed from

$$\rho(\boldsymbol{\xi}) = \prod_{i=1}^{d} \rho_i(\xi_i). \tag{3}$$

From a practical viewpoint, one has to truncate (1) and retain a finite number of terms:

$$y(\boldsymbol{\xi}) \simeq \tilde{y}(\boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha} \in \mathscr{A}} c_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}), \tag{4}$$

where $\mathscr{A} \subset \mathbb{N}_0^d$ is an index set, constructed according to the selected truncation strategy.

Among the available methodologies for the computation of the expansion coefficients, we will use the common least-squares approach. According to the latter, N_t samples from the random space are required, with N_t being larger than the number of unknowns. For each sample $\boldsymbol{\xi}_i$, $i = 1, \ldots, N_t$, the value of the output y is calculated (in our case, via a deterministic solver). By requiring $y = \tilde{y}$ at each sample point, an overdetermined system of equations is formulated,

$$[\mathbf{A}][\mathbf{c}] = [\mathbf{b}], \tag{5}$$

where $[\mathbf{A}]$ is the $N_t \times (P + 1)$ measurement matrix with elements

$$\mathbf{A}_{ij} = \Psi_j(\boldsymbol{\xi}_i), \tag{6}$$

$[\mathbf{b}]$ is the vector with the model's outputs ($[\mathbf{b}] = [y(\boldsymbol{\xi}_1), \ldots, y(\boldsymbol{\xi}_{N_t})]^{\mathrm{T}}$), and $[\mathbf{c}] = [c_0, \ldots, c_P]^{\mathrm{T}}$ is the vector of the expansion coefficients ($P + 1$ is their total number[1]). Then, the approximate solution that minimizes the mean square error of the equations in (5) is given by

$$[\mathbf{c}] \simeq ([\mathbf{A}]^{\mathrm{T}}[\mathbf{A}])^{-1}[\mathbf{A}]^{\mathrm{T}}[\mathbf{b}]. \tag{7}$$

In this work, the number of samples is selected according to $N_t = 2(P+1)$, and a Latin hypercube sampling approach [24] is implemented for their determination.

Definition of the Basis Index Set

As already mentioned, the basis functions are selected according to a specific scheme. Before presenting the proposed strategy, we briefly review the most common truncation rules encountered in the literature:

- In the case of the tensor product (TP) truncation scheme, the corresponding index set is defined as

$$\mathscr{A}^{\mathrm{TP}} = \{\boldsymbol{\alpha} \in \mathbb{N}_0^d : \alpha_i \leq p_i, i = 1, \ldots, d\}, \tag{8}$$

  where, as seen, the polynomial degree for each dimension in the random space is bounded separately. It is easily deduced that the total number of basis functions is $P+1 = \prod_{i=1}^{d}(p_i+1)$ and increases exponentially with the number of dimensions. An example case where $d=2$ and $p_1 = p_2 = 15$ is depicted in Fig. 1a.
- A very common choice is the total degree (TD) truncation rule, where polynomial functions of order lower than or equal to p are retained. The index set in this case is described by

$$\mathscr{A}^{\mathrm{TD}} = \left\{\boldsymbol{\alpha} \in \mathbb{N}_0^d : \|\boldsymbol{\alpha}\|_1 \leq p\right\}, \tag{9}$$

  and it comprises $P+1 = (p+d)!/(p!d!)$ elements ($\|\boldsymbol{\alpha}\|_1 = \alpha_1 + \ldots + \alpha_d$). Evidently, the growth rate of the number of basis functions, with respect to the number of dimensions, is smaller than the case of TP index sets (see, e.g., Fig. 1b).
- The so-called hyperbolic index sets [8] can be constructed by introducing a specific truncating surface that eliminates a higher number of complex contributions, compared to the TD approach, as shown in Fig. 2a. In this case, the basis indices belong to the set

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[1]The number $P+1$ stems from the fact that another common representation of a PC expansion has the form $\tilde{y} = \sum_{\ell=0}^{P} c_\ell \Psi_\ell(\boldsymbol{\xi})$.

