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Erdős-Pósa property of chordless cycles and its applications ☆

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ABSTRACT

A chordless cycle, or equivalently a hole, in a graph G is an induced subgraph of G which is a cycle of length at least 4. We prove that the Erdős-Pósa property holds for chordless cycles, which resolves the major open question concerning the Erdős-Pósa property. Our proof for chordless cycles is constructive: in polynomial time, one can find either $k+1$ vertex-disjoint chordless cycles, or $c_1k^2 \log k + c_2$ vertices hitting every chordless cycle for some constants c_1 and c_2. It immediately implies an approximation algorithm of factor $\mathcal{O}(\mathsf{opt} \log \mathsf{opt})$ for Chordal Vertex Deletion. We complement our main result by showing that chordless cycles of length at least ℓ for any fixed $\ell \geq 5$ do not have the Erdős-Pósa property.

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1. Introduction

All graphs in this paper are finite and have neither loops nor parallel edges. We denote by $\mathbb{N}$ the set of positive integers. A class $\mathcal{C}$ of graphs is said to have the *Erdős-Pósa property* if there exists a function $f : \mathbb{N} \to \mathbb{N}$, called a *gap function*, such that for every graph G and every positive integer k, G contains either

- $k + 1$ pairwise vertex-disjoint subgraphs in $\mathcal{C}$, or
- a vertex set T of G such that $|T| \leq f(k)$ and $G - T$ has no subgraphs in $\mathcal{C}$.

Erdős and Pósa [8] showed that the class of all cycles has this property with a gap function $\mathcal{O}(k \log k)$. This breakthrough result sparked an extensive research on finding min-max dualities of packing and covering for various graph families and combinatorial objects. Erdős and Pósa also showed that the gap function cannot be improved to $o(k \log k)$. The result of Erdős and Pósa has been strengthened for cycles with additional constraints; for example, long cycles [23,4,9,17,5], directed cycles [22,13], cycles with modularity constraints [25,11], or cycles intersecting a prescribed vertex set [15,18,5,11]. Not every variant of cycles has the Erdős-Pósa property; for example, Reed [21] showed that the class of odd cycles does not satisfy the Erdős-Pósa property.

We generally say that a graph class $\mathcal{C}$ has the *Erdős-Pósa property* under a graph containment relation $\leq_\star$ if there exists a gap function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G and every positive integer k, G contains either

- $k+1$ pairwise vertex-disjoint subsets $Z_1, \ldots, Z_k$ such that each subgraph of G induced by Z_i contains a member of $\mathcal{C}$ under $\leq_\star$, or
- a vertex set T of G such that $|T| \leq f(k)$ and $G - T$ contains no member of $\mathcal{C}$ under $\leq_\star$.

Here, $\leq_\star$ can be a graph containment relation such as subgraph, induced subgraph, minor, topological minor, induced minor, or induced subdivision. An edge version and directed version of the Erdős-Pósa property can be similarly defined. In this setting, the Erdős-Pósa properties of diverse undirected and directed graph families have been studied for graph containment relations such as minors [23], immersions [10,14], and (directed) butterfly minors [3]. It is known that the edge-version of the Erdős-Pósa property also holds for cycles [6]. Raymond and Thilikos [20] provided an up-to-date overview on the Erdős-Pósa properties for a range of graph families.

In this paper, we study the Erdős-Pósa property for cycles of length at least 4 under the induced subgraph relation. An induced cycle of length at least 4 in a graph G is called a *hole* or a *chordless cycle*. Considering the extensive study on the topic, it is somewhat surprising that whether the Erdős-Pósa property holds for cycles of length at least 4 under the induced subgraph relation has been left open till now. This question was explicitly asked by Jansen and Pilipczuk [12] in their study of the Chordal Vertex

Deletion problem, and was also asked by Raymond and Thilikos [20] in their survey. We answer this question positively.

Theorem 1.1. *There exist constants c_1, c_2 and a polynomial-time algorithm which, given a graph G and a positive integer k, finds either $k+1$ vertex-disjoint holes or a vertex set of size at most $c_1 k^2 \log k + c_2$ hitting every hole of G.*

One might ask whether Theorem 1.1 can be extended to the class of cycles of length at least ℓ for fixed $\ell \geq 5$. We present a complementary result that for every fixed $\ell \geq 5$, the class of cycles of length at least ℓ does not satisfy the Erdős-Pósa property under the induced subgraph relation.

Theorem 1.2. *Let $\ell \geq 5$ be an integer. Then the class of cycles of length at least ℓ does not have the Erdős-Pósa property under the induced subgraph relation.*

In Section 7, we additionally argue that the class of cycles of length at least ℓ for fixed $\ell \geq 5$ does not have the $1/\alpha$-*integral Erdős-Pósa property* under the induced subgraph relation, for all integers $\alpha \geq 2$.

Theorem 1.1 is closely related to the Chordal Vertex Deletion problem. The Chordal Vertex Deletion problem asks whether, for a given graph G and a positive integer k, there exists a vertex set S of size at most k such that $G - S$ has no holes; in other words, $G - S$ is a chordal graph. In parameterized complexity, whether or not Chordal Vertex Deletion admits a polynomial kernelization was one of major open problems since it was first mentioned by Marx [16]. A *polynomial kernelization* of a parameterized problem is a polynomial-time algorithm that takes an instance (x, k) and outputs an instance (x', k') such that (1) (x, k) is a Yes-instance if and only if (x', k') is a Yes-instance, and (2) $k' \leq k$, and $|x'| \leq g(k)$ for some polynomial function g.

Jansen and Pilipczuk [12], and independently Agrawal et al. [2], presented polynomial kernelizations for the Chordal Vertex Deletion problem. In both works, an approximation algorithm for the optimization version of this problem emerges as an important subroutine. Jansen and Pilipczuk [12] obtained an approximation algorithm of factor $\mathcal{O}(\mathsf{opt}^2 \log \mathsf{opt} \log n)$ using iterative decomposition of the input graph and linear programming. Agrawal et al. [2] obtained an algorithm of factor $\mathcal{O}(\mathsf{opt} \log^2 n)$ based on divide-and-conquer. As one might expect, the factor of an approximation algorithm for the Chordal Vertex Deletion is intrinsically linked to the quality of the polynomial kernels attained in [12] and [2]. We point out that the polynomial-time algorithm of Theorem of 1.1 can be easily converted into an approximation algorithm of factor $O(\mathsf{opt} \log \mathsf{opt})$.

Theorem 1.3. *There is an approximation algorithm of factor $O(\mathsf{opt} \log \mathsf{opt})$ for* Chordal Vertex Deletion.

It should be noted that an $\mathcal{O}(\log^2 n)$-factor approximation algorithm was presented recently by Agrawal et al. [1], which outperforms the approximation algorithm of Theorem 1.3, when applied to the kernelization algorithm.

Our result has another application on packing and covering for weighted cycles. For a graph G and a non-negative weight function $w : V(G) \to \mathbb{N} \cup \{0\}$, let $\mathsf{pack}(G, w)$ be the maximum number of cycles (repetition is allowed) such that each vertex v is used at most $w(v)$ times, and let $\mathsf{cover}(G, w)$ be the minimum value $\sum_{v \in X} w(v)$ where X hits all cycles in G. Ding and Zang [7] characterized *cycle Mengerian graphs* G, which satisfy the property that for all non-negative weight function w, $\mathsf{pack}(G, w) = \mathsf{cover}(G, w)$. Up to our best knowledge, it was not previously known whether $\mathsf{cover}(G, w)$ can be bounded by a function of $\mathsf{pack}(G, w)$.

As a corollary of Theorem 1.1, we show the following.

Corollary 1.4. *For a graph G and a non-negative weight function $w : V(G) \to \mathbb{N} \cup \{0\}$, $\mathsf{cover}(G, w) \leq \mathcal{O}(k^2 \log k)$, where $k = \mathsf{pack}(G, w)$.*

The paper is organized as follows. Section 2 provides basic notations and previous results that are relevant to our result. In Section 3, we explain how to reduce the proof of Theorem 1.1 to a proof under a specific premise, in which we are given a shortest hole C of G such that C has length more than $d_1 k \log k + d_2$ for some constants d_1, d_2 and $G - V(C)$ is chordal. In this setting, we introduce further technical notations and terminology. An outline of our proof will be also given in this section. We present some structural properties of a shortest hole C and its neighborhood in Section 4. In Sections 5 and 6, we prove the Erdős-Pósa property for different types of holes intersecting C step by step, and we conclude Theorem 1.1 at the end of Section 6. Section 7 demonstrates that the class of cycles of length at least ℓ, for every fixed $\ell \geq 5$, does not have the Erdős-Pósa property under the induced subgraph relation. Section 8 illustrates the implications of Theorem 1.1 to weighted cycles and to the Chordal Vertex Deletion problem.

2. Preliminaries

For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. Let G be a graph. For a vertex set S of G, let $G[S]$ denote the subgraph of G induced by S, and let $G - S$ denote the subgraph of G obtained by removing all vertices in S. For $v \in V(G)$, we let $G - v := G - \{v\}$. If $uv \in E(G)$, we say that u is a *neighbor* of v. The set of neighbors of a vertex v is denoted by $N_G(v)$, and the *degree* of v is defined as the size of $N_G(v)$. The *open neighborhood* of a vertex set $A \subseteq V(G)$ in G, denoted by $N_G(A)$, is the set of vertices in $V(G) \setminus A$ having a neighbor in A. The set $N_G(A) \cup A$ is called the *closed neighborhood* of A, and denoted by $N_G[A]$. For convenience, we define these neighborhood operations for subgraphs as well; that is, for a subgraph H of G, let $N_G(H) := N_G(V(H))$ and $N_G[H] := N_G[V(H)]$. When the underlying graph is clear from the context, we drop the subscript G. A vertex set S of a graph is a *clique* if every pair of vertices in S is adjacent, and it is an *independent set* if every pair of vertices in S is non-adjacent. For two subgraphs H and F of G, the *restriction* of F on H is defined as the graph $F[V(F) \cap V(H)]$.

A *walk* is a non-empty alternating sequence of vertices and edges of the form $(x_0, e_0, \ldots, e_{\ell-1}, x_\ell)$, beginning and ending with vertices, such that for every $0 \le i \le \ell-1$, x_i and x_{i+1} are endpoints of e_i. A *path* is a walk in which vertices are pairwise distinct. For a path P on vertices $x_0, \ldots, x_\ell$ with edges $x_i x_{i+1}$ for $i = 0, 1, \ldots, \ell - 1$, we write $P = x_0 x_1 \cdots x_\ell$. It is also called an (x_0, x_ℓ)-path. We say x_i is the i-th neighbor of x_0, and similarly, $x_{\ell-i}$ is the i-th neighbor of x_ℓ in P. A *cycle* is a walk $(x_0, e_0, \ldots, e_{\ell-1}, x_\ell)$ in which vertices are pairwise distinct except $x_0 = x_\ell$. For a cycle C on $x_0, x_1, \ldots, x_\ell$ with edges $x_i x_{i+1}$ for $i = 0, 1, \ldots, \ell - 1$ and $x_\ell x_0$, we write $C = x_0 x_1 \cdots x_\ell x_0$. If a cycle or a path H is an induced subgraph of the given graph G, then we say that H is an induced cycle or an induced path in G, respectively.

A subpath of a path P starting at x and ending at y is denoted as xPy. In the notation xPy, we may replace x or y with $\mathring{x}$ or $\mathring{y}$, to obtain a subpath starting from the neighbor of x in P closer to y or ending at the neighbor of y in P closer to x, respectively. For instance, $xP\mathring{y}$ refers to the subpath of P starting at x and ending at the neighbor of y in P closer to x. Given two walks $P = (v_0, e_0, \ldots, e_{p-1}, v_p)$ and $Q = (u_0, f_0, \ldots, f_{q-1}, u_q)$ such that $v_p = u_0$, the *concatenation* of P and Q is defined as the walk $(v_0, e_0, \ldots, e_{p-1}, v_p(= u_0), f_0, \ldots, f_{q-1}, u_q)$, which we denote as $P \odot Q$. Note that for two internally vertex-disjoint paths P_1 and P_2 from v to w, $vP_1w \odot wP_2v$ denotes the cycle passing through P_1 and P_2.

Given a graph G, the *distance* between two vertices x and y in G is defined as the length of a shortest (x, y)-path and denoted as $\mathrm{dist}_G(x, y)$. If $x = y$, then we define $\mathrm{dist}_G(x, y) = 0$, and $\mathrm{dist}_G(x, y) = \infty$ if there is no (x, y)-path in G. The distance between two vertex sets $X, Y \subseteq V(G)$, written as $\mathrm{dist}_G(X, Y)$, is the minimum $\mathrm{dist}_G(x, y)$ over all $x \in X$ and $y \in Y$. If $X = \{x\}$, then we write $\mathrm{dist}_G(X, Y)$ as $\mathrm{dist}_G(x, Y)$. For a vertex subset S of G, a vertex set U is the *r-neighborhood* of S in G if it is the set of all vertices w such that $\mathrm{dist}_G(w, S) \le r$. We denote the r-neighborhood of S in G as $N_G^r[S]$. When the underlying graph G is clear from the context, we omit the subscript G.

Given a cycle $C = x_0 x_1 \cdots x_\ell x_0$, an edge e of G is a *chord* of C if both endpoints of e are contained in $V(C)$ but e is not an edge of C. A *hole* in a graph G is an induced cycle of length at least 4 in G. A hole is also called as a *chordless cycle*. A graph is *chordal* if it has no holes. A vertex set T of a graph G is called a *chordal deletion set* if $G - T$ is chordal.

Given a vertex set $S \subseteq V(G)$, a path P is called an *S-path* if the endpoints of P are vertices of S and all internal vertices are contained in $V(G) \setminus S$. An S-path is called *proper* if it has at least one internal vertex. An (A, B)-path of a graph G is a path $v_0 v_1 \cdots v_\ell$ such that $v_0 \in A$, $v_\ell \in B$ and all internal vertices are in $V(G) \setminus (A \cup B)$. Observe that every path from A to B contains an (A, B)-path. If A or B is a singleton, then we omit the bracket from the set notation. A vertex set S is an *(A, B)-separator* if S disconnects all (A, B)-paths in G.

We recall Menger's Theorem.

Theorem 2.1 *(Menger's Theorem; See for instance [6]). Let G be a graph and $A, B \subseteq V(G)$. Then the size of a minimum (A, B)-separator in G equals the maximum number of vertex-disjoint (A, B)-paths in G. Furthermore, one can output either one of them in polynomial time.*

A *bipartite graph* is a graph G with a vertex bipartition (A, B) in which each of $G[A]$ and $G[B]$ is edgeless. A set F of edges in a graph is a *matching* if no two edges in F have a common endpoint. A vertex set S of a graph G is a *vertex cover* if $G - S$ has no edges. By Theorem 2.1, given a bipartite graph with a bipartition (A, B), one can find a maximum matching or a minimum vertex cover in polynomial time.

The following result is useful to find many vertex-disjoint cycles in a graph of maximum degree 3. All logarithms in this paper are taken to base 2. We define s_k for $k \in \mathbb{N}$ as

$$s_k = \begin{cases} 4k(\log k + \log\log k + 4) & \text{if } k \geq 2 \\ 2 & \text{if } k = 1. \end{cases}$$

Theorem 2.2 *(Simonovitz [24]). Let G be a graph all of whose vertices have degree 3 and let k be a positive integer. If $|V(G)| \geq s_k$, then G contains at least k vertex-disjoint cycles. Furthermore, such k cycles can be found in polynomial time.*

Lastly, we present lemmas which are useful for detecting a hole.

Lemma 2.3. *Let H be a graph and $x, y \in V(H)$ be two distinct vertices. Let P and Q be internally vertex-disjoint (x, y)-paths such that Q contains an internal vertex w having no neighbor in $V(P) \setminus \{x, y\}$. If Q is an induced path, then $H[V(P) \cup V(Q)]$ has a hole containing w.*

Proof. Let x_1 and x_2 be the neighbors of w in Q. As $xPy \odot yQx$ is a cycle, $H[V(P) \cup V(Q)] - w$ is connected. Let R be a shortest (x_1, x_2)-path in $H[V(P) \cup V(Q)] - w$. As the only neighbors of w contained in $(V(P) \cup V(Q)) \setminus \{w\}$ are x_1 and x_2, w has no neighbors in the internal vertices of R. Note that R has length at least 2 since $x_1, x_2 \in V(Q)$ and Q is an induced path. Therefore, $x_1Rx_2 \odot x_2wx_1$ is a hole containing w, as required. □

A special case of Lemma 2.3 is when there is a vertex w in a cycle C such that w has no neighbors in the internal vertices of $C - w$ and the neighbors of w on C are non-adjacent. In this case, C has a hole containing w by Lemma 2.3.

One can test in polynomial time whether a graph contains a hole or not.

Lemma 2.4. *Given a graph G, one can test in polynomial time whether it has a hole or not. Furthermore, one can find in polynomial time a shortest hole of G, if one exists.*

Proof. We guess three vertices v, w, z where $vw, wz \in E(G)$ and $vz \notin E(G)$, and test whether there is a path from v to z in $G - (N_G[w] \setminus \{v, z\})$. If there is such a path, then we choose a shortest path P from v to z. As w has no neighbors in the set of internal vertices of P, $V(P) \cup \{w\}$ induces a hole. Clearly if G has a hole, then we can find one by the above procedure.

To find a shortest one, for every such a guessed tuple (v, w, z), we keep the length of the obtained hole. Then it is sufficient to output a hole with minimum length among all obtained holes. $\square$

3. Terminology and a proof overview

The proof of Theorem 1.1 begins by finding a sequence of shortest holes. Let G be the input graph and let $G_1 = G$. For each $i = 1, 2, \ldots$, we iteratively find a shortest hole C_i in G_i and set $G_{i+1} := G_i - V(C_i)$. If the procedure fails to find a hole at j-th iteration, then G_j is a chordal graph. This iterative procedure leads us to the following theorem, which is the core component of our result.

For $k \in \mathbb{N}$, we define $\mu_k = 76s_{k+1} + 3217k + 1985$.

Theorem 3.1. *Let G be a graph, k be a positive integer, and C be a shortest hole of G such that C has length strictly larger than μ_k and $G - V(C)$ is chordal. Given such G, k, and C, one can find in polynomial time either $k + 1$ vertex-disjoint holes or a vertex set $X \subseteq V(G)$ of size at most μ_k hitting every hole of G.*

It is easy to derive our main result from Theorem 3.1.

Proof of Theorem 1.1. We construct sequences $G_1, \ldots, G_{\ell+1}$ and $C_1, \ldots, C_\ell$ such that

- $G_1 = G$,
- for each $i \in \{1, 2, \ldots, \ell\}$, C_i is a shortest hole of G_i,
- for each $i \in \{1, 2, \ldots, \ell\}$, $G_{i+1} = G_i - V(C_i)$, and
- $G_{\ell+1}$ is chordal.

Such a sequence can be constructed in polynomial time repeatedly applying Lemma 2.4 to find a shortest hole. If $\ell \geq k + 1$, then we have found a packing of $k + 1$ holes. Hence, we assume that $\ell \leq k$.

We prove the following claim for $j = \ell + 1$ down to $j = 1$.

> One can find in polynomial time either $k+1$ vertex-disjoint holes, or a chordal deletion set T_j of G_j of size at most $(\ell + 1 - j)\mu_k$.

