A half-integral Erdős-Pósa theorem for directed odd cycles

Ken-ichi Kawarabayashi *† Stephan Kreutzer‡ O-joung Kwon§ Qiqin Xie¶

Abstract

We prove that there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that every directed graph G contains either k directed odd cycles where every vertex of G is contained in at most two of them, or a set of at most $f(k)$ vertices meeting all directed odd cycles. We also give a polynomial-time algorithm for fixed k which outputs one of the two outcomes. Using this algorithmic result, we give a polynomial-time algorithm for fixed k to decide whether such k directed odd cycles exist, or there are no k vertex-disjoint directed odd cycles.

This extends the half-integral Erdős-Pósa theorem for undirected odd cycles by Reed [Combinatorica 1999] to directed graphs.

1 Introduction

Erdős and Pósa [5] proved that for every undirected graph G and every positive integer k, G either contains k pairwise vertex-disjoint cycles, or a set of $\mathcal{O}(k \log k)$ vertices that meets all cycles of G. This result has been extended to cycles satisfying various constraints: long cycles [25, 2, 7, 20, 3], cycles with modularity constraints [26, 11, 27], cycles intersecting a prescribed vertex set [13, 21, 3, 11], and holes [17]. We refer to a survey of Raymond and Thilikos [22] for more examples. On the other hand, such a duality does not exist for odd cycles: Lovász and Schrijver (see [26]) found a class of graphs, called Escher walls, where they have no two vertex-disjoint odd cycles but there is no constant c such that every Escher wall admits a set of c vertices meeting all odd cycles. Escher walls are illustrated in Figure 1.

In 1999, Reed [23] obtained a half-integral analogue of the Erdős-Pósa theorem for odd cycles, by relaxing the vertex-disjoint packing to a half-integral packing. A family of subgraphs in an undirected graph or a directed graph G is a *half-integral packing* if every vertex of G is contained in at most two of the subgraphs. This theorem of Reed has been recently generalized to group-labelled graphs by Huynh, Joos, and Wollan [11], Gollin et al. [9], and Gollin et al. [10].

Theorem 1.1. (Reed [23]) *There is a function $g\colon \mathbb{N} \to \mathbb{R}$ such that for every undirected graph G and every positive integer k, G contains a half-integral packing of k odd cycles, or a set of at most $g(k)$ vertices meeting all odd cycles.*

For directed graphs, the situations become much more complicated, and not many results are known. Reed, Robertson, Seymour, and Thomas [24] showed that an analogue of the Erdős-Pósa theorem holds for directed cycles, which confirms a long standing conjecture of Younger [28]. As an application of the directed grid theorem, Kawarabayashi and Kreutzer [16] proved that an analogue of the Erdős-Pósa theorem holds for directed cycles of length at least ℓ for some fixed ℓ. Amiri et al. [1] further extended so that if H is a strongly connected directed graph such that any H-subdivision can be obtained as a subgraph of some cylindrical wall (see Figure 3), then an analogue of the Erdős-Pósa theorem holds for H-subdivisions. Kakimura and Kawarabayashi [12] showed that

*All authors are corresponding authors.

†National Institute of Informatics, 2-1-2, Hitotsubashi, Chiyoda-ku, Tokyo, Japan. Email: k_keniti@nii.ac.jp. Supported by JSPS Kakenhi Grant Number JP18H05291.

‡Logic and Semantics, TU Berlin, Berlin, Germany. Email: stephan.kreutzer@tu-berlin.de. Supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (ERC consolidator grant DISTRUCT, agreement No. 648527).

§Department of Mathematics, Hanyang University, Seoul, South Korea and Discrete Mathematics Group, Institute for Basic Science (IBS), Daejeon, South Korea. Email: ojoungkwon@hanyang.ac.kr. Supported by the National Research Foundation of Korea (NRF) grant funded by the Ministry of Education (No. NRF-2018R1D1A1B07050294) and (No. NRF-2021K2A9A2A11101617) and supported by the Institute for Basic Science (IBS-R029-C1).

¶Fudan University, Shanghai Center for Mathematical Science, 2005 Songhu Rd., Shanghai, China, 200433. Email: qqxie@fudan.edu.cn.