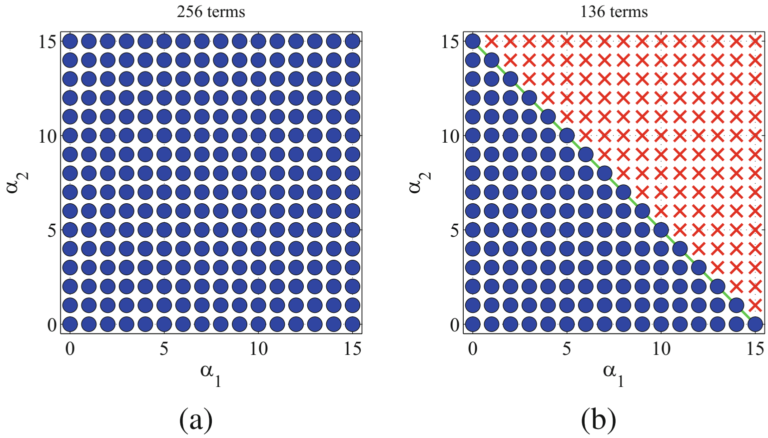


Fig. 1 Retained (blue circles) and rejected (red crosses) basis functions by the standard truncation schemes for different cases: (**a**) TP truncation scheme with $p_1 = p_2 = 15$ and (**b**) TD truncation scheme with $p_1 = p_2 = 15$

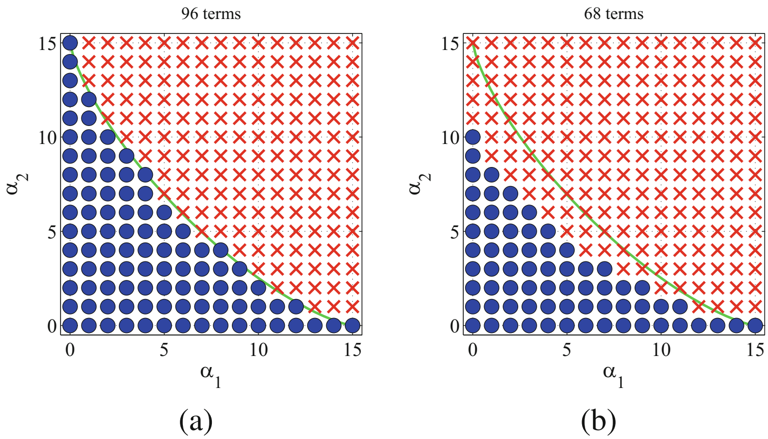


Fig. 2 Retained (blue circles) and rejected (red crosses) basis functions by the standard truncation schemes for different cases: (**a**) hyperbolic truncation scheme with $p_1 = p_2 = 15$, $q = 0.75$ and (**b**) anisotropic hyperbolic truncation scheme with $p_1 = p_2 = 15$, $q = 0.75$, and $w_1 = 1$, $w_2 = 1.5$

$$\mathscr{A}^{\mathrm{HYP}} = \left\{ \boldsymbol{\alpha} \in \mathbb{N}_0^d : \|\boldsymbol{\alpha}\|_q \leq p \right\}, \tag{10}$$

where $\|\boldsymbol{\alpha}\|_q$ denotes the norm

$$\|\boldsymbol{\alpha}\|_q = \left(\sum_{\ell=1}^{d} \alpha_\ell^q\right)^{1/q} \tag{11}$$

with $q < 1$. Anisotropy regarding the maximum polynomial order for each dimension can be considered, by applying additional rules of the form $\alpha_\ell \le p_\ell$, $\ell = 1, \ldots, d$ [25]. Another alternative, the weighted version of the hyperbolic index set, is described in [26], where the modified norm

$$\|\boldsymbol{\alpha}\|_{q,\mathbf{w}} = \left(\sum_{\ell=1}^{d} |w_\ell \alpha_\ell|^q\right)^{1/q} \tag{12}$$

is implemented, instead of $\|\boldsymbol{\alpha}\|_q$. The weight vector $\mathbf{w} = [w_1, \ldots, w_d]^{\mathrm{T}}$ is chosen with the aid of certain indices, so that variables producing the strongest sensitivity are favored. An example is shown in Fig. 2b, where $\mathbf{w} = [1\ \ 1.5]^{\mathrm{T}}$.

