Received: 20 April 2022 | Revised: 25 October 2022 | Accepted: 20 November 2022

DOI: 10.1002/oca.2960

RESEARCH ARTICLE

WILEY

A sufficient condition for optimal control problem of fully coupled forward-backward stochastic systems with jumps: A state-constrained control approach

Hyun Jong Yang[1] | **Jun Moon**[2]

[1]Department of Electrical Engineering, Pohang University of Science and Technology, Pohang, Gyeongsangbuk-do, South Korea

[2]Department of Electrical Engineering, Hanyang University, Seoul, South Korea

Correspondence
Jun Moon, Department of Electrical Engineering, Hanyang University, Seoul 04763, South Korea.
Email: junmoon@hanyang.ac.kr

Funding information
Institute of Information & communications Technology Planning, Grant/Award Numbers: 2020-0-01373, 2018-0-00958; Ministry of Trade, Industry and Energy, Grant/Award Number: 20018112; National Research Foundation of Korea, Grant/Award Numbers: NRF-2021R1A2C2094350, NRF-2017R1A5A1015311

Abstract

We study the stochastic optimal control problem for fully coupled forward-backward stochastic differential equations (FBSDEs) with jump diffusions. A major technical challenge of such problems arises from the dependence of the (forward) diffusion term on the backward SDE and the presence of jump diffusions. Previously, this class of problems has been solved via only the stochastic maximum principle, which guarantees only the necessary condition of optimality and requires identifying unknown parameters in the corresponding variational inequality. Our paper provides an alternative approach, which constitutes the sufficient condition for optimality. Specifically, the original fully coupled FBSDE control problem (referred to as **(P)**) is converted into the terminal state-constrained forward stochastic control problem (referred to as $(\mathbf{P}')$) that includes additional (possibly unbounded) control variables. Then $(\mathbf{P}')$ is solved via the backward reachability analysis, by which the value function of $(\mathbf{P}')$ is expressed as the zero-level set of the value function for the auxiliary unconstrained (forward) control problem (referred to as $(\mathbf{P}'')$). Unlike $(\mathbf{P}')$, $(\mathbf{P}'')$ is an unconstrained problem, which includes additional control variables as a consequence of the martingale representation theorem. We show that the value function for $(\mathbf{P}'')$ is the unique viscosity solution to the associated integro-type Hamilton-Jacobi-Bellman (HJB) equation. The viscosity solution analysis presented in our paper requires a new technique due to additional control variables in the Hamiltonian maximization and the presence of the nonlocal integral operator in terms of the (singular) Lévy measure. To solve the original problem **(P)**, we reverse our approach. Specifically, we first solve $(\mathbf{P}'')$ to obtain the value function using the verification theorem and the viscosity solution of the HJB equation. Then $(\mathbf{P}')$ is solved by characterizing the zero-level set of the value function of $(\mathbf{P}'')$, from which the optimal solution of **(P)** can be constructed. To illustrate the theoretical results of this paper, applications to the linear-quadratic problem for fully coupled FBSDEs with jumps are also presented.

KEYWORDS

backward reachability approach, fully coupled FBSDEs with jump diffusions, integro-type HJB equation, state-constrained stochastic control, verification theorem

1 | INTRODUCTION

We first state the notation used in this paper. The precise problem statement and the detailed literature review are then provided.

1.1 | Notation

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. For $x, y \in \mathbb{R}^n$, $x^\top$ denotes the transpose of x, $\langle x, y\rangle$ is the inner product, and $|x| := \langle x, x\rangle^{1/2}$. Let $\mathbb{S}^n$ be the set of $n \times n$ symmetric matrices. Let $\mathrm{Tr}(A)$ be the trace operator for a square matrix $A \in \mathbb{R}^{n\times n}$. Let $\|\cdot\|_F$ be the Frobenius norm, that is, $\|A\|_F := \mathrm{Tr}(AA^\top)^{1/2}$ for $A \in \mathbb{R}^{n\times m}$. Let I_n be an $n \times n$ identity matrix. In various places of the paper, an exact value of a positive constant C can vary from line to line, which mainly depends on the coefficients in the assumptions of this paper, terminal time T, and the initial condition, but independent to a specific choice of control.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with the natural filtration $\mathbb{F} := \{\mathcal{F}_s,\ 0 \leq s \leq t\}$ generated by the following two mutually independent stochastic processes and augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$: (i) a p-dimensional standard Brownian motion B defined on $[t, T]$ and (ii) an E-marked right continuous Poisson random measure (process) N defined on $E \times [t, T]$, where $E := \bar{E} \setminus \{0\}$ with $\bar{E} \subset \mathbb{R}^l$ is a Borel subset of $\mathbb{R}^l$ equipped with its Borel σ-field $\mathcal{B}(E)$. The intensity measure of N is denoted by $\hat{\lambda}(\mathrm{d}e, \mathrm{d}t) := \lambda(\mathrm{d}e)\mathrm{d}t$, satisfying $\lambda(E) < \infty$, where $\{\tilde{N}(A, (t, s]) := (N - \hat{\lambda})(A, (t, s])\}_{s\in(t,T]}$ is an associated compensated $\mathcal{F}_s$-martingale random (Poisson) measure of N for any $A \in \mathcal{B}(E)$. Here, λ is a Lévy measure on $(E, \mathcal{B}(E))$, which satisfies $\lambda(E) < \infty$ and $\int_E (1 \wedge |e|^2)\lambda(\mathrm{d}e) < \infty$. We introduce the following spaces:

- $L^p(\Omega, \mathcal{F}_s; \mathbb{R}^n)$, $s \in [t, T]$, $p \geq 1$: the space of $\mathcal{F}_s$-measurable $\mathbb{R}^n$-valued random vectors, satisfying $\|x\|_{L^p} := \mathbb{E}[|x|^p] < \infty$;
- $\mathcal{L}^p_{\mathbb{F}}(t, T; \mathbb{R}^n)$, $t \in [0, T]$, $p \geq 1$: the space of $\mathbb{F}$-predictable $\mathbb{R}^n$-valued random processes, satisfying $\|x\|_{\mathcal{L}^p_{\mathcal{F}}} := \mathbb{E}[\int_t^T |x_s|^p \mathrm{d}s]^{\frac{1}{p}} < \infty$;
- $G^2(E, \mathcal{B}(E), \lambda; \mathbb{R}^n)$: the space of functions such that for $k \in G^2(E, \mathcal{B}(E), \lambda; \mathbb{R}^n)$, $k : E \to \mathbb{R}^n$ satisfies $\|k\|_{G^2} := (\int_E |k(e)|^2 \lambda(\mathrm{d}e))^{\frac{1}{2}} < \infty$;
- $\mathcal{G}^2_{\mathbb{F}}(t, T, \lambda; \mathbb{R}^n)$, $t \in [0, T]$: the space of stochastic processes such that for $k \in \mathcal{G}^2_{\mathbb{F}}(t, T, \lambda; \mathbb{R}^n)$, $k : \Omega \times [t, T] \times E \to \mathbb{R}^n$ is an $\mathcal{P} \times \mathcal{B}(E)$-measurable and $\mathbb{R}^n$-valued $\mathbb{F}$-predictable stochastic process, which satisfies $\|k\|_{\mathcal{G}^2_{\mathbb{F}}} := \mathbb{E}[\int_t^T \int_E |k_s(e)|^2 \lambda(\mathrm{d}e)\mathrm{d}s]^{\frac{1}{2}} < \infty$, where $\mathcal{P}$ denotes the σ-algebra of $\mathbb{F}$-predictable subsets of $\Omega \times [0, T]$;
- $C([0, T] \times \mathbb{R}^n)$: the set of $\mathbb{R}$-valued continuous functions on $[0, T] \times \mathbb{R}^n$;
- $C_p([0, T] \times \mathbb{R}^n)$, $p \geq 1$: the set of $\mathbb{R}$-valued continuous functions such that $f \in C_p([0, T] \times \mathbb{R}^n)$ holds $|f(t, x)| \leq C(1 + |x|^p)$;
- $C_b^{l,r}([0, T] \times \mathbb{R}^n)$ $l, r \geq 1$: the set of $\mathbb{R}$-valued continuous functions on $[0, T] \times \mathbb{R}^n$ such that for $f \in C^{l,r}([0, T] \times \mathbb{R}^n)$, $\partial_t^l f$ and $D^r f$ exist, and are continuous and uniformly bounded, where $\partial_t^l f$ is the lth-order partial derivative of f with respect to $t \in [0, T]$ and $D^r f$ is the rth-order derivative of f in $x \in \mathbb{R}^n$.

1.2 | Problem statement

Let $\mathcal{U}_{t,T} := \mathcal{L}^p_{\mathbb{F}}(t, T; U)$ be the set of admissible controls, which is the set of square integrable $\mathbb{F}$-predictable U-valued processes, where $U \subset \mathbb{R}^l$ is compact. The aim of this paper is to minimize the following objective functional over $u \in \mathcal{U}_{t,T}$:

$$\textbf{(P)} \quad J(t, a; u) = \mathbb{E}\left[\int_t^T l(s, X^u_{t,a}(s), u(s), Y^u_{t,a}(s), Z^u_{t,a}(s), \int_E K^u_{t,a}(s, e)\lambda(\mathrm{d}e))\mathrm{d}s + m(X^u_{t,a}(T))\right], \tag{1}$$

subject to the constraint given by

$$
\begin{cases}
\mathrm{d}X_{t,a}^u(s) = f(s, X_{t,a}^u(s-), u(s), Y_{t,a}^u(s-), Z_{t,a}^u(s), \int_E K_{t,a}^u(s,e)\lambda(\mathrm{d}e))\mathrm{d}s \\
\quad +\sigma(s, X_{t,a}^u(s-), u(s), Y_{t,a}^u(s-), Z_{t,a}^u(s), \int_E K_{t,a}^u(s,e)\lambda(\mathrm{d}e))\mathrm{d}B(s) \\
\quad + \int_E \chi(s, e, X_{t,a}^u(s-), u(s), Y_{t,a}^u(s-), Z_{t,a}^u(s), K_{t,a}^u(s,e))\tilde{N}(\mathrm{d}s, \mathrm{d}e) \\
\mathrm{d}Y_{t,a}^u(s) = -g(s, X_{t,a}^u(s-), u(s), Y_{t,a}^u(s-), Z_{t,a}^u(s), \int_E K_{t,a}^u(s,e)\lambda(\mathrm{d}e))\mathrm{d}s \\
\quad +Z_{t,a}^u(s)^\top \mathrm{d}B(s) + \int_E K_{t,a}^u(s,e)\tilde{N}(\mathrm{d}s, \mathrm{d}e) \\
X_{t,a}^u(t) = a, \ Y_{t,a}^u(T) = h(X_{t,a}^u(T)).
\end{cases} \tag{2}
$$

Here, (2) is known as the (controlled) fully coupled forward-backward stochastic differential equation (FBSDE) with jump diffusions. In (2), the quadruple (X, Y, Z, K) are $(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{p\times m} \times \mathbb{R}^m)$-valued stochastic processes, which define the solution to (2). Note that $u \in \mathcal{U}_{t,T}$ in (2) is the corresponding control process. We mention that in (2), X is the forward process with the initial condition, while (Y, Z, K) the backward process with the terminal condition. In fact, "the fully coupled" means that the diffusion part of X (σ in (2)) explicitly depends on the backward SDE (BSDE) (Y, Z, K).

Assumption 1. The fully coupled FBSDE in (2) is well posed, that is, (2) admits a unique solution.

Remark 1. We mention that there are several different conditions for Assumption 1, which can be found in References 1-7 and the references therein. In our paper, we do not need specific conditions for Assumption 1, as the standard Lipschitz-type and linear growth conditions in Assumption 2 given below are sufficient to solve $(\mathbf{P}')$, the equivalent forward problem of **(P)** defined below. In general, Assumption 1 is stronger than Assumption 2 (see Remark 5).

Remark 2. Based on the formulation, **(P)** is known as the stochastic control problem for (fully coupled) FBSDEs with jump diffusions. Since (2) involves coupled forward and backward parts, it is difficult to solve **(P)** directly. Below, we convert **(P)** into the forward optimization with the terminal state constraint.

Let $\mathcal{Z}_{t,T}^{(1)} := \mathcal{L}_{\mathbb{F}}^2(t, T; \mathbb{R}^{p\times m})$ and $\mathcal{K}_{t,T}^{(1)} := \mathcal{G}_{\mathbb{F}}^2(t, T, \lambda; \mathbb{R}^m)$. Consider the minimization the following objective functional over $(u, z, k) \in \mathcal{U}_{t,T} \times \mathcal{Z}_{t,T}^{(1)} \times \mathcal{K}_{t,T}^{(1)}$:

$$
(\mathbf{P}') \quad J(t, a, b; u, z, k) = \mathbb{E}\left[\int_t^T l(s, X_{t,a}^{u,z,k}(s), u(s), Y_{t,b}^{u,z,k}(s), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))\mathrm{d}s + m(X_{t,a}^{u,z,k}(T))\right], \tag{3}
$$

subject to

$$
\begin{cases}
\mathrm{d}X_{t,a}^{u,z,k}(s) = f(s, X_{t,a}^{u,z,k}(s-), u(s), Y_{t,b}^{u,z,k}(s-), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))\mathrm{d}s \\
\quad +\sigma(s, X_{t,a}^{u,z,k}(s-), u(s), Y_{t,b}^{u,z,k}(s-), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))\mathrm{d}B(s) \\
\quad + \int_E \chi(s, e, X_{t,a}^{u,z,k}(s-), u(s), Y_{t,b}^{u,z,k}(s-), z(s), k(s,e))\tilde{N}(\mathrm{d}s, \mathrm{d}e) \\
\mathrm{d}Y_{t,b}^{u,z,k}(s) = -g(s, X_{t,a}^{u,z,k}(s-), u(s), Y_{t,b}^{u,z,k}(s-), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))\mathrm{d}s \\
\quad +z(s)^\top \mathrm{d}B(s) + \int_E k(s,e)\tilde{N}(\mathrm{d}s, \mathrm{d}e) \\
X_{t,a}^{u,z,k}(t) = a, \ Y_{t,b}^{u,z,k}(t) = b,
\end{cases} \tag{4}
$$

and *the terminal state constraint*

$$
\rho^2(X_{t,a}^{u,z,k}(T), Y_{t,b}^{u,z,k}(T)) := \left|h(X_{t,a}^{u,z,k}(T)) - Y_{t,b}^{u,z,k}(T)\right|^2 = 0, \ \mathbb{P}-\text{a.s.} \tag{5}
$$

This problem is referred to as $(\mathbf{P}')$. Notice that unlike (2), (4) has only initial conditions, which corresponds to the the forward SDE (with jump diffusions) with the control process $(u, z, k) \in \mathcal{U}_{t,T} \times \mathcal{Z}_{t,T}^{(1)} \times \mathcal{K}_{t,T}^{(1)}$. While (1) has only one optimization (decision) variable $u \in \mathcal{U}_{t,T}$, the minimization of (3) is taken with respect to $(u, z, k) \in \mathcal{U}_{t,T} \times \mathcal{Z}_{t,T}^{(1)} \times \mathcal{K}_{t,T}^{(1)}$, where $(z, k) \in \mathcal{Z}_{t,T}^{(1)} \times \mathcal{K}_{t,T}^{(1)}$ can be regarded as additional (possibly unbounded) control variables induced due to the conversion from **(P)** to $(\mathbf{P}')$ discussed below.

Remark 3.

(i) It is easy to see that **(P)** is embedded in $(\mathbf{P}')$ in that if $(u^*, z^*, k^*) \in \mathcal{U}_{t,T} \times \mathcal{Z}_{t,T} \times \mathcal{K}_{t,T}$ is the optimal solution of $(\mathbf{P}')$, then u^* is the optimal solution of **(P)** with $(Z, K) = (z^*, k^*)$ in (2). By (5), the forward process (4) with $(u^*, Z, K) = (u^*, z^*, k^*)$ becomes equivalent to the FBSDE in (2). Note that $(\mathbf{P}')$ is the forward optimization problem, where the two martingale terms (Z, K) in **(P)** become the additional control processes. The additional terminal state constraint in (5) is induced due to the terminal condition of the FBSDE in (2). Indeed, $(\mathbf{P}')$ can be regarded as a class of stochastic control problems for jump-diffusion models with the (terminal) state constraint.

(ii) Since **(P)** can be analyzed through $(\mathbf{P}')$, we study $(\mathbf{P}')$. Although $(\mathbf{P}')$ is the forward optimal control problem, the standard optimal control technique cannot be applied directly due to the (terminal) state constraint and additional (possibly unbounded) control variables. Our paper develops a new approach to solve $(\mathbf{P}')$ via the backward reachability analysis.

Remark 4. Note that the terminal state constraint (5) can be replaced by one of the following equivalent constraints: (i) $Y_{t,b}^{u,z,k}(T) = h(X_{t,a}^{u,z,k}(T))$, $\mathbb{P}$-.a.s; (ii) $\mathbb{1}_{Y_{t,b}^{u,z,k}(T)=h(X_{t,a}^{u,z,k}(T))} = 1$, $\mathbb{P}$-.a.s; and (iii) $\mathbb{E}[|Y_{t,b}^{u,z,k}(T) - h(X_{t,a}^{u,z,k}(T))|^2]^{\frac{1}{2}} = 0$.

Assumption 2.

(i) $f : [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^m \times \mathbb{R}^{p\times m} \times \mathbb{R}^m \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^m \times \mathbb{R}^{p\times m} \times \mathbb{R}^m \to \mathbb{R}^{n\times p}$, $\chi : [0,T] \times E \times \mathbb{R}^n \times U \times \mathbb{R}^m \times \mathbb{R}^{p\times m} \times G(E, \mathcal{B}(E), \lambda; \mathbb{R}^m) \to \mathbb{R}^n$, and $g : [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^m \times \mathbb{R}^{p\times m} \times \mathbb{R}^m \to \mathbb{R}^m$. For any $s \in [0,T]$, $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}^m$, and $(u,z,k) \in U \times \mathbb{R}^{p\times m} \times \mathbb{R}^m$, there is a constant $L \geq 0$ such that $|f(s,x,u,y,z,k) - f(s,x',u,y',z,k)| \leq L(|x-x'| + |y-y'|)$, $|\sigma(s,x,u,y,z,k) - \sigma(s,x',u,y',z,k)| \leq L(|x-x'| + |y-y'|)$, $\|\chi(s,\cdot,x,u,y,z,k) - \chi(s,\cdot,x',u,y',z,k)\|_{G_2} \leq L(|x-x'| + |y-y'|)$, $|g(s,x,y,y,z,k) - g(s,x',y',y,z,k)| \leq L(|x-x'| + |y-y'|)$, $|f(s,x,u,y,z,k)| + |\sigma(s,x,u,y,z,k)| + \|\chi(s,\cdot,x,u,y,z,k)\|_{G_2} \leq L(1 + |x| + |y|)$, and $|g(s,x,y,y,z,k)| \leq L(1 + |x| + |y|)$;

(ii) $l : [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^m \times \mathbb{R}^{p\times m} \times \mathbb{R}^m \to \mathbb{R}$, $m : \mathbb{R}^n \to \mathbb{R}$, and $h : \mathbb{R}^n \to \mathbb{R}^m$. For any $s \in [0,T]$, $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}^m$, and $(u,z,k) \in U \times \mathbb{R}^{p\times m} \times \mathbb{R}^m$, there is a constant $L \geq 0$ such that $|l(s,x,u,y,z,k) - l(s,x',u,y',z,k)| \leq L(|x-x'| + |y-y'|)$, $|m(x) - m(x')| + |h(x) - h(x')| \leq L|x-x'|$, and $|l(s,x,u,y,z,k)| + |m(x)| + |h(x)| \leq L(1 + |x| + |y|)$;

(iii) l and m are nonnegative functions.

Remark 5.

(i) We note that Assumption 2 along may not guarantee that the existence and uniqueness of the solution to the FBSDE in (2) for the whole interval $[0,T]$. That is, in some situations, Assumption 2 guarantees the well-posedness of the FBSDE in (2), but in most cases we need additional assumptions in addition to Assumption 2. This is why we impose Assumption 1 for **(P)**. In fact, much stronger assumptions than Assumption 2 are needed in Assumption 1 for (2). We should note that Assumption 1 requires additional assumptions in addition to Assumption 1 (e.g. the monotonicity assumptions), where different sets of additional assumptions needed in Assumption 1 can be found in References 1-7 and the references therein.

(ii) The main purpose not introducing specific conditions in Assumption 1 for **(P)** is that in our paper, only Assumption 2 is sufficient to solve $(\mathbf{P}')$, since the converted forward SDE in (4) admits a unique (strong) solution (X, Y) on $[0,T]$ under Assumption 2 only. This implies that $(\mathbf{P}')$ can be solved under Assumption 2 only, which provides the equivalent optimal solution of **(P)**, provided that Assumption 1 holds.

Lemma 1. *Assume that Assumption 2 holds. Then for any $(a,b) \in \mathbb{R}^{n+m}$ and $(u,z,k) \in \mathcal{U}_{t,T} \times \mathcal{Z}_{t,T}^{(1)} \times \mathcal{K}_{t,T}^{(1)}$, there is a unique $\mathbb{F}$-adapted cádlág (strong) solution of (4). Furthermore, for any $(a,b), (a',b') \in \mathbb{R}^{m+m}$, $(u,z,k) \in \mathcal{U}_{t,T} \times \mathcal{Z}_{t,T}^{(1)} \times \mathcal{K}_{t,T}^{(1)}$, and $t, t' \in [0,T]$ with $t \leq t'$, there exists a constant C such that (i) $\mathbb{E}[\sup_{s\in[t,T]} |X_{t,a}^{u,z,k}(s) - X_{t,a'}^{u,z,k}(s)|^2] \leq C(|a-a'|^2 + |b-b'|^2)$, (ii) $\mathbb{E}[\sup_{s\in[t,T]} |Y_{t,b}^{u,z,k}(s) - Y_{t,b'}^{u,z,k}(s)|^2] \leq C(|a-a'|^2 + |b-b'|^2)$,*

(iii) $\mathbb{E}[\sup_{s\in[t',T]} |X_{t,a}^{u,z,k}(s) - X_{t',a}^{u,z,k}(s)|^2] \leq C(1 + |a|^2 + |b|^2)|t'-t|$, and (iv) $\lim_{t'\to t} \mathbb{E}[|Y_{t,b}^{u,z,k}(T) - Y_{t',b}^{u,z,k}(T)|] = 0$ for $t' \in [0,T]$.

Proof. Let $\widehat{X}^{u,z,k}_{t,a,b}(\cdot) = \begin{bmatrix} X^{u,z,k}_{t,a}(\cdot)^\top & Y^{u,z,k}_{t,b}(\cdot)^\top \end{bmatrix}^\top \in \mathbb{R}^{n+m}$. Then (4) becomes the $n+m$-dimensional (forward) SDE with jump diffusions, where the triple $(u,z,k) \in \mathcal{U}_{t,T} \times \mathcal{Z}^{(1)}_{t,T} \times \mathcal{K}^{(1)}_{t,T}$ is the corresponding control process. By Assumption 2 and (Reference 8, theorem 6.2.3; see also theorem 1.19, Reference 9 and Reference 10), (4) admits a unique $\mathbb{F}$-adapted cádlág (strong) solution. The estimates in (i)–(iv) can be shown via Assumption 2 and the Burkholder-Davis-Gundy inequality (see References 8-11 and the references therein).

We now prove (iv). Without loss of generality, we assume that $t' \geq t$. Consider

$$
\begin{aligned}
Y^{u,z,k}_{t,b}(T) - Y^{u,z,k}_{t',b}(T) = &-\int_t^T [g(s, X^{u,z,k}_{t,a}(s-), u(s), Y^{u,z,k}_{t,b}(s-), z(s), \int_E k(s,e)\lambda(\mathrm{d}e)) \\
&- g(s, X^{u,z,k}_{t',a}(s-), u(s), Y^{u,z,k}_{t',b}(s-), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))]\mathrm{d}s \\
&- \int_t^{t'} g(s, X^{u,z,k}_{t',a}(s-), u(s), Y^{u,z,k}_{t',b}(s-), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))\mathrm{d}s \\
&+ \int_t^{t'} z(s)^\top \mathrm{d}B(s) + \int_t^{t'} \int_E k(s,e)\tilde{N}(\mathrm{d}s,\mathrm{d}e).
\end{aligned}
$$

Notice that as $(z,k) \in \mathcal{Z}^{(1)}_{t,T} \times \mathcal{K}^{(1)}_{t,T}$,

$$
\mathbb{E}\left[\left|\int_t^{t'} z(s)^\top \mathrm{d}B_s\right|\right]^2 \leq \mathbb{E}\left[\int_t^{t'} |z(s)|^2 \mathrm{d}s\right] \to 0 \text{ as } t' \to t,
$$

and similarly using Kunita's formula (theorem 4.4.23 of Reference 8),

$$
\mathbb{E}\left[\left|\int_t^{t'} \int_E k(s,e)\tilde{N}(\mathrm{d}e,\mathrm{d}s)\right|\right]^2 \leq \mathbb{E}\left[\int_t^{t'} \int_E |k(s,e)|^2 \lambda(\mathrm{d}e)\mathrm{d}s\right] \to 0 \text{ as } t' \to t.
$$

Then using the estimates in (i)-(iii), the Burkholder–Davis–Gundy inequality, and the Gronwall's inequality, (iv) follows. This completes the proof. ∎

Remark 6. Based on the estimates in Lemma 1, we can easily show that $\|z\|^2_{\mathcal{L}^2_{\mathbb{F}}} \leq C(1+|a|^2+|b|^2)$ and $\|k\|^2_{\mathcal{G}^2_{\mathbb{F}}} \leq C(1+|a|^2+|b|^2)$. Specifically, note that by (4),

$$
\int_t^T z(s)^\top \mathrm{d}B(s) + \int_t^T \int_E k(s,e)\tilde{N}(\mathrm{d}s,\mathrm{d}e) = Y^{u,z,k}_{t,b}(T) - b + \int_t^T g(s, X^{u,z,k}_{t,a}(s-), u(s), Y^{u,z,k}_{t,b}(s-), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))\mathrm{d}s.
$$

By Assumption 2 and the Burkholder–Davis–Gundy inequality, together with the Kunita's formula (theorem 4.4.23 of Reference 8) and the fact that B and $\tilde{N}$ are (mutually) independent stochastic processes, we can show that

$$
\|z\|^2_{\mathcal{L}^2_{\mathbb{F}}} + \|k\|^2_{\mathcal{G}^2_{\mathbb{F}}} \leq C\left(1 + |b|^2 + \mathbb{E}\left[\sup_{s\in[t,T]} |X^{u,z,k}_{t,a}(s)|^2\right] + \mathbb{E}\left[\sup_{s\in[t,T]} |Y^{u,z,k}_{t,b}(s)|^2\right]\right).
$$

Since Assumption 2 and Lemma 1 imply that $\mathbb{E}\left[\sup_{s\in[t,T]} |X^{u,z,k}_{t,a}(s)|^2\right] \leq C(1+|a|^2)$ and $\mathbb{E}\left[\sup_{s\in[t,T]} |Y^{u,z,k}_{t,b}(s)|^2\right] \leq C(1+|b|^2)$ (see corollary 6.2.4 of Reference 8) and section 3 of Reference 11), we have $\|z\|^2_{\mathcal{L}^2_{\mathbb{F}}} \leq C(1+|a|^2+|b|^2)$ and $\|k\|^2_{\mathcal{G}^2_{\mathbb{F}}} \leq C(1+|a|^2+|b|^2)$. This shows that the additional control variables can be restricted to be $\mathcal{L}^2_{\mathbb{F}} \times \mathcal{G}^2_{\mathbb{F}}$ bounded controls.