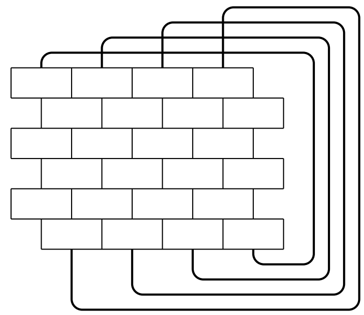


Figure 1: An Escher wall, where the middle wall W is bipartite and each thick path P links from one vertex of the top row to the opposite vertex of the bottom row in the middle wall so that the union of W and P has an odd cycle.

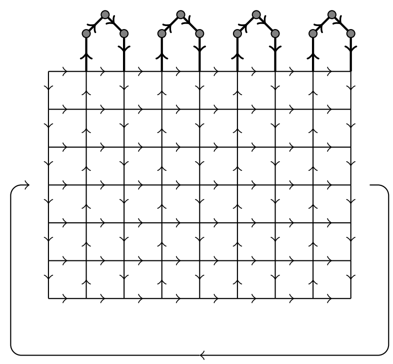


Figure 2: A bipartite cylindrical grid with some parity-changing paths on the top. It is not difficult to see that there are no two vertex-disjoint directed odd cycles, but one can increase the minimum size of a hitting set by taking a larger construction.

an analogue of the Erdős-Pósa theorem does not hold for directed cycles meeting a prescribed set S (so called directed S-cycles), but a 1/5-integral analogue of the Erdős-Pósa theorem holds (this result is further improved to a half-integral analogue in [14]). On the other hand, so far, directed cycles with modularity constraint have not been considered in this context.

The main contribution of this paper is to show that a half-integral analogue of the Erdős-Pósa theorem holds for directed odd cycles. We construct an example, illustrated in Figure 2, showing that an analogue of the Erdős-Pósa theorem does not hold for directed odd cycles even on planar directed graphs. This contrasts with the undirected case; it is known that an analogue of the Erdős-Pósa theorem holds for odd cycles on planar graphs [23, 6, 18].

Theorem 1.2. *There is a function $f : \mathbb{N} \to \mathbb{R}$ such that for every directed graph G and every positive integer k, G contains a half-integral packing of k directed odd cycles, or a set of at most $f(k)$ vertices meeting all directed odd cycles. For every fixed positive integer k, there is a polynomial-time algorithm that given a graph G, outputs one of the two outcomes.*

Sketch of proof We sketch the proof of Theorem 1.2.

To obtain Erdős-Pósa type results for various graph families in the undirected setting, the grid minor theorem [25] has been importantly used, see [25, 26, 23, 11, 9] for examples. For directed graphs, Kawarabayashi and Kreutzer [16] obtained the directed grid theorem, which shows that every directed graph of sufficiently large directed tree-width contains a cylindrical grid of large order as a butterfly minor. They observed that if a directed graph contains a cylindrical grid of large order as a butterfly minor, then it contains a cylindrical wall of large order as a subgraph. Therefore, we will mostly use a cylindrical wall of large order, which is depicted in Figure 3.

A set S of vertices in a directed graph G is a *hitting set* for directed odd cycles, if S meets all directed odd cycles of G. For a directed graph G, we denote by $\nu_2(G)$ the maximum size of a half-integral packing of directed odd cycles in G, and denote by $\tau(G)$ the minimum size of a hitting set for directed odd cycles in G. For each positive integer k, we define α_k as the minimum integer such that for every directed graph G with $\nu_2(G) < k$, we have $\tau(G) \leq \alpha_k$, if such an integer exists, and otherwise α_k is defined to be ∞. It is sufficient to show that $\alpha_k \neq \infty$ for every positive integer k. Clearly, $\alpha_1 = 0$. We will prove it by induction on k.

A set T of vertices in a directed graph G is an *r-externally-well-linked set* if for all disjoint sets A and B of vertices in T with $|A| = |B| \geq r$, there is a set of $|A|$ vertex-disjoint paths from A to B in $G - (T \setminus (A \cup B))$ (and also from B to A). We show in Lemma 5.1 that if $\alpha_{k-1} \neq \infty$ and a directed graph G with $\nu_2(G) < k$ has a hitting set T of directed odd cycles with $|T| = \tau(G)$, then T is $2\alpha_{k-1}$-externally-well-linked. So, we can argue that if $\tau(G)$ is sufficiently large, then G has large directed tree-width, and it contains a cylindrical wall of large order by the directed grid theorem. However, for our purpose, we need a special cylindrical wall of large order that cannot be separated from T by removing a small set of vertices.