- Hyperbolic Cross (HC) index sets [27] are defined according to

$$\mathscr{A}^{\mathrm{HC}} = \{\boldsymbol{\alpha} \in \mathbb{N}_0^d : \prod_{i=1}^{d} (\alpha_i + 1) \le p + 1\} \tag{13}$$

and are characterized by a slower increase of basis functions, compared to the TD scheme, as this is exemplified in Fig. 3.

In the present work, we introduce a modified definition for the basis index set that generalizes the construction of the hyperbolic truncation scheme. Given that the latter involves functions with indices defined by $\|\boldsymbol{\alpha}\|_q \le p$, which can be equivalently written as

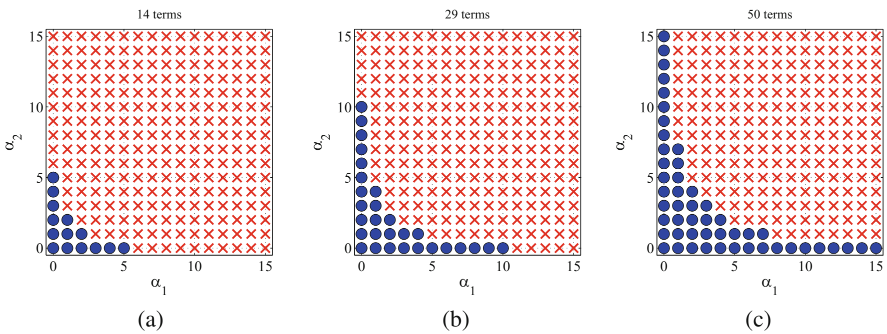


Fig. 3 Retained (blue circles) and rejected (red crosses) basis functions by the HC truncation scheme for different cases: (**a**) $p_1 = p_2 = 5$, (**b**) $p_1 = p_2 = 10$, and (**c**) $p_1 = p_2 = 15$

$$\sum_{\ell=1}^{d} \alpha_\ell^q \leq p^q, \tag{14}$$

we propose the construction of the index set according to the following:

$$\mathscr{A}_{\mathbf{p},\mathbf{q}} = \left\{ \boldsymbol{\alpha} \in \mathbb{N}_0^d : \sum_{\ell=1}^{d} \left(\frac{\alpha_\ell}{p_\ell} \right)^{q_\ell} \leq 1 \right\}. \tag{15}$$

As seen, the aforementioned truncation strategy allows the explicit definition of the maximum polynomial order for each dimension separately, without introducing additional constraints. Furthermore, unlike the standard form of hyperbolic truncation, the parameters q_i, which control the amount and type of retained basis functions, can be selected differently for each random variable, thus allowing further flexibility.

To comprehend the impact of the involved parameters in (15) on the number and properties of the retained basis functions, we examine the simple case of $d = 2$ and different combinations of the (p_1, p_2) and (q_1, q_2) parameters. Some representative results are displayed in Fig. 4. In the first three illustrations (Fig. 4a–c), we keep $q_1 = q_2 = 0.75$ and modify the polynomial order assigned to each variable. Specifically, we set $p_1 = 15$ and change p_2 to 15, 10, or 5. As seen, the preserved indices satisfy the per-dimension restrictions regarding the polynomial order, and a non-trivial reduction of the included terms is observed. In addition, by selecting q_i values smaller than 1, certain functions that correspond to complex interactions are eliminated, while simpler forms are preserved, in accordance with the "sparsity-of-effects" principle [28]. In the next three illustrations (Fig. 4d–f), we assign the same polynomial order to both dimensions ($p_1 = p_2 = 15$) and experiment with different combinations of q values. It is reminded that in a typical hyperbolic index set, the number of terms is reduced for higher q values. For instance, if $p_1 = p_2 = 15$, then the cardinality of the corresponding $\mathscr{A}$ set is equal to 136, 96, 58, or 31, when the q parameter ($q_1 = q_2 = q$) is set to 1, 0.75, 0.5, or 0.25, respectively. By tuning separately the q_1 and q_2 parameters, this reduction of terms becomes anisotropic (even when $p_1 = p_2$) and the remaining polynomials are biased, in the sense that the number of functions where $\alpha_1 > \alpha_2$ is not longer equal to those with $\alpha_1 < \alpha_2$. In the cases depicted in Fig. 4d–f, selecting $q_1 = 1$ and $q_2 = 0.6$ produces 46 terms with $\alpha_1 > \alpha_2$ and 49 terms with $\alpha_1 < \alpha_2$. These numbers become 28 and 37, respectively, if q_2 is lowered to 0.3.