Under Assumptions 1 and 2, we define the value function of ($\mathbf{P}'$), where $V : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$,

$$
V(t,a,b) = \inf_{\substack{u\in U_{t,T} \\ (z,k)\in \mathcal{Z}^{(1)}_{t,T} \times \mathcal{K}^{(1)}_{t,T}}} \{J(t,a,b;u,z,k) \mid \rho^2(X^{u,z,k}_{t,a}(T), Y^{u,z,k}_{t,b}(T)) = 0,\ \mathbb{P}-\text{a.s.}\}. \tag{6}
$$

Remark 7. Under Assumption 1, according to Assumption 2(iii), we have $J \geq 0$, which implies $V \geq 0$. Due to the terminal state constraint, V might be discontinuous.

The main results of this paper can be summarized as follows:

a. The first main result (see Theorems 1 and 2) shows that by using the backward reachability approach, the value function of ($\mathbf{P}'$) in (6) can be expressed as the zero-level set of W, that is, $V(t,a,b) = \inf\{d \geq 0 \mid (a,b,d) \in \mathcal{R}(t)\} = \inf\{d \geq 0 \mid W(t,a,b,d) = 0,\ (a,b,d) \in \mathbb{R}^{n+m+1}\}$, where $\mathcal{R}$ is the backward reachable set and W is the value function of the auxiliary unconstrained (forward) control problem with jump diffusions (referred to as ($\mathbf{P}''$) in (11)). Although ($\mathbf{P}''$) is the unconstrained control problem, it includes additional martingale control variables, as a consequence of the martingale representation theorem. Hence, the standard optimal control theory cannot be applied to solve ($\mathbf{P}''$). We show that W is continuous (see Lemma 5), which implies that V, the (possibly discontinuous function) value function of ($\mathbf{P}'$), has the continuous representation in terms of W. Since ($\mathbf{P}''$) is the unconstrained forward control problem and its value function W is continuous, the characterization of W is more beneficial than the direct characterization of (possibly discontinuous function) V in (6).
b. The second main result provides the approach for the explicit characterization of W, the value function of ($\mathbf{P}''$) (see Theorem 3). Specifically, we prove that W is the unique viscosity solution to the associated (integro-type) Hamilton–Jacobi–Bellman (HJB) equation (see (12)). Unlike the viscosity solution analysis established in existing literatures, our paper requires a new approach, since the (integro-type) HJB equation in (12) has four different additional (possibly unbounded) Hamiltonian maximizing variables $(z,k,\alpha,\beta) \in \mathbb{R}^{p\times m} \times G^2(E,\mathcal{B}(E),\lambda;\mathbb{R}^m) \times \mathbb{R}^p \times G^2(E,\mathcal{B}(E),\lambda;\mathbb{R})$ due to the structure of ($\mathbf{P}''$) and the nonlocal integral operator in terms of the (singular) Lévy measure. We also provide the verification theorem for ($\mathbf{P}''$) (see Theorem 4), which constitutes the sufficient condition for optimality ($\mathbf{P}''$), provided that the corresponding HJB equation admits a unique solution. Hence, the verification theorem (Theorem 4)), together the viscosity solution property of W (Theorem 3), provides the explicit characterization of the value function W for ($\mathbf{P}''$).

Based on the main results given in the statements of (a) and (b), it is possible to solve the original (fully coupled) FBSDE control problem **(P)** (see Remark 15). Specifically, as stated in the statement of (b), we first solve ($\mathbf{P}''$) to obtain the value function W using the verification theorem (Theorem 4) and the solution to the (integro-type) HJB equation (Theorem 3). Then by Theorem 2, one can solve ($\mathbf{P}'$) by characterizing the zero-level set of W, the value function of ($\mathbf{P}''$) (see the statement of (a)). Finally, using the optimal solution of ($\mathbf{P}'$) and the conversion method discussed in Remark 3, the optimal solution of **(P)** can be constructed. We note that this approach is demonstrated through the linear-quadratic (LQ) problem for FBSDEs. In particular,

a. In Section 4, we apply our theoretical results (see the statements in (a) and (b)) to the LQ problem for FBSDEs. That is, we first convert the original LQ problem **(LQ-P)** into the equivalent forward problem ($\mathbf{LQ} - \mathbf{P}'$) as well as its equivalent unconstrained version ($\mathbf{LQ} - \mathbf{P}''$). Then we apply the main results of the paper (see the statements in (a) and (b)) as well as the conversion method (see the preceding discussion and Remark 15) to characterize the optimal solution of **(LQ-P)** (see Proposition 2).

1.3 | Literature review and organization of the paper

The stochastic differential equation (SDE) given in (2) is known as a class of forward-backward SDEs (with jump diffusions), since the forward process X with the initial condition is coupled with the backward SDE (BSDE) (Y,Z,K), and the terminal condition of the BSDE depends on the whole path of the forward process X. Since the pioneering works in the works,[12,13] various types of FBSDEs have been studied extensively in the literature, as they have strong connection with applications in engineering, mathematical finance, science and economics (see References 14-22 and the references therein). For example, FBSDEs can be used to model the coupled nature between the stock price (forward process) and the wealth process (backward process) in mathematical finance.[14,16,23] Moreover, FBSDEs can be used to describe interacting particle systems in mean-field type problems.[24-26] Various recent theoretical develops for FBSDEs can be found in the works[1-7,20,27-30] and references therein.

Concerning stochastic control problems for FBSDEs, the LQ problems were studied in various literature (see References 1,31-40 and the references therein). In most of LQ problems for FBSDEs, the monotonicity assumption of the coefficients for FBSDEs (e.g. References 1,39,40) is crucial to construct and verify the proposed optimal solution using the stochastic maximum principle. In fact, for LQ control of FBSDEs with the monotonicity assumption, the approach is similar to that for classical LQ problems of forward SDEs (e.g. Reference 18), since the monotonicity assumption simplifies the duality analysis between variational and adjoint processes in the stochastic maximum principle. Without the monotonicity assumption,[36-38] established the completion of squares method, which essentially requires the additional restriction on the corresponding objective functional to obtain the optimal solution. This restriction is quite similar to the convexity condition of the objective functional. As mentioned below (the statement in (c)), we note that the LQ result of this paper (see Section 4) needs neither the monotonicity assumption for the FBSDE nor the additional restriction on the objective functional.

We note that **(P)** is regarded as a general nonlinear stochastic control problem for FBSDEs. A major technical challenge of such problems arises from the dependence of the (forward) diffusion term on the BSDE and the presence of jump diffusions. Previously, for such problems, only stochastic maximum principles have been established, which in general constitute only the necessary conditions for optimality in terms of variational inequalities (see References 34,39-51 and the references therein). The notable result of the stochastic maximum principle for control of FBSDEs was obtained in Reference 41. Specifically, the work[41] converted the original problem into the forward problem as in ($\mathbf{P}'$) and then obtained the stochastic maximum principle of the converted (forward) problem using the Ekeland's variational principle. However, due to additional control variables in the forward problem (e.g. $(z,k) \in \mathcal{Z}_{t,T}^{(1)} \times \mathcal{K}_{t,T}^{(1)}$ in ($\mathbf{P}'$)), the stochastic maximum principle in Reference 41 requires identifying unknown parameters. Hence, it is practically not implementable. The results of the work[41] were extended to the case of jump-diffusion models,[47] the risk-sensitive framework,[34] the problem with state constraints,[44] and the mean-field type problems.[42,48] Moreover, when the FBSDE is not fully coupled, the stochastic maximum principle was also obtained in References 34,45. Recently, References 43,46 considered the different approach of the stochastic maximum principle for FBSDEs, where unlike the work,[41] the variational inequalities in Reference 43,46 do not contain unknown parameters. However, the problems in Reference 43,46 are simplified in that their BSDEs are defined as objective functionals, and their FBSDEs are restricted to one-dimensional processes. We also mention that the work[39] considered the stochastic control for fully coupled mean-field type FBSDEs with jump diffusions, and the work[50] studies the partially observed risk-sensitive control for FBSDEs. Moreover, the work[49] considered the stochastic control problem for mixed FBSDEs, where the "mixed" means that there are deterministic and random controllers. The approaches in References 39,49,50 are also closely related to the stochastic maximum principle framework. However, the control spaces in References 39,49,50 are restricted to be convex sets, and the monotonicity condition is imposed in Reference 39 for the corresponding FBSDE. In addition, the sufficient conditions of the stochastic maximum principle in Reference 39,49,50 require additional convexity assumptions for the coefficients of the corresponding FBSDEs. Note that the FBSDE in Reference 49,50 is not fully coupled in that the forward SDE does not depend on the backward SDE. Moreover, the convex control space assumption in References 39,49,50 simplifies the duality analysis of the optimality condition, with which only the first-order variational analysis needs to be considered. We also mention that Øksendal and Sulem in Reference 51 studied the sufficient condition of the stochastic maximum principle with partial information for optimal control problems of FBSDEs with jumps. However, the FBSDE in Reference 51 is not fully coupled in that the forward SDE does not depend on the backward SDE, and the additional convexity assumptions on the FBSDE and the control space (or the additional assumption on the set of admissible controls) are imposed. Moreover, under the similar convexity assumptions of the coefficients and the control space (as well as the monotonicity assumptions), Wang and Xiao in Reference 40 obtained the sufficient condition of the stochastic maximum principle with partial information for optimal control problems of fully coupled FBSDEs without jumps.

The main comparison of this paper with the existing literature is stated as follows:

(i) As mentioned above, the existing approaches mentioned above (see References 34,39-51) are closely related to the stochastic maximum principle framework, which in general guarantee only the necessary condition for optimality. In some situations, they are also sufficient. However, the sufficient condition for the stochastic maximum principle requires additional convexity assumptions as well as the additional conditions in the problem setting.

(ii) Different from existing literatures mentioned above, our paper provides an alternative approach based on the DPP and HJB theories, which constitutes the sufficient condition for optimality of **(P)** (see the statements of (a) and (b)). In fact, unlike the existing literature mentioned above, our approach requires neither the convexity assumption on the control space nor the convexity condition for the coefficients of the FBSDE in (2). Note that our approach allows to characterize of the optimal solution for control of FBSDEs in **(P)**, which are developed based on the backward reachability approach, the viscosity solution analysis of the associated HJB equation, and the verification theorem (see the statements in (a) and (b)). To the best of our knowledge, the sufficient condition for control of FBSDEs using the DPP and HJB theories has not been studied in existing literatures. We address this problem in the context of general jump-diffusion models given in **(P)**.

(iii) We also note that unlike the existing results on LQ control for FBSDEs mentioned above (see References 1,31-40), our LQ results (see the statement in (c)) need neither the monotonicity assumption for the FBSDE nor the restriction on the objective functional. In fact, our paper provides the sufficient condition to characterize the optimal solution to the LQ problem for FBSDEs without such additional assumptions.

Our paper is organized as follows. The characterization of V via the zero-level set of W is given in Section 2. In Section 3, we study the characterization of W by showing that W is the unique viscosity solution of the associated (integro-type) HJB equation. Section 4 studies the example of **(P)**, which is the LQ control problem for fully coupled FBSDEs with jump diffusions. The proofs of the results in Section 3 are provided in Section 5. We conclude this paper in Section 6. Appendices A–F are required for the proofs in Section 5. Appendix G is needed for Section 4.

2 | CHARACTERIZATION OF V

In this section, we characterize (6) as the zero-level set of the value function for the auxiliary unconstrained stochastic control problem with jump diffusions. The results of this section are summarized in the statement of (a) in Section 1.2.

2.1 | Characterization of V via backward reachable set

We introduce another (forward) SDE with jump diffusions

$$\begin{cases} \mathrm{d}\zeta_{t,a,b,d}^{u,z,k,\alpha,\beta}(s) = -l(s, X_{t,a}^{u,z,k}(s-), u(s), Y_{t,b}^{u,z,k}(s-), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))\mathrm{d}s + \alpha(s)^\top \mathrm{d}B(s) + \int_E \beta(s,e)\tilde{N}(\mathrm{d}s, \mathrm{d}e) \\ \zeta_{t,a,b,d}^{u,z,k,\alpha,\beta}(t) = d, \end{cases} \tag{7}$$

where $d \in \mathbb{R}$ is the initial condition and $(\alpha, \beta) \in \mathcal{Z}_{t,T}^{(2)} \times \mathcal{K}_{t,T}^{(2)}$ with $\mathcal{Z}_{t,T}^{(2)} := \mathcal{L}_{\mathbb{F}}^2(t, T; \mathbb{R}^p)$ and $\mathcal{K}_{t,T}^{(2)} := \mathcal{G}_{\mathbb{F}}^2(t, T, \lambda; \mathbb{R})$ can be viewed as additional control variables. Note that (7) captures the objective functional of (2). Let $\pi^{(1)} := (z, \alpha) \in \mathcal{Z}_{t,T}^{(1)} \times \mathcal{Z}_{t,T}^{(2)} =: \Pi_{t,T}^{(1)}$ and $\pi^{(2)} := (k, \beta) \in \mathcal{K}_{t,T}^{(1)} \times \mathcal{K}_{t,T}^{(2)} =: \Pi_{t,T}^{(2)}$. The following results hold, where the proof is similar to that of Lemma 1.

Lemma 2. *Suppose that Assumptions 1 and 2 hold. Then for any $(a, b, d) \in \mathbb{R}^{m+m+1}$ and $(u, \pi^{(1)}, \pi^{(2)}) \in \mathcal{U}_{t,T} \times \Pi_{t,T}^{(1)} \times \Pi_{t,T}^{(2)}$, there is a unique $\mathbb{F}$-adapted cádlág (strong) solution of (7). In addition, for any $(a, b, d), (a', b', d') \in \mathbb{R}^{m+m+1}$ and $(u, \pi^{(1)}, \pi^{(2)}) \in \mathcal{U}_{t,T} \times \Pi_{t,T}^{(1)} \times \Pi_{t,T}^{(2)}$, there exists a constant $C \geq 0$ such that $\mathbb{E}[\sup_{s\in[t,T]} |\zeta_{t,a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(s) - \zeta_{t,a',b',d'}^{u,u,\pi^{(1)},\pi^{(2)}}(s)|^2] \leq C(|a - a'|^2 + |b - b'|^2 + |d - d'|^2)$ and $\lim_{t'\to t} \mathbb{E}[|\zeta_{t,a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(T) - \zeta_{t',a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(T)|^2] = 0$ for $t' \in [0, T]$.*

Remark 8. We may impose bounds on additional control variables. Specifically, let us define $\tilde{J}(t, a, b; u, z, k) := \int_t^T l(s, X_{t,a}^{u,z,k}(s), u(s), Y_{t,b}^{u,z,k}(s), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))\mathrm{d}s + m(X_{t,a}^{u,z,k}(T))$. Note that we have $J(t, a, b; u, z, k) = \mathbb{E}[\tilde{J}(t, a, b; u, z, k)]$. Since $\tilde{J} \in L^2(\Omega, \mathcal{F}_T; \mathbb{R})$, it follows from the martingale representation theorem (see theorem 5.3.6 of Reference 8) that there are unique $(\alpha, \beta) \in \mathcal{Z}_{t,T}^{(2)} \times \mathcal{K}_{t,T}^{(2)}$ such that

$$\begin{aligned} \int_t^T \alpha(s)^\top \mathrm{d}B(s) + \int_t^T \int_E \beta(s,e)\tilde{N}(\mathrm{d}s, \mathrm{d}e) &= \tilde{J}(t, a, b; u, z, k) - J(t, a, b; u, z, k) \\ &= \int_t^T l(s, X_{t,a}^{u,z,k}(s), u(s), Y_{t,b}^{u,z,k}(s), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))\mathrm{d}s + m(X_{t,a}^{u,z,k}(T)) \\ &\quad - \mathbb{E}\left[\int_t^T l(s, X_{t,a}^{u,z,k}(s), u(s), Y_{t,b}^{u,z,k}(s), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))\mathrm{d}s + m(X_{t,a}^{u,z,k}(T))\right]. \end{aligned}$$

By Assumption 2 and the Burkholder–Davis–Gundy inequality, together with the Kunita's formula (theorem 4.4.23 of Reference 8) and the fact that B and $\tilde{N}$ are (mutually) independent stochastic processes (see Remark 6), we can show that

$$\|\alpha\|^2_{\mathcal{L}^2_{\mathbb{F}}} + \|\beta\|^2_{\mathcal{G}^2_{\mathbb{F}}} \leq C\left(1 + \mathbb{E}\left[\sup_{s\in[t,T]} |X^{u,z,k}_{t,a}(s)|^2\right] + \mathbb{E}\left[\sup_{s\in[t,T]} |Y^{u,z,k}_{t,b}(s)|^2\right]\right).$$

Then by Lemma 1 (see Remark 6), we can easily show that $\|\alpha\|^2_{\mathcal{L}^2_{\mathbb{F}}} \leq C(1+|a|^2+|b|^2)$ and $\|\beta\|^2_{\mathcal{G}^2_{\mathbb{F}}} \leq C(1+|a|^2+|b|^2)$. Analogous to Remark 6, the additional control variables are restricted to be $\mathcal{L}^2_{\mathbb{F}} \times \mathcal{G}^2_{\mathbb{F}}$ bounded controls.

We define the *epigraph* of m: $\mathtt{Epi}(m) := \{(a,d) \in \mathbb{R}^n \times \mathbb{R} \mid d \geq m(a)\}$. Then the following result holds:

Lemma 3. *Suppose that Assumptions 1 and 2 hold. Then V in (6) is equivalent to*

$$\begin{aligned} V(t,a,b) = \inf\{d \geq 0 \mid \exists (u,\pi^{(1)},\pi^{(2)}) \in \mathcal{U}_{t,T} \times \Pi^{(1)}_{t,T} \times \Pi^{(2)}_{t,T} \text{ such that} \\ (X^{u,z,k}_{t,a}(T), \zeta^{u,\pi^{(1)},\pi^{(2)}}_{t,a,b,d}(T)) \in \mathtt{Epi}(m) \ \&\ Y^{u,z,k}_{t,b}(T) = h(X^{u,z,k}_{t,a}(T)),\ \mathbb{P}-\text{a.s.}\}. \end{aligned} \tag{8}$$

Proof. By (6) and (7), it is straightforward to see that

$$\begin{aligned} V(t,a,b) = \inf\{d \geq 0 \mid \exists (u,\pi^{(1)},\pi^{(2)}) \in \mathcal{U}_{t,T} \times \Pi^{(1)}_{t,T} \times \Pi^{(2)}_{t,T} \text{ such that} \\ d \geq J(t,a,b;u,z,k) \ \&\ Y^{u,z,k}_{t,b}(T) = h(X^{u,z,k}_{t,a}(T)),\ \mathbb{P}-\text{a.s.}\}. \end{aligned} \tag{9}$$

Then it is necessary to prove the equivalence of the following two statements: (a) $\exists (u,z,k) \in \mathcal{U}_{t,T} \times \mathcal{Z}^{(1)}_{t,T} \times \mathcal{K}^{(1)}_{t,T}$ such that $d \geq J(t,a,b;u,z,k)$ and $Y^{u,z,k}_{t,b}(T) = h(X^{u,z,k}_{t,a}(T))$, $\mathbb{P}$-a.s. and (b) $\exists (u,\pi^{(1)},\pi^{(2)}) \in \mathcal{U}_{t,T} \times \Pi^{(1)}_{t,T} \times \Pi^{(2)}_{t,T}$ such that $\zeta^{u,z,k,\alpha,\beta}_{t,a,b,d}(T) \geq m(X^{u,z,k}_{t,a}(T))$ and $Y^{u,z,k}_{t,b}(T) = h(X^{u,z,k}_{t,a}(T))$, $\mathbb{P}$-a.s. We can easily observe that (b) corresponds to (8), while (a) is equivalent to (9).

By the martingale representation theorem in theorem 5.3.6 of Reference 8 and the definition of $\tilde{J}$ in Remark 8, there are unique $(\tilde{\alpha},\tilde{\beta}) \in \mathcal{Z}^{(2)}_{t,T} \times \mathcal{K}^{(2)}_{t,T}$ such that

$$\tilde{J}(t,a,b;u,z,k) = J(t,a,b;u,z,k) + \int_t^T \tilde{\alpha}(s)^\top \mathrm{d}B(s) + \int_t^T \int_E \tilde{\beta}(s,e)\tilde{N}(\mathrm{d}s,\mathrm{d}e).$$

Since $J \geq 0$, it follows that

$$d \geq \tilde{J}(t,a,b;u,z,k) - \int_t^T \tilde{\alpha}(s)^\top \mathrm{d}B(s) - \int_t^T \int_E \tilde{\beta}(s,e)\tilde{N}(\mathrm{d}s,\mathrm{d}e) \geq 0,\ \mathbb{P}-\text{a.s.}$$

This implies that $\zeta^{u,z,k,\tilde{\alpha},\tilde{\beta}}_{t,a,b,d}(T) \geq m(X^{u,z,k}_{t,a}(T))$. Moreover, from (a), we have $Y^{u,z,k}_{t,b}(T) = h(X^{u,z,k}_{t,a}(T))$, $\mathbb{P}$-a.s. Hence, (a) implies (b).

Note that (b) is equivalent to that there exist $(u,\pi^{(1)},\pi^{(2)}) \in \mathcal{U}_{t,T} \times \Pi^{(1)}_{t,T} \times \Pi^{(2)}_{t,T}$ such that $Y^{u,z,k}_{t,b}(T) = h(X^{u,z,k}_{t,a}(T))$, $\mathbb{P}$-a.s. and

$$\begin{aligned} d \geq & \int_t^T l(s, X^{u,z,k}_{t,a}(s), u(s), Y^{u,z,k}_{t,b}(s), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))\mathrm{d}s + m(X^{u,z,k}_{t,a}(T)) \\ & - \int_t^T \alpha(s)^\top \mathrm{d}B(s) - \int_t^T \int_E \beta(s,e)\tilde{N}(\mathrm{d}s,\mathrm{d}e),\ \mathbb{P}-\text{a.s.} \end{aligned}$$

As the stochastic integrals above are $\mathcal{F}_s$-martingales, by taking the expectation, (b) implies (a). We complete the proof. ∎

We define *the backward reachable set* associated with the epigraph and the (terminal) state constraint in (5):

$$\begin{aligned} \mathcal{R}(t) := \{(a,b,d) \in \mathbb{R}^{n+m+1} \mid \exists (u,\pi^{(1)},\pi^{(2)}) \in \mathcal{U}_{t,T} \times \Pi^{(1)}_{t,T} \times \Pi^{(2)}_{t,T} \text{ such that} \\ X^{u,z,k}_{t,a}(T), \zeta^{u,\pi^{(1)},\pi^{(2)}}_{t,a,b,d}(T)) \in \mathtt{Epi}(m) \ \&\ \rho^2(Y^{u,z,k}_{t,b}(T), X^{u,z,k}_{t,a}(T)) = 0,\ \mathbb{P}-\text{a.s.}\}. \end{aligned}$$

Then by Lemma 3, we state the following result:

Theorem 1. *Let Assumptions 1 and 2 hold. Then for $t \in [0,T]$ and $(a,b) \in \mathbb{R}^{n+m}$, $V(t,a,b) = \inf\{d \geq 0 \mid (a,b,d) \in \mathcal{R}(t)\}$.*

Remark 9. Theorem 1 shows that the original value function V can be expressed in terms of the backward reachable set. The next subsection focuses on the characterization of the backward reachable set via the auxiliary control problem penalized by the (terminal) state constraint.

2.2 | Characterization of backward reachable set via auxiliary control problem

We introduce the auxiliary objective functional

$$\hat{J}(t,a,b,d;u,\pi^{(1)},\pi^{(2)}) = \hat{J}(t,a,b,d;u,z,k,\alpha,\beta) = \mathbb{E}\left[\max\{m(X_{t,a}^{u,z,k}(T)) - \zeta_{t,a,b,d}^{u,z,k,\alpha,\beta}(T), 0\} + \rho^2(X_{t,a}^{u,z,k}(T), Y_{t,b}^{u,z,k}(T))\right], \quad (10)$$

where ρ is defined in (5). We may use the other equivalent terminal constraints given in Remark 4.

Remark 10. By definition, $\rho^2(X_{t,a}^{u,z,k}(T), Y_{t,b}^{u,z,k}(T)) = 0$ if and only if $Y_{t,b}^{u,z,k}(T) = h(X_{t,a}^{u,z,k}(T))$, $\mathbb{P}$-a.s., and $\rho^2(X_{t,a}^{u,z,k}(T), Y_{t,b}^{u,z,k}(T)) \geq 0$ otherwise. Moreover, it follows from Lemma 1 that $\mathbb{E}[|\rho^2(X_{t,a}^{u,z,k}(T), Y_{t,b}^{u,z,k}(T)) - \rho^2(X_{t,a'}^{u,z,k}(T), Y_{t,b'}^{u,z,k}(T))|]^{\frac{1}{2}} \leq C(|a-a'| + |b-b'|)$ for $(a,b),(a',b') \in \mathbb{R}^{n+m}$ and $\lim_{t' \to t} \mathbb{E}[|\rho^2(X_{t',a}^{u,z,k}(T), Y_{t',b}^{u,z,k}(T)) - \rho^2(X_{t,a}^{u,z,k}(T), Y_{t,b}^{u,z,k}(T))|]^{\frac{1}{2}} = 0$ for $t' \in [0,T]$.