The claim trivially holds for $j = \ell + 1$ with $T_{\ell+1} = \emptyset$ because $G_{\ell+1}$ is chordal. Let us assume that for some $j \leq \ell$, we obtained a chordal deletion set T_{j+1} of G_{j+1} of size at

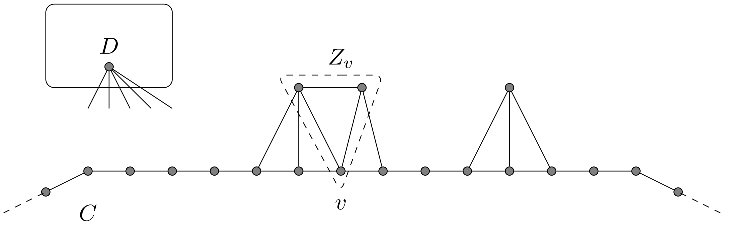

Fig. 1. The set of vertices adjacent to all vertices of C is denoted by D, and for each $v \in V(C)$, Z_v denotes the set $\{v\} \cup (N(v) \setminus V(C) \setminus D)$. Using the fact that C is chosen as a shortest hole and it is long, we will prove in Lemma 4.1 that each vertex in $N(C) \setminus D$ has at most 3 neighbors on C and they are consecutive in C.

most $(\ell - j)\mu_k$. Then in $G_j - T_{j+1}$, C_j is a shortest hole, and $(G_j - T_{j+1}) - V(C_j)$ is chordal. If C_j has length at most μ_k, then we set $T_j := T_{j+1} \cup V(C_j)$. Clearly, $|T_j| \leq (\ell - j + 1)\mu_k$. Otherwise, by applying Theorem 3.1 to $G_j - T_{j+1}$ and C_j, one can find in polynomial time either $k+1$ vertex-disjoint holes or a chordal deletion set X of size at most μ_k of $G_j - T_{j+1}$. In the former case, we output $k+1$ vertex-disjoint holes, and we are done. If we obtain a chordal deletion set X, then we set $T_j := T_{j+1} \cup X$. Observe that the set T_j is a chordal deletion set of G_j and $|T_j| \leq (\ell - j + 1)\mu_k$ as claimed.

From the claim with $j = 1$, we conclude that in polynomial time, one can find either $k+1$ vertex-disjoint holes, or a chordal deletion set of $G_1 = G$ of size at most $\ell\mu_k \leq k\mu_k$. So, in the latter case, there exist constants c_1, c_2 such that G admits a chordal deletion set of size $c_1 k^2 \log k + c_2$ (c_2 is necessary for $k = 1$). $\square$

The rest of this section and Sections 4-6 are devoted to establish Theorem 3.1. Throughout these sections, we shall consider the input tuple (G, k, C) of Theorem 3.1 as fixed.

Let us introduce the notations that are frequently used (see Fig. 1). A vertex $v \in N(C)$ is *C-dominating* if v is adjacent to every vertex on C. We reserve D to denote the set of all C-dominating vertices. For each vertex v in C, we denote by $Z_v := \{v\} \cup (N(v) \setminus V(C) \setminus D)$, and for a subset S of $V(C)$, we denote by $Z_S := \bigcup_{v \in S} Z_v$. We also define

- $G_{deldom} := G - D$ and $G_{nbd} := G[N[C] \setminus D]$.

For a subpath Q of C, the subgraph of G induced by $Z_{V(Q)}$ is called a *Q-tunnel*. By definition of $Z_{V(Q)}$, a Q-tunnel is an induced subgraph of G_{nbd}. When q, q' are endpoints of Q, we say that Z_q and $Z_{q'}$ are *entrances* of the Q-tunnel.

For a subgraph H of G, the *support* of H, denoted by $\mathsf{sp}(H)$, is the set of all vertices $v \in V(C)$ such that $(Z_v \cup D) \cap V(H) \neq \emptyset$. Observe that if H contains a vertex of D, then trivially $\mathsf{sp}(H) = V(C)$.

We distinguish between two types of holes, namely *sunflowers* and *tulips*. A hole H is said to be a *sunflower* if $V(H) \subseteq N[C]$, that is, its entire vertex set is placed within the closed neighborhood of C. A hole that is not a sunflower is called a *tulip*. Every tulip contains at least one vertex not contained in $N[C]$. Also, we classify holes depending on

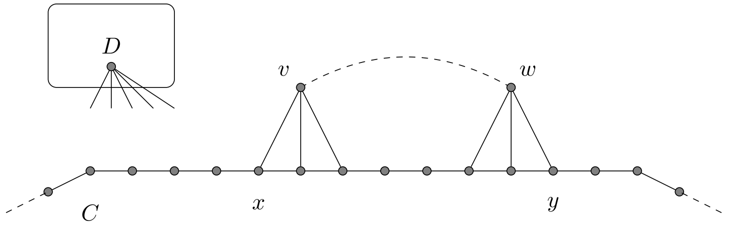

Fig. 2. Illustration of Lemma 4.2: if $\mathrm{dist}_C(x, y) \geq 4$, then there are no edges between Z_x and Z_y. For instance, suppose v and w are adjacent. Note that v and w have at most 3 consecutive neighbors in C. If the distance from $N(v) \cap V(C)$ to $N(w) \cap V(C)$ in C is at least 1, then we can find a hole shorter than C using the shortest path from $N(v) \cap V(C)$ to $N(w) \cap V(C)$ in C. Otherwise, we have $\mathrm{dist}_C(x, y) = 4$ and $|N(v) \cap V(C)| = |N(w) \cap V(C)| = 3$ and thus, the longer path between $N(v) \cap V(C)$ to $N(w) \cap V(C)$ in C creates a hole shorter than C.

whether one contains a C-dominating vertex or not. A hole is *D-traversing* it contains a C-dominating vertex (which is a vertex of D), and *D-avoiding* otherwise.

In the remainder of this section, we present a proof outline of Theorem 3.1. Here are three basic observations, necessary to give the ideas of our proofs.

- (Lemma 4.1) For every vertex v of $N(C)$, either it has at most 3 neighbors in C and they are consecutive in C, or it is C-dominating.
- (Lemma 4.2) Let x, y be two vertices in C such that $\mathrm{dist}_C(x, y) \geq 4$. Then there is no edge between Z_x and Z_y. See Fig. 2 for an illustration.
- (Lemma 4.5) Let H be a connected subgraph in G_{nbd}. Then $C[\mathsf{sp}(H)]$ is connected.
- (Lemma 4.8) D is a clique.

3.1. D-avoiding sunflowers

The set of D-avoiding sunflowers is categorized into two subgroups, *petals* and *full sunflowers*. Note that by Lemma 4.5, the support of every D-avoiding sunflower consists of consecutive vertices of C. For a D-avoiding sunflower H, we say that

- it is a *petal* if $|\mathsf{sp}(H)| \leq 7$, and
- it is *full* if $\mathsf{sp}(H) = V(C)$.

See Fig. 3 for an illustration of a petal.

[Subsection 5.1.] We first obtain a small hitting set of petals, unless G has $k+1$ vertex-disjoint holes. For this, we greedily pack petals and mark their supports on C. Clearly, if there are $k+1$ petals whose supports are pairwise disjoint, then we can find $k+1$ vertex-disjoint holes. Thus, we can assume that there are at most k petals whose supports are pairwise disjoint. We take the union of all those supports and call it T_1. By construction, for every petal H, $\mathsf{sp}(H) \cap T_1 \neq \emptyset$. Then we take the 6-neighborhood of T_1 in C and call it T_{petal}. It turns out that

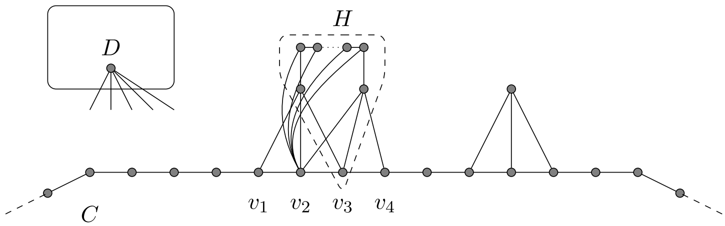

Fig. 3. The cycle H is a petal having the support $\{v_1, v_2, v_3, v_4\}$. A petal can be arbitrarily long.

(∗) for every petal H, $\mathsf{sp}(H)$ is fully contained in T_{petal},

and in particular, $V(H) \cap T_{petal} \neq \emptyset$. The size of T_{petal} is at most $19k$.

[Subsection 5.2.] Somewhat surprisingly, we show that every D-avoiding sunflower that does not intersect T_{petal} is a full sunflower. It is possible that there is a sunflower with support of size at least 8 and less than $|V(C)|$. We argue that if such a sunflower H exists, then there is a vertex $v \in V(C) \cap V(H)$ and a petal whose support contains v. But the property (∗) of T_{petal} implies that T_{petal} contains v, which implies that such a sunflower should be hit by T_{petal}. Therefore, it is sufficient to hit full sunflowers for hitting all remaining D-avoiding sunflowers.

[Subsection 5.3.] We obtain a small hitting set of full sunflowers, when G has no $k+1$ vertex-disjoint holes. For this, we consider two vertex sets Z_v and Z_w for some v and w on C, and apply Menger's theorem for two directions, say 'north' and 'south' hemispheres around C, between Z_v and Z_w in the graph G_{nbd}. We want to argue that if there are many paths in both directions, then we can find many vertex-disjoint holes. However, it is not clear how to link two families of paths.

To handle this issue, we find two families of paths whose supports cross on constant number of vertices. Since C is much larger than the obtained hitting set T_{petal} for petals, we can find 25 consecutive vertices that contain no vertices in T_{petal}. Let $v_{-2}, v_{-1}, v_0, \dots, v_{22}$ be such a set of consecutive vertices. Let $\mathcal{P}$ be the family of vertex-disjoint paths from Z_{v_0} to $Z_{v_{20}}$ whose supports are contained in $\{v_{-2}, v_{-1}, v_0, v_1, \dots, v_{20}, v_{21}, v_{22}\}$, and let $\mathcal{Q}$ be the family of vertex-disjoint paths from Z_{v_5} to $Z_{v_{16}}$ whose supports do not contain v_8 and v_{13}. Non-existence of petals with support intersecting $\{v_{-2}, v_{-1}, \dots, v_5\}$ implies that paths in $\mathcal{P}$ and $\mathcal{Q}$ are 'well-linked' at Z_{v_0} except for few paths, and a symmetric argument holds at $Z_{v_{20}}$. This allows us to link any pair of paths from $\mathcal{P}$ and $\mathcal{Q}$. If one of $\mathcal{P}$ and $\mathcal{Q}$ is small, then we can output a hitting set of full sunflowers using Menger's theorem. The size of the obtained set T_{full} will be at most $3k+14$.

3.2. D-traversing sunflowers

[Subsection 5.4.] It is easy to see that every D-traversing hole H contains exactly one vertex of D (since D is a clique and H contains a vertex of C), and every vertex of $V(C) \cap V(H)$ is a neighbor of the vertex in $D \cap V(H)$. Let $v \in V(C) \cap V(H)$ and $d \in D \cap V(H)$ be such an adjacent pair. We argue that H contains a subpath Q in $N[C] \setminus D$ that starts from v and is contained in $Z_{\{v,v_2,v_3\}}$ for some three consecutive vertices v, v_2, v_3 of C, such that

- $G[V(Q) \cup \{v, v_2, v_3, d\}]$ contains a D-traversing sunflower containing d and v.

In other words, even if $H - d$ has large support, we can find another D-traversing sunflower H' containing d and v where $H' - d$ has support on small number of vertices. The fact that H and H' share v is important as we will take one of d and v as a hitting set for such H', and this will hit H as well.

To this end, we create an auxiliary bipartite graph in which one part is D and the other part consists of sets of 3 consecutive vertices v_1, v_2, v_3 of C, and we add an edge between $d \in D$ and $\{v_1, v_2, v_3\}$ if $G[Z_{\{v_1,v_2,v_3\}} \cup \{d\}]$ contains a D-traversing hole. We argue that if this bipartite graph has a large matching, then we can find many vertex-disjoint holes, and otherwise, we have a small vertex cover. The union of all vertices involved in the vertex cover suffices to cover all D-traversing sunflowers. The hitting set $T_{trav:sunf}$ will have size at most $15k + 9$.

3.3. D-avoiding tulips

We follow the approach of constructing a subgraph of maximum degree 3 used in proving the Erdős-Pósa property for various types of cycles: roughly speaking, if there is a cycle after removing the vertices of degree 3 in the subgraph constructed so far, we augment the construction by adding some path or cycle. Simonovitz [24] first proposed this idea and proved that if the number of degree 3 vertices is s_{k+1}, then there are $k+1$ vertex-disjoint cycles. If a maximal construction has less than s_{k+1} vertices of degree 3, then we can hit all cycles of the input graph by taking all vertices of degree 3 and a few more vertices.

[Subsection 6.1.] The major obstacle for employing Simonovitz' approach is that for our purpose, we need to guarantee that every cycle of such a construction gives a hole. For this reason, we will carefully add a path so that every cycle in a construction contains some hole. We arrive at a notion of an *F-extension*, which is a path to be added iteratively with C at the beginning. By adding F-extensions recursively, we will construct a subgraph such that all vertices have degree 2 or 3 and it contains C. For a subgraph F of G_{deldom} such that all vertices have degree 2 or 3 and it contains C, an F-extension is a proper $V(F)$-path P in G_{deldom}, but has additional properties that

(i) both endpoints of P are vertices of degree 2 in F,
(ii) one of its endpoints should be in C, and
(iii) P has at least one endpoint in C whose second neighbor on P has no neighbors in F, and the path obtained from P by removing its endpoints is induced.

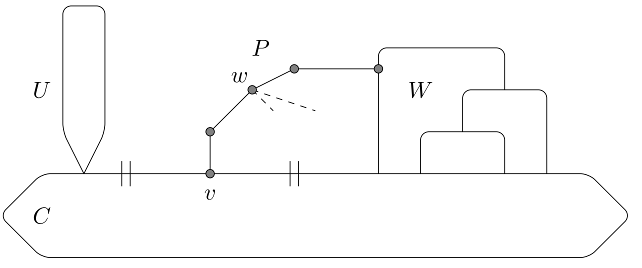

Fig. 4. A brief description of the construction W. Each extension contains at least one endpoint in C whose second neighbor in the extension has no neighbor in W hitherto constructed. For instance, P is a W-extension, and v is the vertex in $V(C) \cap V(P)$, and its second neighbor w in P has no neighbors in W. The subgraph U depicts an almost W-extension.

An almost F-extension is a similar object, but its endpoints on F are the same. Note that an almost F-extension is a cycle and is not an F-extension. We depict an (almost) F-extension in Fig. 4. When we recursively choose an F-extension to add, we apply two priority rules:

- We choose a shortest F-extension among all possible F-extensions.
- We choose an F-extension Q with maximum $|V(Q) \cap V(C)|$ among all shortest F-extensions.

Following these rules, we recursively add F-extensions until there are no F-extensions.

Let W be the resulting graph. The properties (ii), (iii) together with Lemma 2.3 guarantee that the subgraph induced by the vertex set of every cycle of W contains a hole. Therefore, in case when W contains s_{k+1} vertices of degree 3, we can find $k+1$ vertex-disjoint holes by Theorem 2.2. We may assume that it contains less than s_{k+1} vertices of degree 3. Let T_{branch} be the set of degree 3 vertices in W. We also separately argue that we can hit all of almost W-extensions by at most $5k+4$ vertices. Let T_{almost} be the hitting set for almost W-extensions.

Now, let T_{ext} be the union of

$$T_{petal} \cup T_{full} \cup T_{trav:sunf} \cup T_{branch} \cup T_{almost}$$

and

$$N_C^{20}[(T_{petal} \cup T_{full} \cup T_{trav:sunf} \cup T_{branch} \cup T_{almost}) \cap V(C)].$$

Note that

$$|T_{ext}| \leq 41(19k + (3k+14) + (15k+9) + (s_{k+1} - 1) + (5k+4)) \leq 41(s_{k+1} + 42k + 26).$$

Furthermore, $C - (T_{petal} \cup T_{full} \cup T_{trav:sunf} \cup T_{branch} \cup T_{almost})$ contains at most $s_{k+1} + 42k + 26$ connected components, and thus $C - T_{ext}$ does as well.

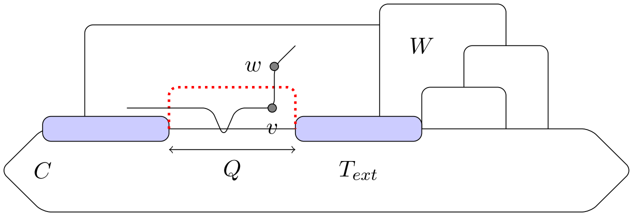

Fig. 5. A description of the set T_{ext} and a component Q of $C - T_{ext}$. When there is an edge vw where $v \in Z_{V(Q)} \setminus V(Q)$ and $w \notin N[C]$, we will prove in Lemma 6.4 that w has no neighbors in W. In particular, if there is a D-avoiding tulip containing such an edge, then we can find a W-extension or an almost W-extension starting with qvw for some $q \in V(Q)$.

[Subsection 6.2.] We discuss the patterns of the remaining tulips in $G_{deldom} - T_{ext}$. Since we will add all components of $C - T_{ext}$ having at most 35 vertices to the deletion set for D-avoiding tulips, we focus on components of $C - T_{ext}$ with at least 36 vertices. Let H be a D-avoiding tulip in $G_{deldom} - T_{ext}$. Let $Q = q_1 q_2 \cdots q_m$ be a connected component of $C - T_{ext}$, and we consider the Q-tunnel R.

We argue that there is no edge vw in H such that v is in the Q-tunnel, and w is not in $N[C]$. See Fig. 5 for an illustration. Suppose there is such a pair, and let $q \in V(Q)$ be a neighbor of v. We mainly prove that since q is sufficiently far from degree 3 vertices of W in C, w has no neighbors in W (this is why we take the 20-neighborhood of $V(C) \cap (T_{petal} \cup T_{full} \cup T_{trav:sunf} \cup T_{branch} \cup T_{almost})$ in C). This is formulated in Lemma 6.3. Note that qvw is a path where $q \in V(C)$ and w has no neighbors in W, and furthermore, H contains a vertex in $V(C) \setminus T_{ext}$ which is a vertex of degree 2 in W. Therefore, if we traverse in H following the direction from v to w, we meet some vertex having a neighbor which is a vertex of degree 2 in W. This gives a W-extension or an almost W-extension. But it is a contradiction as there is no W-extension, and T_{ext} hits all of almost W-extensions. So, there are no such edges vw.

This argument leads to an observation that if H contains some vertex q_i with $6 \leq i \leq m - 5$, then the restriction of H on R should be a path from Z_{q_1} to Z_{q_m}. Let P be the restriction of H on R. We additionally remove $\{q_j : 1 \leq j \leq 15, m - 14 \leq j \leq m\}$ and assume H is not removed. Then there are two ways that P can be placed inside the Q-tunnel R: either the endpoints of P are in the same connected component of $R - (T_{ext} \cup V(Q))$ or not. In the former case, we could reroute this path so that this part does not contain a vertex of Q. So, we could obtain a D-avoiding tulip containing less vertices of C. However, since $G - V(C)$ is chordal, there should be some subpath Q' of $C - T_{ext}$ such that the restriction of H on the Q'-tunnel is of the second type. We will show that such a path can be hit by removing 5 more vertices in Q'. This will give a vertex set $T_{avoid:tulip}$ of size at most $35(s_{k+1} + 42k + 26)$ hitting all the remaining D-avoiding tulips.

3.4. D-traversing tulips

[Subsection 6.3.] This case can be handled similarly as the case of D-traversing sunflowers. It turns out that T_{ext} hits every D-traversing tulip that contains precisely two vertices of C. Using a matching argument between D and the set of three consecutive vertices of C, we show that an additional set $T_{trav:tulip}$ of size at most $25k+9$ plus $T_{avoid:tulip}$ hits all the remaining D-traversing tulips unless G contains $k+1$ vertex-disjoint holes.

In total, we can output in polynomial time either $k+1$ vertex-disjoint holes in G, or a vertex set of size at most

$$\begin{aligned}&|T_{ext} \cup T_{avoid:tulip} \cup T_{trav:tulip}|\\ &\leq 41(s_{k+1} + 42k + 26) + 35(s_{k+1} + 42k + 26) + 25k + 9\\ &\leq 76(s_{k+1} + 42k + 26) + 25k + 9 = 76s_{k+1} + 3217k + 1985\end{aligned}$$

hitting all holes.

4. Structural properties of G

In this section, we present structural properties of a graph G with a shortest hole C. In Subsection 4.1, we derive a relationship between the distance between Z_v and Z_w in G_{deldom} for two vertices $v, w \in V(C)$ and the distance between v and w in C. Briefly, we show that the distance between Z_v and Z_w in G_{deldom} is at least some constant times the distance between v and w in C. We also prove that every connected subgraph in G_{nbd} has a connected support. In Subsection 4.2, we obtain some basic properties of C-dominating vertices. Recall that we assume that the length of C exceeds μ_k.