Approximate Variance-Based Analysis

It becomes evident that the truncation scheme described in (15) features several parameters, whose consistent selection is of crucial importance for the reliability

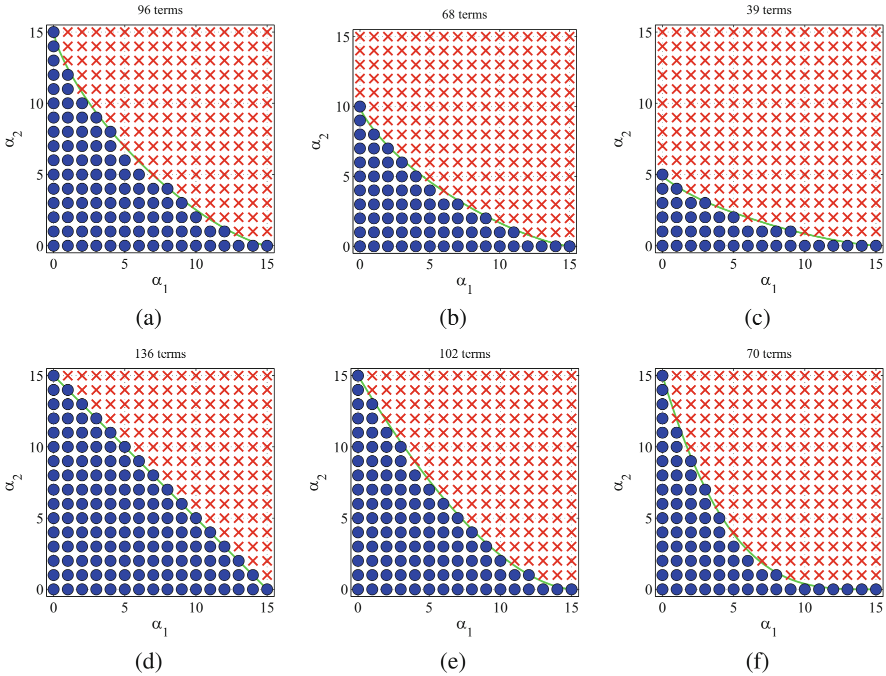


Fig. 4 Retained (blue circles) and rejected (red crosses) basis functions by the proposed truncation scheme for different cases: (**a**) $p_1 = p_2 = 15$, $q_1 = q_2 = 0.75$, (**b**) $p_1 = 15$, $p_2 = 10$, $q_1 = q_2 = 0.75$, (**c**) $p_1 = 15$, $p_2 = 5$, $q_1 = q_2 = 0.75$, (**d**) $p_1 = p_2 = 15$, $q_1 = q_2 = 1$, (**e**) $p_1 = p_2 = 15$, $q_1 = 1$, $q_2 = 0.6$, and (**f**) $p_1 = p_2 = 15$, $q_1 = 1$, $q_2 = 0.3$

of the corresponding PC representations. Instead of selecting them in an intuitive fashion, we propose performing an initial screening analysis, so that the importance of each input random variable is roughly estimated. We exploit for this task the concept of Sobol indices, which constitute sensitivity measures indicating the part of the variance due to one or more factors [29]. It is reminded that the Sobol decomposition of a function $y = f(\boldsymbol{\xi})$ is described by

$$f(\boldsymbol{\xi}) = f_0 + \sum_{\ell=1}^{d} f_\ell(\xi_\ell) + \sum_{1 \leqslant \ell_1 < \ell_2 \leqslant d} f_{\ell_1 \ell_2}(\xi_{\ell_1}, \xi_{\ell_2}) + \ldots + f_{1,\ldots,d}(\boldsymbol{\xi}), \quad (16)$$

where f_0 is the mean value of f, and all the involved terms are orthogonal to each other. The variance of y, which is computed as

$$D = \int_{\Omega} f^2(\boldsymbol{\xi}) \rho(\boldsymbol{\xi}) d\boldsymbol{\xi} - f_0^2, \quad (17)$$