We define the auxiliary value function for (10):

$$(\mathbf{P}'') \quad W(t,a,b,d) = \inf_{\substack{u \in \mathcal{U}_{t,T} \\ (\pi^{(1)},\pi^{(2)}) \in \Pi_{t,T}^{(1)} \times \Pi_{t,T}^{(2)}}} \hat{J}(t,a,b,d;u,\pi^{(1)},\pi^{(2)}), \quad (11)$$

where $W : [0,T] \times \mathbb{R}^{n+m+1} \to \mathbb{R}$. Note that (11) is the *unconstrained* (forward) stochastic control problem, which is penalized by the terminal constraint.

Theorem 2. *Let Assumptions 1 and 2 hold. Suppose that there exists an optimal control that attains the minimum of ($\mathbf{P}''$). Then (i) For $t \in [0,T]$, $\mathcal{R}(t) = \{(a,b,d) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mid W(t,a,b,d) = 0\}$ and (ii) for $t \in [0,T]$, $V(t,a,b) = \inf\{d \geq 0 \mid (a,b,d) \in \mathcal{R}(t)\} = \inf\{d \geq 0 \mid W(t,a,b,d) = 0,\ (a,b,d) \in \mathbb{R}^{n+m+1}\}$.*

Remark 11. In view of Theorems 1 and 2, V, the value function of ($\mathbf{P}'$), can be expressed as the zero-level set of the value function of ($\mathbf{P}''$). As shown in the next subsection, W is continuous. This, together with the discussion in Remark 3, means that instead of seeking for a possibly discontinuous function V, we can use W to solve ($\mathbf{P}'$) as well as **(P)**.

Proof of Theorem 2. Note first that (ii) follows from (i) and Theorem 1. Hence, we prove (i) by showing that $\mathcal{R}(t) \subseteq \hat{\mathcal{R}}(t)$ and $\mathcal{R}(t) \supseteq \hat{\mathcal{R}}(t)$, where $\hat{\mathcal{R}}(t) := \{(a,b,d) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mid W(t,a,b,d) = 0,\ (a,b,d) \in \mathbb{R}^{n+m+1}\}$.

We take $(\bar{a},\bar{b},\bar{d}) \in \mathcal{R}(t)$. Then by definition, there exist $(u,\pi^{(1)},\pi^{(2)}) \in \mathcal{U}_{t,T} \times \Pi_{t,T}^{(1)} \times \Pi_{t,T}^{(2)}$ such that

$$\max\{m(X_{t,\bar{a}}^{u,z,k}(T)) - \zeta_{t,\bar{a},\bar{b},\bar{d}}^{u,\pi^{(1)},\pi^{(2)}}(T), 0\} = 0,\ \rho^2(X_{t,\bar{a}}^{u,z,k}(T), Y_{t,\bar{b}}^{u,z,k}(T)) = 0,\ \mathbb{P}-\text{a.s.}$$

This shows that $W(t,\bar{a},\bar{b},\bar{d}) = 0$ for $t \in [0,T]$. Hence, $(\bar{a},\bar{b},\bar{d}) \in \hat{\mathcal{R}}(t)$, which implies $\mathcal{R}(t) \subseteq \hat{\mathcal{R}}(t)$.

Let $(\bar{a},\bar{b},\bar{d}) \in \hat{\mathcal{R}}(t)$ and $(\hat{u},\hat{z},\hat{k},\hat{\alpha},\hat{\beta}) \in \mathcal{U}_{t,T} \times \Pi_{t,T}^{(1)} \times \Pi_{t,T}^{(2)}$ be the optimal solution to $(\mathbf{P})''$, whose existence is assumed by the statement of the theorem. This leads to

$$W(t,\bar{a},\bar{b},\bar{d}) = \mathbb{E}\left[\max\{m(X_{t,\bar{a}}^{\hat{u},\hat{z},\hat{k}}(T)) - \zeta_{t,\bar{a},\bar{b},\bar{d}}^{\hat{u},\hat{z},\hat{k},\hat{\alpha},\hat{\beta}}(T), 0\} + \rho^2(X_{t,\bar{a}}^{\hat{u},\hat{z},\hat{k}}(T), Y_{t,\bar{b}}^{\hat{u},\hat{z},\hat{k}}(T))\right] = 0.$$

Then $\max\{X_{t,\bar{a}}^{\hat{u},\hat{z},\hat{k}}(T) - \zeta_{t,\bar{a},\bar{b},\bar{d}}^{\hat{u},\hat{z},\hat{k},\hat{\alpha},\hat{\beta}}(T), 0\} + \rho^2(X_{t,\bar{a}}^{\hat{u},\hat{z},\hat{k}}(T), Y_{t,\bar{b}}^{\hat{u},\hat{z},\hat{k}}(T)) < 0$ is not possible due to the definition of ρ, and $\max\{X_{t,\bar{a}}^{\hat{u},\hat{z},\hat{k}}(T) - \zeta_{t,\bar{a},\bar{b},\bar{d}}^{\hat{u},\hat{z},\hat{k},\hat{\alpha},\hat{\beta}}(T), 0\} + \rho^2(X_{t,\bar{a}}^{\hat{u},\hat{z},\hat{k}}(T), Y_{t,\bar{b}}^{\hat{u},\hat{z},\hat{k}}(T)) > 0$ is not possible, as it contradicts $W(t,\bar{a},\bar{b},\bar{d}) = 0$.

This implies that

$$\max\{m(X_{t,\bar{a}}^{\hat{u},\hat{z},\hat{k}}(T)) - \zeta_{t,\bar{a},\bar{b},\bar{d}}^{\hat{u},\hat{z},\hat{k},\hat{\alpha},\hat{\beta}}(T), 0\} + \rho^2(X_{t,\bar{a}}^{\hat{u},\hat{z},\hat{k}}(T), Y_{t,\bar{b}}^{\hat{u},\hat{z},\hat{k}}(T)) = 0,\ \mathbb{P}-\text{a.s.}$$

As ρ is nonnegative, by the definition of the epigraph,

$$(m(X_{t,\bar{a}}^{\hat{u},\hat{z},\hat{k}}(T)), \zeta_{t,\bar{a},\bar{b},\bar{d}}^{\hat{u},\hat{z},\hat{k},\hat{\alpha},\hat{\beta}}(T)) \in \mathtt{Epi}(m)\ \&\ Y_{t,\bar{b}}^{\hat{u},\hat{z},\hat{k}}(T) = h(X_{t,\bar{a}}^{\hat{u},\hat{z},\hat{k}}(T)),\ \mathbb{P}-\text{a.s.}$$

This proves that $(\bar{a}, \bar{b}, \bar{d}) \in \mathcal{R}(t)$, which leads to $\widehat{\mathcal{R}}(t) \subseteq \mathcal{R}(t)$. ∎

2.3 | Properties of W

We state the dynamic programming principle (DPP) of W. Note that since ($\mathbf{P}''$) is the Mayer stochastic optimal control problem, that is, ($\mathbf{P}''$) does not have the running cost, the DPP of ($\mathbf{P}''$) is simplified.[9,18,52]

Lemma 4. *Suppose that Assumptions 1 and 2 hold. Then for any $t \in [0, T]$ and $(a, b, d) \in \mathbb{R}^{n+m+1}$, it holds that* $W(t, a, b, d) = \mathbb{E}\left[W(t+\tau, X_{t,a}^{u,z,k}(t+\tau), Y_{t,b}^{u,z,k}(t+\tau), \zeta_{t,a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(t+\tau))\right]$.

The next result shows the continuity of W.

Lemma 5. *Assume that Assumptions 1 and 2 hold. Then for any $t \in [0, T]$, there is a constant C such that (i) for $t \in [0, T]$ and $(a, b, d), (a', b', d') \in \mathbb{R}^{n+m+1}$, $|W(t, a, b, d) - W(t, a', b', d')| \le C(|a - a'| + |b - b'| + |d - d'|)$ and (ii) W is continuous in $t \in [0, T]$.*

Proof. We prove (i). Note that $|\inf f(x) - \inf g(x)| \le \sup |f(x) - g(x)|$. From Assumption 2, Lemmas 1 and 2 and Hölder inequality,

$$\begin{aligned}
&|W(t, a, b, d) - W(t, a', b', d')| \\
&\le C \sup_{\substack{u \in U_{t,T} \\ (\pi^{(1)},\pi^{(2)}) \in \Pi_{t,T}^{(1)} \times \Pi_{t,T}^{(2)}}} \{\mathbb{E}[|m(X_{t,a}^{u,z,k}(T)) - m(X_{t,a}^{u,z,k}(T))|^2]^{\frac{1}{2}} + \mathbb{E}[|X_{t,a}^{u,z,k}(T) - X_{t,a}^{u,z,k}(T)|^2]^{\frac{1}{2}} \\
&\quad + \mathbb{E}[|Y_{t,b}^{u,z,k}(T) - Y_{t,b}^{u,z,k}(T)|^2]^{\frac{1}{2}} + \mathbb{E}[|\zeta_{t,a,b,d}^{u,z,k,\alpha,\beta}(T) - \zeta_{t,a',b',d'}^{u,z,k,\alpha,\beta}(T)|^2]^{\frac{1}{2}}\} \le C(|a - a'| + |b - b'| + |d - d'|).
\end{aligned}$$

Note that the last inequality is due to Remarks 6 and 8. Hence, (ii) follows.

To prove (ii), using the similar technique as in (i), together with Lemmas 1 and 2, we have

$$\begin{aligned}
&|W(t+\tau, a, b, d) - W(t, a, b, d)| \\
&\le C\tau^{\frac{1}{2}} + C \sup_{\substack{u \in U_{t,T} \\ (\pi^{(1)},\pi^{(2)}) \in \Pi_{t,T}^{(1)} \times \Pi_{t,T}^{(2)}}} \{\mathbb{E}[|Y_{t+\tau,b}^{u,z,k}(T) - Y_{t,b}^{u,z,k}(T)|^2]^{\frac{1}{2}} + \mathbb{E}[|\zeta_{t+\tau,a,b,d}^{u,z,k,\alpha,\beta}(T) - \zeta_{t,a',b',d'}^{u,z,k,\alpha,\beta}(T)|^2]^{\frac{1}{2}}\}.
\end{aligned}$$

Recall from Lemmas 1 and 2 that

$$\lim_{\tau \downarrow 0} \mathbb{E}[|Y_{t+\tau,b}^{u,z,k}(T) - Y_{t,b}^{u,z,k}(T)|^2]^{\frac{1}{2}} = 0,\ \lim_{\tau \downarrow 0} \mathbb{E}[|\zeta_{t+\tau,a,b,d}^{u,z,k,\alpha,\beta}(T) - \zeta_{t,a',b',d'}^{u,z,k,\alpha,\beta}(T)|^2]^{\frac{1}{2}} = 0.$$

Therefore, (ii) holds. This completes the proof. ∎

The boundary conditions of W can be stated as follows:

Lemma 6. *Suppose that Assumptions 1 and 2 hold. Then (i) for $(a, b, d) \in \mathbb{R}^{n+m+1}$, $W(T, a, b, d) = \max\{m(a) - d, 0\} + \rho^2(a, b)$ and (ii) for $d \le 0$, $W(t, a, b, d) = W_0(t, a, b) - d$, where W_0 is the value function of the unconstrained stochastic control problem*:

$$W_0(t,a,b) := \inf_{\substack{u\in U_{t,T} \\ (z,k)\in \mathcal{Z}^{(1)}_{t,T}\times \mathcal{K}^{(1)}_{t,T}}} \left\{ J(t,a,b;u,z,k) + \mathbb{E}\left[\rho^2(X^{u,z,k}_{t,a}(T), Y^{u,z,k}_{t,b}(T))\right]\right\}.$$

Proof. Note that (i) follows from the definition of W in (11). To prove (ii), using the definition of W,

$$\begin{aligned}
W(t,a,b,d) &\geq \inf_{\substack{u\in U_{t,T} \\ (\pi^{(1)},\pi^{(2)})\in \Pi^{(1)}_{t,T}\times \Pi^{(2)}_{t,T}}} \mathbb{E}[m(X^{u,z,k}_{t,a}(T)) - \zeta^{u,z,k,\alpha,\beta}_{t,a,b,d}(T) + \rho^2(X^{u,z,k}_{t,a}(T), Y^{u,z,k}_{t,b}(T))] \\
&\geq \inf_{\substack{u\in U_{t,T} \\ (z,k)\in \mathcal{Z}^{(1)}_{t,T}\times \mathcal{K}^{(1)}_{t,T}}} \mathbb{E}[m(X^{u,z,k}_{t,a}(T)) - d + \rho^2(X^{u,z,k}_{t,a}(T), Y^{u,z,k}_{t,b}(T)) \\
&\quad + \int_t^T l(s, X^{u,z,k}_{t,a}(s-), u(s), Y^{u,z,k}_{t,b}(s-), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))\mathrm{d}s] = W_0(t,a,b) - d,
\end{aligned}$$

where we have used the fact that the stochastic integrals with respect to B and $\tilde{N}$ are $\mathcal{F}_s$-martingales.

On the other hand, note that since $d \leq 0$, and l and m are nonnegative, it follows that

$$\begin{aligned}
&\max\{m(X^{u,z,k}_{t,a}(T)) - \zeta^{u,z,k,0,0}_{t,a,b,d}(T), 0\} \\
&= m(X^{u,z,k}_{t,a}(T)) - d + \int_t^T l(s, X^{u,z,k}_{t,a}(s-), u(s), Y^{u,z,k}_{t,b}(s-), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))\mathrm{d}s \geq 0, \ \mathbb{P}-\text{a.s.}
\end{aligned}$$

Then we have

$$\begin{aligned}
W(t,a,b,d) &\leq \inf_{\substack{u\in U_{t,T} \\ (z,k)\in \mathcal{Z}^{(1)}_{t,T}\times \mathcal{K}^{(1)}_{t,T}}} \mathbb{E}[\max\{m(X^{u,z,k}_{t,a}(T)) - \zeta^{u,z,k,0,0}_{t,a,b,d}(T), 0\} + \rho^2(X^{u,z,k}_{t,a}(T), Y^{u,z,k}_{t,b}(T))] \\
&= \inf_{\substack{u\in U_{t,T} \\ (z,k)\in \mathcal{Z}^{(1)}_{t,T}\times \mathcal{K}^{(1)}_{t,T}}} \mathbb{E}[\max\{m(X^{u,z,k}_{t,a}(T)) - d + \rho^2(X^{u,z,k}_{t,a}(T), Y^{u,z,k}_{t,b}(T)) \\
&\quad + \int_t^T l(s, X^{u,z,k}_{t,a}(s-), u(s), Y^{u,z,k}_{t,b}(s-), z(s), \int_E k(s,e)\lambda(\mathrm{d}e))\mathrm{d}s] = W_0(t,a,b) - d.
\end{aligned}$$

We complete the proof. ■

3 | CHARACTERIZATION OF W VIA VISCOSITY SOLUTION OF (INTEGRO-TYPE) HJB EQUATION

As stated in Remarks 3 and 11, the original (fully coupled) FBSDE control problem **(P)** can be solved via the unconstrained forward optimization problem (**P**″). This section focuses on solving (**P**″) by showing that W, the value function of (**P**″), can be characterized by the viscosity solution of the associated (integro-type) HJB equation. The results of this section are summarized in the statement of (b) in Section 1.2.

Let $f(u,z,k) := f(t,a,u,b,z,\int_E k(e)\lambda(\mathrm{d}e))$ and the similar notation applies to g, l, σ, and χ. Define

$$\mathbf{f}(u,z,k) := \mathbf{f}(t,a,u,b,z,k) := \begin{bmatrix} f(u,z,\int_E k(e)\lambda(\mathrm{d}e)) \\ -g(u,z,\int_E k(e)\lambda(\mathrm{d}e)) \\ -l(u,z,\int_E k(e)\lambda(\mathrm{d}e)) \end{bmatrix}$$

$$\boldsymbol{\sigma}(u,z,k,\alpha) := \boldsymbol{\sigma}(t,a,u,b,z,k,\alpha) := \begin{bmatrix} \sigma(u,z,\int_E k(e)\lambda(\mathrm{d}e)) \\ z^\top \\ \alpha^\top \end{bmatrix}$$

$$\boldsymbol{\chi}(e,u,z,k(e),\beta(e)) := \boldsymbol{\chi}(t,e,a,u,b,z,k(e),\beta(e)) := \begin{bmatrix} \chi(e,u,z,k(e)) \\ k(e) \\ \beta(e) \end{bmatrix}.$$

We define $\mathbf{a} := \begin{bmatrix} a^\top & b^\top & d \end{bmatrix}^\top$, $\mathcal{O} = [0,T) \times \mathbb{R}^{n+m} \times (0,\infty)$, $\overline{\mathcal{O}} = [0,T] \times \mathbb{R}^{n+m} \times [0,\infty)$, $G^{(1)} := G^2(E, \mathcal{B}(E), \lambda; \mathbb{R}^m)$, and $G^{(2)} := G^2(E, \mathcal{B}(E), \lambda; \mathbb{R})$. Moreover, by a slight abuse of notation, $\pi^{(1)} := (z, \alpha) \in \mathbb{R}^{p\times m} \times \mathbb{R}^p =: \Pi^{(1)}$ and $\pi^{(2)} := (k, \beta) \in G^{(1)} \times G^{(2)} =: \Pi^{(2)}$.

We introduce the integro-type HJB equation

$$\begin{cases} -\partial_t W(t,\mathbf{a}) + \mathbb{H}(t,\mathbf{a},(W,DW,D^2W)(t,\mathbf{a})) = 0, \ (t,\mathbf{a}) \in \mathcal{O} \\ W(T,\mathbf{a}) = \max\{m(a)-d, 0\} + \rho^2(a,b), \ (a,b,d) \in \mathbb{R}^{n+m} \times (0,\infty) \\ W(t,a,b,0) = W_0(t,a,b), \ (t,a,b) \in [0,T) \times \mathbb{R}^{n+m}, \end{cases} \tag{12}$$

where $\mathbb{H} : [0,T] \times \mathbb{R}^{n+m+1} \times \mathbb{R} \times \mathbb{R}^{n+m+1} \times \mathbb{S}^{n+m+1} \to \mathbb{R}$ is the Hamiltonian defined by

$$\mathbb{H}(t,\mathbf{a},(W,DW,DW^2)(t,\mathbf{a})) := \sup_{u\in U_{(\pi^{(1)},\pi^{(2)})\in\Pi^{(1)}\times\Pi^{(2)}}} \widehat{\mathbb{H}}(t,\mathbf{a},(W,DW,DW^2)(t,\mathbf{a})), \tag{13}$$

with $\widehat{\mathbb{H}}$ given by

$$\begin{aligned} &\widehat{\mathbb{H}}(t,\mathbf{a},(W,DW,DW^2)(t,\mathbf{a}); u, \pi^{(1)}, \pi^{(2)}) \\ &:= -\langle DW(t,\mathbf{a}), \mathbf{f}(u,z,k)\rangle - \frac{1}{2}\mathrm{Tr}(DW^2(t,\mathbf{a})\boldsymbol{\sigma}\boldsymbol{\sigma}^\top(u,z,k,\alpha)) \\ &\quad - \int_E \left[W(t,\mathbf{a}+\chi(e,u,z,k(e),\beta(e))) - W(t,\mathbf{a}) - \langle DW(t,\mathbf{a}), \chi(e,u,z,k(e),\beta(e))\rangle \right] \lambda(\mathrm{d}e). \end{aligned} \tag{14}$$

Note that the boundary conditions in (12) are due to Lemma 6.

Remark 12. The (integro-type) HJB equation in (12) is a class of nonlinear second-order partial differential equations (PDEs). Note that (12) also includes the nonlocal integral term in terms of the Lévy measure due to the jump-diffusion process. When there are no jumps (i.e., $\chi \equiv 0$ in (4)), the HJB equation in (12) becomes the local nonlinear second-order PDE.

The notion of viscosity solutions for (12) is given as follows:[11,53,54]

Definition 1. A real-valued function $W \in C(\overline{\mathcal{O}})$ is said to be a viscosity subsolution (resp. supersolution) of (12) if

- (i) $W(T,\mathbf{a}) \leq \max\{m(a)-d,0\} + \rho^2(a,b)$ (resp. $W(T,\mathbf{a}) \geq \max\{m(a)-d,0\} + \rho^2(a,b)$) for $(a,b,d) \in \mathbb{R}^{n+m} \times (0,\infty)$ and $W(t,a,b,0) \leq W_0(t,a,b)$ (resp. $W(t,a,b,0) \geq W_0(t,a,b)$) for $(t,a,b) \in [0,T) \times \mathbb{R}^{n+m}$;
- (ii) For all test functions $\phi \in C_b^{1,3}(\overline{\mathcal{O}}) \cap C_2(\overline{\mathcal{O}})$, the following inequality holds at the global maximum (resp. minimum) point $(t,\mathbf{a}) \in \mathcal{O}$ of $W - \phi$:

$$-\partial_t\phi(t,\mathbf{a}) + H(t,\mathbf{a},(\phi,D\phi,D^2\phi)(t,\mathbf{a})) \leq 0 \quad (\text{resp. } -\partial_t\phi(t,\mathbf{a}) + H(t,\mathbf{a},(\phi,D\phi,D^2\phi)(t,\mathbf{a})) \geq 0).$$

A real-valued function $W \in C(\overline{\mathcal{O}})$ is said to be a viscosity solution of (12) if it is both a viscosity subsolution and a viscosity supersolution of (12).

We state the main result of this section, whose proof is given in Section 5.

Theorem 3 (Existence and uniqueness of viscosity solution of (12)). *Suppose that Assumptions 1 and 2 hold. Then W defined in (10), the value function of ($\mathbf{P}''$), is a unique viscosity solution of the integro-type HJB equation in (12).*

Remark 13. Theorem 3, together with Theorem 2, implies that the solution of the HJB equation in (12) and its zero-level set characterize V, that is, the value function of ($\mathbf{P}'$).

Another main result is the verification theorem, which constitutes the sufficient condition of ($\mathbf{P}''$). Its concise proof is given in Section 5.

Theorem 4 (Verification theorem). *Let Assumptions 1 and 2 hold. If $W \in C_b^{1,3}(\overline{\mathcal{O}}) \cap C_2(\overline{\mathcal{O}})$ solves the HJB equation in (12), then for any $(t,\mathbf{a}) \in \overline{\mathcal{O}}$,*

$$W(t,\mathbf{a}) \leq \hat{J}(t,\mathbf{a};u,\pi^{(1)},\pi^{(2)}),\ \forall(u,\pi^{(1)},\pi^{(2)}) \in \mathcal{U}_{t,T} \times \Pi_{t,T}^{(1)} \times \Pi_{t,T}^{(2)}.$$

In addition, assume that for $t \in [0,T]$ almost everywhere, the following holds for $(\hat{u},\hat{\pi}^{(1)},\hat{\pi}^{(2)}) = (\hat{u},\hat{z},\hat{\alpha},\hat{k},\hat{\beta}) \in \mathcal{U}_{t,T} \times \Pi_{t,T}^{(1)} \times \Pi_{t,T}^{(2)}$:

$$\hat{\mathbb{H}}(t,\mathbf{a},(W,DW,DW^2)(t,\mathbf{a});\hat{u},\hat{\pi}^{(1)},\hat{\pi}^{(2)}) = \mathbb{H}(t,\mathbf{a},(W,DW,DW^2)(t,\mathbf{a})). \tag{15}$$

Then $(\hat{u},\hat{\pi}^{(1)},\hat{\pi}^{(2)}) = (\hat{u},\hat{z},\hat{\alpha},\hat{k},\hat{\beta}) \in \mathcal{U}_{t,T} \times \Pi_{t,T}^{(1)} \times \Pi_{t,T}^{(2)}$ is the optimal solution of $(\mathbf{P}'')$ and $W(t,\mathbf{a}) = \hat{J}(t,\mathbf{a};\hat{u},\hat{\pi}^{(1)},\hat{\pi}^{(2)})$ is the corresponding value function.

Remark 14. Under the same assumption as in Theorem 4, the verification theorem can be extended to the nonsmooth case when the HJB equation in (12) admits only the viscosity solution in $C(\overline{\mathcal{O}})$; see for a related discussion in theorem 4.1 of Reference 52. Note that by Theorem 3, the (integro-type) HJB equation in (12) admits the unique viscosity solution. Hence, we can still apply the nonsmooth verification theorem to obtain the optimal solution of $(\mathbf{P}'')$ as in Theorem 4.

Remark 15. In this remark, we discuss the construction of the optimal solution of **(P)** through $(\mathbf{P}'')$ and $(\mathbf{P}')$:

(i) Based on Theorems 3 and 4 (see also Remark 14), we solve $(\mathbf{P}'')$, where W, the (smooth or viscosity) solution to the integro-type HJB equation in (12), is the value function of $(\mathbf{P}'')$;
(ii) By Theorem 2, one can solve $(\mathbf{P}')$ by characterizing the zero-level set of W obtained from (i), where we have to find $(\overline{a},\overline{b},\overline{d}) \in \mathbb{R}^{n+m+1}$ such that $V(t,\overline{a},\overline{b}) = \overline{d} = \inf\{d \geq 0 \mid W(t,\overline{a},\overline{b},d) = 0\}$ (see Remark 16 for the approximated construction of V as well as the triple $(\overline{a},\overline{b},\overline{d})$);
(iii) Finally, using the solution of $(\mathbf{P}')$ in (ii) with $(\overline{a},\overline{b},\overline{d}) \in \mathbb{R}^{n+m+1}$, and the conversion method discussed in Remark 3, the optimal solution of **(P)** can be constructed, where $(\overline{a},\overline{b}) \in \mathbb{R}^{n+m}$ is the given initial condition for the (forward) SDE in (4) controlled by the optimal solution of $(\mathbf{P}')$ (see Remark 3).

In summary, we can easily see that through steps (i)–(iii) and using the conversion method discussed in Remark 3, the optimal solution of **(P)** can be constructed.

The following remark states the construction of the approximated solutions in Remark 15(ii).