4.1. Distance lemmas

The following lemma classifies vertices in $N(C)$ with respect to the number of neighbors in C.

Lemma 4.1. *For every vertex v of $N(C)$, either it has at most 3 neighbors in C and these vertices are consecutive in C, or it is C-dominating.*

Proof. Let us write $N_i := \{v \in N(C) : |N(v) \cap V(C)| = i\}$ for $i \geq 1$. We first show that $N(C) = N_1 \uplus N_2 \uplus N_3 \uplus D$. Let $v \in N(C) \setminus D$.

We claim that v has no two neighbors w_1 and w_2 in C such that

(∗) there is a (w_1, w_2)-subpath Q of C where Q has length at least 2 and at most $|V(C)| - 3$ and v has no neighbor in the internal vertices of Q.

If there is such a path Q, then $w_1vw_2 \odot w_2Qw_1$ is a hole of length at most $|V(C)| - 1 < |V(C)|$, which contradicts the assumption that C is a shortest hole. So the claim holds.

This implies that v has no neighbors z_1 and z_2 with $\text{dist}_C(z_1, z_2) \geq 3$. Indeed, if such neighbors exist, then let Q be a (z_1, z_2)-subpath of C containing at least one internal vertex non-adjacent to v. Since $v \notin D$, such Q exists. Note that the length of Q is at least 3 and at most $|V(C)| - 3$. Then there exist two neighbors w_1 and w_2 of v in $V(Q)$ satisfying $(*)$, a contradiction. Therefore, the neighbors of v in C are contained in three consecutive vertices of C. This implies that $v \in N_1 \uplus N_2 \uplus N_3$.

Furthermore, if $v \in N(C) \setminus D$ has exactly two neighbors with distance 2 in C, then G contains a hole of length 4, a contradiction. Therefore, such a vertex has at most 3 neighbors in C that are consecutive in C, as required. $\square$

The next lemma is illustrated in Fig. 2.

Lemma 4.2. *Let x and y be two vertices in C such that $\mathrm{dist}_C(x, y) \geq 4$. Then there is no edge between Z_x and Z_y.*

Proof. By Lemma 4.1, the neighbors of any vertex in $N(C) \setminus D$ lie within distance at most 2, and thus x has no neighbors in Z_y and y has no neighbors in Z_x. Suppose $v \in Z_x \setminus \{x\}$ and $w \in Z_y \setminus \{y\}$ are adjacent. Let P and Q be (x, y)-subpaths of C such that the length of P is not greater than the length of Q. Since C has length at least 9, we may assume that Q has length at least 5.

Since the length of Q is at least 5, $N(v) \cap V(Q)$ is included in $N^2_C(x) \cap V(Q)$ and likewise we have $N(w) \cap V(Q) \subseteq N^2_C(y) \cap V(Q)$. Thus, Q does not contain a common neighbor of v and w.

Since the length of Q is at most $|V(C)| - 4$, a shortest (v, w)-path Q' in $G[\{v, w\} \cup V(Q)] - vw$ has length at most $|V(C)| - 2$. Moreover, Q' has length at least three due to the above assumption. Therefore, $vQ'w \odot wv$ is a hole strictly shorter than C, a contradiction. $\square$

We prove a generalization of Lemma 4.2.

Lemma 4.3. *Let m be a positive integer, and let P be a $V(C)$-path in G_{deldom} with endpoints x and y. If P has length at most $m + 2$, then $\mathrm{dist}_C(x, y) \leq 4m - 1$.*

Proof. We prove by induction on m. Lemma 4.2 settles the case when $m = 1$. Let us assume $m \geq 2$. Let $P = p_1p_2 \cdots p_n$ be a $V(C)$-path of length at most $m + 2$ from $p_1 = x$ and $p_n = y$ such that all of $p_2, \ldots, p_{n-1}$ are contained in $V(G_{deldom}) \setminus V(C)$, and suppose that $\mathrm{dist}_C(x, y) \geq 4m$. For $\mathrm{dist}_C(x, y) \geq 4m \geq 4$, Lemma 4.2 implies that p_2 is not adjacent to p_{n-1}. Therefore, P contains at least 5 vertices. We distinguish cases depending on whether $\{p_3, \ldots, p_{n-2}\}$ contains a vertex in $N(C)$ or not.

Case 1. $\{p_3, \ldots, p_{n-2}\}$ contains a vertex in $N(C)$.

We choose an integer $i \in \{3, \ldots, n-2\}$ such that $p_i \in N(C)$, and choose a neighbor z of p_i in C. Since there is a $V(C)$-path from p_1 to z of length i, by induction hypothesis, $\mathrm{dist}_C(p_1, z) < 4(i - 2)$. By the same reason, we have $\mathrm{dist}_C(z, p_n) < 4(n - i + 1 - 2) = 4(n - i - 1)$. Therefore, we have

$$\mathrm{dist}_C(p_1, p_n) \leq \mathrm{dist}_C(p_1, z) + \mathrm{dist}_C(z, p_n) < 4(n - 3) \leq 4m,$$

a contradiction.

Case 2. $\{p_3, \dots, p_{n-2}\}$ contains no vertices in $N(C)$.

Let Q be a shortest path from $N(p_2) \cap V(C)$ to $N(p_{n-1}) \cap V(C)$ in C, and let q, q' be its endpoints. Observe that $p_2 P p_{n-1}$ and $p_2 q \odot qQq' \odot q' p_{n-1}$ are two paths from p_2 to p_{n-1} where there are no edges between their internal vertices. Therefore, $G[V(Q) \cup (V(P) \setminus \{p_1, p_n\})]$ is a hole.

Since $\text{dist}_C(x, y) \le \frac{|V(C)|}{2}$, we have $m \le \frac{|V(C)|}{8}$. Therefore, the hole $G[V(Q) \cup (V(P) \setminus \{p_1, p_n\})]$ has length at most

$$|V(Q)| + |V(P)| \le \frac{|V(C)|}{2} + \frac{|V(C)|}{8} + 1 < |V(C)|.$$

This contradicts the assumption that C is a shortest hole of G.

This concludes the proof. $\square$

Next, we show that every connected subgraph in G_{nbd} has a connected support. The following observation is useful.

Lemma 4.4. *Let a, b be vertices of C with $\text{dist}_C(a, b) \in \{2, 3\}$ and let S be the set of internal vertices of the shortest (a, b)-path of C. Then there is no edge between $Z_a \setminus Z_S$ and $Z_b \setminus Z_S$.*

Proof. Suppose there is an edge between $x \in Z_a \setminus Z_S$ and $y \in Z_b \setminus Z_S$. If x is adjacent to b, then $x \ne a$, and by Lemma 4.1, x has a neighbor in S, contradicting the assumption that $x \notin Z_S$. Therefore, x is not adjacent to b. For the same reason, y is not adjacent to a. Therefore, the distance between $N(x) \cap V(C)$ and $N(y) \cap V(C)$ in C is 2 or 3, and the vertex set of the shortest path from $N(x) \cap V(C)$ to $N(y) \cap V(C)$ in C with $\{x, y\}$ induces a hole of length 5 or 6. This contradicts the assumption that C is a shortest hole in G and it has length greater than 6. $\square$

Lemma 4.5. *Let H be a connected subgraph in G_{nbd}. Then $C[\mathsf{sp}(H)]$ is connected.*

Proof. Suppose $\mathsf{sp}(H)$ is not connected. Then H contains an edge xy such that $\mathsf{sp}(G[\{x\}])$ and $\mathsf{sp}(G[\{y\}])$ are contained in distinct components of $C[\mathsf{sp}(H)]$. Notice that $\mathsf{sp}(G[\{x\}]) = N(x) \cap V(C)$ and $\mathsf{sp}(G[\{y\}]) = N(y) \cap V(C)$, and it follows that $x, y \notin V(C)$ by Lemma 4.1. We choose $a \in \mathsf{sp}(G[\{x\}])$ and $b \in \mathsf{sp}(G[\{y\}])$ with minimum $\text{dist}_C(a, b)$. By Lemma 4.2, we have $\text{dist}_C(a, b) \le 3$, and since $\mathsf{sp}(G[\{x\}])$ and $\mathsf{sp}(G[\{y\}])$ are contained in distinct components of $C[\mathsf{sp}(H)]$, we have $\text{dist}_C(a, b) \ge 2$. Let S be the set of internal vertices of the shortest (a, b)-path in C. By the choice, $x \in Z_a \setminus Z_S$ and $y \in Z_b \setminus Z_S$. Then by Lemma 4.4, there is no edge between $Z_a \setminus Z_S$ and $Z_b \setminus Z_S$. This contradicts the assumption that x is adjacent to y. $\square$

The following lemma provides a structure of a (Z_x, Z_y)-path in G_{nbd} for two vertices $x, y \in V(C)$.

Lemma 4.6. *Let x, y be two distinct vertices in C and P_1, P_2 be two (x, y)-paths in C. Let Q be a (Z_x, Z_y)-path in G_{nbd} such that $\mathsf{sp}(Q) \neq V(C)$. Then either Q is contained in $Z_{V(P_1)}$ or $Z_{V(P_2)}$.*

Proof. By Lemma 4.5, $\mathsf{sp}(Q)$ is connected, and since $\mathsf{sp}(Q) \neq V(C)$, $\mathsf{sp}(Q)$ contains either $V(P_1)$ or $V(P_2)$. Without loss of generality, we assume that $\mathsf{sp}(Q)$ contains $V(P_1)$. By the definition of a (Z_x, Z_y)-path, Q contains no vertex of $Z_{\{x,y\}}$ as an internal vertex. Let s and t be the two endpoints of Q contained in Z_x and Z_y, respectively.

We claim that Q contains no vertex of $V(G_{nbd}) \setminus Z_{V(P_1)}$, which immediately implies the statement. Suppose for contradiction that Q contains a vertex $v \in V(G_{nbd}) \setminus Z_{V(P_1)}$. Clearly, we have $v \neq s$ and $v \neq t$. Let u be a vertex in C such that $v \in Z_u$. Observe that $u \neq x$ and $u \neq y$, as Q contains no vertex of $Z_{\{x,y\}}$ as an internal vertex. Let Q_s and Q_t be the (s, v)- and (t, v)-subpath of Q, respectively.

By Lemma 4.5, $\mathsf{sp}(Q_s)$ contains an (x, u)-subpath of C. Assume $\mathsf{sp}(Q_s)$ contains the (x, u)-subpath of C containing y. This means that Q_s contains a vertex of Z_y, other than t, contradicting the fact that Q contains no vertex of $Z_{\{x,y\}}$ as an internal vertex. Therefore, $\mathsf{sp}(Q_s)$ contains the vertex set of the (x, u)-subpath of C avoiding y. Similarly, $\mathsf{sp}(Q_t)$ contains the vertex set of the (y, u)-subpath of C avoiding x. Now, observe that $\mathsf{sp}(Q) = \mathsf{sp}(Q_s) \cup \mathsf{sp}(Q_t) \supseteq V(P_2)$ and also by assumption, we have $V(P_1) \subseteq \mathsf{sp}(Q)$. Consequently, we have $\mathsf{sp}(Q) = V(C)$, a contradiction. This completes the proof. $\square$

The following lemma is useful to find a hole with a small support.

Lemma 4.7. *Let P and Q be two vertex-disjoint induced paths of G_{nbd} such that*

- *there are no edges between $V(P)$ and $V(Q)$, and*
- *$\mathsf{sp}(P) \neq V(C)$ and $\mathsf{sp}(Q) \neq V(C)$.*

If $|\mathsf{sp}(P) \cap \mathsf{sp}(Q)| \geq 3$ and $x, y, z \in \mathsf{sp}(P) \cap \mathsf{sp}(Q)$ are three consecutive vertices on C, then $Z_{\{x,y,z\}}$ contains a hole.

Proof. Since $x, y, z \in \mathsf{sp}(P) \cap \mathsf{sp}(Q)$ and there is no edge between $V(P)$ and $V(Q)$, P and Q contains no vertex of $\{x, y, z\}$. Let P' be a shortest (Z_x, Z_z)-subpath of P, and let Q' be a shortest (Z_x, Z_z)-subpath of Q. As $\mathsf{sp}(P) \neq V(C)$ and $\mathsf{sp}(Q) \neq V(C)$, $V(P')$ and $V(Q')$ are contained in $Z_{\{x,y,z\}}$ by Lemma 4.6. By the preconditions, P' and Q' are vertex-disjoint and there are no edges between P' and Q'. Thus, by Lemma 2.3, $G[V(P') \cup V(Q') \cup \{x, z\}]$ contains a hole, which is in $Z_{\{x,y,z\}}$. $\square$

4.2. C-dominating vertices

We recall that D is the set of C-dominating vertices. We observe that D is a clique because G does not contain a hole of length 4.

Lemma 4.8. *The set D is a clique. Furthermore, every hole contains at most one vertex of D.*

Proof. Note that G contains no hole of length 4. This implies that any two vertices of D are adjacent, which proves the first statement. To see the second statement, suppose that H is a hole containing two distinct vertices u, v of D and let $x \in V(H) \cap V(C)$ (there are no holes in $G - V(C)$). Then $\{x, u, v\}$ forms a triangle, contradicting the assumption that H is a hole. $\square$

Lemma 4.9. *If H is a D-traversing hole, then it contains at most two vertices of C. Furthermore, every vertex of $V(H) \cap V(C)$ is adjacent to the unique C-dominating vertex on H.*

Proof. By Lemma 4.8, H contains exactly one vertex of D, say v. If there is a vertex $x \in V(C) \cap V(H)$, then x is adjacent to v as v is C-dominating. Therefore, any vertex of $V(C) \cap V(H)$ is adjacent to v on H. Since H is a cycle, H contains at most two vertices of C. $\square$

5. Hitting all sunflowers

In this section, we obtain a hitting set for sunflowers, unless G contains $k+1$ vertex-disjoint holes. Like in the previous section, we assume that (G, k, C) is given as an input such that C is a shortest hole of G of length strictly greater than μ_k, and $G - V(C)$ is chordal.

5.1. Hitting all petals

We recall that a D-avoiding sunflower H is a petal if $|\mathsf{sp}(H)| \leq 7$. By Lemma 4.5, the vertices in the support of a petal is consecutive in C.

Lemma 5.1. *There is a polynomial-time algorithm which finds either $k+1$ vertex-disjoint holes in G or a vertex set $T_{petal} \subseteq V(C)$ of at most $19k$ vertices such that*

- *for every petal H, we have $\mathsf{sp}(H) \subseteq T_{petal}$.*

Proof. Set $X := \emptyset$, $\mathcal{C} = \emptyset$, and *counter* $:= 0$ at the beginning. We recursively do the following until the counter reaches $k+1$. For every set of nine consecutive vertices

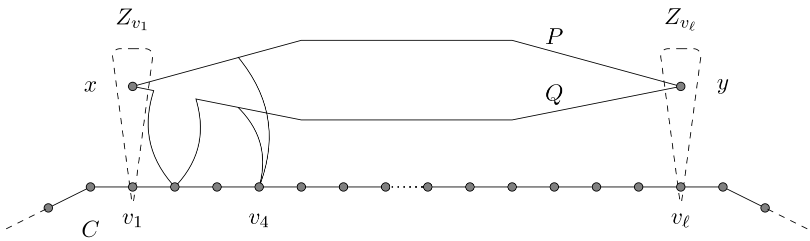

Fig. 6. A D-avoiding sunflower H in $G - T_{petal}$ with $\mathsf{sp}(H) \neq V(C)$ in Lemma 5.2, which is not a petal. This hole H has to intersect one of $\{v_1, v_2, v_3\}$ and $\{v_{\ell-2}, v_{\ell-1}, v_\ell\}$. If H intersects $\{v_1, v_2, v_3\}$ as in the figure, then there is a petal with support contained in $\{v_1, \dots, v_6\}$. The existence of the intersection on $\{v_1, v_2, v_3\}$ implies that T_{petal} does not contain one of v_1, v_2, and v_3. This contradicts the fact that T_{petal} contains the support of every petal.

$v_0, v_1, v_2, \dots, v_7, v_8$ of C with $\{v_1, v_2, \dots, v_7\} \cap X = \emptyset$, we test if $G[Z_{\{v_1,v_2,\dots,v_7\}} \setminus Z_{\{v_0,v_8\}}]$ contains a hole H, and if so, add vertices in $\{v_1, v_2, \dots, v_7\}$ to X, and add H to $\mathcal{C}$ and increase the counter by 1. If the counter reaches $k+1$, then we stop. If the counter does not reach $k+1$, then we have $|X| \leq 7k$. In this case, we set T_{petal} as the 6-neighborhood of X in C. So, $|T_{petal}| \leq (7+12)k = 19k$.

By construction, any hole $H \in \mathcal{C}$ has a support that is fully contained in the considered set $\{v_1, v_2, \dots, v_7\}$. Observe that we choose this set to be disjoint from X constructed thus far. Therefore, holes in $\mathcal{C}$ are pairwise vertex-disjoint; otherwise, their supports have a common vertex. This implies that if the counter reaches $k+1$, then we can output $k+1$ vertex-disjoint holes.

Assume the counter does not reach $k+1$. In this case, we claim that for every petal H, $\mathsf{sp}(H) \subseteq T_{petal}$. Let H be a petal. By the definition of a petal and by Lemma 4.5, there is a set of 7 consecutive vertices $w_1, w_2, \dots, w_7$ in C such that $\mathsf{sp}(H) \subseteq \{w_1, w_2, \dots, w_7\}$. If the set $\{w_1, w_2, \dots, w_7\}$ is disjoint from X, then the above procedure must have considered this set and added it to X, a contradiction. Therefore, $\{w_1, w_2, \dots, w_7\} \cap X \neq \emptyset$. Then during the step of adding 6-neighborhood of X to T_{petal}, $\{w_1, w_2, \dots, w_7\}$ is added to T_{petal}, and thus we have $\mathsf{sp}(H) \subseteq \{w_1, w_2, \dots, w_7\} \subseteq T_{petal}$ as claimed. $\square$

In what follows, we reserve T_{petal} to denote a vertex subset of $V(C)$ that contains the support of every petal.

5.2. Polarization of D-avoiding sunflowers

We show that every D-avoiding sunflower in $G - T_{petal}$ is full. This will imply that, in order to hit every D-avoiding sunflower it is sufficient to find a hitting set for full sunflowers. We illustrate Lemma 5.2 in Fig. 6.

Lemma 5.2. *Every D-avoiding sunflower H in $G - T_{petal}$ is full, that is, $\mathsf{sp}(H) = V(C)$.*

Proof. Suppose H is a D-avoiding sunflower in $G - T_{petal}$ such that $8 \leq |\mathsf{sp}(H)| < |V(C)|$. By Lemma 4.5, $C[\mathsf{sp}(H)]$ is a subpath of C. Let $C[\mathsf{sp}(H)] = v_1 v_2 \cdots v_\ell$. Choose $x \in$

$V(H) \cap Z_{v_1}$ and $y \in V(H) \cap Z_{v_\ell}$, and let P and Q be the two (x,y)-paths on H. As each of $C[\mathsf{sp}(P)]$ and $C[\mathsf{sp}(Q)]$ is connected by Lemma 4.5, we have $\mathsf{sp}(P) = \mathsf{sp}(Q) = \mathsf{sp}(H)$.

Recall that $V(H) \cap V(C) \neq \emptyset$ and $V(H) \cap V(C)$ must be contained in $\mathsf{sp}(H)$. We argue that any $v_i \in \mathsf{sp}(H)$ with $i \in \{4, 5, \ldots, \ell-3\}$ does not lie on H. Suppose $v_i \in V(H) \cap V(C)$ for some $4 \leq i \leq \ell - 3$. Notice that both x and y are distinct from v_i. Therefore, v_i belongs to exactly one of P and Q. Without loss of generality, we assume $v_i \in V(P)$. Since $v_i \in \mathsf{sp}(Q)$, $Z_{v_i} \cap V(Q) \neq \emptyset$ and thus we can choose a vertex v_i' from the set $Z_{v_i} \cap V(Q)$. Lemma 4.1 and $v_i' \notin D$ imply that v_i' is not adjacent to v_1 or v_ℓ, and thus $v_i' \notin Z_{v_1}$ and $v_i' \notin Z_{v_\ell}$. This means that v_i' is distinct from x and y, especially v_i' is an internal vertex of Q. However, $v_i v_i' \in E(G)$ is a chord of H, a contradiction.