Such a result was obtained in [15] (which is the journal version of [16]) for ordinary well-linked sets. A set X of vertices in a directed graph G is a *well-linked set* if for all sets A and B of vertices in X with $|A| = |B|$, there is a set of $|A|$ vertex-disjoint paths from A to B in G (and also from B to A). Kawarabayashi and Kreutzer [15, Theorem 7.1] showed that if G contains a sufficiently large well-linked set X, then it contains a large cylindrical wall of order w, such that for every set F of w vertices that are out-degree 2 or in-degree 2 in the wall, there are w vertex-disjoint paths from F to X in G and from X to F in G.

To relate the $2\alpha_{k-1}$-externally-well-linked set T to some cylindrical wall, we prove in Lemma 4.2 that there is a well-linked set X such that T and X cannot be separated by removing a small set of vertices. Combining with the directed grid theorem, we obtain a required cylindrical wall W of large order that is not separated from T by removing a small set of vertices.

We take k vertex-disjoint subwalls of W in a natural way, and we may assume that one of them, say W', has no directed odd cycles. As any wall is strongly connected, we can argue that the underlying undirected graph of W' is bipartite. Let N be a large set of vertices of W' such that they have out-degree 2 or in-degree 2 in the wall, and they are in the same part of the bipartition of W'.

We prove in Section 3 that given a directed graph F and a set X of vertices in F, F contains either a half-integral packing of k directed odd cycles, or a half-integral packing of k directed odd X-paths whose endvertices are pairwise distinct, or a set of at most $4k-1$ vertices hitting all odd X-walks. We apply this lemma to the set $X = N$ of W' with $F = G$.

In case when there is a small set Y of vertices meeting all odd N-walks, there is a strong component H of $G-Y$ containing most part of the set T. We can argue that more than half of the columns of W are also contained in H. On the other hand, if H has a directed odd cycle, then one can find a directed odd N-walk, which is a contradiction. So, Y together with $T \setminus V(H)$ gives a hitting set for directed odd cycles, which is small. In the case when there are many directed odd N-paths, we show in Section 5, that we can use the bipartite cylindrical wall to find a half-integral packing of k odd cycles, which contradicts the assumption that $\nu_2(G) < k$. This will complete the proof.

Algorithmic applications Reed et al. [24] used their Erdős-Pósa result to show the following: for every fixed positive integer k, there is a polynomial-time algorithm to test whether or not G contains k vertex-disjoint directed cycles. As in [24], a bounded (by function of k) size set that hits all directed cycles is a key to obtain this algorithm. Therefore, we could expect that Theorem 1.2 could give such a result.

To this end, we first discuss how this combinatorial result in Theorem 1.2 and its proof can be turned into a polynomial-time algorithm for fixed k, which outputs one of the outcomes in Section 6. Using this algorithmic result, we give the following:

THEOREM 1.3. *For every fixed positive integer k, there is a polynomial-time algorithm that given a graph G, either*

1. *outputs a half-integral packing of k directed odd cycles, or*
2. *correctly decides that there are no k vertex-disjoint directed odd cycles in G.*

Ideally, we want to replace the second conclusion by "it concludes that there is no half-integral packing of k directed odd cycles.", and indeed we conjecture that this should be the case. However, there is some technical

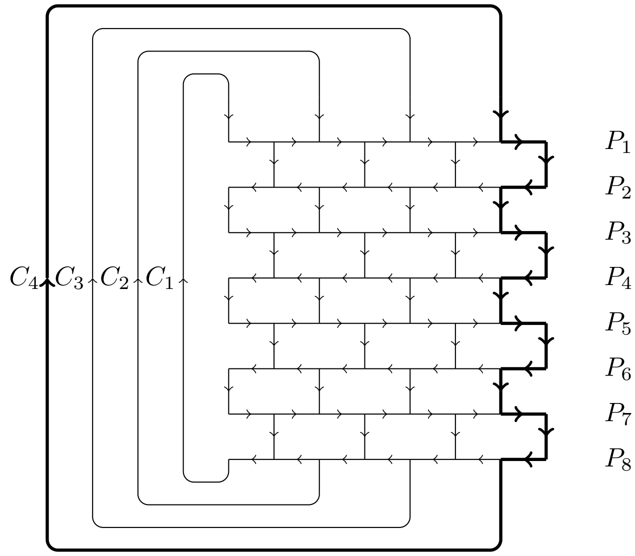


Figure 3: The cylindrical wall of order 4. The cycle C_4 is depicted using thick edges.