Remark 16. Under the conditions of Theorem 2, the approximation of V and the triple $(\overline{a},\overline{b},\overline{d}) \in \mathbb{R}^{n+m+1}$ can be obtained by the following construction:

$$V(t,\overline{a},\overline{b}) = \overline{d} := \min\{\overline{d}' \in [0,\grave{d}] \mid W(t,\overline{a},\overline{b},\overline{d}') = \min_{(a,b,d)\in D\times[0,\grave{d}]} W(t,a,b,d)\}, \tag{16}$$

where $D \subset \mathbb{R}^{n+m}$ is a sufficiently large compact subset, and $\grave{d} \geq 0$ is also a sufficiently large constant. To see this, by definition of $(\mathbf{P}'')$ and Assumption 2, for any $t \in [0,T]$, $(a,b) \in \mathbb{R}^{n+m}$ and $d \geq 0$, it follows that

$$W(t,a,b,d) = \inf_{\substack{u\in U_{t,T} \\ (\pi^{(1)},\pi^{(2)})\in\Pi_{t,T}^{(1)}\times\Pi_{t,T}^{(2)}}} \hat{J}(t,a,b,d;u,\pi^{(1)},\pi^{(2)}) \geq 0.$$

Note that since W is continuous on $D \times [0,\grave{d}]$ by Lemma 5 and $D \times [0,\grave{d}]$ is compact, by the Weierstrass extreme value theorem, the triple $(\overline{a},\overline{b},\overline{d}') \in D \times [0,\grave{d}]$ exists such that $W(t,\overline{a},\overline{b},\overline{d}') = \min_{(a,b,d)\in D\times[0,\grave{d}]} W(t,a,b,d) \geq 0$. Hence, the triple $(\overline{a},\overline{b},\overline{d}) \in \mathbb{R}^{n+m+1}$ in (16) always exists, which implies that (16) is the valid approximation scheme for Remark 15(ii). Note that if $W(t,\overline{a},\overline{b},\overline{d}) = 0$, then V and the triple $(\overline{a},\overline{b},\overline{d}) \in \mathbb{R}^{n+m+1}$ obtained from (16) are the (exact) solutions of Remark 15(ii).

4 | AN EXAMPLE: LQ CONTROL FOR FULLY COUPLED LINEAR FBSDES

We study an example of **(P)**, the LQ problem of **(P)**, to demonstrate steps (i)–(iii) in Remark 15. Specifically, we consider the LQ control problem of **(P)**, which is referred to as **(LQ-P)**. As noted, **(LQ-P)** is embedded in $(\mathbf{LQ}-\mathbf{P}')$

(that will be defined below). Then by Theorems 1–4, $(\mathbf{LQ}-\mathbf{P}')$ can be analyzed via $(\mathbf{LQ}-\mathbf{P}'')$ using the backward reachability analysis and the associated HJB equation. Finally, **(LQ-P)** can be solved using the solutions of $(\mathbf{LQ}-\mathbf{P}')$ and $(\mathbf{LQ}-\mathbf{P}'')$ as stated in Remark 15. The results of this section are summarized in the statement of (c) in Section 1.2.

The linear fully coupled FBSDE with jump diffusions, $(X,Y,Z,K)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{p\times m}\times\mathbb{R}^m$, is given by

$$\begin{cases} \mathrm{d}X(s) = \left[A_1X(s-)+B_1u(s)+A_2Y(s-)+B_2Z(s)^\top+\lambda B_3K(s)\right]\mathrm{d}s \\ \quad +\left[C_1X(s-)+D_1u(s)+C_2Y(s-)+D_2Z(s)^\top+\lambda D_3K(s)\right]\mathrm{d}B(s) \\ \quad +\left[E_1X(s-)+F_1u(s)+E_2Y(s-)+F_2Z(s)^\top+F_3K(s)\right]\mathrm{d}\tilde{N}(s) \\ \mathrm{d}Y(s) = -\left[G_1X(s-)+H_1u(s)+G_2Y(s-)+H_2Z(s)^\top+\lambda H_3K(s)\right]\mathrm{d}s+Z(s)^\top\mathrm{d}B(s)+K(s)\mathrm{d}\tilde{N}(s) \\ X(t)=X,\ \ Y(T)=M_2X(T), \end{cases} \tag{17}$$

and the objective functional of **(LQ-P)** is as follows

$$J'(t,X;u) = \frac{1}{2}\mathbb{E}\Bigg[\int_t^T [\langle X(s),Q_1X(s)\rangle+\langle Y(s),Q_2Y(s)\rangle+\langle u(s),R_1u(s)\rangle \\ +\langle Z(s)^\top,R_2Z(s)^\top\rangle+\lambda\langle K(s),R_3K(s)\rangle]\mathrm{d}s+\langle X(T),M_1X(T)\rangle\Bigg].$$

In (17), we assume $p=1$, that is, B is the one-dimensional Brownian motion, and $E=\{1\}$, that is, the Poisson process N has jumps of unit size. It should be noted that this restriction is only for notational convenience, and we can easily extend the result of this section to the general situation.*

Remark 17. As in Assumption 1, we assume that there is a unique solution of (17), which requires the additional monotonicity assumptions for the coefficients of (17) in addition to Assumption 2[1,31-33,35] (see Remarks 19 and 21). We should mention that $(\mathbf{LQ}-\mathbf{P}')$, which is the equivalent forward problem of **(LQ-P)** defined below, does not need such additional monotonicity assumptions. This remark is closely related to Remarks 1 and 5.

Based on (17) and **(P)**, we consider $(\mathbf{LQ}-\mathbf{P}')$, an LQ version of $(\mathbf{P}')$, that is, the minimization of

$$J'(t,X,Y;u,z,k) = \frac{1}{2}\mathbb{E}\Bigg[\int_t^T [\langle X(s),Q_1X(s)\rangle+\langle Y(s),Q_2Y(s)\rangle \\ +\langle u(s),R_1u(s)\rangle+\langle z(s)^\top,R_2z(s)^\top\rangle+\lambda\langle k(s),R_3k(s)\rangle]\mathrm{d}s+\langle X(T),M_1X(T)\rangle\Bigg], \tag{18}$$

subject to the forward controlled state process $(X,Y)\in\mathbb{R}^n\times\mathbb{R}^m$:

$$\begin{cases} \mathrm{d}X(s) = [A_1X(s-)+B_1u(s)+A_2Y(s-)+B_2z(s)^\top+\lambda B_3k(s)]\mathrm{d}s \\ \quad +[C_1X(s-)+D_1u(s)+C_2Y(s-)+D_2z(s)^\top+\lambda D_3k(s)]\mathrm{d}B(s) \\ \quad +[E_1X(s-)+F_1u(s)+E_2Y(s-)+F_2z(s)^\top+F_3k(s)]\mathrm{d}\tilde{N}(s) \\ \mathrm{d}Y(s) = -[G_1X(s-)+H_1u(s)+G_2Y(s-)+H_2z(s)^\top+\lambda H_3k(s)]\mathrm{d}s+z(s)^\top\mathrm{d}B(s)+k(s)\mathrm{d}\tilde{N}(s) \\ X(t)=X,\ \ Y(t)=Y, \end{cases} \tag{19}$$

and the terminal state constraint†

$$\left|Y(T)-M_2X(T)\right|^2 = 0,\ \mathbb{P}-\text{a.s.} \tag{20}$$

The value function of $(\mathbf{LQ}-\mathbf{P}')$ is defined by

$$(\mathbf{LQ\text{-}P}')\quad V(t,X,Y) = \inf_{\substack{u\in\mathcal{U}_{t,T} \\ (z,k)\in\mathcal{Z}^{(1)}_{t,T}\times\mathcal{K}^{(1)}_{t,T}}} \{J'(t,X,Y;u,z,k)\mid (20)\text{ holds }\mathbb{P}-\text{a.s.}\}. \tag{21}$$

Remark 18.

(i) It is easy to obtain $(\mathbf{LQ-P''})$ from $(\mathbf{LQ-P'})$, where $(\mathbf{LQ-P''})$ can be regarded as the LQ version of $(\mathbf{P''})$ in (11), that is, the unconstrained LQ stochastic control problem with additional control variables.
(ii) It should be noted that in (**LQ-P**$''$), Theorems 1–4 can be applied to solve (**LQ-P**$'$). Then by Remark 15, the results of $(\mathbf{LQ-P''})$ and $(\mathbf{LQ-P'})$ can be used to construct the optimal solution of **(LQ-P)**.

Assumption 3. In (18) and (19), A_i, C_i, E_i, G_i, Q_i, M_i, $i=1,2$, and B_j, D_j, F_j, H_j, R_j, $j=1,2,3$, are t-dependent (time-dependent) deterministic matrices with appropriate dimensions, which are continuous and bounded. We assume that M_1, Q_i, $i=1,2$, and $R_j, j=1,2,3$, are positive definite for all $t\in[0,T]$.

Remark 19. Note that under Assumption 3, (18) and (19) hold Assumption 2.‡

By using the associated HJB equation for $(\mathbf{LQ-P''})$ and applying Theorem 4, we can state the following result:

Proposition 1. *Suppose that Assumption 3 holds and that $\frac{1}{2}\langle X, M_1X\rangle \geq d$. Then the value function of $(\mathbf{LQ-P''})$ is given by*

$$\begin{cases} W(t,X,Y,d)=\frac{1}{2}\begin{bmatrix}X\\Y\end{bmatrix}^\top \mathcal{P}(t)\begin{bmatrix}X\\Y\end{bmatrix}+C(t)d \\ W(T,X,Y,d)=\frac{1}{2}\langle X,M_1X\rangle - d + |M_2X-Y|^2,\ (X,Y,d)\in\mathbb{R}^{n+m}\times(0,\infty) \\ W(t,X,Y,0)=W_0(t,X,Y),\ (t,X,Y)\in[0,T)\times\mathbb{R}^{n+m}, \end{cases} \tag{22}$$

where $\mathcal{P}(\cdot)=\begin{bmatrix}\mathcal{P}_{11}(\cdot) & \mathcal{P}_{12}(\cdot)\\ \mathcal{P}_{12}(\cdot)^\top & \mathcal{P}_{22}(\cdot)\end{bmatrix}$ with $\mathcal{P}_{11}\in\mathbb{S}^n$ and $\mathcal{P}_{22}\in\mathbb{S}^m$ is the solution to the $\mathbb{S}^{n+m}$-valued matrix differential equation given in (G1) of Appendix G, and $C\in\mathbb{R}$ is a continuous and bounded process on $[0,T]$ with $C(T)=-1$. Moreover, the optimal solution of $(\mathbf{LQ-P''})$, $(\hat{u},\hat{z},\hat{k})\in\mathcal{U}_{t,T}\times\mathcal{Z}^{(1)}_{t,T}\times\mathcal{K}^{(1)}_{t,T}$ with arbitrary $(\alpha,\beta)\in\mathcal{Z}^{(2)}_{t,T}\times\mathcal{K}^{(2)}_{t,T}$, can be written as

$$\hat{u}(t)=\mathscr{A}_1[\mathcal{P}(t)]\begin{bmatrix}X(t)\\Y(t)\end{bmatrix},\ \hat{z}(t)^\top=\mathscr{A}_2[\mathcal{P}(t)]\begin{bmatrix}X(t)\\Y(t)\end{bmatrix},\ \hat{k}(t)=\mathscr{A}_3[\mathcal{P}(t)]\begin{bmatrix}X(t)\\Y(t)\end{bmatrix}, \tag{23}$$

provided that

$$-C(t)\hat{\mathcal{R}}+\hat{\mathcal{D}}^\top\mathcal{P}_{11}(t)\hat{\mathcal{D}}+\hat{\mathcal{D}}^\top\widetilde{\mathcal{P}}_{12}(t)+\hat{\mathcal{P}}_{22}(t)+\lambda\hat{\mathcal{F}}^\top\mathcal{P}_{11}(t)\hat{\mathcal{F}}+\lambda\mathcal{P}'_{22}(t)>0, \tag{24}$$

where the coefficients in (24) and $\mathscr{A}_i$, $i=1,2,3$, are defined in Appendix G.

Proof. The HJB equation of $(\mathbf{LQ-P''})$ with the corresponding Hamiltonian can be obtained from (12) and using the coefficients in (18) and (19). Consider the quadratic function W in (22). Note that $\frac{1}{2}\langle X,M_1X\rangle\geq d$ ensures that $W(T,X,Y,d)\geq 0$ in (22). Then it is easy to obtain that

$$W(T,X,Y,d)=\frac{1}{2}\begin{bmatrix}X\\Y\end{bmatrix}^\top\begin{bmatrix}M_1+2M_2^\top M_2 & -2M_2^\top\\ -2M_2 & 2I\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}-d\geq 0. \tag{25}$$

Hence, based on the conditions of the proposition, $\mathcal{P}(\cdot)$ has to satisfy the terminal condition $\mathcal{P}(T)$ given in (G1) of Appendix G. Then under (24), we can show that (23) is the unique maximizing solution of the Hamiltonian of $(\mathbf{LQ-P''})$, where the explicit expression of $\mathscr{A}_i$, $i=1,2,3$, can be obtained from Appendix G. By substituting (23) into the same Hamiltonian, $\mathcal{P}$ in Appendix G can be obtained. Hence, W in (22) solves the HJB equation of $(\mathbf{LQ-P''})$. This, together with the verification theorem in Theorem 4, implies that (23) is the optimal solution and W in (22) is the value function of (**LQ-P**$''$). This completes the proof. ∎

Remark 20. Similar to the forward indefinite LQ control problems (see References 18,56,57), (24) is needed to get the unique optimal solution of $(\mathbf{LQ-P''})$. We also mention that the condition $\frac{1}{2}\langle X,M_1X\rangle\geq d$ in Proposition 1 is essential to construct the quadratic value function W of $(\mathbf{LQ}-\mathbf{P}'')$ in (22), which, together with $C(T)=-1$, ensures that the terminal condition of W at T (see (22)) satisfies (25). Without this condition, the quadratic form of the value function W as in (22) cannot be guaranteed, and we may have to rely on numerical techniques (see References 14,18,58 and the references therein) to solve the integro-type HJB equation of $(\mathbf{LQ}-\mathbf{P}'')$.

Based on Theorem 2 and Proposition 1, (21), the value function for $(\mathbf{LQ}-\mathbf{P}')$, can equivalently written as for $t\in[0,T]$,

$$\begin{aligned} V(t,X,Y) &= \inf_{\substack{u\in U_{t,T}\\ (z,k)\in\mathcal{Z}^{(1)}_{t,T}\times\mathcal{K}^{(1)}_{t,T}}} \{J'(t,X,Y;u,z,k)\mid (20)\text{ holds }\mathbb{P}-\text{a.s.}\} \\ &= \inf\{d\ge 0\mid W(t,X,Y,d)=0,\ (X,Y)\in\mathbb{R}^{n+m}\} =: \overline{d}, \end{aligned} \tag{26}$$

where W is the value function of $(\mathbf{LQ}-\mathbf{P}'')$ in (22). Let $(\overline{X},\overline{Y})\in\mathbb{R}^{n+m}$ be the argument of W such that $W(t,\overline{X},\overline{Y},\overline{d})=0$ in (26). Hence, under (24), in view of (23) and Proposition 1, the optimal solution of $(\mathbf{LQ}-\mathbf{P}')$ can be written by

$$\begin{cases} \widehat{u}_{\overline{d}}(t) := \widehat{u}(t)|_{d=\overline{d}} = \mathscr{A}_1[\mathcal{P}(t)]|_{d=\overline{d}} \begin{bmatrix} X_{\overline{X},\overline{Y}}(t)\\ Y_{\overline{X},\overline{Y}}(t)\end{bmatrix} \\ \widehat{z}_{\overline{d}}(t)^\top := \widehat{z}(t)^\top|_{d=\overline{d}} = \mathscr{A}_2[\mathcal{P}(t)]|_{d=\overline{d}} \begin{bmatrix} X_{\overline{X},\overline{Y}}(t)\\ Y_{\overline{X},\overline{Y}}(t)\end{bmatrix} \\ \widehat{k}_{\overline{d}}(t) := \widehat{k}(t)|_{d=\overline{d}} = \mathscr{A}_3[\mathcal{P}(t)]|_{d=\overline{d}} \begin{bmatrix} X_{\overline{X},\overline{Y}}(t)\\ Y_{\overline{X},\overline{Y}}(t)\end{bmatrix}, \end{cases} \tag{27}$$

where $|_{d=\overline{d}}$ means that $(\widehat{u},\widehat{z},\widehat{k})$ are evaluated at $d=\overline{d}$ in (26), that is, $\mathscr{A}_i$, $i=1,2,3$, are evaluated at $d=\overline{d}$. In (27), the subscript pair $(\overline{X},\overline{Y})$ indicates that the initial condition of (19) is $(\overline{X},\overline{Y})\in\mathbb{R}^{n+m}$. We note that with (27), the terminal state constraint in (20) is satisfied (see (26)).

By Remark 15 and (26), under (24), the optimal solution to the original linear (fully coupled) FBSDE control problem **(LQ-P)** can be written as

$$u^*(t) := \widehat{u}_{\overline{d}}(t) = \mathscr{A}_1[\mathcal{P}(t)]|_{d=\overline{d}} \begin{bmatrix} X_{\overline{X},\overline{Y}}(t)\\ Y_{\overline{X},\overline{Y}}(t)\end{bmatrix}, \tag{28}$$

where by defining $Z^* := \widehat{z}_{\overline{d}}$ and $K^* := \widehat{k}_{\overline{d}}$, the corresponding optimal FBSDE with jump diffusions for **(LQ-P)** in (17) controlled by the optimal solution (28) is given by

$$\begin{cases} \mathrm{d}X_{\overline{X},\overline{Y}}(s) = \left[A_1X_{\overline{X},\overline{Y}}(s-)+B_1u^*(s)+A_2Y_{\overline{X},\overline{Y}}(s-)+B_2Z^*(s)^\top+\lambda B_3K^*(s)\right]\mathrm{d}s \\ \qquad +\left[C_1X_{\overline{X},\overline{Y}}(s-)+D_1u^*(s)+C_2Y_{\overline{X},\overline{Y}}(s-)+D_2Z^*(s)^\top+\lambda D_3K^*(s)\right]\mathrm{d}B(s) \\ \qquad +\left[E_1X_{\overline{X},\overline{Y}}(s-)+F_1u^*(s)+E_2Y_{\overline{X},\overline{Y}}(s-)+F_2Z^*(s)^\top+F_3K^*(s)\right]\mathrm{d}\tilde{N}(s) \\ \mathrm{d}Y_{\overline{X},\overline{Y}}(s) = -\left[G_1X_{\overline{X},\overline{Y}}(s-)+H_1u^*(s)+G_2Y_{\overline{X},\overline{Y}}(s-)+H_2Z^*(s)^\top+\lambda H_3K^*(s)\right]\mathrm{d}s+Z^*(s)^\top\mathrm{d}B(s)+K^*(s)\mathrm{d}\tilde{N}(s) \\ X_{\overline{X},\overline{Y}}(t)=\overline{X},\ Y_{\overline{X},\overline{Y}}(T)=M_2X_{\overline{X},\overline{Y}}(T). \end{cases} \tag{29}$$

The preceding analysis shows that by following the steps in Remark 15, one can construct the optimal solution of **(LQ-P)**.

In summary, we state the following result:

Proposition 2. *Assume that the conditions in Proposition 1 hold. Then (28) is the optimal solution of* **(LQ-P)**, *where (29) is the optimal FBSDE of* **(LQ-P)** *controlled by (28).*

Remark 21.

(i) Unlike existing literatures on LQ optimal control for FBSDEs studied in References 1,31-33,35,39, **(LQ-P)** does not need any additional assumptions. In fact, Assumption 3 is standard in various (indefinite) LQ control problems. Note that in the earlier results on LQ control for FBSDEs, the additional monotonicity assumptions for the coefficients of the FBSDE are crucial to construct and verify the corresponding optimal solutions.[1,31-33,35,39]

(ii) As mentioned in Section 1.3, the previous approaches to solve **(LQ-P)** is the maximum principle or the completion of squares method (see References 1,31-33,35-39 and the references therein), where the former is only the necessary condition, while the latter requires additional restrictions on the objective functional. On the other hand, the approach of this paper provides the sufficient condition to characterize the optimal solution of **(LQ-P)**, which does not require any additional restrictions of the objective functional.

5 | PROOF OF THEOREMS 3 AND 4

This section proves Theorems 3 and 4.

5.1 | Proof of Theorem 3: existence

The existence result in Theorem 3 can be stated as follows.

Lemma 7. *Suppose that Assumption 2 holds. Then W defined in (10) is the viscosity solution of the (integro-type) HJB equation in (12).*

Proof. We first prove that W is the viscosity subsolution of (12). By Lemma 5, $W \in C(\overline{\mathcal{O}})$. Also, based on Lemma 6, W satisfies (i) of Definition 1.

To prove (ii) of Definition 1. let $\phi \in C_b^{1,3}(\overline{\mathcal{O}}) \cap C_2(\overline{\mathcal{O}})$ be the test function of W such that $(W-\phi)(t,\mathbf{a}) = \max_{(\bar{t},\bar{\mathbf{a}})\in\mathcal{O}}(W-\phi)(\bar{t},\bar{\mathbf{a}})$. Without loss of generality, we may assume that $W(t,\mathbf{a}) = \phi(t,\mathbf{a})$. This implies $W(\bar{t},\bar{\mathbf{a}}) \leq \phi(\bar{t},\bar{\mathbf{a}})$ for $(\bar{t},\bar{\mathbf{a}}) \in \mathcal{O}$ and $(\bar{t},\bar{\mathbf{a}}) \neq (t,\mathbf{a})$.

Then from Lemma 4,

$$\begin{aligned}
\phi(t,\mathbf{a}) = W(t,\mathbf{a}) &= \mathbb{E}\left[W(t+\tau, X_{t,a}^{u,z,k}(t+\tau), Y_{t,b}^{u,z,k}(t+\tau), \zeta_{t,a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(t+\tau))\right] \\
&\leq \mathbb{E}\left[\phi(t+\tau, X_{t,a}^{u,z,k}(t+\tau), Y_{t,b}^{u,z,k}(t+\tau), \zeta_{t,a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(t+\tau))\right].
\end{aligned}$$

By applying Itô's formula of Lévy-type stochastic integrals (theorem 4.4.7 of Reference 8),

$$\begin{aligned}
&-\mathbb{E}[\int_t^{t+\tau} \partial_t\phi(s, X_{t,a}^{u,z,k}(s), Y_{t,b}^{u,z,k}(s), \zeta_{t,a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(s))\mathrm{d}s] - \mathbb{E}[\int_t^{t+\tau} \langle D\phi(s, X_{t,a}^{u,z,k}(s), Y_{t,b}^{u,z,k}(s), \zeta_{t,a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(s)), \widehat{f}\rangle \mathrm{d}s] \\
&-\frac{1}{2}\mathbb{E}[\int_t^{t+\tau} \mathrm{Tr}(\widehat{\sigma}\widehat{\sigma}^\top D^2\phi(s, X_{t,a}^{u,z,k}(s), Y_{t,b}^{u,z,k}(s), \zeta_{t,a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(s)))\mathrm{d}s] \\
&-\mathbb{E}[\int_t^{t+\tau}\int_E [\phi(s, X_{t,a}^{u,z,k}(s)+\chi(e), Y_{t,b}^{u,z,k}(s)+k(e), \zeta_{t,a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(s)+\beta(e)) - \phi(s, X_{t,a}^{u,z,k}(s), Y_{t,b}^{u,z,k}(s), \zeta_{t,a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(s)) \\
&\qquad - \langle D\phi(s, X_{t,a}^{u,z,k}(s), Y_{t,b}^{u,z,k}(s), \zeta_{t,a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(s)), \chi(e)\rangle]\pi(\mathrm{d}e)\mathrm{d}s] \leq 0.
\end{aligned}$$

Here, we have used the fact that the expectation for the stochastic integrals of B and $\tilde{N}$ are zero, since they are $\mathcal{F}_t$-martingales.

Multiplying $\frac{1}{\tau}$ above and then letting $\tau \downarrow 0$, it follows from (14) that

$$-\partial_t\phi(t,\mathbf{a}) + \widehat{\mathbb{H}}(t,\mathbf{a},(\phi, D\phi, D^2\phi)(t,\mathbf{a}); u, \pi^{(1)}, \pi^{(2)}) \leq 0.$$

By taking sup with respect to $(u, \pi^{(1)}, \pi^{(2)}) \in U \times \Pi^{(1)} \times \Pi^{(2)}$ and using (13),

$$-\partial_t\phi(t,\mathbf{a}) + \mathbb{H}(t,\mathbf{a},(\phi, D\phi, D^2\phi)(t,\mathbf{a})) \leq 0, \tag{30}$$

which shows that W is the viscosity subsolution of (12).

We now prove, by contradiction, the supersolution property. Again, notice that W satisfies the boundary inequalities in (i) of Definition 1.

Let $\phi \in C_b^{1,3}(\overline{\mathcal{O}}) \cap C_2(\overline{\mathcal{O}})$ be the test function of W satisfying the following property: $(W-\phi)(t,\mathbf{a}) = \min_{(\bar{t},\bar{\mathbf{a}})\in\mathcal{O}}(W-\phi)(\bar{t},\bar{\mathbf{a}})$. Again, we may assume $W(t,\mathbf{a}) = \phi(t,\mathbf{a})$. Hence, $W(\bar{t},\bar{\mathbf{a}}) \geq \phi(\bar{t},\bar{\mathbf{a}})$ for $(\bar{t},\bar{\mathbf{a}}) \in \mathcal{O}$ and $(\bar{t},\bar{\mathbf{a}}) \neq (t,\mathbf{a})$.