Therefore, $v_i \notin V(H)$ for every $4 \leq i \leq \ell - 3$. At least one of $\{v_1, v_2, v_3\}$ and $\{v_{\ell-2}, v_{\ell-1}, v_\ell\}$ intersects with $V(H)$, and we assume that $\{v_1, v_2, v_3\}$ intersects with $V(H)$ without loss of generality (a symmetric argument works in the other case). From each of P and Q, choose the first vertex (starting from x) that lies in Z_{v_4} and call them $p \in V(P)$ and $q \in V(Q)$ respectively; the existence of such vertices follows from $v_4 \in \mathsf{sp}(P) = \mathsf{sp}(Q)$. Let H' be the cycle $pPx \odot xQq \odot qv_4p$.

Since p, q are the first vertices contained in Z_{v_4}, v_4 has no neighbors in $(V(xPp) \cup V(xQq)) \setminus \{p, q\}$, and thus H' is a hole. By Lemma 4.1, we have $\mathsf{sp}(H') \subseteq \{v_1, v_2, \ldots, v_6\}$, that is, H' is a petal. Because T_{petal} contains the support of every petal, $\{v_1, v_2, v_3, v_4\} \subseteq \mathsf{sp}(H') \subseteq T_{petal}$. On the other hand, $\{v_1, v_2, v_3\} \cap V(H) \neq \emptyset$ and $V(H) \cap T_{petal} = \emptyset$ implies $\{v_1, v_2, v_3\} \setminus T_{petal} \neq \emptyset$. This is a contradiction. This completes the proof. $\square$

5.3. Hitting all D-avoiding sunflowers

In this subsection we focus on full sunflowers.

Proposition 5.3. *There is a polynomial-time algorithm which finds either $k+1$ vertex-disjoint holes in G or a vertex set $T_{full} \subseteq V(G) \setminus T_{petal}$ of at most $3k + 14$ vertices such that $T_{petal} \cup T_{full}$ hits all full sunflowers.*

Our strategy is to find two collections of many vertex-disjoint paths so that we can link the paths to obtain many vertex-disjoint holes. The following lemma explains how to do this. Note that since C has length greater than μ_k, $V(C) \setminus T_{petal}$ contains 25 vertices that are consecutive in C.

Lemma 5.4. *Let $v_{-2}v_1v_0v_1 \cdots v_{20}v_{21}v_{22}$ be a subpath of C that does not intersect T_{petal}, and X be the (v_0, v_{20})-subpath of C containing v_1 and Y be the (v_5, v_{15})-subpath of C containing v_4. Let $\mathcal{P}$ be a collection of vertex-disjoint $(Z_{v_0}, Z_{v_{20}})$-paths in $G[Z_{V(X)}]$, and let $\mathcal{Q}$ be a collection of vertex-disjoint $(Z_{v_5}, Z_{v_{15}})$-paths in $G[Z_{V(Y)}]$. Given such $\mathcal{P}$ and $\mathcal{Q}$, if $|\mathcal{P}| \geq k + 13$ and $|\mathcal{Q}| \geq 3k + 15$, then one can output $k+1$ vertex-disjoint holes in polynomial time.*

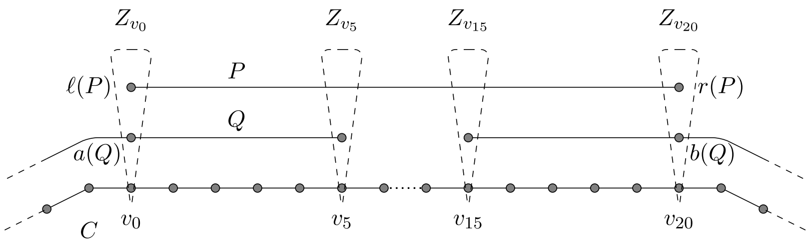

Fig. 7. Paths $P \in \mathcal{P}_1$ and $Q \in \mathcal{Q}_1$ in Lemma 5.4. The vertices $\ell(P)$ and $a(Q)$ for $P \in \mathcal{P}_1$ and $Q \in \mathcal{Q}_1$ are adjacent, otherwise, we can find a hole $Z_{\{v_0,v_1,v_2\}}$, which is a petal. For the same reason, $r(P)$ is adjacent to $b(Q)$ for $P \in \mathcal{P}_1$ and $Q \in \mathcal{Q}_1$.

Proof. We begin with the observation that there is no petal whose support contains a vertex of $\{v_{-2}, v_{-1}, \dots, v_{22}\}$. This is because $\{v_{-2}, v_{-1}, \dots, v_{22}\} \cap T_{petal} = \emptyset$ by assumption and T_{petal} contains the support of every petal of G. We may assume that every path in $\mathcal{P}$ is induced, and similarly every path in $\mathcal{Q}$ is induced.

We take a subset $\mathcal{P}_1$ of $\mathcal{P}$ with $|\mathcal{P}_1| = k + 1$ that consists of paths containing no vertices of $\{v_0, v_1, \dots, v_5\} \cup \{v_{15}, v_{16}, \dots, v_{20}\}$. Such a collection $\mathcal{P}_1$ exists because the paths of $\mathcal{P}$ are vertex-disjoint and at most 12 of them intersect with $\{v_0, v_1, \dots, v_5\} \cup \{v_{15}, v_{16}, \dots, v_{20}\}$. Similarly we take a subset $\mathcal{Q}_1$ of $\mathcal{Q}$ with $|\mathcal{Q}_1| = 3k + 3$ that consists of paths containing no vertices of $\{v_0, v_1, \dots, v_5\} \cup \{v_{15}, v_{16}, \dots, v_{20}\}$.

For each $P \in \mathcal{P}_1$, let $\ell(P)$ and $r(P)$ be the endpoints of P contained in Z_{v_0} and $Z_{v_{20}}$, respectively. For each $Q \in \mathcal{Q}_1$, let $a(Q)$ be the vertex of $Z_{v_0} \cap V(Q)$ that is closest to $Z_{v_5} \cap V(Q)$ in Q, and let $b(Q)$ be the vertex in $Z_{v_{20}} \cap V(Q)$ that is closest to $Z_{v_{15}} \cap V(Q)$ in Q. By definition, no internal vertex of the subpath of Q from $a(Q)$ to the vertex in $V(Q) \cap Z_{v_5}$ contains a neighbor of v_0, and similarly no internal vertex of the subpath of Q from $b(Q)$ to the vertex in $V(Q) \cap Z_{v_{15}}$ contains a neighbor of v_{20}. See Fig. 7 for an illustration.

Claim 1. *Let $w \in \{\ell(P) : P \in \mathcal{P}_1\}$ and $z \in \{a(Q) : Q \in \mathcal{Q}_1\}$. If $w \neq z$, then $wz \in E(G)$. Let $w \in \{r(P) : P \in \mathcal{P}_1\}$ and $z \in \{b(Q) : Q \in \mathcal{Q}_1\}$. If $w \neq z$, then $wz \in E(G)$.*

Proof of the Claim. Suppose $w \in \{\ell(P) : P \in \mathcal{P}_1\}$ and $z \in \{a(Q) : Q \in \mathcal{Q}_1\}$ such that $w \neq z$ and they are not adjacent. Let $P_w \in \mathcal{P}_1$ and $Q_z \in \mathcal{Q}_1$ such that $\ell(P_w) = w$ and $a(Q_z) = z$. Let Q'_z be the subpath of Q_z from z to the vertex in Z_{v_5}. Note that v_0 has no neighbors in $V(P_w) \setminus \{w\}$ and $V(Q'_z) \setminus \{z\}$. In case when P_w and Q'_z meet somewhere in $\bigcup_{i\in\{1,2\}} Z_{v_i}$, we obtain a hole contained in $Z_{\{v_0,v_1,v_2\}}$ by Lemma 2.3. When P_w and Q'_z do not meet in $\bigcup_{i\in\{1,2\}} Z_{v_i}$, there is a hole contained in $Z_{\{v_0,v_1,v_2\}}$ by Lemma 2.3 since v_2 has a neighbor in both P_w and Q'_z. In both cases, there is a petal with support contained in $\{v_i : -2 \leq i \leq 4\}$, a contradiction. We conclude that $wz \in E(G)$. The proof of the latter statement is symmetric. $\diamond$

For every $P \in \mathcal{P}_1$, $\ell(P)$ is the unique vertex of $Z_{v_0} \cap V(P)$. Therefore, for fixed $P \in \mathcal{P}_1$, there is at most one path $Q \in \mathcal{Q}_1$ such that $V(Q) \cap V(P) \cap Z_{v_0} \neq \emptyset$. Similarly, there is at most one path of $\mathcal{Q}_1$ intersecting with P at a vertex of $Z_{v_{20}}$. We construct a new collection $\mathcal{Q}_2$ so that

> for every $Q \in \mathcal{Q}_1$, $\mathcal{Q}_2$ contains the subpath $a(Q)Qb(Q)$ if and only if Q does not intersect with any $P \in \mathcal{P}_1$ at a vertex of $Z_{v_0} \cup Z_{v_{20}}$.

Observe that $\mathcal{Q}_2$ contains at least $k+1$ paths because each path of $\mathcal{P}_1$ can make at most two paths of $\mathcal{Q}_1$ drop out. For our purpose, taking precisely $k+1$ paths is sufficient. Let $\mathcal{P}_1 = \{P_1, \dots, P_{k+1}\}$ and $\mathcal{Q}_2 = \{Q_1, Q_2, \dots, Q_{k+1}\}$. For each $i \in \{1, 2, \dots, k+1\}$, we create a cycle C_i from the disjoint union of $P_i \in \mathcal{P}_1$ and $Q_i \in \mathcal{Q}_2$ by adding two edges $a(Q_i)\ell(P_i)$ and $b(Q_i)r(P_i)$. Such edges exist by Claim 1.

We observe that each C_i contains a hole. To see this, take $v \in Z_{v_{10}} \cap V(P_i)$. As $Q_i \in \mathcal{Q}_2$ is a path of $G[Z_{V(Y)}]$, Lemma 4.1 implies that v is not adjacent to any vertex of Q_i. Note that v is an internal vertex of the induced path P_i. Therefore, $G[V(C_i)]$ contains a hole by Lemma 2.3.

Lastly, we verify that two holes contained in distinct cycles of $\{C_i : 1 \leq i \leq k+1\}$ are vertex-disjoint. To prove this, it is sufficient to show that for two integers $a, b \in \{1, 2, \dots, k+1\}$, no internal vertex of $P_a \in \mathcal{P}_1$ is an internal vertex of $Q_b \in \mathcal{Q}_2$. Suppose the contrary, that is, w is an internal vertex of $P_a \in \mathcal{P}_1$ and $Q_b \in \mathcal{Q}_2$ simultaneously for some a, b. Since Q_b is a path of $G[Z_Y]$ and $w \notin Z_{v_0} \cup Z_{v_{20}}$ is an internal vertex of P_a, we have $w \in Z_{\{v_1,v_2,v_3,v_4,v_5\}} \cup Z_{\{v_{15},v_{16},v_{17},v_{18},v_{19}\}}$. Without loss of generality, we assume $w \in Z_{\{v_1,v_2,v_3,v_4,v_5\}}$. In fact, w cannot be in Z_{v_5} since otherwise, the path $Q'_b \in \mathcal{Q}_1$ having Q_b as a proper subpath contains a vertex of Z_{v_5} as an internal vertex; violating the definition of $(Z_{v_5}, Z_{v_{15}})$-path. Now observe that Q_b contains, as a subpath, a Z_{v_0}-path Q' having all internal vertices in $Z_{\{v_1,v_2,v_3,v_4\}} \setminus Z_{\{v_0,v_5\}}$. Let x, y be the endpoints of Q'. Since v_0 is adjacent to x, y but is not adjacent to any internal vertices of Q', $G[\{v_0\} \cup V(Q')]$ contains a hole by Lemma 2.3, a contradiction. A symmetric argument holds for the case $w \in Z_{\{v_{15},v_{16},v_{17},v_{18},v_{19}\}}$. Therefore, $\{C_i : 1 \leq i \leq k+1\}$ is a collection of vertex-disjoint holes of G, which completes the proof. $\square$

Proof of Proposition 5.3. Note that $|T_{petal}| \leq 19k$ and $|V(C)| > \mu_k > 25|T_{petal}|$ by assumption. Thus, there are 25 consecutive vertices on C having no vertices in T_{petal}. We choose a subpath $v_{-2}v_{-1}v_0v_1 \cdots v_{20}v_{21}v_{22}$ of C that contains no vertices in T_{petal}. Let P_1 be the (v_0, v_{20})-subpath of C containing v_1 and P_2 be the (v_5, v_{15})-subpath of C containing v_4.

We apply Menger's Theorem for $(Z_{v_0}, Z_{v_{20}})$-paths in $G[Z_{V(P_1)}]$, and then for $(Z_{v_5}, Z_{v_{15}})$-paths in $G[Z_{V(P_2)}]$. We have one of the following.

- The first application of Menger's Theorem outputs a vertex set X with $|X| \leq k+12$ hitting all $(Z_{v_0}, Z_{v_{20}})$-paths in $G[Z_{V(P_1)}]$.
- The second application of Menger's Theorem outputs a vertex set X with $|X| \leq 3k+14$ hitting all $(Z_{v_5}, Z_{v_{15}})$-paths in $G[Z_{V(P_2)}]$.

- The first algorithm outputs at least $k+13$ vertex-disjoint paths, and the second algorithm outputs at least $3k+15$ vertex-disjoint paths.

In the third case, by Lemma 5.4, we can construct $k+1$ vertex-disjoint holes in polynomial time.

Suppose we obtained a vertex set X in the first case. We claim that $T_{petal} \cup X$ hits all full sunflowers. Suppose that there is a full sunflower H avoiding every vertex of $T_{petal} \cup X$. By definition, $\mathsf{sp}(H) = V(C)$. In particular, H contains at least one vertex of $Z_{v_{10}}$, say w. Let F be the connected component of the restriction of H on $G[Z_{V(P_1)}]$ containing w. Clearly F is a path and its endpoints are contained in $Z_{\{v_0,v_{20}\}}$ because of Lemma 4.5.

Suppose that the endpoints of F are contained in distinct sets of Z_{v_0} and $Z_{v_{20}}$, respectively. Let F' be a subpath of F that is a $(Z_{v_0}, Z_{v_{20}})$-path. Note that F' is a $(Z_{v_0}, Z_{v_{20}})$-path of $G[Z_{V(P_1)}]$ because F is a path of $G[Z_{V(P_1)}]$. But it contradicts the fact that X hits all such paths.

Suppose that both endpoints of F are contained in one of Z_{v_0} or $Z_{v_{20}}$, say Z_{v_0}. Let F_1 and F_2 be the two subpaths of F from w to its endpoints. Then by Lemma 4.5, both $\mathsf{sp}(F_1)$ and $\mathsf{sp}(F_2)$ contain the (v_0, v_{10})-subpath of $P_1 = v_0v_1 \cdots v_{20}$. This implies that $\mathsf{sp}(F_1 - w) \cap \mathsf{sp}(F_2 - w)$ contains $\{v_0, v_1, v_2\}$. Since there are no edges between $F_1 - w$ and $F_2 - w$, Lemma 4.7 implies that there is a hole contained in $Z_{\{v_0,v_1,v_2\}}$. This is a contradiction because we assumed $\{v_{-2}, v_{-1}, \ldots, v_{22}\} \cap T_{petal} = \emptyset$ while T_{petal} contains the support of every petal of G. Therefore, $T_{petal} \cup X$ hits every full sunflower. The case when both endpoints of F are contained on $Z_{v_{20}}$ follows from a symmetric argument.

The second case when we obtain the vertex set X with $|X| \leq 3k + 14$ can be handled similarly. Hence, in the first or second case, we can output a required vertex set T_{full} of size at most $3k + 14$ hitting every full sunflower in polynomial time. $\square$

5.4. Hitting all D-traversing sunflowers

Our proof builds on the observation that any D-traversing sunflower entails another D-traversing sunflower H' where the support of the path $H' - D$ is 'small'. Then we exploit the min-max duality of vertex cover and matching on bipartite graphs in order to cover such D-traversing sunflowers with small support.

The following lemma describes how to obtain such a sunflower H'. We depict the setting of Lemma 5.5 in Fig. 8.

Lemma 5.5. *Let $v_1v_2 \cdots v_5$ be a subpath of C such that $\{v_1, \ldots, v_5\} \cap T_{petal} = \emptyset$ and let $P = p_1p_2 \cdots p_m$ be a path in $G[D \cup Z_{\{v_1,v_2,\ldots,v_5\}}]$ such that*

(i) p_1 *is a C-dominating vertex and* $p_2 = v_3$,
(ii) $p_m \in Z_{\{v_1,v_5\}} \setminus \{v_1, v_5\}$,
(iii) *all internal vertices of P are in* $Z_{\{v_2,v_3,v_4\}} \setminus Z_{\{v_1,v_5\}}$, *and*

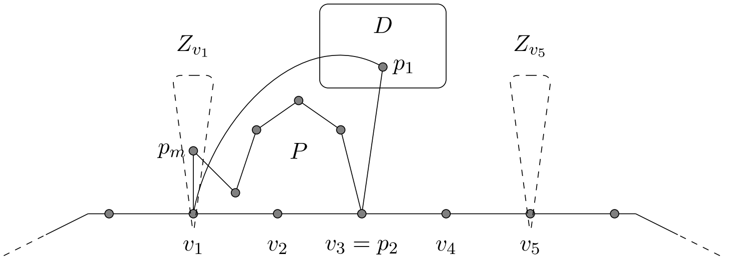

Fig. 8. Obtaining another sunflower in Lemma 5.5. As $v_1p_1p_2$ is an induced path and p_1 has no neighbors in the set of internal vertices of $p_2Pp_m \odot p_mv_1$, $G[V(P) \cup \{v_1\}]$ contains a hole.

(iv) $E(G[V(P)]) \setminus E(P) \subseteq \{p_1p_m\}$*; that is,* $G[V(P)]$ *is either an induced path* $p_1p_2\cdots p_m$ *or an induced cycle* $p_1p_2\cdots p_mp_1$*,*

(v) if $G[V(P)]$ *is an induced cycle, then* $m \geq 4$*.*

Then there exists a D*-traversing sunflower* H *containing* p_1 *and* p_2 *such that* $V(H) \setminus \{p_1\} \subseteq Z_{\{v_1,v_2,v_3\}} \cap (V(P) \cup \{v_1\})$ *or* $V(H) \setminus \{p_1\} \subseteq Z_{\{v_3,v_4,v_5\}} \cap (V(P) \cup \{v_5\})$*.*

Proof. We claim the following:

> If $p_m \in Z_{v_1}$, then $P - p_1$ is contained in $Z_{\{v_1,v_2,v_3\}}$. Likewise, if $p_m \in Z_{v_5}$, then $P - p_1$ is contained in $Z_{\{v_3,v_4,v_5\}}$.

We only prove the first statement; the proof of the second statement will be symmetric. Let us assume $p_m \in Z_{v_1}$. We observe that P contains no vertex in Z_{v_5} because all internal vertices of P are in $Z_{\{v_2,v_3,v_4\}} \setminus Z_{\{v_1,v_5\}}$.

We first show that P contains no vertex in $Z_{v_4} \setminus Z_{v_3}$. Suppose the contrary and let $w \in V(P) \cap (Z_{v_4} \setminus Z_{v_3})$. Since both $C[\mathsf{sp}(p_2Pw)]$ and $C[\mathsf{sp}(wPp_m)]$ are connected by Lemma 4.5, P contains a Z_{v_3}-subpath P' whose internal vertices are all contained in $Z_{v_4} \setminus Z_{v_3}$. Then v_3 is not adjacent to any internal vertex of P', and by Lemma 2.3, $G[V(P') \cup \{v_3\}]$ contains a hole, which is a petal. This contradicts the fact that $v_3 \notin T_{petal}$, because by the construction of T_{petal} in Lemma 5.1, T_{petal} fully contains the support of every petal. Hence, P contains no vertex of $Z_{v_4} \setminus Z_{v_3}$ and we have $V(P) \setminus \{p_1\} \subseteq Z_{\{v_1,v_2,v_3\}}$.