difficulty, and we will mention this in Section 6. Let us remark that it is NP-complete to decide whether or not a directed graph contains two vertex-disjoint odd cycles (there is a straightforward reduction to the directed two disjoint paths problem). Thus we cannot replace the first by "k vertex-disjoint directed odd cycles".

2 Preliminaries

Let $\mathbb{N}$ be the set of all positive integers, and $\mathbb{R}$ be the set of all reals. For an integer m, we write $[m]$ for the set of positive integers at most m. In this paper, all directed graphs have no multiple edges and loops. Directed walks, directed paths, and directed cycles are simply called walks, paths, and cycles respectively.

Let G be a directed graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. If (v, w) is an edge, then v is its *tail* and w is its *head*. For a set A of vertices in G, we denote by $G - A$ the graph obtained from G by removing all the vertices in A, and denote by $G[A]$ the subgraph of G induced by A. For two directed graphs G and H, let $G \cup H := (V(G) \cup V(H), E(G) \cup E(H))$ and $G \cap H := (V(G) \cap V(H), E(G) \cap E(H))$. For a set $\mathcal{G}$ of directed graphs, we denote by $\bigcup \mathcal{G}$ the union of the directed graphs in $\mathcal{G}$.

We say that a directed graph G is *strongly connected* if for any two vertices v and w in G, there is a path from v to w in G and there is a path from w to v in G. A *strong component* of G is a maximal subgraph of G that is strongly connected. It is well known that the set of strong components of G can be labelled $G_1, G_2, \ldots, G_t$ such that there is no edge from G_j to G_i if $j \geq i$. Such an ordering is called an *acyclic ordering* of the strong components of G.

For sets A and B of vertices in a directed graph G, a path is an *(A, B)-path* if it starts at A and ends at B, and all its internal vertices are not in $A \cup B$. For a set A of vertices in G, an *A-walk* P is a walk having at least one edge such that both endvertices of P are in A and all its internal vertices are not in A. Note that the two endvertices of an A-walk may be the same vertex. An A-walk is *closed* if its endvertices are the same. An A-walk is called an *A-path* if it is a path.

Let t be a positive integer. A family $(G_i : i \in [m])$ of subgraphs in a directed graph G is a *$(1/t)$-integral packing* if every vertex of G is contained in at most t of $G_1, G_2, \ldots, G_m$. When $t = 2$, we say that it is a *half-integral packing*.

2.1 Cylindrical walls For an integer $k \geq 2$, a *cylindrical wall* of order k is a directed graph consisting of k pairwise vertex-disjoint cycles $C_1, \ldots, C_k$, called *columns*, and a set of $2k$ pairwise vertex-disjoint paths $P_1, \ldots, P_{2k}$, called *rows*, such that

- for each $i \in [k]$ and $j \in [2k]$, $C_i \cap P_j$ is a path with at least one edge,

- both endvertices of P_i are in $V(C_1) \cup V(C_k)$,
- the paths $P_1 \cap C_i, \ldots, P_{2k} \cap C_i$ appear in this order on each C_i and
- for odd i, the cycles $C_1 \cap P_i, \ldots, C_k \cap P_i$ appear in this order on P_i, and for even i, $C_k \cap P_i, \ldots, C_1 \cap P_i$ appear in this order on P_i.

See Figure 3 for an illustration of a cylindrical wall of order 4. An endvertex of $C_i \cap P_j$ for some $i \in [k]$ and $j \in [2k]$ is called a *nail*, and we denote by N^W the set of all nails of W. Note that an N^W-path in W is a path such that its endvertices are nails, but all the internal vertices are not nails.

We will use cylindrical walls that do not contain odd cycles. Because of the following fact, the underlying undirected graph of such a wall is bipartite.

Proposition 2.1. (Folklore) *Let D be a strongly connected directed graph having no odd cycle. Then, the underlying undirected graph of D is bipartite.*

We say that a cylindrical wall is *bipartite* if its underlying undirected graph is bipartite.