(Ω is the random space), is decomposed with the aid of (16), according to

$$D = \sum_{\ell=1}^{d} D_\ell + \sum_{1 \leqslant \ell_1 < \ell_2 \leqslant d} D_{\ell_1 \ell_2} + \ldots + D_{1,\ldots,d}. \tag{18}$$

Each term in the right-hand side of (18) denotes the partial variance due to one or more variables, identified by the corresponding subscripts, and is used for defining the Sobol index

$$S_{\ell_1,\ldots,\ell_k} = \frac{D_{\ell_1,\ldots,\ell_k}}{D}. \tag{19}$$

The index (19) quantifies the part of the variance D that emanates from the combined interaction of the inputs corresponding to the indices $\ell_1, \ldots, \ell_k$. If a PC representation of y is available, with $\mathscr{A}$ being its basis index set and $\mathscr{A}^{\ell_1,\ldots,\ell_k}$ the subset of $\mathscr{A}$ comprising terms that depend on the variables denoted by the indices $\ell_1, \ldots \ell_k$, then

$$S_{\ell_1,\ldots,\ell_k} \simeq \frac{\sum\limits_{\boldsymbol{\alpha} \in \mathscr{A}^{\ell_1,\ldots,\ell_k}} c_{\boldsymbol{\alpha}}^2 \|\Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\|^2}{\sum\limits_{\boldsymbol{\alpha} \in \mathscr{A}} c_{\boldsymbol{\alpha}}^2 \|\Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\|^2}. \tag{20}$$

The information conveyed by the Sobol indices would be very useful for our purposes, as they can facilitate a consistent formulation of the truncation rule in (15). However, the computation of the Sobol indices requires the availability of a complete PC expansion, as it becomes evident from the denominator in (20).

In this work, we follow an idea originally presented in [26] and adapt it to our specific needs. In essence, we propose performing the rough estimation of the first-order S_ℓ indices only, $\ell = 1, \ldots, d$, and exploit the obtained information for the determination of the various parameters appearing in (15). The partial variances required for calculating the numerators of the S_ℓ indices depend on univariate polynomials of the ℓth variables only. Consequently, the procedure for obtaining the partial variances does not involve significant computational cost, as it is based on an auxiliary PC expansion featuring only $\sum_{i=1}^{d} p_i + 1$ terms, where p_i denotes the preselected polynomial order for the ith dimension. A loss of accuracy seems to emerge due to the computation of the total variance appearing in the denominator of the Sobol indices, as terms due to the interaction of multiple variables are not available. However, since our purpose at this point is to initially sort the input variables according to their impact (i.e., by comparing the corresponding Sobol indices) and, given that all the S_ℓ indices feature the same denominator, the variable sorting should rely on the partial variances appearing in the numerators. Consequently, the proposed approach is expected to suffice for a preliminary estimation of each variable's weight on the output.

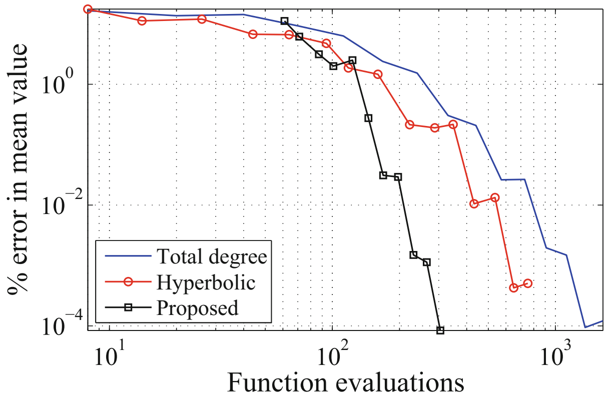
4 Numerical Results

We initially test the performance of the new scheme in the approximation of the Ishigami function, which is defined as