We claim that there is a D-traversing sunflower as claimed. If $p_1p_m \in E(G)$, then $G[V(P)]$ is a hole as claimed by (iv) and due to the previous claim. Hence, we may assume p_1 is not adjacent to p_m. Observe that $v_1p_1p_2$ is an induced path. Also, $p_2Pp_m \odot p_mv_1$ is a path from v_1 to v_3, and it does not contain v_2. Indeed, if the path $p_2Pp_m \odot p_mv_1$ contains v_2, then $\{p_1, p_2 = v_3, v_2\}$ forms a triangle, a contradiction to (iv). Now, p_1 has no neighbors in $V(P) \setminus \{p_1, p_2\}$ as we assumed that p_1 is not adjacent to p_m. Therefore, we can apply Lemma 2.3 with the induced path $v_1p_1p_2$ with p_1 as an internal vertex and the path $p_2Pp_m \odot p_mv_1$. It follows that there is a hole in $G[V(P) \cup \{v_1\}]$ containing p_1, p_2 such that $V(H) \setminus \{p_1\} \subseteq Z_{\{v_1,v_2,v_3\}}$. The statement follows immediately. □

Lemma 5.6. *Let H be a D-traversing sunflower in $G - T_{petal}$ containing a C-dominating vertex d. Then there exist three consecutive vertices x, y, z on C and a D-traversing sunflower H' containing d such that $V(H) \cap \{x, y, z\} \neq \emptyset$ and $V(H') \setminus \{d\} \subseteq Z_{\{x,y,z\}}$.*

Proof. By Lemma 4.9, H contains at most 2 vertices of C, and every vertex in $V(H) \cap V(C)$ is adjacent to the (unique) vertex of $V(H) \cap D$. Let P be the connected component of $H - (V(C) \cup \{d\})$. By Lemma 4.8, every hole contains at most one vertex of D, and it implies that P contains no vertices of D. Let $a \in V(H) \cap V(C)$. If the support of P is contained in $N_C[a]$, then we are done as $|N_C[a]| \leq 3$ and we can take $H' = H$ and $\{x, y, z\} = N_C[a]$. We may assume that the support of P contains a vertex in C whose distance to a in C is 2 by Lemma 4.5. Let v_1, v_2, v_3, v_4, v_5 be the consecutive vertices of C where $a = v_3$. This assumption implies that P contains a vertex in either Z_{v_1} or Z_{v_5}.

Let w be the vertex of $Z_{\{v_1,v_5\}} \cap V(H)$ that is closest to a in P. Let Q be the (d, w)-subpath of H containing a. We verify the preconditions of Lemma 5.5 with $(p_1, p_2, P) = (d, a, Q)$. The first condition is clear. Note that H contains neither v_2 nor v_4; otherwise, dav_2 or dav_4 is a triangle in H, contradicting the assumption that H is a hole. Thus Q contains neither v_2 nor v_4. If w is v_1 or v_5, then the neighbor of w in Q is also in $Z_{\{v_1,v_5\}}$, contradicting the choice of w. Thus, $w \in Z_{\{v_1,v_5\}} \setminus \{v_1, v_5\}$. Clearly, all internal vertices of Q are in $Z_{\{v_2,v_3,v_4\}} \setminus Z_{\{v_1,v_5\}}$; otherwise by Lemma 4.5, Q must contain an internal vertex from $Z_{\{v_1,v_5\}}$, contradicting the choice of w. The last two conditions are satisfied because H is a hole. Then (d, a, Q) meets the preconditions of Lemma 5.5.

Therefore, there exists a D-traversing sunflower H' containing d and a such that $V(H') \setminus \{d\} \subseteq Z_{\{v_1,v_2,v_3\}}$ or $V(H') \setminus \{d\} \subseteq Z_{\{v_3,v_4,v_5\}}$. As $a = v_3 \in V(H)$, we have $V(H) \cap \{v_1, v_2, v_3\} \neq \emptyset$ or $V(H) \cap \{v_3, v_4, v_5\} \neq \emptyset$, respectively. □

Based on Lemma 5.6, we prove the following.

Proposition 5.7. *There is a polynomial-time algorithm which finds either $k + 1$ vertex-disjoint holes in G or a vertex set $T_{trav:sunf} \subseteq (D \cup V(C)) \setminus T_{petal}$ of size at most $15k + 9$ such that $T_{petal} \cup T_{trav:sunf}$ hits all D-traversing sunflowers.*

Proof. Let $C = v_0 v_1 \cdots v_{m-1} v_0$. All additions are taken modulo m. We create an auxiliary bipartite graph $\mathcal{G}_i = (D \uplus \mathcal{A}_i, \mathcal{E}_i)$ for each $0 \leq i \leq 4$, such that

- $\mathcal{A}_i = \{\{v_{5j+i}, v_{5j+i+1}, v_{5j+i+2}\} : j = 0, 1, \ldots, \lfloor \frac{m}{5} \rfloor - 1\}$,
- there is an edge between $d \in D$ and $\{x, y, z\} \in \mathcal{A}_i$ if and only if there is a hole H containing d such that $V(H) \setminus \{d\} \subseteq Z_{\{x,y,z\}}$ (thus, $V(H) \cap \{x, y, z\} \neq \emptyset$).

Clearly, the auxiliary graph $\mathcal{G}_i$ can be constructed in polynomial time using Lemma 2.4.

Now, we apply Theorem 2.1 to each $\mathcal{G}_i$ and output either a matching of size $k + 1$ or a vertex cover of size at most k.

Suppose that there exists $i \in \{0, 1, \ldots, 4\}$ such that $\mathcal{G}_i$ contains a matching M of size at least $k + 1$. We argue that there are $k + 1$ vertex-disjoint holes in this case. Let

$e = (d, \{x, y, z\})$ and $e' = (d', \{x', y', z'\})$ be two distinct edges of M. By construction, there exist two holes H and H' such that

- H contains d and $V(H) \setminus \{d\} \subseteq Z_{\{x,y,z\}}$, and
- H' contains d' and $V(H') \setminus \{d'\} \subseteq Z_{\{x',y',z'\}}$.

Recall that any vertex of $N(C) \setminus D$ has at most three neighbors on C, which are consecutive by Lemma 4.1. On the other hand, the distance between $\{x, y, z\}$ and $\{x', y', z'\}$ on C is at least three by the construction of the family $\mathcal{A}_i$. Therefore the two sets $Z_{\{x,y,z\}}$ and $Z_{\{x',y',z'\}}$ are disjoint, which implies H and H' are vertex-disjoint. We conclude that one can output $k+1$ vertex-disjoint holes when there is a matching M of size $k+1$ in one of $\mathcal{G}_i$'s.

Consider the case when for every $0 \le i \le 4$, $\mathcal{G}_i$ admits a vertex cover S_i of size at most k. For S_i, let S_i^* be the vertex set

$$(S_i \cap D) \cup \bigcup_{\{x,y,z\} \in S_i \cap \mathcal{A}_i} \{x, y, z\}$$

and let $T_{trav:sunf} := \left(\bigcup_{i=0}^{4} S_i^*\right) \cup \{v_{5\lfloor \frac{m}{5} \rfloor + i} : -2 \le i \le 6\}$. Notice that $|T_{trav:sunf}| \le 15k + 9$.

Claim 2. *The vertex set $T_{petal} \cup T_{trav:sunf}$ hits all D-traversing sunflowers.*

Proof of the Claim. Suppose $G - (T_{petal} \cup T_{trav:sunf})$ contains a D-traversing sunflower H having a vertex $d \in D$. By Lemma 5.6, there exist x, y, z that are consecutive vertices on C and a D-traversing sunflower H' containing d such that $V(H) \cap \{x, y, z\} \neq \emptyset$ and $V(H') \setminus \{d\} \subseteq Z_{\{x,y,z\}}$. Clearly, we have either

- d is adjacent to $\{x, y, z\}$ in one of the bipartite graphs $\mathcal{G}_i$, or
- $\{x, y, z\} \subseteq \{v_{5\lfloor \frac{m}{5} \rfloor + i} : -2 \le i \le 6\}$.

In the first case, S_i^* contains $\{d\}$ or $\{x, y, z\}$, as S_i is a vertex cover of $\mathcal{G}_i$. Since $V(H) \cap \{x, y, z\} \neq \emptyset$ and H contains d, S_i^* contains a vertex of H, which contradicts the assumption that H is a D-traversing sunflower in $G - (T_{petal} \cup T_{trav:sunf})$. In the second case, $T_{trav:sunf}$ contains $\{x, y, z\}$, which again contradicts that H is a D-traversing sunflower in $G - (T_{petal} \cup T_{trav:sunf})$. $\diamond$

This completes the proof. $\square$

6. Hitting all tulips

In this section, we show that one can find in polynomial time either $k+1$ vertex-disjoint holes or a vertex set hitting all tulips. Again, we assume that (G, k, C) is given as an input such that C is a shortest hole of G of length strictly greater than μ_k and $G - V(C)$ is chordal.

The first few sections will focus on D-avoiding tulips. In Subsection 6.3, we settle the case of D-traversing tulips. Subsection 6.4 will establish the main theorem for holes in general, Theorem 3.1. For D-avoiding tulips, it is sufficient to consider the graph $G_{deldom} = G - D$.

6.1. Constructing a nested structure of partial tulips

We recursively construct a subgraph of G_{deldom} in which all vertices have degree 2 or 3 and it contains C. A subgraph of G is called a $(2,3)$-*subgraph* if its all vertices have degree 2 or 3. For a $(2,3)$-subgraph F, a vertex v of degree 3 in F is called a *branching point* in F, and other vertices are called *non-branching points*.

Given a $(2,3)$-subgraph F of G_{deldom} containing C, an (x,y)-path P of G_{deldom} is a F-*extension* if it satisfies the following.

(i) x and y are distinct non-branching points of F.
(ii) $\{x,y\} \cap V(C) \neq \emptyset$.
(iii) P is a proper $V(F)$-path and $P - V(F)$ is an induced path of G_{deldom}.
(iv) There exists a vertex $v \in V(P)$ such that $\mathrm{dist}_P(v, \{x,y\} \cap V(C)) = 2$ and $v \notin N[F]$.

Note that by condition (iv), the length of an F-extension is at least 4.

A cycle H of G_{deldom} is an *almost F-extension* if it satisfies the following.

(i') $|V(H) \cap V(C)| = 1$ and the vertex in $V(H) \cap V(C)$ is a non-branching point of F in C.
(ii') $H - V(C)$ is an induced path of G_{deldom} and contains no vertex of F.
(iii') There exists a vertex $v \in V(H)$ such that $\mathrm{dist}_H(v, V(H) \cap V(C)) = 2$ and $v \notin N[F]$.

We call the vertex in $V(H) \cap V(C)$ the *root* of the almost F-extension H.

It is not difficult to see that given a $(2,3)$-subgraph F containing C, there is a polynomial-time algorithm to find a shortest F-extension P or correctly decides that there is no F-extension. For this, we exhaustively guess five vertices x, y, x', y', v such that

- x and y are non-branching points of F such that $x \in V(C)$,
- x' and y' are neighbors of x and y in $V(G_{deldom}) \setminus V(F)$, respectively,
- v is a neighbor of x' in $V(G) \setminus N[F]$.

Since we are looking for an (x,y)-path P where $\mathring{x}P\mathring{y}$ is induced, we check whether there is a path from v to y' in $G_{deldom} - ((V(F) \cup N[x']) \setminus \{v\})$. If there is such a path, then we find a shortest one Q. Then $xx'v \odot vQy' \odot y'y$ is an F-extension. Among all possible choices of five vertices x, y, x', y', v, we find a shortest F-extension using these five vertices. Clearly if there is an F-extension, then we can find a tuple of such five vertices that outputs a shortest F-extension in the above procedure.

Throughout this section, we heavily rely on the structure of a maximal subgraph obtained by adding a sequence of F-extensions exhaustively. We additionally impose a tie break rule for the choice of F-extensions.

Initialize: $W_1 = C$, $B_1 = \emptyset$, and $i = 1$.
At step i: We perform the following.

(1) Find a shortest W_i-extension P_i such that
(**tie break rule**) $|V(P_i) \cap V(C)|$ is maximum.
If no W_i-extension exists, then terminate. Let x_i, y_i be the endpoints of P_i otherwise.

(2) Set $W_{i+1} := (V(W_i) \cup V(P_i), E(W_i) \cup E(P_i))$.

(3) Set $B_{i+1} := B_i \cup \{x_i, y_i\}$ and increase i by one.

We remark that even with the tie break rule, the choice of a W_i-extension in (1) is not unique.

Notice that every vertex of W_i has degree 2 or 3. Let $W_1, W_2, \ldots, W_\ell$ be the sequence of subgraphs constructed exhaustively until there is no W_ℓ-extension. Let $W = W_\ell$ and $T_{branch} = B_\ell$. Throughout this section, we fix those sequences $W_1, W_2, \ldots, W_\ell = W$ and $P_1, P_2, \ldots, P_{\ell-1}$, and $B_1, B_2, \ldots, B_\ell = T_{branch}$. Clearly, the construction of W requires at most n iterations, and thus we can construct these sequences in polynomial time.

The first observation is that if T_{branch} has size at least s_{k+1}, then $G[V(W)]$ contains $k+1$ vertex-disjoint holes. In fact, the construction of W is calibrated so that every cycle of W contains a hole of G. For this, the condition (iv) of W-extension is crucial. Due to the next lemma, we may assume that $|T_{branch}| < s_{k+1}$.

Lemma 6.1. *If W has at least s_{k+1} branching points, then there are $k+1$ vertex-disjoint holes and they can be detected in polynomial time.*

Proof. By Theorem 2.2, $|T_{branch}| \geq s_{k+1}$ implies that W has at least $k+1$ vertex-disjoint cycles, and such a collection of cycles can be found in polynomial time. We shall prove that for each cycle H of W, there is a hole in the subgraph of G induced by $V(H)$. Clearly, this immediately establishes the statement. We fix a cycle H of W. We may assume that $H \neq C$. Recall that for each i, P_i is a W_i-extension added to W_i.

Let i be the minimum integer such that $E(H) \subseteq E(W_{i+1})$. We claim that P_i is entirely contained in H as a subgraph. Notice that for any W_j-extension P_j, every branching point $v \in T_{branch}$ which is an internal vertex of P_j has been added at iteration $j' > j$. Therefore, if P_i is not entirely contained in H as a subgraph, then for some $i' > i$, there exists a subpath of $P_{i'}$ such that $E(P_{i'}) \cap E(H) \neq \emptyset$, contradicting the choice of i.

Let x, y be the endpoints of P_i. Let v be an internal vertex of P_i that is not contained in $N[W_i]$. Such a vertex exists by the condition (iv) of the definition of a W-extension. Let $Q := H - v$. Since $\mathring{x}P_i\mathring{y}$ is induced, the neighbors of v in P_i are not adjacent, and v has no neighbors in the set of internal vertices of Q. Therefore, by Lemma 2.3, $G[V(H)]$ contains a hole, as claimed. $\square$

In the next step, we exhaustively find almost W-extensions and cover them if there are no $k+1$ vertex-disjoint holes. We show that if there are two almost W-extensions with roots x_1 and x_2 and $\text{dist}_C(x_1, x_2) \geq 5$, then these two almost W-extensions do not intersect. This is because if they meet, then we can obtain a W-extension, contradicting the maximality of W. Using this, we can deduce that if there are $5k+5$ almost W-extensions with distinct roots, then there are $k+1$ vertex-disjoint holes.

Proposition 6.2. *There is a polynomial-time algorithm that finds either $k+1$ vertex-disjoint holes or a vertex set $T_{almost} \subseteq V(C) \setminus T_{branch}$ of size at most $5k+4$ such that $T_{almost} \cup T_{branch}$ hits all almost W-extensions.*

Proof. Let $C = v_0 v_1 \cdots v_{m-1} v_0$. All additions are taken modulo m. We greedily construct a collection of almost W-extensions $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_t\}$ (not necessarily vertex-disjoint) with distinct roots $v_{a_1}, v_{a_2}, \ldots, v_{a_t} \in V(C) \setminus T_{branch}$, and stop if t reaches $5k+5$. To construct such a collection, we do the following for each vertex $v \in V(C) \setminus T_{branch}$:

(1) Choose three vertices w_1, w_2, w_3 such that $w_1, w_3 \in Z_v \setminus \{v\}$, $w_2 \notin N[W]$, and w_1 is adjacent to w_2 but not adjacent to w_3.
(2) Test whether there is a path from w_2 to w_3 in $G_{deldom} - ((V(W) \cup N[w_1]) \setminus \{w_2\})$. If there is such a path P, then we add the cycle $H = w_1 w_2 \odot w_2 P w_3 \odot w_3 v w_1$ to $\mathcal{Y}$.

It is not difficult to verify that there is an almost W-extension with root v if and only if the algorithm outputs such a cycle H.

We claim that if v_{a_p} and v_{a_q} have distance at least 5 in C, then Y_p and Y_q do not meet.

Claim 3. *Let $p, q \in \{1, 2, \ldots, t\}$. If $\text{dist}_C(v_{a_p}, v_{a_q}) \geq 5$, then $V(Y_p) \cap V(Y_q) = \emptyset$.*

Proof of the Claim. Suppose for contradiction that Y_p and Y_q meet at a vertex z. Let $Y_p = p_1 p_2 \cdots p_r v_{a_p} p_1$ and $Y_q = q_1 q_2 \cdots q_s v_{a_q} q_1$. For convenience let $p_0 := v_{a_p}$ and $q_0 := v_{a_q}$. By the condition (iii') of an almost W-extension, we may assume that $p_2, q_2 \notin N[W]$. Let t_1 be the minimum integer such that p_{t_1} has a neighbor in Y_q. We choose a neighbor q_{t_2} of p_{t_1} in Y_q with minimum t_2. Let $R := p_0 Y_p p_{t_1} \odot p_{t_1} q_{t_2} \odot q_{t_2} Y_q q_0$. It is not difficult to see that R is an induced path.

Since $\text{dist}_C(p_0, q_0) \geq 5$, the length of R is at least 4 by Lemma 4.3. Therefore, R contains either p_2 or q_2. It implies that R is a W-extension, contradicting the maximality of W. $\diamond$

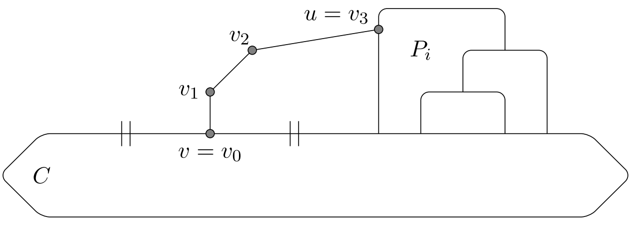

Fig. 9. The setting in Lemma 6.3 where v is a vertex of $C - T_{ext}$ and there is a path of length 3 from v to $u \in V(W) \setminus V(C)$ whose internal vertices are in $V(G_{deldom}) \setminus V(W)$. By Lemma 6.3, v_2 should have a neighbor in C.