2.2 Linkages and separations For a positive integer t and sets A and B of vertices in G, a family $(P_i : i \in [m])$ of (A,B)-paths in G is a *$(1/t)$-integral linkage of order m from A to B* if it is a $(1/t)$-integral packing. When $t = 1$, we simply call it a linkage. A *separation* of a directed graph G is an ordered pair (A,B) of sets of vertices in G such that $A \cup B = V(G)$ and there are no edges from $A \setminus B$ to $B \setminus A$. The *order* of the separation (A,B) is $|A \cap B|$.

Theorem 2.1. (Menger's theorem [19]) *Let A and B be sets of vertices in a directed graph G, and let k be a positive integer. Then G contains either a linkage of order k from A to B, or a separation (X,Y) of order less than k such that $A \subseteq X$ and $B \subseteq Y$.*

We will use the following observation.

Lemma 2.1. *Let t and m be positive integers, and let A and B be sets of vertices in a directed graph G. If there is a $(1/t)$-integral linkage $\mathcal{P}_1$ of order m from A to B, then there is a linkage $\mathcal{P}_2$ of order at least m/t from A to B such that $\bigcup \mathcal{P}_2$ is a subgraph of $\bigcup \mathcal{P}_1$.*

Proof. We may assume that $G = \bigcup \mathcal{P}_1$. Suppose that there is no linkage of order at least m/t from A to B in G. Then by Menger's theorem, there is a separation (C,D) of order less than m/t in G such that $A \subseteq C$ and $B \subseteq D$. Now, since $\mathcal{P}_1$ is $(1/t)$-integral, each vertex of $C \cap D$ is contained in at most t paths of $\mathcal{P}_1$. Since every path in $\mathcal{P}_1$ contains a vertex of $C \cap D$, the order of $\mathcal{P}_1$ is at most $(\lceil m/t \rceil - 1)t$, which is less than m. This contradicts the assumption that $\mathcal{P}_1$ has order m. □

2.3 Well-linked sets We will discuss two versions of well-linked sets. A set T of vertices in a directed graph G is a *well-linked set* if for all sets A and B of vertices in T with $|A| = |B|$, there is a linkage of order $|A|$ from A to B in G and there is a linkage of order $|A|$ from B to A in G. It is known that a directed graph has a large well-linked set if and only if it has large directed tree-width.

A set T of vertices in a directed graph G is an *r-externally-well-linked set* if for all disjoint sets A and B of vertices in T with $|A| = |B| \geq r$, there is a linkage of order $|A|$ from A to B in $G - (T \setminus (A \cup B))$ and there is a linkage of order $|A|$ from B to A in $G - (T \setminus (A \cup B))$. This concept naturally appears in the Erdős-Pósa type results, see [24] for instance.

For a positive integer q, a set S of vertices in a directed graph G is *q-linked* if for every set $X \subseteq V(G)$ with $|X| < q$, there is a unique strong component of $G - X$ that contains more than half of the vertices in S.

We use the following relation between r-externally-well-linked sets and q-linked sets.

Lemma 2.2. *Let q and r be positive integers with $q \geq r$. Every r-externally-well-linked set of order at least $6q-4$ is q-linked.*

Proof. Let T be an r-externally-well-linked set of size at least $6q-4$. To show that T is q-linked, we choose a set X of less than q vertices. Let $H_1, H_2, \ldots, H_m$ be the set of all strong components of $G - X$, and assume that it is ordered in an acyclic ordering. Suppose for contradiction that there is no strong component of $G - X$ containing more than half of the vertices in T.

We choose a minimum integer j such that $\bigcup_{i \in [j]} V(H_i)$ contains at least q vertices of T. As every strong component of $G - X$ has at most $|T|/2$ vertices of T, $\bigcup_{i \in [j]} V(H_i)$ contains at most $(q-1) + |T|/2$ vertices of T. Thus, $\bigcup_{i \in [m] \setminus [j]} V(H_i)$ contains at least

$$|T| - (q-1) - \left(q - 1 + \frac{|T|}{2}\right) = \frac{|T|}{2} - 2(q-1) \geq q$$

vertices of T. It implies that there is a linkage of order q from $T \cap (\bigcup_{i \in [m] \setminus [j]} V(H_i))$ to $T \cap (\bigcup_{i \in [j]} V(H_i))$. But all these q paths have to contain a vertex of X, which is not possible.