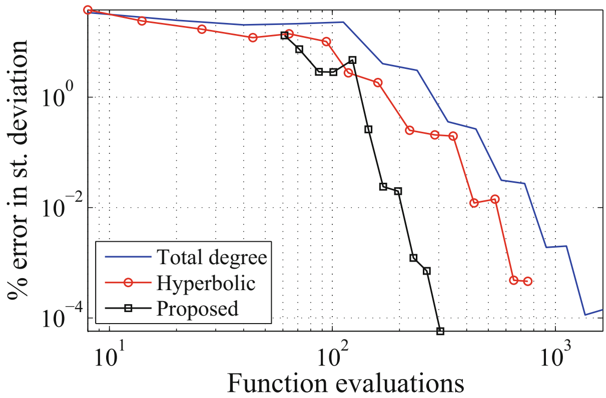
$$f(\boldsymbol{\xi}) = \sin\xi_1 + a\sin^2\xi_2 + b\xi_3^4\sin\xi_1. \tag{21}$$

The variables ξ_i, $i = 1, 2, 3$ are uniformly distributed over $[-\pi, \pi]$ and the a and b parameters are selected as $a = 7$ and $b = 0.1$. The mean value and the variance of (21) are calculated easily and are equal to $\frac{a}{2}$ and $\frac{1}{2} + \frac{a^2}{8} + \frac{b^2\pi^8}{18} + \frac{b\pi^4}{5}$, respectively. The accomplished percentage errors in the mean value and the standard deviation of the Ishigami function are examined, in relation to the required number of function evaluations. It is noted that the Ishigami function is highly nonlinear, posing severe difficulties to any polynomial approximation. In the context of applying the proposed methodology, we first perform a crude sensitivity analysis, by considering only sixth-order 1D basis functions. The total number of the latter is 19 and, consequently, the number of necessary function evaluations is equal to 38. Despite the sparse auxiliary model, the computation of the Sobol indices is proven to be sufficiently reliable, at least for the purpose of sorting the input parameters according to their influence on the output. Specifically, we find that $S_1 = 0.317$, $S_2 = 0.539$, and $S_3 = 0.145$, with the exact values being equal to $S_1^{\text{exact}} = 0.314$, $S_2^{\text{exact}} = 0.442$, and $S_3^{\text{exact}} = 0$. The aforementioned approximate values of the Sobol indices represent the average estimations of 1000 different trials, verifying that consistent conclusions can be extracted, despite the variation characterizing different realizations of the Latin hypercube sampling. From the preliminary data, we deduce that ξ_2 is the most influential parameter, whereas the output is less sensitive to ξ_3. Based on these findings, we select the individual polynomial orders according to $p_1 = p_2 - 1$ and $p_3 = p_2 - 3$, with $p_2 \leq 15$. In addition, given the obtained values of the Sobol indices, we choose an aggressive strategy for the q values, by selecting $q_2 = 0.1$ and $q_1 = q_3 = 1$. The performance of the proposed methodology is compared with those of the TD truncation scheme and the hyperbolic index set with $q = 0.75$. The latter two cases also consider polynomial basis of up to 15th order. The errors produced by the three approaches can be assessed in Fig. 5, where the faster convergence of the proposed approach over the standard solutions is indicative of the efficiency of the corresponding surrogate. Evidently, the initial information regarding the significance of each variable played a significant role in the selection of the proper parameter values in the proposed truncation rule, excluding a nontrivial number of terms that are not likely to affect the output to a significant degree.

The second numerical test pertains to a time-dependent electromagnetic problem and involves a configuration comprising eight lossless dielectric slabs, separated by air (the geometry is shown in Fig. 6). The thickness of each slab is set to $d_s = 1.202$ cm, and the distance between successive slabs is equal to $d_a = 3.606$ cm. We are interested in studying the properties of the aforementioned layout, which



(a)



(b)