Suppose $t \geq 5k + 5$. There exists $M = \{b_1, b_2, \dots, b_{k+1}\} \subseteq \{a_1, a_2, \dots, a_t\} \setminus \{v_i : m - 4 \leq i \leq m - 1\}$ such that for all $b_i, b_j \in M$, $b_i \equiv b_j \pmod 5$. As we exclude the vertices of $\{v_i : m - 4 \leq i \leq m - 1\}$, for every $b_i, b_j \in M$, $\mathrm{dist}_C(v_{b_i}, v_{b_j}) \geq 5$. By Claim 3, for $i, j \in \{1, 2, \dots, k + 1\}$, the corresponding almost W-extensions Y_{b_i} and Y_{b_j} are vertex-disjoint. Thus, we can output $k + 1$ vertex-disjoint holes in polynomial time. Otherwise, $|\mathcal{Y}| \leq 5k+4$, and thus the set of all roots $\{v_{a_1}, v_{a_2}, \dots, v_{a_t}\} \subseteq V(C) \setminus T_{branch}$ of $\mathcal{Y}$ contain at most $5k+4$ vertices. Clearly, the set of all roots hits every element of $\mathcal{Y}$, that is, every almost W-extension. $\square$

6.2. Q-tunnels

We define

$$T_{ext} := T_{petal} \cup T_{full} \cup T_{trav:sunf} \cup T_{branch} \cup T_{almost} \cup \\ N_C^{20}[(T_{petal} \cup T_{full} \cup T_{trav:sunf} \cup T_{branch} \cup T_{almost}) \cap V(C)].$$

Note that

$$|T_{ext}| \leq 41(19k + (3k + 14) + (15k + 9) + (s_{k+1} - 1) + (5k + 4)) \leq 41(s_{k+1} + 42k + 26).$$

Since $|T_{petal} \cup T_{full} \cup T_{trav:sunf} \cup T_{branch} \cup T_{almost}| \leq s_{k+1} + 42k + 26$, $C - (T_{petal} \cup T_{full} \cup T_{trav:sunf} \cup T_{branch} \cup T_{almost})$ contains at most $s_{k+1} + 42k + 26$ connected components, so does $C - T_{ext}$. Let $\mathcal{Q}$ be the set of connected components of $C - T_{ext}$ and we call each element of $\mathcal{Q}$ a *C-fragment*.

We want to show that for every D-avoiding tulip H not hit by T_{ext} and for every $Q \in \mathcal{Q}$, if H contains a vertex of Q far from the endpoints of Q, then H must traverse the Q-tunnel from one entrance to the other entrance. To argue this, we show that H contains no edge vw where $v \in Z_{V(Q)}$ and $w \notin N[C]$. We first need to show that in such a case, we have $w \notin N[W]$. The next lemma states a useful distance property of W. See Fig. 9 for an illustration.

Lemma 6.3. *Let $Q \in \mathcal{Q}$ be a C-fragment, and $v \in V(Q)$ and $u \in V(W)$ with $v \neq u$. Then every $V(W)$-path $R = v_0 v_1 \cdots v_s$ from $v_0 = v$ to $v_s = u$ satisfies one of the following.*

(1) u is a vertex of C such that $\mathrm{dist}_C(u, V(Q)) \le 4$,
(2) R has length 3 *and v_2 is adjacent to a vertex of C,*
(3) R has length at least 4.

Proof. Recall that $W_1, W_2, \dots, W_\ell = W$ is a sequence of subgraphs, $P_1, P_2, \dots, P_{\ell-1}$ is a sequence of W_i-extensions, and $B_1, B_2, \dots, B_\ell = T_{branch}$ is a sequence of branching points during the construction of W.

Suppose $u \in V(C)$. If a $V(W)$-path R between u and v has length at most 3, then there is an edge between Z_u and Z_v or we have $Z_u \cap Z_v \neq \emptyset$. Then Lemma 4.2 implies that $\mathrm{dist}_C(u, v) \le 3$, and R satisfies (1). Therefore, we may assume $u \notin V(C)$. In particular, the following claim for every $i \in \{1, \dots, \ell - 1\}$ establishes the statement immediately. We prove it by induction on i:

($*$) if u is an internal vertex of P_i, then every $V(W)$-path R between v and u satisfies (2) or (3).

Let $P_i = u_0 u_1 \cdots u_p$. By definition of a W_i-extension, we may assume $u_0 \in V(C)$ and $u_2 \notin N[W_i]$. Suppose there exists a $V(W)$-path from v to an internal vertex of P_i violating (3). Such a path has length at most 3. Let $s \in \{1, 2, 3\}$ be the minimum integer such that there is a $V(W)$-path of length s between v and an internal vertex of P_i. We choose the minimum integer $j \in \{1, 2, \dots, p - 1\}$ such that there is a (v, u_j)-path R of length s. Let $R := v_0 v_1 \cdots v_s$ with $v_0 = v$ and $v_s = u_j$, and $R_1 = u_0 P_i u_j \odot R$.

We verify that R_1 is a W_i-extension.

Claim 4. *R_1 is a W_i-extension containing u_2.*

Proof of the Claim. By the choice of s and u_j, every vertex in $\mathring{v}_0 R \mathring{u}_j$ has no neighbors in $\mathring{u}_0 P_i \mathring{u}_j$. Therefore, $\mathring{u}_0 R_1 \mathring{v}_0$ is an induced path. Also, v_0 is a non-branching point of W_i. Hence, R_1 satisfies the conditions (i)-(iii) of W_i-extension. For (iv), it is sufficient to show that $j \ge 2$. Suppose $j = 1$. Then R_1 has length at most 4, and by Lemma 4.3 with $m = 2$, we have $\mathrm{dist}_C(v, u_0) \le 7$. Since $u_0 \in T_{branch} \cap V(C)$, this contradicts the fact that $v \in V(Q)$ and thus $\mathrm{dist}_C(v, T_{branch} \cap V(C)) \ge 20$. We conclude that $j \ge 2$ and thus R_1 contains u_2. Since P_i meets (iv) as a W_i-extension, we have $u_2 \notin N[W_i]$. Therefore, R_1 satisfies all four conditions for being a W_i-extension. $\diamond$

Next, we show that R has length exactly 3. When R has length 1 or 2, we derive a contradiction from the fact that P_i is taken as a i-th W_i-extension.

Claim 5. *$s = 3$; that is, R has length* 3.

Proof of the Claim. First assume that R has length 1. If $j < p - 1$, then R_1 is shorter than P_i, which contradicts the fact that P_i is taken as a shortest W_i-extension. Thus we have $j = p-1$. If u_p is a vertex in C, observe that $u_{p-1} \in Z_{v_0} \cap Z_{u_p}$. Then, by Lemma 4.2 we have $\mathrm{dist}_C(v_0, u_p) \le 3$. However, $u_p \in T_{branch}$ and $v_0 \in V(Q)$ imply $\mathrm{dist}_C(v_0, u_p) \ge 20$, a contradiction. Hence, u_p is not a vertex of C. This means that R_1 should have been chosen as a W_i-extension instead of P_i because of tie break rule, a contradiction.

Suppose now that R has length 2. If $j < p-2$, then R_1 is shorter than P_i, which contradicts the fact that P_i is taken as a shortest W_i-extension. Thus we have $j = \{p-2, p-1\}$. If u_p is a vertex of C, then $\mathrm{dist}_C(v_0, u_p) \le 7$ by Lemma 4.3. This contradicts the fact that $\mathrm{dist}_C(v_0, u_p) \ge 20$. Therefore, u_p is not a vertex of C. If $j = p-2$, then R_1 should be taken instead of P_i because of tie break rule.

Hence, we may assume that $j = p-1$. Then $v_0 v_1 u_{p-1} u_p$ is a path of length 3 from v_0 to $u_p \in V(W_i)$, and by induction hypothesis, u_{p-1} has a neighbor in C. Let z be a neighbor of u_{p-1} in C.

By Lemma 4.3, we have $\mathrm{dist}_C(v_0, z) \le 3$. Therefore,

$$z \notin T_{petal} \cup T_{branch} \cup T_{almost}$$

since otherwise, v_0 is contained in the 20-neighborhood of $V(C) \cap (T_{petal} \cup T_{branch} \cup T_{almost})$ and thus, $v_0 \in T_{ext}$. In particular, z is a non-branching point of W. We choose a neighbor $u_{j'}$ of z such that j' is minimum and let $R_2 = z u_{j'} \odot u_{j'} P_i u_0$. Note that $j' \le j = p-1$.

It is easy to verify that R_2 is a W_i-extension similarly as in Claim 4. Especially, R_2 meets condition (iv) because of $j' \ge 3$, which follows from the fact that z has no neighbors in $\{u_1, u_2\}$ by Lemma 4.3. If $j' < p-1$, then R_2 is shorter than P_i contradicting the fact that P_i is chosen as a shortest W_i-extension. If $j' = p-1$, then since $u_p \notin V(C)$, R_2 should have been chosen instead of P_i because of tie break rule. We conclude that R cannot have length 2, which completes the proof of the claim. $\diamond$

Now, we shall show that v_2 has a neighbor in C, thus R satisfies (2). Suppose the contrary, and observe $v_2 \notin N[C]$ because R is a $V(W)$-path and $v_2 \notin V(C)$. If v_2 has a neighbor in $V(W_i) \setminus V(C)$, then by induction hypothesis, v_2 has a neighbor in C, a contradiction. Therefore, v_2 has no neighbors in W_i and especially, $v_2 \notin N[W_i]$.

If $j < p-3$, then R_1 is shorter than P_i, which contradicts that P_i is taken as a shortest W_i-extension. Thus we have $p-3 \le j \le p-1$. We choose a neighbor $u_{j'}$ of v_2 with maximum j'. By the choice of j and from $v_2 \notin N[W_i]$, we have $p-3 \le j \le j' \le p-1$. Since v_0 or v_1 has no neighbors in P_i by Claim 5, $v_0 v_1 v_2 u_{j'} \odot u_{j'} P_i u_p$ is a W_i-extension in which $v_2 \notin N[W_i]$.

If $j \ge 4$, then $v_0 v_1 v_2 u_{j'} \odot u_{j'} P_i u_p$ is shorter than P_i, contradicting the fact that P_i is chosen as a shortest W_i-extension. If $j \le 3$, then R_1 has length at most 6, and by Lemma 4.3, we have $\mathrm{dist}_C(v_0, u_0) \le 15$, a contradiction to the fact that $\mathrm{dist}_C(v_0, u_0) \ge 20$.

Therefore, we conclude that v_2 is adjacent to a vertex of C. This proves the claim $(*)$, which completes the proof. $\square$

The following is a simple, but important observation. See Fig. 5 for an illustration.

Lemma 6.4. *Let $Q \in \mathcal{Q}$ be a C-fragment, and let H be a D-avoiding tulip in $G_{deldom} - T_{ext}$. Then H contains no two adjacent vertices v and w such that v is in the Q-tunnel and $w \in V(G_{deldom}) \setminus N[C]$.*

Proof. Suppose H contains two adjacent vertices v and w such that $v \in Z_{V(Q)}$ and $w \in V(G_{deldom}) \setminus N[C]$. Since $v \in Z_{V(Q)}$ and $w \notin N[C]$, we have $v \notin V(C)$ and v has a neighbor in Q. Let z be a neighbor of v in Q. We prove that w has no neighbors in W.

Claim 6. $w \notin N[W]$.

Proof of the Claim. By Lemma 6.3, $w \notin V(W)$. Suppose w has a neighbor in W. Since $w \notin N[C]$, w has a neighbor in $V(W) \setminus N[C]$. Let u be such a neighbor. Since $zvwu$ is a path of length 3, by Lemma 6.3, w has a neighbor in C, a contradiction. Therefore, we conclude that w has no neighbors in W. Thus, we have $w \notin N[W]$. $\diamond$

Let $H = v_1v_2 \cdots v_mv_1$ where $v_1 = v$ and $v_2 = w$. Note that H contains at least one non-branching point of W since $(V(C) \setminus T_{ext}) \cap V(H) \neq \emptyset$. We choose a minimum integer $i > 2$ such that v_i has a neighbor that is a non-branching point of W. Clearly $2 < i \leq m$ from the fact that v_1 is not in C and by Lemma 6.3. We also observe that $v_2 = w \notin N[W]$ by Claim 6 and that $v_1v_2 \cdots v_i$ is an induced path. Let z' be a neighbor of v_i which is a non-branching point of W. Then $zv_1v_2 \cdots v_iz'$ is a W-extension or an almost W-extension depending on whether $z = z'$ or not. It contradicts either the maximality of W or that T_{ext} hits all almost W-extensions. This completes the proof. $\square$

Next, we prove that if a D-avoiding tulip contains a vertex of a C-fragment Q that is far from its endpoints v and w, then its restriction on the Q-tunnel should be some path from Z_v to Z_w. Since we will add all vertices of C-fragments having at most 35 vertices to the deletion set for remaining D-avoiding tulips, we focus on C-fragments Q with at least 36 vertices.

Lemma 6.5. *Let $Q = q_1q_2 \cdots q_m \in \mathcal{Q}$ be a C-fragment with at least 36 vertices and let R be the Q-tunnel. Let H be a D-avoiding tulip in $G_{deldom} - T_{ext}$ such that*

- *H contains no vertices in $\{q_i : 1 \leq i \leq 5, m-4 \leq i \leq m\}$,*
- *H contains a vertex v in $V(Q)$.*

Then the connected component of the restriction of H on R containing v is a path from Z_{q_1} to Z_{q_m}.

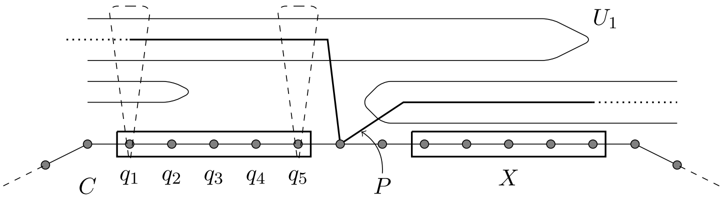

Fig. 10. An illustration of the set X defined in Lemma 6.6. Since X consists of the last 5 vertices of the support of the component U_1, every path from Z_{q_1} to Z_{q_m} should move to a vertex of Q appearing before X, and pass through another connected component of $R - (T_{ext} \cup V(Q))$ to reach Z_{q_m}. But this will lead to a petal whose support is near to X, contradicting the choice of the set T_{ext}.

Proof. Since H is a tulip, H is not fully contained in R. Therefore, the component of the restriction of H on R containing v is a path. Let P be such a path.

We claim that both endpoints of P are contained in $Z_{\{q_1,q_m\}}$. Suppose the contrary, and let w be an endpoint of P such that $w \in (\bigcup_{2\le i\le m-1} Z_{q_i}) \setminus Z_{\{q_1,q_m\}}$. Let $\overline{Q}$ be the (q_1, q_m)-subpath of C that does not contain v_2.

Let $w' \in N_H(w) \setminus V(P)$. Since w is a vertex of R, Lemma 6.4 implies that $w' \in N[C]$. Suppose $w' \in Z_{V(\overline{Q})} \setminus Z_{V(Q)}$. We choose $y \in \mathsf{sp}(G[\{w\}])$ and $y' \in \mathsf{sp}(G[\{w'\}])$ so as to minimize $\mathrm{dist}_C(y, y')$. Then $\mathrm{dist}_C(y, y') \le 3$ by Lemma 4.2. We also have $\mathrm{dist}_C(y, y') \ge 2$ because $y \in V(Q) \setminus \{q_1, q_m\}$ and $y' \in V(\overline{Q})$. Then Lemma 4.4 implies that w is not adjacent to w', a contradiction. Therefore, each endpoint of P is contained in $Z_{\{q_1,q_m\}}$.

Now, we claim that the endpoints of P are contained in distinct sets of Z_{v_1} and Z_{v_m}. Suppose for contradiction that both endpoints of P are contained in the same set of Z_{v_1} or Z_{v_m}. Without loss of generality, they are contained in Z_{v_1}. Since P contains no vertices in $\{q_i : 1 \le i \le 5\}$, P contains two subpaths from $Z_{q_1} \setminus \{q_1\}$ to $Z_{q_5} \setminus \{q_5\}$. But the supports of those two paths share three vertices q_1, q_2, q_3, and by Lemma 4.7, there is a petal contained in $Z_{\{q_1,q_2,q_3\}}$. This contradicts the assumption that $T_{petal} \subseteq T_{ext}$ contains the support of every petal.

This implies that one endpoint is in Z_{q_1} and the other endpoint is in Z_{q_m}, as required. □

Due to Lemma 6.5, we know that for any D-avoiding tulip H in $G_{deldom} - T_{ext}$, there is a subpath P of H and a Q-tunnel R such that P is a path from one entrance of R to the other entrance. The next lemma describes how to find a hitting set for such path P when its two endpoints belong to distinct connected components of $R - (T_{ext} \cup V(Q))$. See Fig. 10 for an illustration.

Lemma 6.6. *Let $Q = q_1q_2 \cdots q_m \in \mathcal{Q}$ be a C-fragment with at least 36 vertices and let R be the Q-tunnel. One can find in polynomial time a vertex set $X \subseteq V(Q) \setminus \{q_i : 1 \le i \le 5, m-4 \le i \le m\}$ of size at most 5 hitting every path P from Z_{q_1} to Z_{q_m} in $R - T_{ext}$ such that*

- *P contains no vertices in $\{q_i : 1 \le i \le 5, m-4 \le i \le m\}$,*

- *the endpoints of P are contained in distinct connected components of $R - (T_{ext} \cup V(Q))$.*

Proof. We begin with the following claim.

Claim 7. *There is exactly one connected component of $R - (T_{ext} \cup V(Q))$ intersecting both $Z_{q_1} \setminus \{q_1\}$ and $Z_{q_5} \setminus \{q_5\}$.*

Proof of the Claim. Since P contains no vertices in $\{q_1, q_2, \dots, q_5\}$, there is at least one component of $R-(T_{ext} \cup V(Q))$ intersecting both $Z_{q_1} \setminus \{q_1\}$ and $Z_{q_5} \setminus \{q_5\}$. Suppose there are two such components C_1 and C_2. For each C_i, we find a path P_i form $Z_{q_1} \setminus \{q_1\}$ and $Z_{q_5} \setminus \{q_5\}$. Clearly P_1 and P_2 are vertex-disjoint, and there are no edges between P_1 and P_2. As $\mathsf{sp}(P_1)$ and $\mathsf{sp}(P_2)$ share 3 vertices q_3, q_4, q_5, by Lemma 4.7, $Z_{\{q_3,q_4,q_5\}}$ contains a petal. This contradicts the assumption that $T_{petal} \subseteq T_{ext}$ contains the support of every petal. $\diamond$

Let U_1 be the connected component of $R - (T_{ext} \cup V(Q))$ intersecting both $Z_{q_1} \setminus \{q_1\}$ and $Z_{q_5} \setminus \{q_5\}$. Likewise, let U_2 be the unique connected component of $R - (T_{ext} \cup V(Q))$ intersecting both $Z_{q_{m-4}} \setminus \{q_{m-4}\}$ and $Z_{q_m} \setminus \{q_m\}$. Note that U_1 and U_2 are distinct since P must intersect $V(U_1) \cap Z_{q_1}$ and $V(U_2) \cap Z_{q_m}$. Let x be the maximum integer such that $Z_{q_x} \cap V(U_1) \neq \emptyset$ and let $X := \{q_i : x-4 \leq i \leq x\}$.

Now, we show that X hits P. Suppose this is not the case. The choice of x implies that $\{q_1, \dots, q_x\}$ is a separator in $R - T_{ext}$ between $V(U_1)$ and $V(U_2)$. Since P intersects both $V(U_1)$ and $V(U_2)$ while avoiding $\{q_1, \dots, q_5\} \cup X$, it must contain a vertex of $\{q_6, \dots, q_{x-5}\}$. (Especially, we have $x - 5 \geq 6$.) Let P_3' be a $(\{q_6, \dots, q_{x-5}\}, Z_{q_m})$-path which is a subpath of P. As a subpath of P_3', we can choose $(Z_{q_{x-4}}, Z_{q_x})$-path P_3. Let P_4 be a path from $Z_{q_{x-4}}$ to Z_{q_x} in U_1.