We assume that W is not a viscosity supersolution. Then there exists a constant $\vartheta > 0$ such that

$$-\partial_t\phi(t,\mathbf{a}) + \mathbb{H}(t,\mathbf{a},(\phi,D\phi,D^2\phi)(t,\mathbf{a})) \leq -\vartheta < 0.$$

Recall the definition of $\widehat{\mathbb{H}}$ in (14) and $\widehat{\mathbb{H}} \leq \sup_{(u,\pi^{(1)},\pi^{(2)})\in U\times\Pi^{(1)}\times\Pi^{(2)}} \widehat{\mathbb{H}} = \mathbb{H}$. Then for any $(u,\pi^{(1)},\pi^{(2)}) \in U\times\Pi^{(1)}\times\Pi^{(2)}$, we have

$$-\partial_t\phi(t,\mathbf{a}) + \widehat{\mathbb{H}}(t,\mathbf{a},(\phi,D\phi,D^2\phi)(t,\mathbf{a});u,\pi^{(1)},\pi^{(2)}) \leq -\vartheta < 0. \tag{31}$$

On the other hand, Lemma 4 implies

$$\begin{aligned}\phi(t,\mathbf{a}) = W(t,\mathbf{a}) &= W(t+\tau, X_{t,a}^{u,z,k}(t+\tau), Y_{t,b}^{u,z,k}(t+\tau), \zeta_{t,a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(t+\tau)) \\ &\geq \phi(t+\tau, X_{t,a}^{u,z,k}(t+\tau), Y_{t,b}^{u,z,k}(t+\tau), \zeta_{t,a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(t+\tau)).\end{aligned}$$

By the regularity of the test function, for each $\epsilon > 0$, there exist $(u_\epsilon, \pi_\epsilon^{(1)}, \pi_\epsilon^{(2)}) \in \mathcal{U}_{t,T} \times \Pi_{t,T}^{(1)} \times \Pi_{t,T}^{(2)}$ such that

$$\epsilon\tau \leq \phi(t,\mathbf{a}) - \phi(t+\tau, X_{t,a}^{u_\epsilon,z_\epsilon,k_\epsilon}(t+\tau), Y_{t,b}^{u_\epsilon,z_\epsilon,k_\epsilon}(t+\tau), \zeta_{t,a,b,d}^{u_\epsilon,\pi_\epsilon^{(1)},\pi_\epsilon^{(2)}}(t+\tau)). \tag{32}$$

Similar to the viscosity subsolution case, by applying Itô's formula to (32). We then multiply $\frac{1}{\tau}$. Note that (31) holds for any $(u,\pi^{(1)},\pi^{(2)}) \in U\times\Pi^{(1)}\times\Pi^{(2)}$. Then by letting $\tau \downarrow 0$ and noticing the arbitrariness of ϵ (we also use the fact that stochastic integrals are $\mathcal{F}_s$-martingales), it follows that

$$0 \leq \partial_t\phi(t,\mathbf{a}) + \widehat{\mathbb{H}}(t,\mathbf{a},(\phi,D\phi,D^2\phi)(t,\mathbf{a});u,\pi^{(1)},\pi^{(2)}) \leq -\vartheta < 0,$$

which is the desired contradiction due to (31). This implies that W is the viscosity supersolution of (12). This, together with (30), shows that W is the continuous viscosity solution of (12). We complete the proof. ■

5.2 | Proof of Theorem 3: uniqueness

The comparison principle for viscosity subsolution and supersolution of (12) is stated as follows.

Lemma 8. *Suppose that Assumption 2 holds. Let $\underline{W} \in C(\overline{\mathcal{O}})$ be the viscosity subsolution of the HJB equation in (12), and $\overline{W} \in C(\overline{\mathcal{O}})$ the viscosity supersolution of (12), where both $\underline{W}$ and $\overline{W}$ satisfy the linear growth condition in $(a,b) \in \mathbb{R}^{n+m}$. Then*

$$\underline{W}(t,\mathbf{a}) \leq \overline{W}(t,\mathbf{a}), \ \forall (t,\mathbf{a}) \in \overline{\mathcal{O}}. \tag{33}$$

Proof. For $\eta, \nu > 0$, we define

$$\Psi_{\nu;\eta,\lambda}(t,\mathbf{a}) := \underline{W}_\nu(t,\mathbf{a}) - \overline{W}(t,\mathbf{a}) - 2\eta e^{-\lambda t}(1+|a|^2+|b|^2+d),$$

where λ will be determined later. We will prove that

$$\Psi_{\nu;\eta,\lambda}(t,\mathbf{a}) \leq 0. \tag{34}$$

In fact, by letting $\eta \downarrow 0$ and then $\nu \downarrow 0$, this implies (33), which is the desired conclusion.

Below, we prove (34) by contradiction. Assume that (34) is not true, that is, $\Psi_{\nu;\eta,\lambda}(t,\mathbf{a}) > 0$ for some $(t,\mathbf{a}) \in \overline{\mathcal{O}}$. We consider

$$\Psi_{\nu;\eta,\lambda}(\tilde{t},\widetilde{\mathbf{a}}) := \Psi_{\nu;\eta,\lambda}(\tilde{t},\tilde{a},\tilde{b},\tilde{d}) := \max_{(t,\mathbf{a})\in\mathcal{O}} \Psi_{\nu;\eta,\lambda}(t,\mathbf{a}) > 0. \tag{35}$$

Note that $(\tilde{t},\widetilde{\mathbf{a}})$ depend on (ν,η,λ), i.e., $(\tilde{t},\widetilde{\mathbf{a}}) := (\tilde{t}_{\nu;\eta,\lambda},\widetilde{\mathbf{a}}_{\nu;\eta,\lambda})$. Indeed, $(\tilde{t},\widetilde{\mathbf{a}})$ exist, since $\overline{W}$, $\underline{W}$ and the log function satisfy the linear growth condition, and $e^{-\lambda t}$ is decreasing.

If $\tilde{t}=T$, then in view of the terminal condition of $\overline{W}$ and $\underline{W}$,

$$\Psi_{\nu;\eta,\lambda}(T,\widetilde{\mathbf{a}}) = \underline{W}(T,\widetilde{\mathbf{a}}) - \nu\log(1+\tilde{d}) - \overline{W}(T,\widetilde{\mathbf{a}}) - 2\eta e^{-\lambda T}(1+|\tilde{a}|^2+|\tilde{b}|^2+\tilde{d}) \leq 0.$$

Hence, it contradicts (35), which implies $\tilde{t}<T$. Similarly, when $\tilde{d}=0$,

$$\Psi_{\nu;\eta,\lambda}(t,\tilde{a},\tilde{b},0) = \underline{W}(t,\tilde{a},\tilde{b},0) - \nu(T-\tilde{t}) - \overline{W}(t,\tilde{a},\tilde{b},0) - 2\eta e^{-\lambda\tilde{t}}(1+|\tilde{a}|^2+|\tilde{b}|^2) \leq 0.$$

which also contradicts (35). Hence, $\tilde{d}>0$. Therefore, $(\tilde{t},\widetilde{\mathbf{a}})\in\mathcal{O}$.

After doubling variables of Ψ, for $\epsilon>0$, we define

$$\Psi^{\epsilon}_{\nu;\eta,\lambda}(t,\mathbf{a},\breve{\mathbf{a}}) := \Psi^{\epsilon}_{\nu;\eta,\lambda}(t,a,b,d,\breve{a},\breve{b},\breve{d}) := \widehat{\Psi}_{\nu;\eta,\lambda}(t,\mathbf{a},\breve{\mathbf{a}}) - \frac{1}{\epsilon}\theta(\mathbf{a},\breve{\mathbf{a}}), \tag{36}$$

where

$$\begin{aligned}\widehat{\Psi}_{\nu;\eta,\lambda}(t,\mathbf{a},\breve{\mathbf{a}}) &:= \underline{W}_{\nu}(t,\mathbf{a}) - \overline{W}(t,\breve{\mathbf{a}}) - \eta e^{-\lambda t}(1+|a|^2+|b|^2+d) - \eta e^{-\lambda t}(1+|\breve{a}|^2+|\breve{b}|^2+\breve{d})\\ &\quad - \frac{\eta e^{-\lambda t}}{2}\left(|a-\tilde{a}|^2+|b-\tilde{b}|^2+(d-\tilde{d})\right) - \frac{1}{2}|t-\tilde{t}|^2\\ \theta(\mathbf{a},\breve{\mathbf{a}}) &:= \frac{1}{2}\left(|a-\breve{a}|^2+|b-\breve{b}|^2+|d-\breve{d}|^2\right).\end{aligned}$$

We can easily see that $\widehat{\Psi}_{\nu;\eta,\lambda}(t,\mathbf{a},\mathbf{a}) \leq \Psi_{\nu;\eta,\lambda}(t,\mathbf{a})$ and $\widehat{\Psi}_{\nu;\eta,\lambda}(\tilde{t},\widetilde{\mathbf{a}},\widetilde{\mathbf{a}}) = \Psi_{\nu;\eta,\lambda}(\tilde{t},\widetilde{\mathbf{a}})$. Then from (35), it follows that

$$\Psi_{\nu;\eta,\lambda}(\tilde{t},\widetilde{\mathbf{a}}) = \max_{(t,\mathbf{a})\in\mathcal{O}}\Psi_{\nu;\eta,\lambda}(t,\mathbf{a}) = \max_{(t,\mathbf{a})\in\mathcal{O}}\widehat{\Psi}_{\nu;\eta,\lambda}(t,\mathbf{a},\mathbf{a}). \tag{37}$$

Consider,

$$\Psi^{\epsilon}_{\nu;\eta,\lambda}(t'_{\epsilon},\mathbf{a}'_{\epsilon},\breve{\mathbf{a}}'_{\epsilon}) := \Psi^{\epsilon}_{\nu;\eta,\lambda}(t'_{\epsilon},a'_{\epsilon},b'_{\epsilon},d'_{\epsilon},\breve{a}'_{\epsilon},\breve{b}'_{\epsilon},\breve{d}'_{\epsilon}) := \max_{(t,\mathbf{a},t,\breve{\mathbf{a}})\in\mathcal{O}\times\mathcal{O}}\left\{\widehat{\Psi}_{\nu;\eta,\lambda}(t,\mathbf{a},\breve{\mathbf{a}}) - \frac{1}{\epsilon}\theta(\mathbf{a},\breve{\mathbf{a}})\right\}, \tag{38}$$

where the maximum points $(t'_{\epsilon},\mathbf{a}'_{\epsilon},\breve{\mathbf{a}}'_{\epsilon})$ exist by (37) and the fact that θ is coercive. Then by proposition 3.7 of Reference 59, it follows that

$$\begin{cases}\lim\limits_{\epsilon\to 0}\frac{1}{\epsilon}\theta(\mathbf{a}'_{\epsilon},\breve{\mathbf{a}}'_{\epsilon}) = 0\\ \lim\limits_{\epsilon\to 0}\Psi^{\epsilon}_{\nu;\eta,\lambda}(t'_{\epsilon},\mathbf{a}'_{\epsilon},\breve{\mathbf{a}}'_{\epsilon}) = \widehat{\Psi}_{\nu;\eta,\lambda}(t',\mathbf{a}',\breve{\mathbf{a}}') = \max\limits_{\theta(\mathbf{a},\breve{\mathbf{a}})=0}\widehat{\Psi}_{\nu;\eta,\lambda}(t,\mathbf{a},\breve{\mathbf{a}}) = \max\widehat{\Psi}_{\nu;\eta,\lambda}(t,\mathbf{a},\mathbf{a})\\ \lim\limits_{\epsilon\to 0}\theta(\mathbf{a}'_{\epsilon},\breve{\mathbf{a}}'_{\epsilon}) = \theta(\mathbf{a}',\breve{\mathbf{a}}') = \theta(a',b',d',\breve{a}',\breve{b}',\breve{d}') = 0.\end{cases}$$

This and (37) imply that as $\epsilon\downarrow 0$, $\mathbf{a}'_{\epsilon},\breve{\mathbf{a}}'_{\epsilon}\to\widetilde{\mathbf{a}}$ and $t'_{\epsilon}\to\tilde{t}$, that is, as $\epsilon\downarrow 0$,

$$\begin{cases}|a'_{\epsilon}-\breve{a}'_{\epsilon}|^2,\ |b'_{\epsilon}-\breve{b}'_{\epsilon}|^2,\ |d'_{\epsilon}-\breve{d}'_{\epsilon}|^2\to 0\\ \frac{1}{\epsilon}|a'_{\epsilon}-\breve{a}'_{\epsilon}|^2,\ \frac{1}{\epsilon}|b'_{\epsilon}-\breve{b}'_{\epsilon}|^2,\ \frac{1}{\epsilon}|d'_{\epsilon}-\breve{d}'_{\epsilon}|^2\to 0\\ t'_{\epsilon}\to\tilde{t},\ a'_{\epsilon},\breve{a}'_{\epsilon}\to\tilde{a},\ b'_{\epsilon},\breve{b}'_{\epsilon}\to\tilde{b},\ d'_{\epsilon},\breve{d}'_{\epsilon}\to\tilde{d}.\end{cases} \tag{39}$$

For simplicity, we let $(t',\mathbf{a}',\breve{\mathbf{a}}') := (t'_{\epsilon},\mathbf{a}'_{\epsilon},\breve{\mathbf{a}}'_{\epsilon})$. Equivalently, $(t',a',b',d',\breve{a}',\breve{b}',\breve{d}') := (t'_{\epsilon},a'_{\epsilon},b'_{\epsilon},d'_{\epsilon},\breve{a}'_{\epsilon},\breve{b}'_{\epsilon},\breve{d}'_{\epsilon})$.

Define

$$\begin{cases}\delta_{\eta,\lambda}(t,\mathbf{a}) := \eta e^{-\lambda t}(1+|a|^2+|b|^2+d)+\frac{1}{2}|t-\tilde{t}|^2+\frac{\eta e^{-\lambda t}}{2}\left(|a-\tilde{a}|^2+|b-\tilde{b}|^2+(d-\tilde{d})\right)\\ \breve{\delta}_{\eta,\lambda}(t,\breve{\mathbf{a}}) := \eta e^{-\lambda t}(1+|\breve{a}|^2+|\breve{b}|^2+\breve{d})\\ \theta_\epsilon(\mathbf{a},\breve{\mathbf{a}}) := \frac{1}{2\epsilon}\left(|a-\breve{a}|^2+|b-\breve{b}|^2+|d-\breve{d}|^2\right).\end{cases}$$

For δ, $\breve{\delta}$ and θ_ϵ and their derivatives (see Appendix F), we use the superscript $'$ when they are evaluated at $(t',\mathbf{a}',\breve{\mathbf{a}}')$, for example, $\delta'_{\eta,\lambda} := \delta_{\eta,\lambda}(t',\mathbf{a}')$ and $D_{\mathbf{a}}\delta'_{\eta,\lambda} := D_{\mathbf{a}}\delta_{\eta,\lambda}(t',\mathbf{a}')$.

By definition of $\Psi^\epsilon_{\nu;\eta,\lambda}$ in (36), we have

$$\Psi^\epsilon_{\nu;\eta,\lambda}(t,\mathbf{a},\breve{\mathbf{a}}) = (\underline{W}_\nu(t,\mathbf{a})-\delta_{\eta,\lambda}(t,\mathbf{a}))-(\overline{W}(t,\breve{\mathbf{a}})+\breve{\delta}_{\eta,\lambda}(t,\breve{\mathbf{a}}))-\theta_\epsilon(\mathbf{a},\breve{\mathbf{a}}). \tag{40}$$

Then based on Crandall–Ishii's lemma in theorem 8.3 and remark 2.7 of Reference 59, there exist

$$\begin{cases} q+\breve{q}=\partial_t\theta'_\epsilon=0\\ (q+\partial_t\delta'_{\eta,\lambda}, D_{\mathbf{a}}(\delta'_{\eta,\lambda}+\theta'_\epsilon), P+D^2_{\mathbf{aa}}\delta'_{\eta,\lambda})\in\overline{\mathcal{P}}^{1,2,+}\underline{W}_\nu(t',\mathbf{a}')\\ (-\breve{q}-\partial_t\breve{\delta}'_{\eta,\lambda}, -D_{\mathbf{a}}(\breve{\delta}'_{\eta,\lambda}+\theta'_\epsilon), -\breve{P}-D^2_{\mathbf{aa}}\breve{\delta}'_{\eta,\lambda})\in\overline{\mathcal{P}}^{1,2,-}\overline{W}(t',\breve{\mathbf{a}}'),\end{cases} \tag{41}$$

such that with $P,\breve{P}\in\mathbb{S}^{n+m+1}$,

$$-\frac{3}{\epsilon}\begin{bmatrix} I_{n+m+1} & 0_{n+m+1}\\ 0_{n+m+1} & I_{n+m+1}\end{bmatrix}\le\begin{bmatrix} P & 0_{n+m+1}\\ 0_{n+m+1} & \breve{P}\end{bmatrix}\le\frac{3}{\epsilon}\begin{bmatrix} I_{n+m+1} & -I_{n+m+1}\\ -I_{n+m+1} & I_{n+m+1}\end{bmatrix}. \tag{42}$$

From Lemmas 9–11 in Appendix A and Lemma 12 in Appendix B, there exists $\phi\in C_b^{1,3}(\overline{\mathcal{O}})\cap C_2(\overline{\mathcal{O}})$ such that

$$\begin{aligned}&\sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}}\{\Lambda^+(\mathcal{Q}_\psi(t',\mathbf{a}',(q+\partial_t\delta'_{\eta,\lambda}, D_{\mathbf{a}}(\delta'_{\eta,\lambda}+\theta'_\epsilon), P+D^2_{\mathbf{aa}}\delta'_{\eta,\lambda}); u,z,k))\\ &\quad+\sup_{\beta\in G^{(2)}}\{\mathbb{H}^{(21)}_\kappa(t',\mathbf{a}',(\phi,D\phi)(t',\mathbf{a}');u,z,k,\beta)+\mathbb{H}^{(22)}_\kappa(t',\mathbf{a}',\underline{W}_\nu(t',\mathbf{a}'),D_{\mathbf{a}}(\delta'_{\eta,\lambda}+\theta'_\epsilon);u,z,k,\beta)\}\}\le-\frac{\nu}{8},\end{aligned}$$

and

$$\begin{aligned}&\sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}}\{\Lambda^+(\mathcal{Q}_\psi(t',\breve{\mathbf{a}}',(-\breve{q}-\partial_t\breve{\delta}'_{\eta,\lambda}, -D_{\breve{\mathbf{a}}}(\breve{\delta}'_{\eta,\lambda}+\theta'_\epsilon), -\breve{P}-D^2_{\breve{\mathbf{a}}}\breve{\delta}'_{\eta,\lambda}); u,z,k))\\ &\quad+\sup_{\beta\in G^{(2)}}\{\mathbb{H}^{(21)}_\kappa(t',\breve{\mathbf{a}}',(\phi,D\phi)(t',\breve{\mathbf{a}}');u,z,k,\beta)+\mathbb{H}^{(22)}_\kappa(t',\breve{\mathbf{a}}',\overline{W}(t',\breve{\mathbf{a}}'),-D_{\breve{\mathbf{a}}}(\breve{\delta}'_{\eta,\lambda}+\theta'_\epsilon);u,z,k,\beta)\}\}\ge 0.\end{aligned}$$

We can show that

$$\Upsilon^{(1)}+\Upsilon^{(2)}+\Upsilon^{(3)}\ge\frac{\nu}{8},$$

where

$$\begin{aligned}\Upsilon^{(1)} &:= \sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}}\{\Lambda^+(\mathcal{Q}_\psi(t',\breve{\mathbf{a}}',(-\breve{q}-\partial_t\breve{\delta}'_{\eta,\lambda}, -D_{\breve{\mathbf{a}}}(\breve{\delta}'_{\eta,\lambda}+\theta'_\epsilon), -\breve{P}-D^2_{\breve{\mathbf{a}}\breve{\mathbf{a}}}\breve{\delta}'_{\eta,\lambda}); u,z,k))\\ &\quad-\Lambda^+(\mathcal{Q}_\psi(t',\mathbf{a}',(q+\partial_t\delta'_{\eta,\lambda}, D_{\mathbf{a}}(\delta'_{\eta,\lambda}+\theta'_\epsilon), P+D^2_{\mathbf{aa}}\delta'_{\eta,\lambda}); u,z,k))\}\\ \Upsilon^{(2)} &:= \sup_{(u,z,\pi^{(2)})\in U\times\mathbb{R}^{p\times m}\times\Pi^{(2)}}\{\mathbb{H}^{(21)}_\kappa(t',\breve{\mathbf{a}}',(\phi,D\phi)(t',\breve{\mathbf{a}}');u,z,k,\beta)\\ &\quad-\mathbb{H}^{(21)}_\kappa(t',\mathbf{a}',(\phi,D\phi)(t',\mathbf{a}');u,z,k,\beta)\}\\ \Upsilon^{(3)} &:= \sup_{(u,z,\pi^{(2)})\in U\times\mathbb{R}^{p\times m}\times\Pi^{(2)}}\{\mathbb{H}^{(22)}_\kappa(t',\breve{\mathbf{a}}',\overline{W}(t',\breve{\mathbf{a}}'),-D_{\breve{\mathbf{a}}}(\breve{\delta}'_{\eta,\lambda}+\theta'_\epsilon);u,z,k,\beta)\\ &\quad-\mathbb{H}^{(22)}_\kappa(t',\mathbf{a}',\underline{W}_\nu(t',\mathbf{a}'),D_{\mathbf{a}}(\delta'_{\eta,\lambda}+\theta'_\epsilon);u,z,k,\beta)\}.\end{aligned}$$

It is shown in Appendices C–E that for a specific choice of λ,

$$\frac{\nu}{8} \leq \lim_{\eta \downarrow 0} \lim_{\epsilon \downarrow 0} \lim_{\kappa \downarrow 0} \{\Upsilon^{(1)} + \Upsilon^{(2)} + \Upsilon^{(3)}\} \leq 0, \tag{43}$$

which leads to the desired contradiction, since $\nu > 0$ (see Lemma 12 in Appendix B). This implies the comparison principle in (33). Indeed, (C1), (D2) and (E1) imply (43) for $\lambda \geq \max\{C_1, C_3 + C_4\}$, where C_1, C_3, and C_4 are defined in Appendices C.0.1-C.0.3. This shows (33); thus completing the proof of Lemma 8. ∎

Proof of Theorem 3. We note that by Lemma 7, the auxiliary value function W defined in (11) is a viscosity solution of (12). This implies the existence. Note also that from Lemma 5, W satisfies the conditions in Lemma 8. Suppose that W and W' are viscosity solutions of (12). This implies that W and W' are both viscosity subsolutions and supersolutions of (12). Then in view of Lemma 8, it follows that $W = W'$, which implies the uniqueness. We complete the proof. ∎

5.3 | Proof of Theorem 4

Proof of Theorem 4. We provide a concise proof of Theorem 4, as it resembles the proof of the standard (smooth) verification theorem[9,18,52] (see corollary 4.1 of Reference 52). Specifically, we first apply the Itô's formula of Lévy-type stochastic integrals (see theorem 4.4.7 of Reference 8) to $W(s, X_{t,a}^{u,z,k}(s), Y_{t,b}^{u,z,k}(s), \zeta_{t,a,b,d}^{u,\pi^{(1)},\pi^{(2)}}(s))$, and then integrate it from t to T. By taking the expectation, the stochastic integrals with respect to B and $\tilde{N}$ are eliminated, since they are $\mathcal{F}_s$-martingales. By the fact that $(\hat{u}, \hat{\pi}^{(1)}, \hat{\pi}^{(2)}) = (\hat{u}, \hat{z}, \hat{\alpha}, \hat{k}, \hat{\beta}) \in \mathcal{U}_{t,T} \times \Pi_{t,T}^{(1)} \times \Pi_{t,T}^{(2)}$ attains the supremum of $\hat{\mathbb{H}}$ (see (15)) and $\hat{\mathbb{H}} \geq \mathbb{H}$ for any $(u, \pi^{(1)}, \pi^{(2)}) \in \mathcal{U}_{t,T} \times \Pi_{t,T}^{(1)} \times \Pi_{t,T}^{(2)}$, we can show that $W(t, \mathbf{a}) = \hat{J}(t, \mathbf{a}; \hat{u}, \hat{\pi}^{(1)}, \hat{\pi}^{(2)}) \leq \hat{J}(t, \mathbf{a}; u, \pi^{(1)}, \pi^{(2)})$ for $(u, \pi^{(1)}, \pi^{(2)}) \in \mathcal{U}_{t,T} \times \Pi_{t,T}^{(1)} \times \Pi_{t,T}^{(2)}$. This completes the proof. ∎

6 | CONCLUSIONS

In this paper, we have studied the stochastic optimal control problem for fully coupled FBSDEs with jump diffusions. By using the backward reachability approach, the original (fully coupled) FBSDE control problem is converted into the (terminal) state-constrained forward stochastic control problem, where the value function of the latter problem can be expressed by the zero-level set of the value function for the auxiliary unconstrained (forward) stochastic control problem. The auxiliary unconstrained (forward) stochastic control problem includes additional control variables as a consequence of the martingale representation theorem. We show that the value function of the auxiliary unconstrained (forward) stochastic control problem is a unique viscosity solution to the associated integro-type HJB equation that includes the nonlocal integral operator in terms of the (singular) Lévy measure and the additional Hamiltonian maximizing variables. We have applied the theoretical results of this paper to the LQ problem, for which the explicit optimal solution is obtained by solving the corresponding integro-type HJB equation and then characterizing the zero-level set of the auxiliary value function (the solution of the integro-type HJB equation).

In addition to the LQ problem of this paper, it is possible to apply the theoretical results of this paper to various examples of optimal control problems for fully coupled FBSDEs. For example, as in References 40,51, one may study the utility maximization problem, the risk minimization problem, and the cash management problem using the theoretical results of this paper, where the main technical challenge would be to construct the (smooth or viscosity) solution of the corresponding integro-type HJB equation and to characterize its zero-level set (see Remark 15). These problems can also be considered within the LQ framework of this paper and are currently under studying, which we leave for the future research problem.

We believe the assumption that l and m are nonnegative functions can be relaxed. Indeed, one may consider the situation that $l, m \geq -M$ with $M \geq 0$. Then by letting $\tilde{l} = l + M$ and $\tilde{m} = m + M$, we may obtain the same results of this paper by little modifications in the proofs. Studying the problem of this paper under other possible weaker assumptions of l and m would also be an interesting potential future research problem.

AUTHOR CONTRIBUTIONS

Hyun Jong Yang: problem formulation, analysis, writing and revising the paper. **Jun Moon**: problem formulation, writing and revising the paper.

ACKNOWLEDGMENTS

This work was supported in part by the Technology Innovation Program (20018112) funded by the Ministry of Trade, Industry and Energy (MOTIE, Korea), in part by the National Research Foundation of Korea (NRF) Grant funded by the Ministry of Science and ICT, South Korea (NRF-2021R1A2C2094350, NRF-2017R1A5A1015311), and in part by Institute of Information & Communications Technology Planning and Evaluation (IITP) grant funded by the Korea government (MSIT) (No. 2020-0-01373 and 2018-0-00958). The authors would like to thank the reviewer for careful reading and helpful suggestions on the earlier version of the article.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

ENDNOTES

*We mention that when $E = \{1\}$, $\{\tilde{N}((0,t]) := (N - \lambda')((0,t])\}_{t\in(0,T]}$ is the corresponding compensated Poisson process, where $\lambda'(\mathrm{d}t) := \lambda \mathrm{d}t$ and $\lambda > 0$ is the intensity of N.[8,9] This implies that $\int_E \phi(s,e)\tilde{N}(\mathrm{d}s,\mathrm{d}e) = \phi(s)\mathrm{d}\tilde{N}(s)$ and $\int_E \phi(s,e)\lambda(\mathrm{d}e)\mathrm{d}s = \lambda\phi(s)\mathrm{d}s$.