Fig. 5 Percentage error in (**a**) the mean value and (**b**) the standard deviation of the Ishigami function versus the required number of function evaluations

acts as a filtering structure, in the case of normal incidence of a pulsed plane wave. Specifically, the relative dielectric constant of each slab is treated as an independent random variable that is uniformly distributed within the range $\epsilon_r = 3.5 \times (1 \pm 0.1)$. Consequently, we now have to deal with an eight-dimensional problem. The QoI herein is the transmission coefficient T of the structure, typically computed as $T(\omega) = |\mathscr{E}_t(\omega)/\mathscr{E}_i(\omega)|$, where $\mathscr{E}_t(\omega)$ and $\mathscr{E}_i(\omega)$ are the Fourier transforms of the time-dependent intensities of the transmitted $E_t(t)$ and incident $E_i(t)$ electric fields, respectively. As illustrated in Fig. 7, the magnitude of T displays randomness, due to the uncertainty in the electric parameters of the filter. For the calculation of the transmission coefficient, the finite-difference time-domain method is implemented [30] in one-dimensional formulation, which discretizes Maxwell's equations using

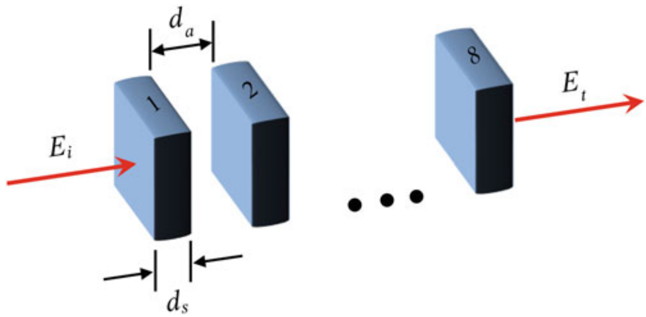
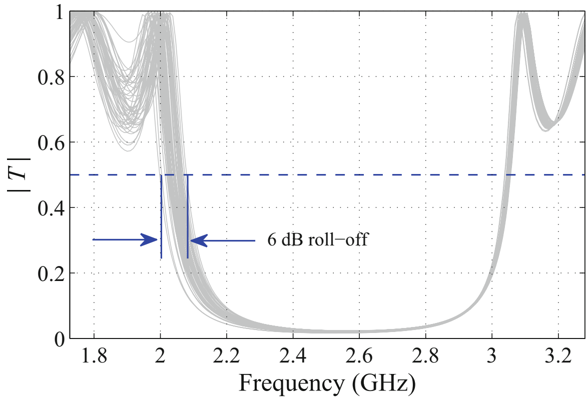


Fig. 6 Geometric configuration of the electromagnetic filter problem

Fig. 7 Uncertainty characterizing the transmission coefficient curves of the electromagnetic filter, as well as the value of the 6-dB roll-off frequency



uniform space-time grids, and directly computes the electric and the magnetic field samples in an explicit and conditionally stable fashion.

In order to develop a PC surrogate for the QoI, we first perform the approximate variance analysis with $p = 3$, and thus a total of 25 basis functions are considered and only $2 \times 25 = 50$ simulations are conducted. Instead of using the Sobol indices, we determine the influence of each input variable considering the corresponding partial variance, averaged over the frequency range of interest. In this way, the ordering of the inputs is mainly affected by the results observed at frequencies where the total variance attains the highest values. Taking into account the result of the initial variance analysis, we construct an anisotropic basis set, by selecting the vector of the polynomial order per direction as $[2\ 3\ 4\ 4\ 4\ 4\ 3\ 2]^{\mathrm{T}}$ and the corresponding q values as $[1\ 0.95\ 0.95\ 0.85\ 0.85\ 0.95\ 0.95\ 1]^{\mathrm{T}}$. Evidently, the maximum polynomial order has been set to 4, which is a common choice for problems of this nature. However, a typical TD basis set would comprise $(4 + 8)!/(4!8!) = 495$ functions, whereas the cardinality of the proposed set is 101. Consequently, the required number of simulations is reduced from $2 \times 495 = 990$ to $2 \times 101 + 50 = 252$, indicating a reduction of the computational cost by almost 75%. Of course, these savings will be meaningful only provided that the model