Since no internal vertex of P_3' belongs to $\{q_6, \dots, q_{x-5}\}$, P_3' and thus P_3 does not contain a vertex of U_1. Hence, P_3 and P_4 are disjoint. Notice that P_3 is contained in $Z_X \setminus X$ due to Lemma 4.6 and the assumption $V(P) \cap X = \emptyset$. Therefore, P_3 is a subpath of P that is contained in some component of $R - (T_{ext} \cup V(Q))$ different from U_1. Therefore, there is no edge between $V(P_3)$ and $V(P_4)$. As $\mathsf{sp}(P_3)$ and $\mathsf{sp}(P_4)$ share three vertices $q_{x-4}, q_{x-3}, q_{x-2}$, by Lemma 4.7, $Z_{\{q_{x-4},q_{x-3},q_{x-2}\}}$ contains a petal. However, this contradicts the assumption that $T_{petal} \subseteq T_{ext}$ contains the support of every petal. We conclude that X hits P. $\square$

The path meeting the conditions of Lemma 6.5 can be hit by a vertex set obtained in Lemma 6.6, unless its endpoints are contained in the same connected component of $R - (T_{ext} \cup V(Q))$. If the endpoints are contained in the same component of $R - (T_{ext} \cup V(Q))$, then clearly the endpoints can be connected via a path of R traversing no vertices of Q. We prove that if its endpoints are contained in the same component of $R - (T_{ext} \cup V(Q))$ and the path contains a vertex of Q, then we could reroute to find another D-avoiding tulip with less vertices of C.

Lemma 6.7. *Let $Q = q_1q_2 \cdots q_m \in \mathcal{Q}$ be a C-fragment of at least 36 vertices and let R be the Q-tunnel. Let H be a D-avoiding tulip in $G_{deldom} - T_{ext}$ such that*

- *H contains no vertices in $\{q_i : 1 \le i \le 15, m - 14 \le i \le m\}$,*
- *H contains a vertex v in $V(Q)$,*
- *the endpoints of the restriction of H on R containing v are contained in the same connected component of $R - (T_{ext} \cup V(Q))$.*

Then there is a D-avoiding tulip H' in $G_{deldom} - T_{ext}$ such that $V(C) \cap V(H') \subsetneq V(C) \cap V(H)$.

Proof. By Lemma 6.5, the connected component of the restriction of H on R is a path from Z_{q_1} to Z_{q_m}. Let $P = p_1p_2 \cdots p_n$ be the path such that $p_1 \in Z_{q_1}$ and $p_n \in Z_{q_m}$. We choose the minimum integer x such that p_x is contained in $Z_{q_{15}}$ and choose the maximum integer y such that p_y is contained in $Z_{q_{m-14}}$. Let U be the connected component of $R - (T_{ext} \cup V(Q))$ containing p_1 and p_n. Then p_x is a vertex of U since otherwise p_1Pp_x traverses a vertex of Q, which must be in $\{q_{16}, \dots, q_{m-15}\}$; this means that p_1Pp_x contains a vertex of $Z_{q_{15}}$ as an internal vertex by Lemma 4.5, contradicting the choice of x. Similarly, p_y is in U.

Let J be a shortest path from p_x to p_y in U. We want to show that p_2 has no neighbors in J. To show this, we claim that J does not intersect Z_{q_9}.

Claim 8. *J does not intersect Z_{q_9}.*

Proof of the Claim. Suppose for contradiction that J contains a vertex $r \in Z_{q_9}$. Let J_1 and J_2 be the two components of $J - r$. Since J is induced, there are no edges between J_1 and J_2. Also, by Lemma 4.2, for each $i \in \{1, 2\}$, the endpoint of J_i adjacent to r should have a neighbor in C which has distance at most 4 from q_9. This implies that $\mathsf{sp}(J_1)$ and $\mathsf{sp}(J_2)$ both contain $\{q_{13}, q_{14}, q_{15}\}$. Then by Lemma 4.7, we can find a petal contained in $Z_{\{q_{13},q_{14},q_{15}\}}$. This contradicts that $T_{petal} \subseteq T_{ext}$ contains the support of all petals. $\diamond$

Suppose that p_2 has a neighbor in J. Consequently, there is a $V(C)$-path from q_1 to a vertex of $\mathsf{sp}(J)$ traversing p_1 and p_2 whose length is 4. By Lemma 4.3, the endpoint of this path contained in $\mathsf{sp}(J)$ is within distance at most 7 in C. Thus, by Lemma 4.5, $\mathsf{sp}(J)$ contains q_9 and this contradicts Claim 8. Therefore, p_2 does not have a neighbor in J.

Let P_{rem} be the subpath of H from p_1 to p_m not containing p_2. Observe that

$$P_{new} = p_3Pp_x \odot p_xJp_y \odot p_yPp_m \odot p_mP_{rem}p_1$$

is a walk from p_3 to p_1 in G_{deldom}. Observe that the only vertices of P_{new} adjacent to p_2 are p_1 and p_3. Also, p_1 is not adjacent to p_3. Thus by applying Lemma 2.3 to $p_1p_2p_3$ and a shortest path from p_1 to p_3 in P_{new}, we derive that $G_{deldom}[\{p_2\} \cup V(P_{new})]$ contains a hole H'. Clearly, H' is a D-avoiding hole. If it is a sunflower, then H' is hit by T_{full}, a contradiction. Thus, H' is a D-avoiding tulip. Clearly, $V(H') \cap V(C) \subsetneq V(H) \cap V(C)$ because the restriction of H' on R containing p_2 does not contain a vertex of Q. $\square$

We are ready to construct a hitting set for all D-avoiding tulips. In the hitting set, we impose an additional condition that will be used for hitting D-traversing tulips.

Proposition 6.8. *There is a polynomial-time algorithm which finds either $k+1$ vertex-disjoint holes in G or a vertex set $T_{avoid:tulip} \subseteq V(G) \setminus T_{ext}$ of at most $35(s_{k+1}+42k+26)$ vertices such that*

- $T_{ext} \cup T_{avoid:tulip}$ *contains $N_C^{15}[T_{ext} \cap V(C)]$, and*
- $T_{ext} \cup T_{avoid:tulip}$ *hits all D-avoiding tulips.*

Proof. We construct a set $T_{avoid:tulip}$ as follows:

(1) for each component of $C - T_{ext}$ having at most 35 vertices, we add all vertices to $T_{avoid:tulip}$,
(2) for each component $Q = q_1 q_2 \cdots q_m$ of $C - T_{ext}$ with $m \geq 36$, we add $\{q_i : 1 \leq i \leq 15, m-14 \leq i \leq m\}$, and the set obtained in Lemma 6.6 to $T_{avoid:tulip}$.

Since $C - T_{ext}$ contains at most $s_{k+1} + 42k + 26$ connected components, we have $|T_{avoid:tulip}| \leq 35(s_{k+1} + 42k + 26)$. Furthermore, $T_{ext} \cup T_{avoid:tulip}$ contains the 15-neighborhood of $T_{ext} \cap V(C)$ in C. We claim that $T_{ext} \cup T_{avoid:tulip}$ hits all D-avoiding tulips.

Suppose for contradiction that there is a D-avoiding tulip H in $G_{deldom} - (T_{ext} \cup T_{avoid:tulip})$. We choose such a tulip with minimum $|V(C) \cap V(H)|$. Since H is a hole, it contains a vertex in $V(C) \setminus T_{ext}$, say v. Let $Q = q_1 q_2 \cdots q_m$ be a connected component of $C - T_{ext}$ containing v. By (1), we have $m \geq 36$.

As $\{q_i : 1 \leq i \leq 15, m-14 \leq i \leq m\}$ was added to $T_{avoid:tulip}$, v is a vertex in $V(Q) \setminus \{q_i : 1 \leq i \leq 15, m-14 \leq i \leq m\}$. Let R be the Q-tunnel, and let P be the restriction of H on the Q-tunnel containing v.

By Lemma 6.5, P is a path from Z_{q_1} to Z_{q_m}. If the endpoints of P are contained in the distinct components of $R - (T_{ext} \cup V(Q))$, then $T_{avoid:tulip}$ hits this path by Lemma 6.6, a contradiction. Assume the endpoints of P are contained in the same component of $R - (T_{ext} \cup V(Q))$. Then by Lemma 6.7, there is a D-avoiding tulip H' with $V(H') \cap V(C) \subsetneq V(H) \cap V(C)$, contradicting the minimality of H.

Therefore, $T_{ext} \cup T_{avoid:tulip}$ intersects all D-avoiding tulips. $\square$

6.3. Handling D-traversing tulips

By Lemmas 4.8 and 4.9, every D-traversing tulip H contains precisely one C-dominating vertex and it contains one or two vertices of C. Also, the vertices in $V(H) \cap V(C)$ are adjacent to the unique C-dominating vertex in H.

Here, we use a technique similar to the one in Theorem 5.7. That is, we construct auxiliary bipartite graphs, and we will find either $k+1$ vertex-disjoint holes, or a set covering all such tulips.

Lemma 6.9. *There is a polynomial-time algorithm which finds either $k+1$ vertex-disjoint holes in G or a vertex set $T_{trav:tulip} \subseteq (D \cup V(C)) \setminus T_{ext} \setminus T_{avoid:tulip}$ of size at most $25k+9$ such that $T_{ext} \cup T_{avoid:tulip} \cup T_{trav:tulip}$ hits every D-traversing tulip.*

Proof. Let $C = v_0 v_1 \cdots v_{m-1} v_0$. All additions are taken modulo m. We create an auxiliary bipartite graph $\mathcal{G}_i = (D \uplus \mathcal{A}_i, \mathcal{E}_i)$ for each $0 \le i \le 4$, such that

- $\mathcal{A}_i = \{v_{5j+i} : j = 0, \ldots, \lfloor \frac{m}{5} \rfloor - 1\}$,
- there is an edge between $d \in D$ and $x \in \mathcal{A}_i$ if and only if there is an (x,d)-path P in $G - ((D \cup V(C)) \setminus \{d,x\}) - T_{ext} - T_{avoid:tulip}$ such that
  - $G[V(P)]$ is a hole,
  - the second neighbor of x in P is not in $N[C]$.

It is not difficult to see that each auxiliary graph $\mathcal{G}_i$ can be constructed in polynomial time. For a pair of $d \in D$ and $x \in V(C)$, we first ensure that $dx \in E(G)$ and $\{d,x\} \cap T_{ext} = \emptyset$. By guessing the vertices y, z such that $z \notin N[C]$ and $dxyz$ forms an induced path, and then computing a shortest (z,d)-path, we can find an (x,d)-path P satisfying the two conditions above. On the other hand, if there is an (x,d)-path meeting the above conditions, then we can find such a path P with the corresponding choice of y and z.

Suppose there exists $i \in \{0,1,\ldots,4\}$ such that $\mathcal{G}_i$ contains a matching M of size at least $k+1$. We argue that there are $k+1$ holes in this case.

Let $e_1 = (d_1, x_1)$ and $e_2 = (d_2, x_2)$ be two distinct edges of M. By construction, we have $\mathrm{dist}_C(x_1, x_2) \ge 5$, and for each $i \in \{1,2\}$, there is an (x_i, d_i)-path P_i in $G - ((D \cup V(C)) \setminus \{d_i, x_i\}) - T_{ext} - T_{avoid:tulip}$ such that $G[V(P_i)]$ is a hole and the second neighbor of x_i in P_i is not in $N[C]$. Let y_i and z_i be the first and second neighbors of x_i in P_i respectively, for each i.

First, we show that $z_i \notin N[W]$. Notice that $x_i \notin T_{ext}$ and x_i is a vertex of a C-fragment. If $y_i \in V(W)$ or $z_i \in V(W)$, then Lemma 6.3 implies that $y_i \in V(C)$ or $z_i \in V(C)$, which is not possible since P_i is a path with $V(P_i) \cap V(C) = \{x_i\}$. Hence, we have $y_i, z_i \notin V(W)$. Suppose that z_i has a neighbor $z_i' \in V(W)$. Since $x_i y_i z_i z_i'$ is a $V(W)$-path of length 3, z_i has a neighbor in C by Lemma 6.3. This contradicts the assumption that $z_i \notin N[C]$. We conclude that $z_i \notin N[W]$.

Next, we show that if P_i contains a non-branching point of W other than x_i, then there is a W-extension.

Claim 9. *P_i contains no point of W other than x_i.*

Proof of the Claim. We prove for P_1; the same proof holds for P_2. Suppose the claim does not hold. Recall that $V(P_1) \cap T_{branch} \subseteq V(P_1) \cap T_{ext} = \emptyset$. Hence, we may assume that P_1 contains a non-branching point of W. Let $P_1 := p_1 p_2 \cdots p_m$ where $p_1 = x_1$ and $p_m = d_1$, and let $j \in \{1, 2, \ldots, m\}$ be the minimum integer such that p_j is a non-branching point of W other than $x_1 = p_1$. Clearly, $p_j \notin V(C)$, as p_1 is the unique vertex of P_i contained in C.

If $j \leq 4$, then $p_1 P_1 p_j$ is a path of length at most 3, and since $p_1 \notin T_{ext}$, by Lemma 6.3, $j = 4$ and p_{j-1} has a neighbor in C. However, it contradicts the choice of P_1 that the second neighbor $y_1 = p_3$ of x_1 is not in $N[W]$. Therefore, $j \geq 5$. It means that $p_1 P_1 p_5$ is a W-extension containing $p_3 \notin N[W]$. This contradicts the maximality of W. $\diamond$

We further claim that if P_1 and P_2 intersect, then there is a W-extension.

Claim 10. *P_1 and P_2 do not share a vertex.*

Proof of the Claim. Suppose that P_1 and P_2 intersect. We choose $f_1 \in V(P_1)$ having a neighbor in P_2 so that $\mathrm{dist}_{P_1}(f_1, x_1)$ is minimized. Among neighbors of f_1 in P_2, we choose f_2 that is closest to x_2 in P_2. By the choice of f_1 and f_2, $x_1 P_1 f_1 \odot f_1 f_2 \odot f_2 P_2 x_2$ is an induced path. Note that there are no edges between Z_{x_1} and Z_{x_2} because $\mathrm{dist}_C(x_1, x_2) \geq 5$. Therefore, $x_1 P_1 f_1 \odot f_1 f_2 \odot f_2 P_2 x_2$ contains at least one of z_1 and z_2, which are not in $N[W]$. By Claim 9, $x_1 P_1 f_1 \odot f_1 f_2 \odot f_2 P_2 x_2$ is a $V(W)$-path. It implies that $x_1 P_1 f_1 \odot f_1 f_2 \odot f_2 P_2 x_2$ is a W-extension, contradicting the maximality of W. Therefore, P_1 and P_2 do not share a vertex. $\diamond$

Claim 10 implies that if a bipartite graph $\mathcal{G}_i$ contains a matching of size $k+1$, then we can construct $k+1$ vertex-disjoint holes in polynomial time. Consider the case when for every $0 \leq i \leq 4$, $\mathcal{G}_i$ admits a vertex cover S_i of size at most k. For S_i, let S_i^* be the vertex set

$$(S_i \cap D) \cup \bigcup_{x \in S_i \cap \mathcal{A}_i} N_C^2[x]$$

and let $T_{trav:tulip} := \left(\bigcup_{i=0}^{4} S_i^*\right) \cup \{v_{5\lfloor \frac{m}{5} \rfloor + i} : -2 \leq i \leq 6\}$. Notice that $|T_{trav:tulip}| \leq 25k + 9$.

In what follows, we will prove that the vertex set $T_{ext} \cup T_{avoid:tulip} \cup T_{trav:tulip}$ indeed hits all D-traversing tulips. Suppose that there is a D-traversing tulip H containing $d \in D$ and $x \in V(C)$ in $G - (T_{ext} \cup T_{avoid:tulip} \cup T_{trav:tulip})$. Clearly $P := H - dx$ is an (x, d)-path such that $G[V(P)]$ is a hole, but it may not certify the existence of the edge dx in the auxiliary bipartite graph, because the second neighbor of x in P can be in $N[C]$. Let $P = p_1p_2 \cdots p_m$ with $p_1 = x$ and $p_m = d$. Let $w_1, w_2, \ldots, w_5$ be the consecutive vertices on C such that $w_3 = x$. Let i be the minimum integer such that $p_{i+1} \notin N[C]$. Such p_{i+1} exists because H is a tulip.

Claim 11. *We have* $\{p_1, \ldots, p_i\} \subseteq Z_{\{w_2,w_3,w_4\}} \setminus Z_{\{w_1,w_5\}}$.

Proof of the Claim. Suppose not. By Lemma 4.5, p_1Pp_i contains an $(x, Z_{\{w_1,w_5\}})$-subpath Q. Then it is easy to see that $dx \circ Q$ meets the preconditions of Lemma 5.5; especially every internal vertex of Q is in $Z_{\{w_2,w_3,w_4\}} \setminus Z_{\{w_1,w_5\}}$ by Lemma 4.6. Therefore, Lemma 5.5 implies that there exists a D-traversing sunflower H' containing v and d such that $V(H') \setminus \{d\}$ is contained in either $Z_{\{w_1,w_2,w_3\}} \cap (V(Q) \cup \{w_1\})$ or $Z_{\{w_3,w_4,w_5\}} \cap (V(Q) \cup \{w_5\})$. Since $T_{petal} \cup T_{trav:sunf}$ hits H' by Proposition 5.7 while $(T_{petal} \cup T_{trav:sunf}) \cap (V(Q) \cup \{d\}) \subseteq T_{ext} \cap V(H) = \emptyset$, either w_1 or w_5 must be contained in $T_{petal} \cup T_{trav:sunf}$. However, by the construction of T_{ext}, this implies $x = w_3 \in T_{ext}$, a contradiction. This proves the claim. ◇

By Claim 11, p_i has a neighbor in $\{w_2, w_3, w_4\}$. Let w be a neighbor of p_i in $\{w_2, w_3, w_4\}$. Note that $\{w_2, w_3, w_4\} \cap T_{ext} = \emptyset$ since otherwise, we have $x = w_3 \in T_{ext} \cup T_{avoid:tulip}$ by the construction in Proposition 6.8, contradicting the assumption that $V(H) \cap (T_{ext} \cup T_{avoid:tulip}) = \emptyset$.

Claim 12. *We have* $p_{i+1} \notin N[W]$.

Proof of the Claim. By Lemma 6.3, p_{i+1} cannot be in W. We show that $p_{i+1} \notin N(W)$. Suppose that p_{i+1} has a neighbor p'_{i+1} in W. Then $wp_ip_{i+1}p'_{i+1}$ is a $V(W)$-path and $p'_{i+1} \notin V(C)$ because of $p_{i+1} \notin N[C]$. Recall that $w \notin T_{ext}$. Now, Lemma 6.3 applies and we have $p_{i+1} \in N(C)$. This contradicts the choice of i and $p_{i+1} \notin N[C]$. It follows that $p_{i+1} \notin N(W)$. ◇

Claim 13. *P contains no point of W other than p_1.*

Proof of the Claim. Suppose the contrary. Clearly, P does not contain a branching-point of W because $V(P) \cap T_{branch} \subseteq V(H) \cap T_{ext} = \emptyset$. Therefore, we may assume that P contains a non-branching point of W other than p_1. Let $j \in \{2, \ldots, m\}$ be the minimum integer such that p_j is a non-branching point of W.

Suppose $j \leq i$. By Claim 11, we have $p_j \in Z_{\{w_2,w_3,w_4\}} \setminus Z_{\{w_1,w_5\}}$. Let $p'_j \in \{w_2, w_3, w_4\}$ be a neighbor of p_j. Then p'_jp_j is a $V(W)$-path and $p'_j \notin T_{ext}$. Therefore, Lemma 6.3 applies to p'_jp_j and we have $p_j \in V(C)$, thus $p_j \in \{w_2, w_3, w_4\}$. Note that P contains no vertices in $\{w_2, w_4\}$ as H contains no triangles. Hence, it follows $p_j = w_3 = p_1$. This contradicts the assumption that $p_j \neq p_1$.

Therefore, we assume $j \geq i + 1$. Notice that $wp_i \odot p_iPp_j$ is a $V(W)$-path. We want to argue that this is a W-extension, deriving a contradiction. If $j \in \{i + 1, i + 2\}$, then from $w \notin T_{ext}$ and by Lemma 6.3 we know that $j = i + 2$ and p_{i+1} has a neighbor in C. However, this again contradicts $p_{i+1} \notin N[C]$. Therefore, we have $j \geq i + 3$. It means that $wp_i \odot p_i P p_j$ has length at least 4, and thus $p_{i+1} \notin N[W]$, which makes $wp_i \odot p_i P p_j$ qualify as a W-extension. This contradicts the maximality of W. We conclude that H contains no non-branching point of W other than p_1. $\diamond$

Let $\ell \geq i+1$ be the minimum integer such that p_ℓ is a neighbor of w. Since $p_m = d$ is a neighbor of w, such ℓ exists. Furthermore, $\ell > i + 1$ because we have $p_{i+1} \notin N[C]$ due to the choice of i. Observe that $p_\ell w p_i$ is an induced path with w as an internal vertex and w is not adjacent to any internal vertex of $p_i P p_\ell$. Now Lemma 2.3 applies, implying that $G[V(p_i P p_\ell) \cup \{w\}]$ has a hole H' containing w. By Claim 13, H' contains no point of W other than w.