We conclude that T is q-linked. ☐

2.4 Directed tree-width We will not explicitly use directed tree-decompositions, but to state the directed grid theorem, we introduce directed tree-decompositions and directed tree-width.

An *arborescence* T is a directed graph obtained from an undirected rooted tree by orienting every edge away from the root. For $s, t \in V(T)$, we write $s <_T t$ if $s \neq t$ and there exists a path in T from s to t, and we write $s \leq_T t$ if $s <_T t$ or $s = t$. If $e \in E(T)$ is an edge with head s, we write $e <_T t$ if either $s = t$ or $s <_T t$.

A *directed tree-decomposition* of a directed graph G is a triple (T, β, γ), where T is an arborescence, $\beta \colon V(T) \to 2^{V(G)}$ and $\gamma \colon E(T) \to 2^{V(G)}$ are functions such that

1. $\{\beta(t) : t \in V(T)\}$ is a partition of $V(G)$ into non-empty sets,

2. if $e \in E(T)$ and $B := \bigcup\{\beta(t) : t \in V(T), e <_T t\}$, then there is no closed walk P in $G - \gamma(e)$ where the first and last vertices of P are in B and P uses a vertex of $G - (B \cup \gamma(e))$.

For any $t \in V(T)$ we define $\Gamma(t) := \beta(t) \cup \bigcup\{\gamma(e) : e \sim t\}$, where $e \sim t$ if e is incident with t.

The *width* of (T, β, γ) is the minimum integer w such that $|\Gamma(t)| \leq w + 1$ for all $t \in V(T)$. The *directed tree-width* of G, denoted by $\mathrm{dtw}(G)$, is the minimum integer w such that G has a directed tree-decomposition of width w.

We will use the following version of the directed grid theorem.

THEOREM 2.2. (KAWARABAYASHI AND KREUTZER, THEOREM 7.1 OF [15]) *There is a function $f_{wall} : \mathbb{N} \to \mathbb{R}$ such that for every positive integer w and every directed graph G, if G contains a well-linked set A of order $f_{wall}(w)$, then it contains a cylindrical wall W of order w, such that for every set F of w nails, there are w vertex-disjoint paths from F to A in G and from A to F in G.*

3 Lemmas on odd X-walks

In this section, we prove the following lemma, which will be used in the proof of Theorem 1.2.

LEMMA 3.1. *Let k be a positive integer, let G be a directed graph, and let $X \subseteq V(G)$. Then G contains either*

1. *a half-integral packing of k odd cycles,*

2. *a half-integral packing of k odd X-paths whose endvertices are pairwise disjoint, or*

3. *a set Y of at most $4k - 1$ vertices such that $G - Y$ has no odd X-walk.*

As a first step, we prove the following.

LEMMA 3.2. *Let ℓ be a positive integer, let G be a directed graph, and let $X \subseteq V(G)$. Then G contains either*

1. *a set of ℓ odd X-walks such that every vertex of G is used in at most two of them including the number of repetitions in each walk, or*

2. *a set Y of at most $\ell - 1$ vertices such that $G - Y$ has no odd X-walk.*

Proof. We obtain a new directed graph from G by splitting each vertex v into two vertices v_1 and v_2, and adding edges $(v_1, w_2), (v_2, w_1)$ if (v, w) is an edge of G. Formally, let D be the bipartite directed graph with bipartition (A, B) such that

- $A = \{v_1 : v \in V(G)\}$ and $B = \{v_2 : v \in V(G)\}$, and
- $E(D) = \{(v_1, w_2), (v_2, w_1) : (v, w) \in E(G)\}$.

Let $X_A := \{v_1 : v \in X\}$ and $X_B := \{v_2 : v \in X\}$. For a vertex $v_i \in V(D)$, we say that v is the original vertex of v_i.

Observe that X_A and X_B lie in distinct parts of D, and therefore, any path from X_A to X_B in D has odd length.