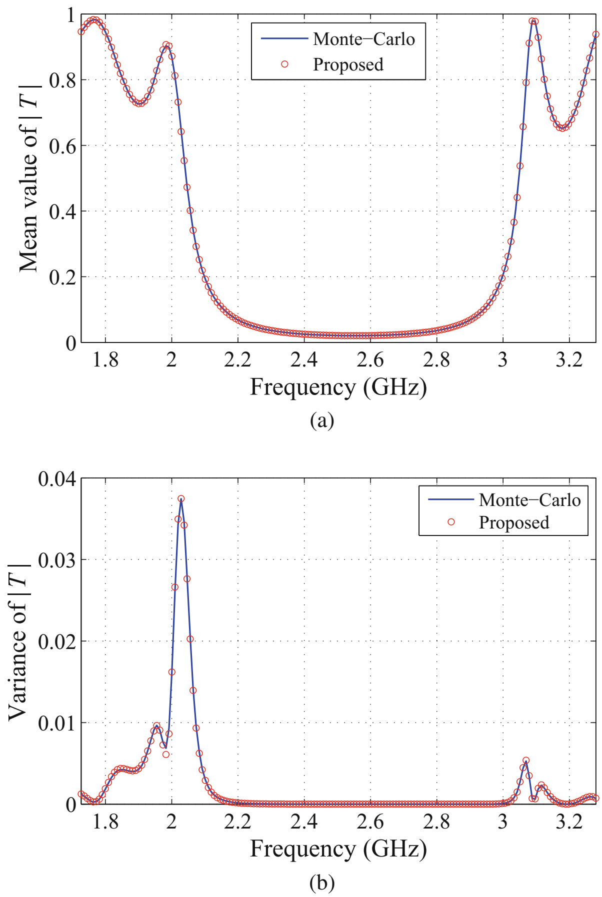


Fig. 8 (**a**) Mean value and (**b**) variance of $|T|$ for the electromagnetic filter problem

based on the fewer basis functions is a credible one. The PC predictions of the mean value and the variance of $|T|$ are depicted in Fig. 8a and b, respectively, where direct comparison with the reference solutions can be performed. The latter are constructed considering the results from 20,000 MC simulations. Evidently, the assessment verifies the validity of the predictions of the PC model, despite the reduction of the basis functions, compared to the standard TD approach.

Furthermore, similar to [16], we are interested in predicting the variability of the 6-dB roll-off frequency of the transmission coefficient (Fig. 7). Figure 9 depicts the pdfs computed by the MC method, the isotropic TD PC surrogate with $p = 4$, and the proposed anisotropic PC scheme. In the case of the PC expansions, 10,000 samples are extracted (this procedure is faster than performing MC simulations, as it involves only polynomial evaluations), and the corresponding pdfs are computed via kernel density estimation [31]. It can be verified from Fig. 9 that the pdf predicted

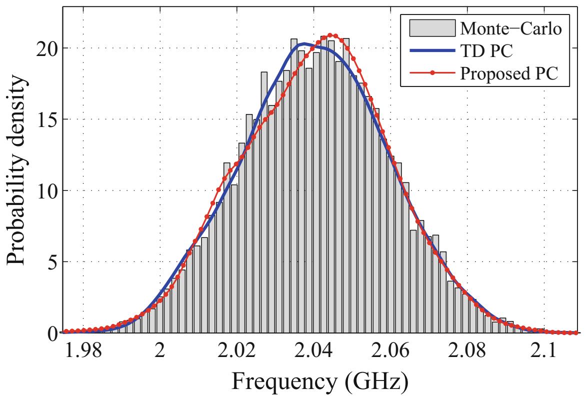


Fig. 9 Pdf of the 6-dB roll-off frequency, calculated by various methods

by the reduced basis set exhibits satisfactory agreement with the reference data of the MC study. In addition, a high degree of similarity is observed with the result computed by the more computationally demanding TD PC model. Although some slight deviation can be observed at the central frequencies, the pdf extracted using the proposed approach can be safely considered as a reliable approximation of the exact one.

5 Conclusions

In this work, we have presented and evaluated an approach for the development of reliable reduced-basis PC models, when studying problems with uncertain inputs. An initial, computationally cheap, approximate variance calculation enables the ordering of the input variables according to their impact on the QoI. This information facilitates the consistent formulation of the truncation rule for the basis index set, which features a multi-parametric form and enables the judicious elimination of unnecessary terms. The performance of the proposed methodology has been validated in two representative test cases, and the results have verified that the suggested technique provides a simple, yet reliable, treatment for uncertainty quantification problems.

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