†See Remark 4 for other equivalent terminal constraints.

‡Assumption 2(iii) holds when we take the sufficiently large compact subsets in $\mathbb{R}^n$ and $\mathbb{R}^m$ that include X and Y; see chapter 6 of Reference 55 for a related discussion.

ORCID

Jun Moon https://orcid.org/0000-0002-8877-9519

REFERENCES

1. Peng S, WZ. Fully coupled forward-backward stochastic differential equations and applications to optimal control. *SIAM J Control Optim*. 1999;37(3):825-843.
2. Li J, Wei Q. L^p estimates for fully coupled FBSDEs with jumps. *Stoch Process Appl*. 2014;124:1582-1611.
3. Yong J. Linear forward-backward stochastic differential equations with random coefficients. *Probab Theory Related Fields*. 2006;135:53-83.
4. Li J, Wei Q. Stochastic differential games for fully coupled FBSDEs with jumps. *Appl Math Optim*. 2015;71:411-448.
5. Wu Z. Forward-backward stochastic differential equations with Brownian motion and Poisson process. *Acta Math Appl Sin*. 1999;15:433-443.
6. Delarue F. On the existence and uniqueness of solutions to FBSDEs in a non-degenerate case. *Stoch Process Appl*. 2002;99:209-286.
7. Wu Z. Fully coupled FBSDE with Brownian motion and Poisson process in stopping time duration. *J Aust Math Soc*. 2003;74:249-266.
8. Applebaum D. *Lévy Processes and Stochastic Calculus*. 2nd ed. Cambridge University Press; 2009.
9. Øksendal B, Sulem A. *Applied Stochastic Control of Jump Diffusions*. 2nd ed. Springer; 2006.
10. Fujiwara T, Kunita H. Stochastic differential equations of jump type and Lévy processes in diffeomorphisms group. *Kyoto J Math*. 1985;25(1):71-106.
11. Buckdahn R, Hu Y, Li J. Stochastic representation for solutions of Isaacs' type integral-partial differential equations. *Stoch Process Appl*. 2011;121:2715-2750.
12. Bismut JM. An introductory approach to duality in optimal stochastic control. *SIAM Rev*. 1978;20(1):62-78.
13. Bismut JM. Linear quadratic optimal stochatic control with random coefficients. *SIAM J Control Optim*. 1976;14(3):419-444.
14. Zhang J. *Backward Stochastic Differential Equations: From Linear to Fully Nonlinear Theory*. Springer; 2017.
15. Ma J, Yong J. *Forward-Backward Stochastic Differential Equations and Their Applications*. Springer-Verlag; 1999.
16. El-Karoui N, Peng S, Quenez MC. Backward stochastic differential equations in finance. *Math Financ*. 1997;7(1):1-71.
17. Pardoux E, Rascanu A. *Stochastic Differential Equations, Backward SDEs, Partial Differential Equations*. Springer; 2014.
18. Yong J, Zhou XY. *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Springer; 1999.
19. Wang S, Xiao H. Individual and mass behavior in large population forward–backward stochastic control problems: centralized and Nash equilibrium solutions. *Opt Control Appl Methods*. 2021;42:1269-1292.
20. Moon J. Necessary and sufficient conditions of risk-sensitive optimal control and differential games for stochastic differential delayed equations. *Int J Robust Nonlinear Control*. 2019;29:4812-4827.
21. Moon J. Risk-sensitive maximum principle for stochastic optimal control of mean-field type Markov regime-switching jump-diffusion systems. *Int J Robust Nonlinear Control*. 2021;31:2141-2167.

22. Caraballo T, Mchiri L. η stability of hybrid neutral stochastic differential equations with infinite delay. *Int J Robust Nonlinear Control*. 2021;32:1973-1989.
23. Moon J. Generalized risk-sensitive optimal control and Hamilton-Jacobi-Bellman equation. *IEEE Trans Automat Contr*. 2021;66(5):2319-2325.
24. Carmona R, Delarue F. Probabilistic analysis of mean-field games. *SIAM J Control Optim*. 2013;51(4):2705-2734.
25. Carmona R, Delarue F. *Probabilistic Theory of Mean Field Games with Applications I*. Springer; 2018.
26. Moon J, Başar T. Risk-sensitive mean field games via the stochastic maximum principle. *Dyn Games Appl*. 2019;9:1100-1125.
27. Zhang J. The wellposedness of FBSDEs. *Discr Contin Dyn Syst Ser B*. 2006;6(4):927-940.
28. Ma J, Wu Z, Zhang D, Zhang J. On well-posedness of forward-backward SDEs–A unified approach. *Ann Appl Probab*. 2015;25(4):2168-2214.
29. Wu S, Shu L. Partially observed linear quadratic control problem with delay via backward separation method. *Opt Control Appl Methods*. 2017;38:814-828.
30. Guerdouh D, Khelfallah N, Mezerdi B. On the well-posedness of coupled forward–backward stochastic differential equations driven by Teugels martingales. *Math Methods Appl Sci*. 2020;43:10296-10318.
31. Wu Z. Forward-backward stochastic differential equations, linear quadratic stochastic optimal control and nonzero sum differential games. *J Syst Sci Complex*. 2005;18(2):179-192.
32. Shi JT, Wu Z. One kind of fully coupled linear quadratic stochastic control problem with random jumps. *Acta Automat Sin*. 2009;35(1):92-97.
33. Li N, Yu Z. Recursive stochastic linear-quadratic optimal control and nonzero-sum differential game problems with random jumps. *Adv Differ Equ*. 2015;144:1-19.
34. Moon J. The risk-sensitive maximum principle for controlled forward-backward stochastic differential equations. *Automatica*. 2020;120:1-14.
35. Huang H, Wang X, Hou T, Xu L. Linear quadratic stochastic optimal control of forward backward stochastic control system associated with lévy process. *Math Probl Eng*. 2017;2541687:1-11.
36. Moon J. Linear-quadratic stochastic stackelberg differential games for jump-diffusion systems. *SIAM J Control Optim*. 2021;59(2):954-976.
37. Moon J. Linear-quadratic mean-field type Stackelberg differential games for stochastic jump-diffusion systems. *Math Control Related Fields*. 2021;12(2):371-404.
38. Yong J. A leader-follower stochastic linear quadratic differential game. *SIAM J Control Optim*. 2002;41(4):1015-1041.
39. Li W, Min H. Fully coupled mean-field FBSDEs with jumps and related optimal control problems. *Opt Control Appl Methods*. 2021;42:305-329.
40. Wang G, Xiao H. Arrow sufficient conditions for optimality of fully coupled forward–backward stochastic differential equations with applications to finance. *J Optim Theory Appl*. 2015;165:639-656.
41. Yong J. Optimality variational principle for controlled forward-backward stochastic differential equations with mixed initial-terminal conditions. *SIAM J Control Optim*. 2010;48(6):4119-4156.
42. Song T, Liu B. A maximum principle for fully coupled controlled forward-backward stochastic difference systems of mean-field type. *Adv Differ Equ*. 2020;188:1-24.
43. Hu M. Stochastic global maximum principle for optimization with recursive utilities. *Probab Uncertain Quant Risk*. 2017;2(1):1-20.
44. Ji S, Wei Q. A maximum principle for fully coupled forward-backward stochastic control systems with terminal state constraints. *J Math Anal Appl*. 2013;407:200-210.
45. Wu Z. A general maximum principle for optimal control of forward-backward stochastic systems. *Automatica*. 2013;49(5):1473-1480.
46. Hu M, Ji S, Xue X. A global stochastic maximum principle for fully coupled forward-backward stochastic systems. *SIAM J Control Optim*. 2018;56(6):4309-4335.
47. Shi JT. Necessary conditions for optimal control of forward-backward stochastic systems with random jumps. *Int J Stoch Anal*. 2012;258674:1-50.
48. Hafayed M, Tabet M, Boukaf S. Mean-field maximum principle for optimal control of forward-backward stochastic systems with jumps and its application to mean-variance portfolio problem. *Commun Math Stat*. 2015;3:163-186.
49. Zhang H. Mixed optimal control of forward-backward stochastic system. *Opt Control Appl Methods*. 2021;42:833-847.
50. Ma H, Wang W. Stochastic maximum principle for partially observed risk-sensitive optimal control problems of mean-field forward-backward stochastic differential equations. *Opt Control Appl Methods*. 2021;43:532-553.
51. Øksendal B, Sulem A. Maximum principles for optimal control of forward-backward stochastic differential equations with jumps. *SIAM J Control Optim*. 2009;48(5):2945-2976.
52. Shi JT, Wu Z. Relationship between MP and DPP for the stochastic optimal control problem of jump diffusions. *Appl Math Optim*. 2011;63:151-189.
53. Barles G, Buckdahn R, Pardoux E. Backward stochastic differential equations and integral-partial differential equations. *Stoch Stoch Rep*. 1997;60:57-83.
54. Barles G, Imbert C. Second-order elliptic integro-differential equations: viscosity solutions' theory revisited. Annales de l'Institut Henri Poincare (C) non linear. *Analysis*. 2008;25(3):567-585.
55. Başar T, Olsder GJ. *Dynamic Noncooperative Game Theory*. 2nd ed. SIAM; 1999.
56. Zhang F, Dong Y, Meng Q. Backward stochastic Riccati equation with jumps associated with stochastic linear quadratic optimal control with jump and random coefficients. *SIAM J Control Optim*. 2020;58(1):393-424.

57. Moon J, Duncan TE. A simple proof of indefinite linear-quadratic stochastic optimal control with random coefficients. *IEEE Trans Automat Contr*. 2020;65(12):5422-5428.
58. Kushner H. Jump-diffusions with controlled jumps: existence and numerical methods. *J Math Anal Appl*. 2000;249:179-198.
59. Crandall MG, Ishii H, Lions P-L. User's guide to viscosity solutions of second order partial differential equations. *Bull Am Math Soc*. 1992;27:1-67.
60. Peng S, Zhu XH. The viability property of controlled jump diffusion processes. *Acta Math Sin*. 2008;24(8):1351-1368.

How to cite this article: Yang HJ, Moon J. A sufficient condition for optimal control problem of fully coupled forward-backward stochastic systems with jumps: A state-constrained control approach. *Optim Control Appl Meth*. 2022;1-36. doi: 10.1002/oca.2960

APPENDIX A. EQUIVALENT DEFINITIONS OF VISCOSITY SOLUTIONS

To prove Lemma 8, it is necessary to decompose the nonlocal integral operator into singular and nonsingular parts. This appendix introduces the equivalent form of the HJB equation in (12) as well as the equivalent definition of viscosity solutions.

Recall the integro-type HJB equation in (12):

$$\begin{cases} -\partial_t W(t,\mathbf{a}) + \mathbb{H}(t,\mathbf{a},(W,DW,D^2W)(t,\mathbf{a})) = 0, \ (t,\mathbf{a}) \in \mathcal{O} \\ W(T,\mathbf{a}) = \max\{m(a)-d,0\} + \rho^2(a,b), \ (a,b,d) \in \mathbb{R}^{n+m} \times (0,\infty) \\ W(t,a,b,0) = W_0(t,a,b), \ (t,a,b) \in [0,T) \times \mathbb{R}^{n+m}, \end{cases} \tag{A1}$$

where $DW \in \mathbb{R}^{n+m+1}$ and $D^2W \in \mathbb{S}^{n+m+1}$ can be decomposed by

$$DW = \begin{bmatrix} D_aW \\ D_bW \\ D_dW \end{bmatrix}, \ D^2W = \begin{bmatrix} D^2_{aa}W & D^2_{ab}W & D^2_{ad}W \\ D^2_{ab}W^\top & D^2_{bb}W & D^2_{bd}W \\ D^2_{ad}W^\top & D^2_{bd}W & D^2_{dd}W \end{bmatrix} \tag{A2}$$

with $D_aW \in \mathbb{R}^n, D_bW \in \mathbb{R}^m, D_dW \in \mathbb{R}, D^2_{aa}W \in \mathbb{S}^n, D^2_{ab} \in \mathbb{R}^{n\times m}, D^2_{ad}W \in \mathbb{R}^n, D^2_{bb}W \in \mathbb{S}^m, D^2_{bd}W \in \mathbb{R}^m$, and $D^2_{dd}W \in \mathbb{R}$. Then the HJB equation in (A1) is equivalent to

$$\begin{cases} \sup\limits_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \left\{ \sup\limits_{\alpha\in\mathbb{R}^p} \mathbb{H}^{(1)}(t,\mathbf{a},(\partial_t W, DW, DW^2)(t,\mathbf{a}); u,z,k,\alpha) + \sup\limits_{\beta\in G^{(2)}} \mathbb{H}^{(2)}(t,\mathbf{a},(W,DW)(t,\mathbf{a}); u,z,k,\beta) \right\} = 0, \ (t,\mathbf{a}) \in \mathcal{O} \\ W(T,\mathbf{a}) = \max\{m(a)-d,0\} + \rho^2(a,b), \ (a,b,d) \in \mathbb{R}^{n+m} \times (0,\infty) \\ W(t,a,b,0) = W_0(t,a,b), \ (t,a,b) \in [0,T) \times \mathbb{R}^{n+m}, \end{cases} \tag{A3}$$

where

$$\begin{aligned} \mathbb{H}^{(1)}(t,\mathbf{a},(\partial_t W, DW, D^2W)(t,\mathbf{a}); u,z,k,\alpha) := & -\partial_t W(t,\mathbf{a}) - \langle D_aW(t,\mathbf{a}), f(u,z,k)\rangle - \langle D_bW(t,\mathbf{a}), -g(u,z,k)\rangle \\ & - \langle D_dW(t,\mathbf{a}), -l(u,z,k)\rangle - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(u,z,k)D^2_{aa}W(t,\mathbf{a})) - \frac{1}{2}\mathrm{Tr}(z^\top z D^2_{bb}W(t,\mathbf{a})) \\ & - \frac{1}{2}D^2_{dd}W(t,\mathbf{a})\alpha^\top\alpha - \mathrm{Tr}(z^\top\sigma(u,z,k)^\top D^2_{ab}W(t,\mathbf{a})) \\ & - \langle \alpha, \sigma(u,z,k)^\top D^2_{ad}W(t,\mathbf{a})\rangle - \langle \alpha, zD^2_{bd}W(t,\mathbf{a})\rangle, \end{aligned}$$

and

$$\mathbb{H}^{(2)}(t,\mathbf{a},(W,DW,D^2W)(t,\mathbf{a});u,z,k,\beta) := -\int_E [W(t,\mathbf{a}+\chi(e,u,z,k(e),\beta(e))) - W(t,\mathbf{a}) - \langle DW(t,\mathbf{a}),\chi(e,u,z,k(e),\beta(e))\rangle]\lambda(\mathrm{d}e).$$

Let (for simplicity $(t,\mathbf{a})$ is omitted)

$$\hat{x} := -\partial_t W - \langle D_a W, f(u,z,k)\rangle - \langle D_b W, -g(u,z,k)\rangle - \langle D_d W(t,\mathbf{a}), -l(u,z,k)\rangle - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top(u,z,k)D^2_{aa}W) - \frac{1}{2}\mathrm{Tr}(z^\top z D^2_{bb}W) - \mathrm{Tr}(z^\top \sigma(u,z,k)^\top D^2_{ab}W)$$

$$\hat{S} := -\frac{1}{2}\sigma(u,z,k)^\top D^2_{ad}W = \begin{bmatrix}\hat{s}_1 & \cdots & \hat{s}_p\end{bmatrix}^\top,\ \tilde{s} := -\frac{1}{2}D^2_{dd}W,\ \hat{B} := -\frac{1}{2}zD^2_{bd}W = \begin{bmatrix}\hat{b}_1 & \cdots & \hat{b}_p\end{bmatrix}^\top.$$

We define

$$Q_\psi(t,\hat{a};\partial_t W,DW,D^2W;u,z,k) := \begin{bmatrix}\hat{x} & \psi(d)(\hat{S}+\hat{B})^\top \\ \psi(d)(\hat{S}+\hat{B}) & \psi^2(d)\tilde{s}I_p\end{bmatrix},$$

where $\psi : [0,\infty) \to (0,\infty)$ are continuous nondecreasing functions. Note that $Q_\psi \in \mathbb{S}^{p+1}$.

Remark 22. In the proof of Lemma 8, $\psi(d) := \frac{1}{2}e^{\frac{1}{2}d} > 0$ for $d \in [0,\infty)$.

Lemma 9. $\mathbb{H}^{(1)}$ *can be expressed by*

$$\sup_{\alpha\in\mathbb{R}^p}\mathbb{H}^{(1)}(t,\mathbf{a},(\partial_t W,DW,D^2W);u,z,k,\alpha) \le 0 \ \Leftrightarrow\ \Lambda^+(Q_\psi(t,\mathbf{a};(\partial_t W,DW,D^2W);u,z,k)) \le 0,$$

where $\Lambda^+(A) := \sup_{|v|=1}|Av|$, *that is, the largest eigenvalue of* $A \in \mathbb{S}^n$.

Proof. We first consider the situation with $\psi = 1$. It follows that

$$\begin{aligned}\sup_{\alpha\in\mathbb{R}^p}\mathbb{H}^{(1)}(t,\mathbf{a},(\partial_t W,DW,D^2W);u,z,k,\alpha) \le 0 \ &\Leftrightarrow \sup_{\alpha\in\mathbb{R}^p}\left\{\hat{x} + 2\langle\alpha,\hat{S}\rangle + 2\langle\alpha,\hat{B}\rangle + \tilde{s}|\alpha|^2\right\}\\ &= \sup_{\hat{\alpha}\in\mathbb{R}^{p+1},\hat{\alpha}_1\neq 0}\left\{\hat{x} + 2\sum_{i=1}^p\frac{\hat{\alpha}_{i+1}}{\hat{\alpha}_1}\hat{s}_i + 2\sum_{i=1}^p\frac{\hat{\alpha}_{i+1}}{\hat{\alpha}_1}\hat{b}_i + \tilde{s}\sum_{i=1}^p\frac{\hat{\alpha}^2_{i+1}}{\hat{\alpha}^2_1}\right\}\\ &= \sup_{\hat{\alpha}\in\mathbb{R}^{p+1},\hat{\alpha}_1\neq 0,|\hat{\alpha}|=1}\left\{\hat{\alpha}_1^2\hat{x} + 2\sum_{i=1}^p\hat{\alpha}_1\hat{\alpha}_{i+1}\hat{s}_i + 2\sum_{i=1}^p\hat{\alpha}_1\hat{\alpha}_{i+1}\hat{b}_i + \tilde{s}\sum_{i=1}^p\hat{\alpha}^2_{i+1}\right\}\\ &= \sup_{\hat{\alpha}\in\mathbb{R}^{p+1},\hat{\alpha}_1\neq 0,|\hat{\alpha}|=1}\hat{\alpha}^\top Q_1(t,\mathbf{a};\partial_t W,DW,D^2W;u,z,k)\hat{\alpha}\\ &= \Lambda^+(Q_1(t,\mathbf{a};\partial_t W,DW,D^2W;u,z,k)) \le 0.\end{aligned}$$

Hence, the result holds when $\psi = 1$. Since $\psi(d) > 0$ for $d \in [0,\infty)$ (equivalently, $\psi : [0,\infty) \to (0,\infty)$) and $\hat{x}$ is independent of α, using a similar approach as above, we can show that

$$\begin{aligned}\Lambda^+(Q_1(t,\mathbf{a};\partial_t W,DW,D^2W;u,z,k)) \le 0 \ &\Leftrightarrow \sup_{\alpha\in\mathbb{R}^p}\left\{\hat{x} + 2\langle\alpha,\hat{S}\rangle + 2\langle\alpha,\hat{B}\rangle + \tilde{s}|\alpha|^2\right\} \le 0\\ &\Leftrightarrow \sup_{\hat{\alpha}\in\mathbb{R}^{p+1},\frac{\hat{\alpha}_1}{\psi(d)}\neq 0}\\ &\left\{\hat{x} + 2\sum_{i=1}^p\frac{\psi(d)\hat{\alpha}_{i+1}}{\hat{\alpha}_1}\hat{s}_i + 2\sum_{i=1}^p\frac{\psi(d)\hat{\alpha}_{i+1}}{\hat{\alpha}_1}\hat{b}_i + \tilde{s}\sum_{i=1}^p\frac{\psi^2(d)\hat{\alpha}^2_{i+1}}{\hat{\alpha}^2_1}\right\} \le 0\\ &\Leftrightarrow \sup_{\hat{\alpha}\in\mathbb{R}^{p+1},\frac{\hat{\alpha}_1}{\psi(d)}\neq 0,|\hat{\alpha}|=1}\end{aligned}$$

$$\left\{ \hat{\alpha}_1^2\hat{x} + 2\sum_{i=1}^{p}\psi(d)\hat{\alpha}_1\hat{\alpha}_{i+1}\hat{s}_i + 2\sum_{i=1}^{p}\psi(d)\hat{\alpha}_1\hat{\alpha}_{i+1}\hat{b}_i + \tilde{s}\sum_{i=1}^{p}\psi^2(d)\hat{\alpha}_{i+1}^2 \right\} \le 0$$
$$\Leftrightarrow \sup_{\hat{\alpha}\in\mathbb{R}^{p+1},\frac{\hat{\alpha}_1}{\psi(d)}\neq 0,|\hat{\alpha}|=1} \hat{\alpha}^\top Q_\psi(t,\mathbf{a};\partial_t W, DW, D^2W;u,z,k)\hat{\alpha} \le 0$$
$$\Leftrightarrow \Lambda^+(Q_\psi(t,\mathbf{a};(\partial_t W, DW, D^2W);u,z,k)) \le 0.$$

This completes the proof. ∎

We decompose of $\mathbb{H}^{(2)}$ in (A3) into singular and nonsingular parts. Specifically, for $\kappa > 0$, let $E_\delta := \{e \in E \mid |e| < \kappa\}$; hence, $E = E_\kappa \cup E_\kappa^C$. We then define

$$\mathbb{H}^{(2)}(t,\mathbf{a},(W,DW)(t,\mathbf{a});u,z,k,\beta) := \mathbb{H}_\kappa^{(21)}(t,\mathbf{a},(W,DW)(t,\mathbf{a});u,z,k,\beta) + \mathbb{H}_\kappa^{(22)}(t,\mathbf{a},(W,DW)(t,\mathbf{a});u,z,k,\beta),$$

where

$$\mathbb{H}_\kappa^{(21)}(t,\mathbf{a},(W,DW)(t,\mathbf{a});u,z,k,\beta) := -\int_{E_\kappa}\left[W(t,\mathbf{a}+\chi(e,u,z,k(e),\beta(e))) - W(t,\mathbf{a})\right]\lambda(\mathrm{d}e) + \int_{E_\kappa}\langle DW(t,\mathbf{a}), \chi(e,u,z,k(e),\beta(e))\rangle\lambda(\mathrm{d}e),$$

and

$$\mathbb{H}_\kappa^{(22)}(t,\mathbf{a},(W,DW)(t,\mathbf{a});u,z,k,\beta) := -\int_{E_\kappa^C}[W(t,\mathbf{a}+\chi(e,u,z,k(e),\beta(e))) - W(t,\mathbf{a})]\lambda(\mathrm{d}e) + \int_{E_\kappa^C}\langle DW(t,\mathbf{a}), \chi(e,u,z,k(e),\beta(e))\rangle\lambda(\mathrm{d}e),$$

By Lemma 9 and the above decomposition, we rewrite the HJB equation in (A3) as follows:

$$\begin{cases} \sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \{\Lambda^+(Q_\psi(t,\mathbf{a};(\partial_t W, DW, D^2W)(t,\mathbf{a});u,z,k)) \\ \quad + \sup_{\beta\in G^{(2)}}\{\mathbb{H}_\kappa^{(21)}(t,\mathbf{a},(W,DW)(t,\mathbf{a});u,z,k,\beta) + \mathbb{H}_\kappa^{(22)}(t,\mathbf{a},(W,DW)(t,\mathbf{a});u,z,k,\beta)\}\} = 0, \ (t,\mathbf{a}) \in \mathcal{O} \\ W(T,\mathbf{a}) = \max\{m(a) - d, 0\} + \rho^2(a,b), \ (a,b,d) \in \mathbb{R}^{n+m}\times(0,\infty) \\ W(t,a,b,0) = W_0(t,a,b), \ (t,a,b) \in [0,T)\times\mathbb{R}^{n+m}. \end{cases} \tag{A4}$$

Remark 23. In the proof of Lemma 8, we use the equivalent HJB equation in (A4).

Based on References 53,54 (see proposition 1 of Reference 54), the first equivalent definition of Definition 1 is as follows:

Lemma 10. *Suppose that W is a viscosity subsolution (resp. supersolution) of the HJB equation in (A4). Then,*

(i) $W(T,\mathbf{a}) \le \max\{m(a)-d,0\} + \rho^2(a,b)$ *(resp.* $W(T,\mathbf{a}) \ge \max\{m(a)-d,0\} + \rho^2(a,b)$*) for* $(a,b,d) \in \mathbb{R}^{n+m}\times(0,\infty)$ *and* $W(t,a,b,0) \le W_0(t,a,b)$ *(resp.* $W(t,a,b,0) \ge W_0(t,a,b)$*) for* $(t,a,b) \in [0,T)\times\mathbb{R}^{n+m}$;

(ii) For all $\kappa \in (0,1)$ *and test functions* $\phi \in C_b^{1,3}(\overline{\mathcal{O}}) \cap C_2(\overline{\mathcal{O}})$*, the following inequality holds at the global maximum (resp. minimum) point* $(t,\mathbf{a}) \in \mathcal{O}$ *of* $W - \phi$:

$$\sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \{\Lambda^+(Q_\psi(t,\mathbf{a};(\partial_t\phi, D\phi, D^2\phi)(t,\mathbf{a});u,z,k)) + \sup_{\beta\in G^{(2)}}\{\mathbb{H}_\kappa^{(21)}(t,\mathbf{a},(\phi,D\phi)(t,\mathbf{a});u,z,k,\beta) + \mathbb{H}_\kappa^{(22)}(t,\mathbf{a},(W,D\phi)(t,\mathbf{a});u,z,k,\beta)\}\} \le 0$$
$$(\text{resp. } \sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \{\Lambda^+(Q_\psi(t,\mathbf{a};(\partial_t\phi, D\phi, D^2\phi)(t,\mathbf{a});u,z,k)) + \sup_{\beta\in G^{(2)}}\{\mathbb{H}_\kappa^{(21)}(t,\mathbf{a},(\phi,D\phi)(t,\mathbf{a});u,z,k,\beta) + \mathbb{H}_\kappa^{(22)}(t,\mathbf{a},(W,D\phi)(t,\mathbf{a});u,z,k,\beta)\}\} \ge 0).$$

The definition of parabolic superjet and subjet is given as follows:[59]

Definition 2.