Assume that there is a family $\mathcal{Q}$ of ℓ vertex-disjoint paths from X_A to X_B in D. We obtain from each path $Q \in \mathcal{Q}$, a walk Q^* in G by taking the sequence of corresponding original vertices. Then $(Q^* : Q \in \mathcal{Q})$ is a family of ℓ odd X-walks in G such that every vertex of G is used in at most two of them including the number of repetitions in each walk. In this case, we get the first conclusion. Otherwise, by Menger's theorem, there is a separation (S, T) in D of order at most $\ell - 1$ such that $X_A \subseteq S$ and $X_B \subseteq T$. Let Y be the set of all vertices v in G for which v_1 or v_2 is in $S \cap T$. Then $|Y| \le \ell - 1$. Let $Y' := \{v_1, v_2 : v \in Y\}$. Clearly, $S \cap T \subseteq Y'$.

We claim that $G - Y$ has no odd X-walk. Assume there is an odd X-walk $(q_1, q_2, \ldots, q_m)$ in $G - Y$. Then $((q_1)_1, (q_2)_2, (q_3)_1, \ldots, (q_m)_2)$ is a walk in $D - Y'$ from X_A to X_B. Thus, there is an (X_A, X_B)-path in $D - Y'$. It is a contradiction, as $D - Y'$ is a subgraph of $D - (S \cap T)$. We conclude that $G - Y$ has no odd X-walk. ∎

Now, we prove Lemma 3.1.

Proof. [Proof of Lemma 3.1] We apply Lemma 3.2 to G and X with $\ell = 4k$. If G contains a set of at most $4k - 1$ vertices hitting all odd X-walks, then we are done. Thus, we may assume that there are $4k$ odd X-walks such that every vertex of G is used at most twice, including the number of repetitions in each walk. If there are k odd X-walks such that each of them contains an odd cycle, then we get a half-integral packing of k odd cycles. So we may assume that there is a set $\mathcal{Q}$ of at least $3k$ odd X-walks containing no odd cycles.

We verify that every closed odd walk contains an odd cycle. Let Q be a closed odd walk, and let $Q' = (q_1, q_2, \ldots, q_m)$ be a shortest closed odd walk in Q with $q_1 = q_m$. If there are no repeated vertices except endvertices, then Q' is an odd cycle. Assume that there is a pair of repeated vertices. We choose such a pair (q_i, q_j) with $|j - i|$ being minimum. If the length from q_i to q_j is odd, then Q' contains an odd cycle. Otherwise, it has even length, and by removing this part, we can find a shorter closed odd walk, a contradiction. It implies that each walk in $\mathcal{Q}$ is not closed.

Let $W \in \mathcal{Q}$, and let $W' = (w_1, w_2, \ldots, w_t)$ be a shortest odd walk in W where W and W' have the same endvertices. We claim that W' is an odd X-path. If W' has no repeated vertices, then W' is an odd X-path. Assume that there is a pair of repeated vertices. We choose such a pair (w_i, w_j) with $|j - i|$ being minimum. If the length from w_i to w_j is odd, then W' contains an odd cycle, a contradiction. Otherwise, it has even length, and by removing this part, we can find a shorter odd walk with the same endvertices. It contradicts the minimality of W'. As the endvertices of W' are distinct, we deduce that W' is an odd X-path.

So, G contains $3k$ odd X-paths such that each vertex of G is used in at most two of them. By greedily choosing one X-path and removing two possible X-paths sharing an endvertex with it, we can find k of them that have pairwise disjoint endvertices. ∎

4 Well-linked sets and r-externally-well-linked sets

In this section, we construct a useful structure from a large r-externally-well-linked set. A *bramble* in a directed graph G is a set $\mathcal{B}$ of strongly connected subgraphs of G such that for all $B_1, B_2 \in \mathcal{B}$, $V(B_1) \cap V(B_2) \neq \emptyset$. A *cover* of $\mathcal{B}$ is a set X of vertices in G such that $V(B) \cap X \neq \emptyset$ for all $B \in \mathcal{B}$. The *order* of $\mathcal{B}$ is the minimum size of a cover of $\mathcal{B}$.