Observe that H' qualifies as an almost W-extension if $p_\ell \neq d$; especially we have $p_{i+1} \notin N[W]$ by Claim 12. Therefore T_{almost} hits H'. On the other hand, $T_{almost} \cap (V(H') \setminus \{w\}) \subseteq T_{ext} \cap V(H) = \emptyset$, which implies $w \in T_{almost}$. Then by the construction of T_{ext}, we have $x \in T_{ext}$, a contradiction. If $p_\ell = d$, then $H' - dw$ is a path certifying an edge in an auxiliary bipartite graph. Therefore either one of $\{d, w\}$ is contained in the vertex cover or $w = v_{5\lfloor \frac{m}{5} \rfloor + a}$ with $0 \leq a \leq 4$. In both cases, x is included in $T_{trav:tulip}$, a contradiction. This completes the proof. $\square$

6.4. Proof of our main result

We prove Theorem 3.1.

We apply Lemma 5.1, Proposition 5.3, and Proposition 5.7. Over all, we can in polynomial time either output $k+1$ vertex-disjoint holes or vertex sets $T_{petal}, T_{full}, T_{trav:sunf}$ hitting petals, full sunflowers, and D-avoiding sunflowers, respectively.

We construct W with the set T_{branch} of branching points as described in Subsection 6.1. By Lemma 6.1, if W has at least s_{k+1} branching points, then there are $k + 1$ vertex-disjoint holes and they can be detected in polynomial time. We apply Proposition 6.2. If it outputs $k+1$ vertex-disjoint holes in G, then we are done. We may assume it outputs a vertex set T_{almost} of at most $5k + 4$ vertices where T_{almost} hits all almost W-extensions.

Let T_{ext} be the union of $T_{petal} \cup T_{full} \cup T_{trav:sunf} \cup T_{branch} \cup T_{almost}$ and the 20-neighborhood of $V(C) \cap (T_{petal} \cup T_{full} \cup T_{trav:sunf} \cup T_{branch} \cup T_{almost})$.

By Proposition 6.8, we can in polynomial time either find $k + 1$ vertex-disjoint holes or find a set $T_{avoid:tulip} \subseteq V(G) \setminus T_{ext}$ of at most $35(s_{k+1} + 42k + 26)$ vertices such that $T_{ext} \cup T_{avoid:tulip}$ hits all D-avoiding tulips. By Lemma 6.9, we can either find $k + 1$ holes or find a set $T_{trav:tulip} \subseteq V(G) \setminus (T_{ext} \cup T_{avoid:tulip})$ of size $25k + 9$ such that $T_{ext} \cup T_{avoid:tulip} \cup T_{trav:tulip}$ hits all D-traversing tulips. Therefore, we can either find $k + 1$ vertex-disjoint holes, or output a vertex set with at most

$$|T_{ext} \cup T_{avoid:tulip} \cup T_{trav:tulip}|$$

$$\leq 41(s_{k+1} + 42k + 26) + 35(s_{k+1} + 42k + 26) + 25k + 9$$
$$\leq 76s_{k+1} + 3217k + 1985$$

vertices hitting all holes. This completes the proof of Theorem 3.1.

7. Cycles of length at least 5 do not have the Erdős-Pósa property under the induced subgraph relation

In this section, we show that the class of cycles of length at least ℓ for every fixed $\ell \geq 5$ has no Erdős-Pósa property under the induced subgraph relation.

A *hypergraph* is a pair $(X, \mathcal{E})$ such that X is a set and $\mathcal{E}$ is a family of non-empty subsets of X, called *hyperedges*. A subset Y of X is called a *hitting set* if for every $F \in \mathcal{E}$, $Y \cap F \neq \emptyset$. For positive integers a, b with $a \geq b$, let an (a, b)*-uniform hypergraph*, denote it by $U_{a,b}$, be the hypergraph $(X, \mathcal{E})$ such that $|X| = a$ and $\mathcal{E}$ is the set of all subsets of X of size b. It is not hard to observe that in $U_{2k-1,k}$, every two hyperedges intersect and the minimum size of a hitting set of $U_{2k-1,k}$ is precisely k.

Theorem 1.2. *Let $\ell \geq 5$ be a positive integer. Then the class of cycles of length at least ℓ has no Erdős-Pósa property under the induced subgraph relation.*

Proof. Suppose for contradiction that there is a function $f_\ell : \mathbb{N} \to \mathbb{N}$ such that for every graph G and a positive integer k, either

- G contains $k+1$ pairwise vertex-disjoint holes of length at least ℓ or
- there exists $T \subseteq V(G)$ with $|T| \leq f_\ell(k)$ such that $G - T$ contains no holes of length at least ℓ.

Let $x = \max\{f_\ell(1) + 1, \ell\}$. From the hypergraph $U_{2x-1,x} = (X, \mathcal{E})$, we construct a graph G on the vertex set $S \uplus \bigcup_{F \in \mathcal{E}} Y_F$, where

- $S = \{s_v : v \in X\}$ is an independent set of size $|X|$,
- $Y_F = \{y_v : v \in F\}$ is an independent set of size x for each $F \in \mathcal{E}$.

The edge set of G is created as follows.

- For each hyperedge $F \in \mathcal{E}$ with $F = \{v_i : 1 \leq i \leq x\}$, we add the edge set

$$\{y_{v_1}s_{v_1}, s_{v_1}y_{v_2}, \ldots, y_{v_x}s_{v_x}, s_{v_x}y_{v_1}\}.$$

- For each pair of two distinct hyperedges $F_1, F_2 \in \mathcal{E}$, we add all possible edges between Y_{F_1} and Y_{F_2}.

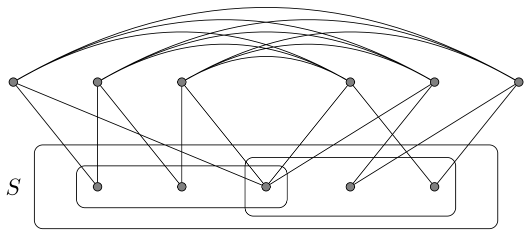

Fig. 11. An illustration of two holes constructed from two hyperedges.

Note that for each $F \in \mathcal{E}$, $G[Y_F \cup S]$ contains precisely one hole, which has length $2x(\geq \ell)$. We denote this hole as C_F. Fig. 11 depicts the construction.

We verify that every hole of length at least ℓ is one of the holes in $\{C_F : F \in \mathcal{E}\}$.

Claim 14. *Every hole of length at least ℓ is exactly one of the holes in $\{C_F : F \in \mathcal{E}\}$.*

Proof of the Claim. Suppose C is a hole of length at least $\ell \geq 5$. We show that $V(C) \subseteq V(C_F)$ for some $F \in \mathcal{E}$. Clearly, it implies the claim as each C_F is a hole.

Suppose for contradiction that C is not contained in one of $\{C_F : F \in \mathcal{E}\}$. Then there are two distinct hyperedges $F, F' \in \mathcal{E}$ such that $V(C) \cap Y_F \neq \emptyset$ and $V(C) \cap Y_{F'} \neq \emptyset$. Let $v \in V(C) \cap Y_F$ and $v' \in V(C) \cap Y_{F'}$. Due to the construction of G, we have $vv' \in E(G)$. Furthermore, this also implies that for every $F'' \in \mathcal{E} \setminus \{F, F'\}$, we have $V(C) \cap Y_{F''} = \emptyset$.

Since S is independent, among the vertices of $V(C) \setminus \{v, v'\}$ there are at least $\lfloor (|V(C)| - 2)/2 \rfloor$ vertices of $Y_F \cup Y_{F'}$. Suppose $V(C) \setminus \{v, v'\} \setminus S$ has two vertices w and w'. If both of w and w' are in Y_F, then v' is adjacent to at least three vertices of C, a contradiction. Therefore, we may assume that $w \in Y_F$ and $w' \in Y_{F'}$. Then $G[\{v, v', w, w'\}]$ is a cycle of length 4, contradicting the assumption that C is a hole of length at least $\ell(\geq 5)$. If $V(C) \setminus \{v, v'\} \setminus S$ contains a unique vertex, say $w \in Y_F$, observe that $|V(C)| = 5$ and wv' is a chord of C, a contradiction. $\diamond$

By Claim 14, $\{C_F : F \in \mathcal{E}\}$ is precisely the set of all holes of length at least ℓ in G. One can observe that two holes in $\{C_F : F \in \mathcal{E}\}$ intersect because $(X, \mathcal{E})$ is the hypergraph $U_{2x-1,x}$, in which every two hyperedges intersect. Therefore, by the property of the function f_ℓ, there exists a vertex subset $T \subseteq V(G)$ with $|T| \leq f_\ell(1) < x$ such that $G - T$ contains no holes of length at least ℓ. We may assume that T is a subset of S; a vertex of Y_F hits no hole of G other than the hole C_F, which can be hit by choosing the corresponding vertex of S instead. However, there is always a hyperedge avoiding a set of $x - 1$ elements in the hypergraph $U_{2x-1,x}$, and it implies that $G - T$ contains a hole in $\{C_F : F \in \mathcal{E}\}$. This is a contradiction. We conclude that such a function f_ℓ does not exist. $\square$

For an integer $\alpha \geq 2$, a graph class $\mathcal{C}$ has the $1/\alpha$*-integral Erdős-Pósa property* under a graph containment relation $\leq_\star$ if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G and a positive integer k, G contains either

- $k+1$ pairwise distinct subsets $Z_1, \dots, Z_k$ such that each subgraph of G induced by Z_i contains a member of $\mathcal{C}$ under $\leq_\star$ and each vertex of G is contained in at most α sets of $Z_1, \dots, Z_k$, or
- a vertex set T of G such that $|T| \leq f(k)$ and $G - T$ contains no member of $\mathcal{C}$ under $\leq_\star$.

Sometimes, a class of graphs that does not have the Erdős-Pósa property has the 1/2-integral Erdős-Pósa property. For example, the class of odd cycles has the 1/2-integral Erdős-Pósa property (under the subgraph relation), while it has no Erdős-Pósa property [21].

Simply modifying the proof of Theorem 1.2, we can show that for any fixed integers $\alpha \geq 2$ and $\ell \geq 5$, the class of cycles of length at least ℓ does not have the $1/\alpha$-integral Erdős-Pósa property under the induced subgraph relation. The main idea we used in Theorem 1.2 is that the set of hyperedges in the uniform hypergraph $U_{2x-1,x}$ satisfies that two hyperedges always intersect and the size of a hitting set for $U_{2x-1,x}$ is at least x.

Let us consider the uniform hypergraph $U_{(\alpha+1)x-1,\alpha x}$. We claim that any tuple of $\alpha+1$ hyperedges in $U_{(\alpha+1)x-1,\alpha x}$ has a common intersection. Inductively, one can verify that for all integers $2 \leq t \leq \alpha+1$, any tuple of t hyperedges has at least $(\alpha+1-t)x+(t-1)$ common intersections. Thus, any tuple of $\alpha+1$ hyperedges has a non-empty intersection. But the size of a hitting set for $U_{(\alpha+1)x-1,\alpha x}$ is at least x. Thus, by taking $x = \max\{f_\ell(\alpha)+1, \ell\}$ and replacing $U_{2x-1,x}$ with $U_{(\alpha+1)x-1,\alpha x}$ in the proof of Theorem 1.2, we obtain that the class of cycles of length at least ℓ does not have the $1/\alpha$-integral Erdős-Pósa property under the induced subgraph relation.

8. Applications of the Erdös-Pósa property for holes

8.1. Packing and covering weighted cycles

We show the weighted version of Erdös-Pósa property of cycles. We recall that for a graph G and a non-negative weight function $w : V(G) \to \mathbb{N} \cup \{0\}$, let $\mathsf{pack}(G, w)$ be the maximum number of cycles (repetition is allowed) such that each vertex v is used at most $w(v)$ times, and let $\mathsf{cover}(G, w)$ be the minimum value $\sum_{v \in X} w(v)$ where X hits all cycles in G.

Corollary 8.1. *For a graph G and a non-negative weight function $w : V(G) \to \mathbb{N} \cup \{0\}$, $\mathsf{cover}(G, w) \leq O(k^2 \log k)$ where $k = \mathsf{pack}(G, w)$.*

Proof. We may assume that $w(v)$ is positive for every $v \in V(G)$. Let $k = \mathsf{pack}(G, w)$. We construct a new graph H from G as follows:

(1) We obtain a graph G' from G by subdividing each edge uv into an induced path $u - e_{uv} - v$, and give the weight $w(e_{uv}) := \min\{w(u), w(v)\}$.
(2) For each vertex v in G', let Q_v be a complete graph on $w(v)$ vertices.
(3) Let H be the graph obtained from the vertex-disjoint union of all graphs in $\{Q_v : v \in V(G')\}$ by adding all edges between Q_v and Q_x for each edge vx in G'.

We will show that

- $\mathsf{pack}(G, w)$ is the same as the number of maximum pairwise vertex-disjoint holes in H, and
- $\mathsf{cover}(G, w)$ is the same as the size of a minimum vertex set S in H such that $H - S$ has no holes.

By Theorem 1.1, this implies $\mathsf{cover}(G, w) \leq O(k^2 \log k)$.

First, each hole C of H intersects a complete subgraph Q_v at most once for every $v \in V(G')$; otherwise, H contains a triangle. Also, if a hole C intersects a complete subgraph Q_v for some vertex v in G', then C traverses precisely two complete subgraphs among $\{Q_x : x \in N_{G'}(v)\}$. This means that each hole C of H corresponds to a cycle of G', and thus to a cycle of G. It easily follows that $\mathsf{pack}(G, w)$ is as large as the number of maximum pairwise vertex-disjoint holes in H. Conversely, let $\mathcal{P}$ be a packing of cycles of G in which every vertex $v \in V(G)$ is used at most $w(v)$ times. Clearly, each cycle of G yields a canonical hole of H due to the subdivision in the intermediate graph G'. It is easy to see that one can build a packing $\mathcal{P}'$ of vertex-disjoint holes of H from $\mathcal{P}$ such that $|\mathcal{P}'| = |\mathcal{P}|$. Therefore, $\mathsf{pack}(G, w)$ equals the number of maximum pairwise vertex-disjoint holes in H.

Let S be a minimum-sized vertex set in H such that $H - S$ has no holes. We observe that if S contains a vertex of Q_v for some $v \in V(G')$, then $V(Q_v) \subseteq S$ because of the minimality of S and the fact that all vertices of Q_v have the same neighborhood.

If S contains $Q_{e_{xy}}$ for a subdividing vertex e_{xy} of G' such that $w(e_{xy}) = w(x)$, then the set $(S \cup V(Q_x)) \setminus V(Q_{e_{xy}})$ hits every hole of H. Hence, we can assume that S contains only vertices of Q_v for $v \in V(G)$. Now, the vertex subset $S' := \{v \in V(G) : V(Q_v) \subseteq S\}$ of $V(G)$ hits every cycle of G and $\sum_{v \in S'} w(v) = |S|$. This implies that $\mathsf{cover}(G, w) \leq |S|$. Conversely, if G contains a solution S, then we can simply take $\bigcup_{v \in S} Q_v$ to hit every hole of H. Therefore, $\mathsf{cover}(G, w)$ equals the size of a minimum vertex set S of H such that $H - S$ has no holes. $\square$

8.2. *Approximations for* Chordal Vertex Deletion

Theorem 1.1 can be converted into an approximation algorithm of factor $O(\mathsf{opt} \log \mathsf{opt})$ for Chordal Vertex Deletion, where opt is the minimum number of vertices whose deletion makes the input graph G chordal. We may assume that the input graph G is not chordal. The approximation algorithm works as follows. We first greedily construct a maximal packing of p vertex-disjoint holes. For $k = p, \dots$, we apply the algorithm of Theorem 1.1. If it outputs $k+1$ holes, then we increase k by one and recurse. If a hitting set X of size $O(k^2 \log k)$ is returned, then we return X as an approximate solution for Chordal Vertex Deletion and terminate the algorithm. To see that this procedure achieves the claimed approximation factor, notice that performing the algorithm of Theorem 1.1 for k means that in the previous step (whether it was the greedy packing step or the run of the algorithm of Theorem 1.1) found k vertex-disjoint holes. Therefore, we have $\mathsf{opt} \geq k$. In particular, when a hitting set X is returned, we have $|X| \leq c_1 k^2 \log k + c_2 \leq c_1 \mathsf{opt}^2 \log \mathsf{opt} + c_2$. Here c_1, c_2 are the constants as in Theorem 1.1. As we run the polynomial-time algorithm of Theorem 1.1 at most n times, clearly this approximation runs in polynomial-time.

The following statement summarizes the result.

Theorem 8.2 *(Restatement of Theorem 1.3). There is an approximation algorithm of factor $O(\mathsf{opt} \log \mathsf{opt})$ for* Chordal Vertex Deletion.

9. Concluding remarks

We show that the class of cycles of length at least 4 has the Erdős-Pósa property under the induced subgraph relation, with a gap function $f(k) = c_1 k^2 \log k + c_2$ for some constants c_1 and c_2. A natural question is whether an improved bound can be obtained. A lower bound $c' k \log k$ for some constant c' is known for the Erdős-Pósa property on cycles [8]. The following reduction shows that this is also a lower bound on a gap function for holes. Given a graph G, let G' be a graph obtained by subdividing each edge of G once. Then the girth of G' is at least 4, and there is an obvious one-to-one correspondence between cycles of G and holes of G'. Since we may assume that a minimum-sized vertex set hitting every hole of G' contains no subdividing vertex, the packing and covering numbers for cycles of G equals the packing and covering numbers for holes of G', respectively.

One might ask whether or not variants of cycles satisfy the Erdős-Pósa property under the induced subgraph relation. We list some of open problems.

- It was shown in [15,18] that any graph with a vertex set S contains either $k+1$ vertex-disjoint S-cycles or a vertex set of size $O(k \log k)$ hitting all S-cycles, where S-cycles are cycles intersecting S. As a generalization, Bruhn, Joos, and Schaudt [5] showed that the class of long S-cycles also has the Erdős-Pósa property. It is easy to see that the class of S-cycles has the Erdős-Pósa property under the induced subgraph relation, because every S-cycle contains an induced S-cycle. Determining whether or not the class of S-cycles of length at least 4 has the Erdős-Pósa property under the induced subgraph relation is an open problem.

- Huynh, Joos, and Wollan [11] generalized the results of [15,18] to (S_1, S_2)-cycles, which intersect two prescribed vertex sets S_1 and S_2. The same question can be asked for the cycles of length at least 4 intersecting two prescribed sets S_1 and S_2.
- Thomassen [25] proved that the class of even cycles has the Erdős-Pósa property. One might ask whether the class of even cycles has the Erdős-Pósa property under the induced subgraph relation.

Our construction for no Erdős-Pósa property in Section 7 creates many holes of length exactly 4, and we are not aware of any construction which does not feature (many) holes of length 4. Does the class of cycles of length at least ℓ, for fixed $\ell \geq 6$, has the Erdős-Pósa property on C_4-free graphs under the induced subgraph relation? The answer is not clear to us. When $\ell = 5$, the Erdős-Pósa property holds as an immediate consequence of our result.

Our result can be reformulated as the Erdős-Pósa property for the class of C_4-subdivisions under the induced subgraph relation. Investigating the Erdős-Pósa property of H-subdivisions under the induced subgraph relation for other graphs H, and the computational aspect of related covering/packing problems can be a fruitful research direction. Recently, the second author and Raymond [19] determined, for various graphs H, whether the class of H-subdivisions has the Erdős-Pósa property under the induced subgraph relation or not.

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