(i) For W, the superjet of W at the point of $(t,\mathbf{a}) \in \mathcal{O}$ is defined by

$$\mathcal{P}^{1,2,+}W(t,\mathbf{a}) := \{(q,p,P) \in \mathbb{R}\times\mathbb{R}^{n+m+1}\times\mathbb{S}^{n+m+1} \mid W(t',\mathbf{a}') \leq W(t,\mathbf{a}) + q(t'-t) + \langle p, \mathbf{a}'-\mathbf{a}\rangle + \frac{1}{2}\langle P(\mathbf{a}'-\mathbf{a}), \mathbf{a}'-\mathbf{a}\rangle + o(|t'-t| + |\mathbf{a}'-\mathbf{a}|^2)\}.$$

1. The closure of $\mathcal{P}^{1,2,+}W(t,\mathbf{a})$ is defined by

$$\overline{\mathcal{P}}^{1,2,+}W(t,\mathbf{a}) := \{(q,p,P) \in \mathbb{R}\times\mathbb{R}^{n+m+1}\times\mathbb{S}^{n+m+1} \mid (q,p,P) = \lim_{n\to\infty}(q_n,p_n,P_n) \text{ with } (q_n,p_n,P_n) \in \mathcal{P}^{1,2,+}W(t_n,\mathbf{a}_n) \text{ and } \lim_{n\to\infty}(t_n,\mathbf{a}_n,W(t_n,\mathbf{a}_n)) = (t,\mathbf{a},W(t,\mathbf{a}))\}.$$

2. For W, the subjet of W at the point of $(t,\mathbf{a}) \in \mathcal{O}$ and its closure are defined by

$$\mathcal{P}^{1,2,-}W(t,\mathbf{a}) := -\mathcal{P}^{1,2,+}(-W(t,\mathbf{a})),\ \overline{\mathcal{P}}^{1,2,-}W(t,\mathbf{a}) := -\overline{\mathcal{P}}^{1,2,+}(-W(t,\mathbf{a})).$$

In view of Definition 2 and Lemma 10, the second equivalent definition of Definition 1 can be stated as follows (see also lemma 3.5 of Reference 60 and proposition 1 of Reference 54):

Lemma 11. *Suppose that W is a viscosity subsolution (resp. supersolution) of the HJB equation in (A4). Then,*

(i) $W(T,\mathbf{a}) \leq \max\{m(a)-d,0\} + \rho^2(a,b)$ (resp. $W(T,\mathbf{a}) \geq \max\{m(a)-d,0\} + \rho^2(a,b)$) for $(a,b,d) \in \mathbb{R}^{n+m}\times(0,\infty)$ and $W(t,a,b,0) \leq W_0(t,a,b)$ (resp. $W(t,a,b,0) \geq W_0(t,a,b)$) for $(t,a,b) \in [0,T)\times\mathbb{R}^{n+m}$;
(ii) For all $\kappa \in (0,1)$ and test functions $\phi \in C_b^{1,3}(\overline{\mathcal{O}}) \cap C_2(\overline{\mathcal{O}})$ with the local maximum (resp. minimum) point $(t,\mathbf{a}) \in \mathcal{O}$ of $W-\phi$, if $(q,p,P) \in \overline{\mathcal{P}}^{1,2,+}W(t,\mathbf{a})$ (resp. $(q,p,P) \in \overline{\mathcal{P}}^{1,2,-}W(t,\mathbf{a})$) with $p = D\phi(t,\mathbf{a})$ and $P = D^2\phi(t,\mathbf{a})$, then the following inequality holds:

$$\sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}}\{\Lambda^+(Q_\psi(t,\mathbf{a};(q,p,P);u,z,k)) + \sup_{\beta\in G^{(2)}}\{\mathbb{H}_\kappa^{(21)}(t,\mathbf{a},(\phi,D\phi)(t,\mathbf{a});u,z,k,\beta) + \mathbb{H}_\kappa^{(22)}(t,\mathbf{a},W(t,\mathbf{a}),p;u,z,k,\beta)\}\} \leq 0$$

$$(\text{resp.} \sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}}\{\Lambda^+(Q_\psi(t,\mathbf{a};(q,p,P);u,z,k)) + \sup_{\beta\in G^{(2)}}\{\mathbb{H}_\kappa^{(21)}(t,\mathbf{a},(\phi,D\phi)(t,\mathbf{a});u,z,k,\beta) + \mathbb{H}_\kappa^{(22)}(t,\mathbf{a},W(t,\mathbf{a}),p;u,z,k,\beta)\}\} \geq 0).$$

Remark 24. Lemma 11 is required due to the presence of the singularity of the Lévy measure in zero, appearing in the nonlocal operator $\mathbb{H}_\kappa^{(21)}$. In the proof of Lemma 8, we will use the regularity of the test function to pass the limit of $\mathbb{H}_\kappa^{(21)}$ with respect to κ around the singular point of the measure.

APPENDIX B. STRICT VISCOSITY SUBSOLUTION

Lemma 12. *Suppose that $\underline{W}$ is the viscosity subsolution of (A4). Define*

$$\underline{W}_\nu(t,\mathbf{a}) := \underline{W}(t,\mathbf{a}) + \nu\gamma(t,d),$$

where for $\nu > 0$, $\gamma(t,d) := -(T-t) - (1-e^{-d})$. Then $\underline{W}_\nu$ is the strict viscosity subsolution of (A4) in the sense that ≤ 0 is replaced by $\leq -\frac{\nu}{8}$ in Definition 1 (equivalently Lemma 11).

Proof. Notice that $\underline{W}_\nu(T,\mathbf{a}) = \underline{W}(T,\mathbf{a}) - \nu\log(1+d) \leq \max\{m(a)-d,0\} + \rho^2(a,b)$ and $\underline{W}_\nu(t,a,b,0) = \underline{W}(t,a,b,0) - \nu(T-t) \leq W_0(t,a,b)$. Hence, $\underline{W}_\nu$ satisfies the boundary inequalities of the viscosity subsolution.

Let $\phi_\nu \in C_b^{1,3}(\overline{\mathcal{O}})$ be the test function such that

$$(\underline{W}_\nu - \phi_\nu)(t, \mathbf{a}) = \max_{(t',\mathbf{a}')\in\mathcal{O}} (\underline{W}_\nu - \phi_\nu)(t', \mathbf{a}').$$

It is necessary to show that

$$\begin{aligned}
&\sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \{\Lambda^+(Q_\psi(t,\mathbf{a};(\partial_t\phi_\nu, D\phi_\nu, D^2\phi_\nu)(t,\mathbf{a});u,z,k)) \\
&\quad + \sup_{\beta\in G^{(2)}} \{\mathbb{H}_\kappa^{(21)}(t,\mathbf{a},(\phi_\nu, D\phi_\nu)(t,\mathbf{a});u,z,k,\beta) + \mathbb{H}_\kappa^{(22)}(t,\mathbf{a},(\phi_\nu, D\phi_\nu)(t,\mathbf{a});u,z,k,\beta)\}\} \leq -\frac{\nu}{8}.
\end{aligned} \tag{B1}$$

We define $\underline{\phi}(t,\mathbf{a}) := -\nu\gamma(t,d) + \phi_\nu(t,\mathbf{a})$, where it can be seen that $\underline{\phi} \in C_b^{1,3}(\overline{\mathcal{O}})$. Notice that $(\underline{W}_\nu - \phi_\nu)(t,\mathbf{a}) = \underline{W}(t,\mathbf{a}) - (\nu\gamma(t,d) + \phi_\nu(t,\hat{a})) = (\underline{W} - \underline{\phi})(t,\mathbf{a})$. This implies that

$$\max_{(t',\mathbf{a}')\in\mathcal{O}} (\underline{W}_\nu - \phi_\nu)(t',\mathbf{a}') = (\underline{W}_\nu - \phi_\nu)(t,\mathbf{a}) = (\underline{W} - \underline{\phi})(t,\mathbf{a}) = \max_{(t',\mathbf{a}')\in\mathcal{O}} (\underline{W} - \underline{\phi})(t',\mathbf{a}'). \tag{B2}$$

By the linearity of $\mathbb{H}^{(2)}$ in $D\phi_\mu$ and the triangular inequality property of Λ^+, it follows that

$$\begin{aligned}
&\sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \{\Lambda^+(Q_\psi(t,\mathbf{a};(\partial_t\phi_\nu, D\phi_\nu, D^2\phi_\nu)(t,\mathbf{a});u,z,k)) \\
&\quad + \sup_{\beta\in G^{(2)}} \{\mathbb{H}_\kappa^{(21)}(t,\mathbf{a},(\phi_\nu, D\phi_\nu)(t,\mathbf{a});u,z,k,\beta) + \mathbb{H}_\kappa^{(22)}(t,\mathbf{a},(\phi_\nu, D\phi_\nu)(t,\mathbf{a});u,z,k,\beta)\}\} \\
&\leq F^{(1)} + F^{(2)},
\end{aligned}$$

where

$$\begin{aligned}
F^{(1)} :=& \sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \{\Lambda^+(Q_\psi(t,\mathbf{a};(\partial_t\underline{\phi}, D\underline{\phi}, D^2\underline{\phi})(t,\mathbf{a});u,z,k)) + \sup_{\beta\in G^{(2)}} \{\mathbb{H}_\kappa^{(21)}(t,\mathbf{a},(\underline{\phi}, D\underline{\phi})(t,\mathbf{a});u,z,k,\beta) \\
&+ \mathbb{H}_\kappa^{(22)}(t,\mathbf{a},(\underline{\phi}, D\underline{\phi})(t,\mathbf{a});u,z,k,\beta)\}\} \\
F^{(2)} :=& \nu \sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \{\Lambda^+(Q_\psi(t,\mathbf{a};(\partial_t\gamma, D\gamma, D^2\gamma)(t,\mathbf{a});u,z,k)) + \sup_{\beta\in G^{(2)}} \\
&\{\mathbb{H}_\kappa^{(21)}(t,\mathbf{a},(\gamma, D\gamma)(t,\mathbf{a});u,z,k,\beta) + \mathbb{H}_\kappa^{(22)}(t,\mathbf{a},(\gamma, D\gamma)(t,\mathbf{a});u,z,k,\beta)\}\}.
\end{aligned}$$

Since $\underline{W}$ is the viscosity subsolution of (A4) and $\underline{\phi}$ is the corresponding test function by (B2), we have

$$F^{(1)} \leq 0. \tag{B3}$$

Regarding $F^{(2)}$, since $d \in [0,\infty)$, we can show that for any $(u,z,k) \in U \times \mathbb{R}^{p\times m} \times G^{(1)}$,

$$\begin{aligned}
&\sup_{\beta\in G^{(2)}} \mathbb{H}_\kappa^{(21)}(t,\mathbf{a},(\gamma,D\gamma)(t,\mathbf{a});u,z,k,\beta) = \sup_{\beta\in G^{(2)}} \int_{E_\kappa} \left[(1-e^{-(d+\beta(e))}) - (1-e^{-d}) - e^{-d}\beta(e)\right] \lambda(\mathrm{d}e) \leq 0 \\
&\sup_{\beta\in G^{(2)}} \mathbb{H}_\kappa^{(22)}(t,\mathbf{a},(\gamma,D\gamma)(t,\mathbf{a});u,z,k,\beta) = \sup_{\beta\in G^{(2)}} \int_{E_\kappa^C} \left[(1-e^{-(d+\beta(e))}) - (1-e^{-d}) - e^{-d}\beta(e)\right] \lambda(\mathrm{d}e) \leq 0.
\end{aligned}$$

Moreover, recall $\psi(d) = \frac{1}{2}e^{\frac{1}{2}d}$ in Remark 22. By definition of Q_ψ,

$$Q_\psi(t,\mathbf{a};(\partial_t\gamma, D\gamma, D^2\gamma)(t,d);u,z,k) = \begin{bmatrix} -1-e^{-d}l(u,z,k) & 0 \\ 0 & -\frac{1}{8}I_p \end{bmatrix}.$$

Since l is nonnegative and $d \in [0,\infty)$, we have $-1-e^{-d}l(u,z,k) \leq -1$. Hence, we have

$$F^{(2)} \leq -\frac{1}{8}\nu. \tag{B4}$$

(B3) and (B4) lead to the desired result; thus completing the proof. ■

APPENDIX C. ESTIMATE OF $\Upsilon^{(1)}$

By linearity of $\mathcal{Q}_\psi$, we denote

$$\mathcal{Q}_\psi(t', \breve{\mathbf{a}}', (-\breve{q} - \partial_t \breve{\delta}'_{\eta,\lambda}, -D_{\breve{\mathbf{a}}}(\breve{\delta}'_{\eta,\lambda} + \theta'_\epsilon), -\breve{P} - D^2_{\breve{\mathbf{a}}\breve{\mathbf{a}}}\breve{\delta}'_{\eta,\lambda}); u, z, k) = \breve{\mathcal{Q}}^{(1)}_\psi + \breve{\mathcal{Q}}^{(2)}_\psi + \breve{\mathcal{Q}}^{(3)}_\psi,$$

where

$$\begin{aligned}
\breve{\mathcal{Q}}^{(1)}_\psi &:= \mathcal{Q}_\psi\left(t', \breve{\mathbf{a}}', \left(-\breve{q} - \frac{1}{2}\partial_t \breve{\delta}'_{\eta,\lambda}, -D_{\breve{\mathbf{a}}}(\breve{\delta}'_{\eta,\lambda} + \theta'_\epsilon), 0); u, z, k\right)\right. \\
\breve{\mathcal{Q}}^{(2)}_\psi &:= \mathcal{Q}_\psi(t', \breve{\mathbf{a}}', (0, 0, -\breve{P}); u, z, k) \\
\breve{\mathcal{Q}}^{(3)}_\psi &:= \mathcal{Q}_\psi\left(t', \breve{\mathbf{a}}', \left(-\frac{1}{2}\partial_t \breve{\delta}'_{\eta,\lambda}, 0, -D^2_{\breve{\mathbf{a}}\breve{\mathbf{a}}}\breve{\delta}'_{\eta,\lambda}\right); u, z, k\right).
\end{aligned}$$

Similarly,

$$\mathcal{Q}_\psi(t', \mathbf{a}', (q + \partial_t \delta'_{\eta,\lambda}, D_{\mathbf{a}}(\delta'_{\eta,\lambda} + \theta'_\epsilon), P + D^2_{\mathbf{a}\mathbf{a}}\delta'_{\eta,\lambda}); u, z, k) = \mathcal{Q}^{(1)}_\psi + \mathcal{Q}^{(2)}_\psi + \mathcal{Q}^{(3)}_\psi,$$

where

$$\begin{aligned}
\mathcal{Q}^{(1)}_\psi &:= \mathcal{Q}_\psi\left(t', \mathbf{a}', \left(q + \frac{1}{2}\partial_t \delta'_{\eta,\lambda}, D_{\mathbf{a}}(\delta'_{\eta,\lambda} + \theta'_\epsilon), 0\right); u, z, k\right) \\
\mathcal{Q}^{(2)}_\psi &:= \mathcal{Q}_\psi(t', \mathbf{a}', (0, 0, P); u, z, k) \\
\mathcal{Q}^{(3)}_\psi &:= \mathcal{Q}_\psi\left(t', \mathbf{a}', \left(\frac{1}{2}\partial_t \delta'_{\eta,\lambda}, 0, D^2_{\mathbf{a}\mathbf{a}}\delta'_{\eta,\lambda}\right); u, z, k\right).
\end{aligned}$$

We then have

$$\Upsilon^{(1)} = \sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \left\{\Lambda^+(\breve{\mathcal{Q}}^{(1)}_\psi + \breve{\mathcal{Q}}^{(2)}_\psi + \breve{\mathcal{Q}}^{(3)}_\psi) - \Lambda^+(\mathcal{Q}^{(1)}_\psi + \mathcal{Q}^{(2)}_\psi + \mathcal{Q}^{(3)}_\psi)\right\} \leq \Upsilon^{(11)} + \Upsilon^{(12)} + \Upsilon^{(13)},$$

where

$$\begin{aligned}
\Upsilon^{(11)} &:= \sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \Lambda^+(\breve{\mathcal{Q}}^{(1)}_\psi - \mathcal{Q}^{(1)}_\psi) \\
\Upsilon^{(12)} &:= \sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \Lambda^+(\breve{\mathcal{Q}}^{(2)}_\psi - \mathcal{Q}^{(2)}_\psi) \\
\Upsilon^{(13)} &:= \sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \Lambda^+(\breve{\mathcal{Q}}^{(3)}_\psi - \mathcal{Q}^{(3)}_\psi).
\end{aligned}$$

In Appendices C.0.1–C.0.3 below, it is shown that for $\lambda \geq \max\{C_1, C_3 + C_4\}$, where C_1, C_3, and C_4 are defined in Appendices C.0.1–C.0.3,

$$\lim_{\eta\downarrow 0}\lim_{\epsilon\downarrow 0}\lim_{\kappa\downarrow 0}\Upsilon^{(1)} \leq \lim_{\eta\downarrow 0}\lim_{\epsilon\downarrow 0}\lim_{\kappa\downarrow 0}\{\Upsilon^{(11)} + \Upsilon^{(12)} + \Upsilon^{(13)}\} \leq 0. \tag{C1}$$

C.0.1 Estimate of $\Upsilon^{(11)}$

By definition of $\mathcal{Q}_\psi$ and Λ^+ and noting that $\breve{q} + q = 0$ from (41),

$$\Upsilon^{(11)} = \sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \max\left\{\frac{1}{2}\partial_t(\breve{\delta}'_{\eta,\lambda} + \delta'_{\eta,\lambda}) + \langle D_{\breve{\mathbf{a}}}(\breve{\delta}'_{\eta,\lambda} + \theta'_\epsilon), \breve{\mathbf{f}}'(u, z, k)\rangle + \langle D_{\mathbf{a}}(\delta'_{\eta,\lambda} + \theta'_\epsilon), \mathbf{f}'(u, z, k)\rangle, 0\right\}$$

where $\mathbf{f}'(u, z, k) := \mathbf{f}(t, a', u, b', z, k)$ and $\breve{\mathbf{f}}'(u, z, k) := \mathbf{f}(t, \breve{a}', u, \breve{b}', z, k)$ (see the notation in Section 3).

Using the derivatives given in Appendix F, it follows that (note that $(t', \mathbf{a}', \breve{\mathbf{a}}') := (t'_\epsilon, \mathbf{a}'_\epsilon, \breve{\mathbf{a}}'_\epsilon)$, that is, $(t', a', b', d', \breve{a}', \breve{b}', \breve{d}') := (t'_\epsilon, a'_\epsilon, b'_\epsilon, d'_\epsilon, \breve{a}'_\epsilon, \breve{b}'_\epsilon, \breve{d}'_\epsilon)$; see the statement below (39))

$$\begin{aligned}\frac{1}{2}\partial_t(\breve{\delta}'_{\eta,\lambda} + \delta'_{\eta,\lambda}) &= -\frac{\eta}{2}\lambda e^{-\lambda t'}(1 + |\breve{a}'|^2 + |\breve{b}'|^2 + \breve{d}') - \frac{\eta}{2}\lambda e^{-\lambda t'}(1 + |a'|^2 + |b'|^2 + d') \\ &\quad + (t' - \tilde{t}) - \frac{\eta\lambda e^{-\lambda t'}}{4}\left(|a' - \tilde{a}|^2 + |b' - \tilde{b}|^2 + (d' - \tilde{d})\right) \\ &\to -\eta\lambda e^{-\lambda\tilde{t}}(1 + |\tilde{a}|^2 + |\tilde{b}|^2 + \tilde{d}) \text{ as } \epsilon \downarrow 0 \text{ due to (39).}\end{aligned} \tag{C2}$$

Moreover, from Appendix F and Assumption 2 and using Cauchy-Schwarz inequality (note that $(t', \mathbf{a}', \breve{\mathbf{a}}') := (t'_\epsilon, \mathbf{a}'_\epsilon, \breve{\mathbf{a}}'_\epsilon)$, i.e., $(t', a', b', d', \breve{a}', \breve{b}', \breve{d}') := (t'_\epsilon, a'_\epsilon, b'_\epsilon, d'_\epsilon, \breve{a}'_\epsilon, \breve{b}'_\epsilon, \breve{d}'_\epsilon)$; see the statement below (39)),

$$\begin{aligned}&\langle D_{\breve{\mathbf{a}}}(\breve{\delta}'_{\eta,\lambda} + \theta'_\epsilon), \breve{\mathbf{f}}'(u,z,k)\rangle + \langle D_{\mathbf{a}}(\delta'_{\eta,\lambda} + \theta'_\epsilon), \mathbf{f}'(u,z,k)\rangle \\ &\quad \le (|2\eta e^{-\lambda t'}\breve{a}' - \frac{1}{\epsilon}(a' - \breve{a}')| + |2\eta e^{-\lambda t'}\breve{b}' \\ &\quad\quad - \frac{1}{\epsilon}(b' - \breve{b}')| + |\eta e^{-\lambda t'} - \frac{1}{\epsilon}(d' - \breve{d}')|)(1 + |\breve{a}'| + |\breve{b}'|) \\ &\quad\quad + (|2\eta e^{-\lambda t'}a' + \eta e^{-\lambda t'}(a' - \tilde{a}') + \frac{1}{\epsilon}(a' - \breve{a}')| + |2\eta e^{-\lambda t'}b' \\ &\quad\quad + \eta e^{-\lambda t'}(b' - \tilde{b}') + \frac{1}{\epsilon}(b' - \breve{b}')| + |\frac{3}{2}\eta e^{-\lambda t'} \\ &\quad\quad + \frac{1}{\epsilon}(d' - \breve{d}')|)(1 + |a'| + |b'|) \to C_1\eta e^{-\lambda\tilde{t}}(1 + |\tilde{a}|^2 + |\tilde{b}|^2 + \tilde{d}) \text{ as } \epsilon \downarrow 0 \text{ due to (39).}\end{aligned}$$

Hence,

$$\lim_{\epsilon\downarrow 0}\Upsilon^{(11)} \le \max\{(-\lambda + C_1)\eta e^{-\lambda\tilde{t}}(1 + |\tilde{a}|^2 + |\tilde{b}|^2 + \tilde{d}), 0\},$$

and for any $\lambda \ge C_1$ with $\lambda > 0$, we have $\lim_{\epsilon\downarrow 0}\Upsilon^{(11)} \le 0$.

C.0.2 Estimate of $\Upsilon^{(12)}$

In (42), $P, \breve{P} \in \mathbb{S}^{n+m+1}$, where

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^\top & P_{22} & P_{23} \\ P_{13}^\top & P_{23}^\top & P_{33}\end{bmatrix}, \ \breve{P} = \begin{bmatrix} \breve{P}_{11} & \breve{P}_{12} & \breve{P}_{13} \\ \breve{P}_{12}^\top & \breve{P}_{22} & \breve{P}_{23} \\ \breve{P}_{13}^\top & \breve{P}_{23}^\top & \breve{P}_{33}\end{bmatrix}.$$

Note that the dimension of P_{ij} and $\breve{P}_{ij}$ is the same as those of D^2W (see (A2) in Appendix A). Let $\sigma'(u,z,k) := \sigma(t', a', u, b', z, k)$ and $\breve{\sigma}'(u,z,k) := \sigma(t', \breve{a}', u, \breve{b}', z, k)$.

By definition of $\mathcal{Q}_\psi$,

$$\breve{\mathcal{Q}}_\psi^{(2)} = \begin{bmatrix}\breve{\mathcal{Q}}_\psi^{(2,11)} & (\breve{\mathcal{Q}}_\psi^{(2,12)})^\top \\ \breve{\mathcal{Q}}_\psi^{(2,12)} & \breve{\mathcal{Q}}_\psi^{(2,22)}\end{bmatrix}, \ \mathcal{Q}_\psi^{(2)} = \begin{bmatrix}\mathcal{Q}_\psi^{(2,11)} & (\mathcal{Q}_\psi^{(2,12)})^\top \\ \mathcal{Q}_\psi^{(2,12)} & \mathcal{Q}_\psi^{(2,22)}\end{bmatrix},$$

where

$$\begin{aligned}\breve{\mathcal{Q}}_\psi^{(2,11)} &:= \frac{1}{2}\mathrm{Tr}(\breve{\sigma}'(\breve{\sigma}')^\top(u,z,k)\breve{P}_{11}) + \frac{1}{2}\mathrm{Tr}(zz^\top\breve{P}_{22}) + \mathrm{Tr}(z^\top\breve{\sigma}'(u,z,k)^\top\breve{P}_{12}) \\ \breve{\mathcal{Q}}_\psi^{(2,12)} &:= \psi(\breve{d}')\frac{1}{2}\breve{\sigma}'(u,z,k)^\top\breve{P}_{13} + \psi(\breve{d}')\frac{1}{2}z\breve{P}_{23}, \ \breve{\mathcal{Q}}_\psi^{(2,22)} := \psi^2(\breve{d}')\frac{1}{2}\breve{P}_{33}I_p \\ \mathcal{Q}_\psi^{(2,11)} &:= -\frac{1}{2}\mathrm{Tr}(\sigma'(\sigma')^\top(u,z,k)P_{11}) - \frac{1}{2}\mathrm{Tr}(zz^\top P_{22}) - \mathrm{Tr}(z^\top\sigma'(u,z,k)^\top P_{12}) \\ \mathcal{Q}_\psi^{(2,12)} &:= -\psi(d')\frac{1}{2}\sigma'(u,z,k)^\top P_{13} - \psi(d')\frac{1}{2}zP_{23}, \ \mathcal{Q}_\psi^{(2,22)} := -\psi^2(d')\frac{1}{2}P_{33}I_p.\end{aligned}$$

We define

$$\Delta := \begin{bmatrix} \sigma'(u,z,k)^\top & 0_{p\times m} & 0_p \\ 0_{p\times n} & z & 0_p \\ 0_n^\top & 0_m^\top & \psi(d') \end{bmatrix}, \ \breve{\Delta} := \begin{bmatrix} \breve{\sigma}'(u,z,k)^\top & 0_{p\times m} & 0_p \\ 0_{p\times n} & z & 0_p \\ 0_n^\top & 0_m^\top & \psi(\breve{d}') \end{bmatrix}.$$

Note that $\Delta, \breve{\Delta} \in \mathbb{R}^{(2p+1)\times(n+m+1)}$. Then for any $r \in \mathbb{R}^{2p+1}$, it follows from (42) and Assumption 2 that

$$r^\top \begin{bmatrix} \Delta & \breve{\Delta} \end{bmatrix} \begin{bmatrix} P & 0_{n+m+1} \\ 0_{n+m+1} & \breve{P} \end{bmatrix} \begin{bmatrix} \Delta^\top \\ \breve{\Delta}^\top \end{bmatrix} r \leq \frac{3}{\epsilon} r^\top \begin{bmatrix} \Delta & \breve{\Delta} \end{bmatrix} \begin{bmatrix} I_{n+m+1} & -I_{n+m+1} \\ -I_{n+m+1} & I_{n+m+1} \end{bmatrix} \begin{bmatrix} \Delta^\top \\ \breve{\Delta}^\top \end{bmatrix} r \tag{C3}$$

$$\leq \frac{3}{\epsilon}\|\Delta - \breve{\Delta}\|_F^2 |r|^2 \leq C\frac{3}{\epsilon}(|a' - \breve{a}'|^2 + |b' - \breve{b}'|^2 + |d' - \breve{d}'|^2)|r|^2, \tag{C3}$$

where we use the Cauchy–Schwarz inequality to get the second inequality.

We define $r_{(i)}$, $i \in \{1, \dots, p\}$, by

$$r_{(i)} := \begin{bmatrix} \tilde{r}_{(i)}^\top & \tilde{r}_{(i)}^\top & r_i \end{bmatrix}^\top \in \mathbb{R}^{p+1}, \ \tilde{r}_{(i)} := \begin{bmatrix} 0 & \cdots 0 & \tilde{r} & 0 & \cdots & 0 \end{bmatrix}^\top_{i\text{th component}} \in \mathbb{R}^p, \ r_i \in \mathbb{R}.$$

Using (C3), it can be shown that

$$\begin{aligned} &\frac{1}{2} r_{(i)}^\top \begin{bmatrix} \Delta & \breve{\Delta} \end{bmatrix} \begin{bmatrix} P & 0_{n+m+1} \\ 0_{n+m+1} & \breve{P} \end{bmatrix} \begin{bmatrix} \Delta^\top \\ \breve{\Delta}^\top \end{bmatrix} r_{(i)} \\ &= \frac{1}{2}\tilde{r}^2(\sigma'(u,z,k)^\top P_{11}\sigma'(u,z,k) + \breve{\sigma}'(u,z,k)^\top \breve{P}_{11}\breve{\sigma}'(u,z,k) + zP_{22}z^\top + z\breve{P}_{22}z^\top)_{ii} \\ &\quad + \tilde{r}^2(\sigma'(u,z,k)^\top P_{12} z^\top)_{ii} + \tilde{r}^2(\breve{\sigma}'(u,z,k)^\top \breve{P}_{12} z^\top)_{ii} \\ &\quad + \tilde{r}(\psi(d')P_{13}^\top \sigma'(u,z,k) + \psi(d')P_{23}^\top z^\top + \breve{\psi}(\breve{d}')\breve{P}_{13}^\top \breve{\sigma}'(u,z,k) + \breve{\psi}(\breve{d}')\breve{P}_{23}^\top z^\top)_i r_i + \frac{1}{2} r_i^2(\psi^2(d')P_{33} + \breve{\psi}^2(\breve{d}')\breve{P}_{33}) \\ &\leq C\frac{3}{2\epsilon}(|a' - \breve{a}'|^2 + |b' - \breve{b}'|^2 + |d' - \breve{d}'|^2)(|\tilde{r}|^2 + 2|r_i|^2). \end{aligned}$$

where $(\cdot)_{ii}$ denotes the ith row and ith column of the square matrix, and $(\cdot)_i$ indicates the ith element of the vector.

We define

$$\hat{r} := \begin{bmatrix} \tilde{r} & \bar{r}^\top \end{bmatrix}^\top \in \mathbb{R}^{p+1}, \ \bar{r} := \begin{bmatrix} r_1 & r_2 & \cdots & r_p \end{bmatrix}^\top \in \mathbb{R}^p.$$

Then using the definition of the trace operator,

$$\begin{aligned} \hat{r}^\top(\breve{Q}_\psi^{(2)} - Q_\psi^{(2)})\hat{r} &= \tilde{r}^2(\breve{Q}_\psi^{(2,11)} - Q_\psi^{(2,11)}) + 2\tilde{r}(\breve{Q}_\psi^{(2,12)} - Q_\psi^{(2,12)})\bar{r} + \bar{r}^\top(\breve{Q}_\psi^{(2,22)} - Q_\psi^{(2,22)})\bar{r} \\ &= \frac{1}{2}\sum_{i=1}^p \{\tilde{r}^2(\sigma'(u,z,k)^\top P_{11}\sigma'(u,z,k) + \breve{\sigma}'(u,z,k)^\top \breve{P}_{11}\breve{\sigma}'(u,z,k) + zP_{22}z^\top + z\breve{P}_{22}z^\top)_{ii} \\ &\quad + \tilde{r}^2(\sigma'(u,z,k)^\top P_{12}z^\top)_{ii} + \tilde{r}^2(\breve{\sigma}'(u,z,k)^\top \breve{P}_{12}z^\top)_{ii}\} \\ &\quad + \sum_{i=1}^p \tilde{r}(\psi(d')P_{13}^\top\sigma'(u,z,k) + \psi(d')P_{23}^\top z^\top + \breve{\psi}(\breve{d}')\breve{P}_{13}^\top\breve{\sigma}'(u,z,k) + \breve{\psi}(\breve{d}')\breve{P}_{23}^\top z^\top)_i r_i \\ &\quad + \frac{1}{2}\sum_{i=1}^p r_i^2(\psi^2(d')P_{33} + \breve{\psi}^2(\breve{d}')\breve{P}_{33}) \\ &\leq C\frac{3}{2\epsilon}(|a' - \breve{a}'|^2 + |b' - \breve{b}'|^2 + |d' - \breve{d}'|^2)\sum_{i=1}^p(|\tilde{r}|^2 + 2|r_i|^2) \leq C\frac{3}{2\epsilon}(|a' - \breve{a}'|^2 + |b' - \breve{b}'|^2 + |d' - \breve{d}'|^2)p|\hat{r}|^2. \end{aligned} \tag{C4}$$

By the arbitrariness of $\tilde{r}, r_i \in \mathbb{R}$ and $(u, z, k) \in U \times \mathbb{R}^{p\times m} \times G^{(1)}$, it follows from (C4) that

$$\sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \sup_{|r|=1} \hat{r}^\top(\check{Q}_\psi^{(2)} - Q_\psi^{(2)})\hat{r} = \sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \Lambda^+(\check{Q}_\psi^{(2)} - Q_\psi^{(2)}) \le C\frac{3}{2\epsilon}(|a' - \check{a}'|^2 + |b' - \check{b}'|^2 + |d' - \check{d}'|^2)p.$$

Hence, by (39), we have $\lim_{\epsilon\downarrow 0} \Upsilon^{(12)} \le 0$.

C.0.3 Estimate of $\Upsilon^{(13)}$

By definition

$$\Upsilon^{(13)} = \sup_{(u,z,k)\in U\times\mathbb{R}^{p\times m}\times G^{(1)}} \max\{\frac{1}{2}\partial_t(\check{\delta}'_{\eta,\lambda} + \delta'_{\eta,\lambda}) + \eta e^{-\lambda t'}(\mathrm{Tr}(\check{\sigma}'(\check{\sigma}')^\top(u,z,k)) + \mathrm{Tr}(z^\top z))$$
$$+ \frac{3}{2}\eta e^{-\lambda t'}(\mathrm{Tr}(\sigma'(\sigma')^\top(u,z,k)) + \mathrm{Tr}(z^\top z)), 0\}.$$

Notice that by (C2),

$$\frac{1}{2}\partial_t(\check{\delta}'_{\eta,\lambda} + \delta'_{\eta,\lambda}) \to -\eta\lambda e^{-\lambda\tilde{t}}(1 + |\tilde{a}|^2 + |\tilde{b}|^2 + \tilde{d}) \text{ as } \epsilon \downarrow 0 \text{ due to (39).}$$

Moreover, by Assumption 2 (note that $(t', \mathbf{a}', \check{\mathbf{a}}') := (t'_\epsilon, \mathbf{a}'_\epsilon, \check{\mathbf{a}}'_\epsilon)$, that is, $(t', a', b', d', \check{a}', \check{b}', \check{d}') := (t'_\epsilon, a'_\epsilon, b'_\epsilon, d'_\epsilon, \check{a}'_\epsilon, \check{b}'_\epsilon, \check{d}'_\epsilon)$; see the statement below (39)),

$$\left|\eta e^{-\lambda t'}\mathrm{Tr}(\check{\sigma}'(\check{\sigma}')^\top(u,z,k)) + \frac{3\eta e^{-\lambda t'}}{2}\mathrm{Tr}(\sigma'(\sigma')^\top(u,z,k))\right|$$
$$\le C_2\eta e^{-\lambda t'}(1 + |a'|^2 + |b'|^2 + |\check{a}'|^2 + |\check{b}'|^2) \to C_3\eta e^{-\lambda\tilde{t}}(1 + |\tilde{a}|^2 + |\tilde{b}|^2) \text{ as } \epsilon \downarrow 0 \text{ due to (39).}$$

Finally, by Remark 6 (note that $(t', \mathbf{a}', \check{\mathbf{a}}') := (t'_\epsilon, \mathbf{a}'_\epsilon, \check{\mathbf{a}}'_\epsilon)$, that is, $(t', a', b', d', \check{a}', \check{b}', \check{d}') := (t'_\epsilon, a'_\epsilon, b'_\epsilon, d'_\epsilon, \check{a}'_\epsilon, \check{b}'_\epsilon, \check{d}'_\epsilon)$; see the statement below (39)),

$$\eta e^{-\lambda t'}\left|\frac{3}{2}\mathrm{Tr}(z^\top z) + \mathrm{Tr}(z^\top z)\right| \le C_4\eta e^{-\lambda t'}(1 + |a'|^2 + |b'|^2 + |\check{a}'|^2 + |\check{b}'|^2) \to C_4\eta e^{-\lambda\tilde{t}}(1 + |\tilde{a}|^2 + |\tilde{b}|^2) \text{ as } \epsilon \downarrow 0 \text{ due to (39).}$$

For $\lambda \ge C_3 + C_4$, this leads to

$$\Upsilon^{(13)} \le \max\{((C_3 + C_4) - \lambda)e^{-\lambda\tilde{t}}(1 + |\tilde{a}|^2 + |\tilde{b}|^2 + \tilde{d}), 0\} \le 0.$$

Hence, $\lim_{\epsilon\downarrow 0} \Upsilon^{(13)} \le 0$.

APPENDIX D. ESTIMATE OF $\Upsilon^{(2)}$

By definition of $\Upsilon^{(2)}$ and $\mathbb{H}_\kappa^{(21)}$,

$$\Upsilon^{(2)} \le \sup_{(u,z,\pi^{(2)})\in U\times\mathbb{R}^{p\times m}\times\Pi^{(2)}} \Upsilon^{(21)} + \sup_{(u,z,\pi^{(2)})\in U\times\mathbb{R}^{p\times m}\times\Pi^{(2)}} \Upsilon^{(22)},$$

where

$$\Upsilon^{(21)} := -\int_{E_\kappa} [\phi(t', \check{\mathbf{a}}' + \check{\chi}'(e,u,z,k(e),\beta(e))) - \phi(t', \check{\mathbf{a}}')]\lambda(de) + \int_{E_\kappa} \langle D_{\mathbf{a}}\phi(t, \check{\mathbf{a}}'), \check{\chi}'(e,u,z,k(e),\beta(e))\rangle\lambda(de)$$
$$\Upsilon^{(22)} := \int_{E_\kappa} [\phi(t', \mathbf{a}' + \chi'(e,u,z,k(e),\beta(e))) - \phi(t', \mathbf{a}')]\lambda(de) - \int_{E_\kappa} \langle D_{\mathbf{a}}\phi(t, \mathbf{a}'), \chi'(e,u,z,k(e),\beta(e))\rangle\lambda(de),$$

where $\chi'(e,u,z,k(e),\beta(e)) := \chi(t',e,a',u,b',z,k(e),\beta(e))$ and $\breve{\chi}'(e,u,z,k(e),\beta(e))$ is defined similarly. It follows from the standard result of the Taylor series expansion and the Cauchy–Schwarz inequality that

$$\Upsilon^{(22)} \leq \int_{E_\kappa}\int_0^1 (1-\tau)\left\| D^2_{\mathbf{aa}}\phi(t',\mathbf{a}'+\tau\chi'(e,u,z,k(e),\beta(e)))\right\|_F (|\chi'(e,u,z,k(e))| + |k(e)| + |\beta(e)|)\lambda(\mathrm{d}e). \quad \text{(D1)}$$

Then the regularity of the test function and Assumption 2 imply

$$\Upsilon^{(22)} \leq C(\int_{E_\kappa} (1+|a'|^2+|b'|^2)\lambda(\mathrm{d}e)\,)^{\frac{1}{2}} + C(\int_{E_\kappa} |k(e)|^2\lambda(\mathrm{d}e)\,)^{\frac{1}{2}} + C(\int_{E_\kappa} |\beta(e)|^2\lambda(\mathrm{d}e)\,)^{\frac{1}{2}}.$$

By Remarks 6 and 8, $\lim_{\kappa\downarrow 0}\Upsilon^{(22)} \leq 0$. Similarly, we can show that $\lim_{\kappa\downarrow 0}\Upsilon^{(21)} \leq 0$. This implies

$$\lim_{\kappa\downarrow 0}\Upsilon^{(2)} \leq 0. \quad \text{(D2)}$$

APPENDIX E. ESTIMATE OF $\Upsilon^{(3)}$

By definition of (40),

$$\underline{W}_\nu(t,\mathbf{a}) - \overline{W}(t,\breve{\mathbf{a}}) = \Psi^{\epsilon}_{\nu;\eta,\lambda}(t,\mathbf{a},\breve{\mathbf{a}}) + \delta_{\eta,\lambda}(t,\mathbf{a}) + \breve{\delta}_{\eta,\lambda}(t,\breve{\mathbf{a}}) + \theta_\epsilon(\mathbf{a},\breve{\mathbf{a}}).$$

Then due to the fact that $(t',\mathbf{a}',\breve{\mathbf{a}}')$ is the maximum point of $\Psi^{\epsilon}_{\nu;\eta,\lambda}$ by (38), it follows that

$$\Upsilon^{(3)} \leq \sup_{(u,z,\pi^{(2)})\in U\times\mathbb{R}^{p\times m}\times\Pi^{(2)}} \{\Upsilon^{(31)}+\Upsilon^{(32)}+\Upsilon^{(33)}\},$$

where

$$\begin{aligned}
\Upsilon^{(31)} &:= \int_{E_\kappa^C} [\delta_{\eta,\lambda}(t',\mathbf{a}'+\chi'(e,u,z,k(e),\beta(e))) - \delta_{\eta,\lambda}(t',\mathbf{a}') - \langle D_{\mathbf{a}}\delta'_{\eta,\lambda}, \chi'(e,u,z,k(e),\beta(e))\rangle]\lambda(\mathrm{d}e)\\
\Upsilon^{(32)} &:= \int_{E_\kappa^C} [\breve{\delta}_{\eta,\lambda}(t',\breve{\mathbf{a}}'+\breve{\chi}'(e,u,z,k(e),\beta(e))) - \breve{\delta}_{\eta,\lambda}(t',\breve{\mathbf{a}}') - \langle D_{\mathbf{a}}\breve{\delta}'_{\eta,\lambda}, \breve{\chi}'(e,u,z,k(e),\beta(e))\rangle]\lambda(\mathrm{d}e)\\
\Upsilon^{(33)} &:= \int_{E_\kappa^C} [\theta_\epsilon(\mathbf{a}'+\chi'(e,u,z,k(e),\beta(e)),\breve{\mathbf{a}}'+\breve{\chi}'(e,u,z,k(e),\beta(e)))\\
&\quad - \langle D_{\mathbf{a}}\theta_\epsilon(\mathbf{a}',\breve{\mathbf{a}}'),\chi'(e,u,z,k(e),\beta(e))\rangle + \langle D_{\breve{\mathbf{a}}}\theta_\epsilon(\mathbf{a}',\breve{\mathbf{a}}'),\breve{\chi}'(e,u,z,k(e),\beta(e))\rangle]\lambda(\mathrm{d}e).
\end{aligned}$$

Using the derivatives in Appendix F and the approach analogous to get (D1),

$$\Upsilon^{(31)} = \int_{E_\kappa^C}\int_0^1 (1-\tau)\mathrm{Tr}(\begin{bmatrix} 3\eta e^{-\lambda t}I_n & 0 & 0\\ 0 & 3\eta e^{-\lambda t}I_m & 0\\ 0 & 0 & 0\end{bmatrix}\chi'(\chi')^\top(e,u,z,k(e),\beta(e))) \leq C(n+m)\eta e^{-\lambda t'}(1+|a'|^2+|b'|^2),$$

and similarly,

$$\Upsilon^{(32)} \leq C(n+m)\eta e^{-\lambda t'}(1+|\breve{a}'|^2+|\breve{b}'|^2).$$

Furthermore, using the derivatives in Appendix F and Assumption 2,

$$\Upsilon^{(33)} = \frac{1}{2\epsilon}\int_{E_\kappa^C} |\chi'(e,u,z,k(e),\beta(e)) - \breve{\chi}'(e,u,z,k(e),\beta(e))|^2\lambda(\mathrm{d}e) \leq \frac{1}{2\epsilon}C(|a'-\breve{a}'|+|b'-\breve{b}'|^2) \to 0 \text{ as } \epsilon\downarrow 0 \text{ due to (39).}$$

By the above estimates, it follows that

$$\lim_{\eta\downarrow 0}\lim_{\epsilon\downarrow 0}\lim_{\kappa\downarrow 0}\Upsilon^{(3)} \leq 0. \tag{E1}$$

APPENDIX F. DERIVATIVES OF δ, $\breve{\delta}$ AND θ_ϵ

We compute

$$\begin{cases}
\partial_t \delta_{\eta,\lambda}(t,\mathbf{a}) = -\eta\lambda e^{-\lambda t}(1+|a|^2+|b|^2+d) + (t-\tilde{t}) - \frac{\eta\lambda e^{-\lambda t}}{2}(|a-\tilde{a}|^2+|b-\tilde{b}|^2+(d-\tilde{d})) \\
\partial_t \breve{\delta}_{\eta,\lambda}(t,\breve{\mathbf{a}}) = -\eta\lambda e^{-\lambda t}(1+|\breve{a}|^2+|\breve{b}|^2+\breve{d}) \\
D_{\mathbf{a}}\delta_{\eta,\lambda}(t,\mathbf{a}) = \begin{bmatrix} 2\eta e^{-\lambda t}a + \eta e^{-\lambda t}(a-\tilde{a}) \\ 2\eta e^{-\lambda t}b + \eta e^{-\lambda t}(b-\tilde{b}) \\ \frac{3}{2}\eta e^{-\lambda t} \end{bmatrix}, \; D_{\breve{\mathbf{a}}}\breve{\delta}_{\eta,\lambda}(t,\breve{\mathbf{a}}) = \begin{bmatrix} 2\eta e^{-\lambda t}\breve{a} \\ 2\eta e^{-\lambda t}\breve{b} \\ \eta e^{-\lambda t} \end{bmatrix} \\
D^2_{\mathbf{aa}}\delta_{\eta,\lambda}(t,\mathbf{a}) = \begin{bmatrix} 3\eta e^{-\lambda t}I_n & 0 & 0 \\ 0 & 3\eta e^{-\lambda t}I_m & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; D^2_{\breve{\mathbf{a}}\breve{\mathbf{a}}}\breve{\delta}_{\eta,\lambda}(t,\breve{\mathbf{a}}) = \begin{bmatrix} 2\eta e^{-\lambda t}I_n & 0 & 0 \\ 0 & 2\eta e^{-\lambda t}I_m & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
D_{\mathbf{a}}\theta_\epsilon(\mathbf{a},\breve{\mathbf{a}}) = \begin{bmatrix} \frac{1}{\epsilon}(a-\breve{a}) \\ \frac{1}{\epsilon}(b-\breve{b}) \\ \frac{1}{\epsilon}(d-\breve{d}) \end{bmatrix}, \; D_{\breve{\mathbf{a}}}\theta_\epsilon(\mathbf{a},\breve{\mathbf{a}}) = \begin{bmatrix} -\frac{1}{\epsilon}(a-\breve{a}) \\ -\frac{1}{\epsilon}(b-\breve{b}) \\ -\frac{1}{\epsilon}(d-\breve{d}) \end{bmatrix}.
\end{cases}$$

As mentioned, for δ, $\breve{\delta}$, and θ_ϵ and their derivatives, we use the superscript $'$ when they are evaluated at $(t',\mathbf{a}',\breve{\mathbf{a}}')$, for example, $\delta'_{\eta,\lambda} := \delta_{\eta,\lambda}(t',\mathbf{a}')$ and $D_{\mathbf{a}}\delta'_{\eta,\lambda} := D_{\mathbf{a}}\delta_{\eta,\lambda}(t',\mathbf{a}')$.

APPENDIX G. EXPLICIT EXPRESSION OF $\mathcal{P}$, AND $\mathscr{A}_i$, $i = 1,2,3$, IN PROPOSITION 1

Let us define

$$\begin{cases}
\widehat{\mathcal{A}} = \begin{bmatrix} A_1 & A_2 \\ G_1 & G_2 \end{bmatrix}, \; \widehat{\mathcal{B}} = \begin{bmatrix} B_1 & B_2 & \lambda B_2 \\ H_1 & H_2 & \lambda H_3 \end{bmatrix}, \; \widehat{\mathcal{Q}} = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \; \widehat{\mathcal{R}} = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & \lambda R_3 \end{bmatrix} \\
\widehat{\mathcal{C}} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \; \widehat{\mathcal{D}} = \begin{bmatrix} D_1 & D_2 & \lambda D_3 \end{bmatrix}, \; \widehat{\mathcal{E}} = \begin{bmatrix} I+E_1 & E_2 \end{bmatrix} \\
\widehat{\mathcal{F}} = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}, \; \widetilde{\mathcal{E}} = \begin{bmatrix} E_1 & E_2 \\ 0 & 0 \end{bmatrix}, \; \widetilde{\mathcal{F}} = \begin{bmatrix} F_1 & F_2 & F_3 \\ 0 & 0 & I \end{bmatrix}, \; \widehat{\mathcal{P}}_{22}(\cdot) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathcal{P}_{22}(\cdot) & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\widetilde{\mathcal{P}}_{12}(\cdot) = \begin{bmatrix} 0 & \mathcal{P}_{12}(\cdot) & 0 \end{bmatrix}, \; \mathcal{P}'_{11}(\cdot) \begin{bmatrix} \mathcal{P}_{11}(\cdot) & 0 \\ 0 & 0 \end{bmatrix}, \; \mathcal{P}'_{22}(\cdot) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{P}_{22}(\cdot) \end{bmatrix}, \; \mathcal{P}''_{22}(\cdot) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \mathcal{P}_{22}(\cdot) \end{bmatrix}.
\end{cases}$$

The optimal solution in Proposition 1 can be written as

$$\begin{bmatrix} \widehat{u}(t) \\ \widehat{z}(t)^\top \\ \widehat{k}(t) \end{bmatrix} = -\mathscr{R}[\mathcal{P}(t)]^{-1}\mathscr{B}[\mathcal{P}(t)]^\top \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \begin{bmatrix} \mathscr{A}_1[\mathcal{P}(t)] \\ \mathscr{A}_2[\mathcal{P}(t)] \\ \mathscr{A}_3[\mathcal{P}(t)] \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix},$$

where

$$\mathscr{R}[\mathcal{P}(t)] := -\mathcal{C}(t)\widehat{\mathcal{R}} + \widehat{\mathcal{D}}^\top \mathcal{P}_{11}(t)\widehat{\mathcal{D}} + \widehat{\mathcal{D}}^\top \widetilde{\mathcal{P}}_{12}(t) + \widehat{\mathcal{P}}_{22}(t) + \lambda \widehat{\mathcal{F}}^\top \mathcal{P}_{11}(t)\widehat{\mathcal{F}} + \lambda \mathcal{P}'_{22}(t)$$
$$\mathscr{B}[\mathcal{P}(t)] := \mathcal{P}(t)\widehat{\mathcal{B}} + \widehat{\mathcal{C}}^\top \mathcal{P}_{11}(t)\widehat{\mathcal{D}} + \widehat{\mathcal{C}}^\top \widetilde{\mathcal{P}}_{12}(t) + \lambda \widehat{\mathcal{E}}^\top \mathcal{P}_{11}(t)\widehat{\mathcal{F}} + \lambda \mathcal{P}''_{22}(t)^\top - \widetilde{\mathcal{F}}^\top \mathcal{P}(t).$$

Here, $\mathcal{P}$ in Proposition 1 (see (22)) holds the following $\mathbb{S}^{n+m}$-valued matrix differential equation:

$$\begin{cases} -\dfrac{\mathrm{d}\mathcal{P}(t)}{\mathrm{d}t} = \widehat{\mathcal{A}}^\top \mathcal{P}(t) + \mathcal{P}(t)\widehat{\mathcal{A}} - \mathcal{C}(t)\mathcal{Q} + \widehat{\mathcal{C}}^\top \mathcal{P}_{11}(t)\widehat{\mathcal{C}} + \lambda \widehat{\mathcal{E}}^\top \mathcal{P}_{11}(t)\widehat{\mathcal{E}} - \lambda \mathcal{P}'_{11}(t) - \lambda \widetilde{\mathcal{E}}^\top \mathcal{P}(t) - \lambda \mathcal{P}(t)\widetilde{\mathcal{E}} - \mathscr{B}[\mathcal{P}(t)]\mathscr{R}[\mathcal{P}(t)]^{-1}\mathscr{B}[\mathcal{P}(t)]^\top \\ \mathcal{P}(T) = \begin{bmatrix} M_1 + 2M_2^\top M_2 & -2M_2^\top \\ -2M_2 & 2I \end{bmatrix}. \end{cases} \tag{G1}$$