Lemma 4.1. (Lemma 4.3 of [16]) *Let G be a directed graph and $\mathcal{B}$ be a bramble of G. Then there is a path P intersecting every set in $\mathcal{B}$.*

Lemma 4.2. *Let r and p be positive integers with $2p(p+1) \ge r$. If a directed graph G contains an r-externally-well-linked set T of size at least $12p(p+1)+1$, then there exist a path P in G and $A \subseteq V(P)$ with $|A| = p$ such that*

- *A is well-linked, and*
- *for every subset Z of T of size at least $|T|/2$, there is a linkage of order p from A to Z, and there is a linkage of order p from Z to A.*

Proof. Let T be an r-externally-well-linked set of size $m \geq 12p(p+1)+1$ in a directed graph G. As $2p(p+1) \geq r$, by Lemma 2.2, T is $2p(p+1)$-linked. We construct a bramble $\mathcal{B}$ of order at least $2p(p+1)$ as follows. By definition of a k-linked set, for every set X of less than $2p(p+1)$ vertices in G, $G-X$ has a unique strong component, say C_X, containing more than half of the vertices of T. We define

$$\mathcal{B} := \{C_X : X \subseteq V(G), |X| < 2p(p+1)\}.$$

Since any two distinct sets in $\mathcal{B}$ intersect on T, $\mathcal{B}$ is a bramble. The order of $\mathcal{B}$ is at least $2p(p+1)$, because for every set Y of less than $2p(p+1)$ vertices, Y does not hit C_Y in $\mathcal{B}$.

By Lemma 4.1, there is a path P intersecting every element of $\mathcal{B}$. We now find the required set A in P. We construct sequences of subpaths $P_1, \ldots, P_{2p}$ of P and brambles $\mathcal{B}_1, \ldots, \mathcal{B}_{2p} \subseteq \mathcal{B}$.

For a subpath Q of P, we consider some subfamily $\mathcal{B}_Q$ of $\mathcal{B}$ such that $\mathcal{B}_Q \subseteq \{B \in \mathcal{B} : V(B) \cap V(Q) \neq \emptyset\}$. Clearly, $\mathcal{B}_Q$ is a bramble. We will use the fact that if

- Q^* is another subpath of P with $V(Q^*) \setminus V(Q) = \{z\}$, and
- $\mathcal{B}_Q \subseteq \mathcal{B}_{Q^*} \subseteq \{B \in \mathcal{B} : V(B) \cap V(Q^*) \neq \emptyset\}$,

then the order of $\mathcal{B}_{Q^*}$ is at most the order of $\mathcal{B}_Q$ plus one, because all sets in $\mathcal{B}_{Q^*} \setminus \mathcal{B}_Q$ can be hit by z.

Let P_1 be the minimal initial subpath of P such that $\mathcal{B}_1 = \{B \in \mathcal{B} : V(B) \cap V(P_1) \neq \emptyset\}$ is a bramble of order $p+1$.

Now, suppose that for some $i < 2p$, sequences $P_1, \ldots, P_i$ and $\mathcal{B}_1, \ldots, \mathcal{B}_i$ have been constructed. Let v be the last vertex of P_i and s be the successor of v in P. Let P_{i+1} be the minimal subpath of P starting at s such that

$$\mathcal{B}_{i+1} = \left\{ B \in \mathcal{B} : V(B) \cap \left(\bigcup_{j \in [i]} V(P_j) \right) = \emptyset \text{ and } V(B) \cap V(P_{i+1}) \neq \emptyset \right\}$$

has order $p+1$. As $\mathcal{B}$ has order $2p(p+1)$, such sequences $P_1, \ldots, P_{2p}$ and $\mathcal{B}_1, \ldots, \mathcal{B}_{2p} \subseteq \mathcal{B}$ exist. For each $i \in [p]$, let a_i be the first vertex of P_{2i}, and let $A = \{a_i : i \in [p]\}$.

We verify that A is well-linked. Let X and Y be subsets of A with $|X| = |Y| = q$. Let $X = \{a_{i_t} : t \in [q]\}$ and $Y = \{a_{j_t} : t \in [q]\}$. Note that $q \leq p$. We claim that there is a linkage from X to Y of order q.

Suppose for contradiction that there is no linkage of order q from X to Y. Then by Menger's theorem, there is a separation (C, D) of order less than q in G such that $X \subseteq C$ and $Y \subseteq D$. As $|C \cap D| < q \leq p$, for each $j \in [2p]$, $C \cap D$ is not a hitting set of $\mathcal{B}_j$. Also, $C \cap D$ does not meet one of the paths in $\{P_{2i_t} : t \in [q]\}$. So, there exist $\ell \in [q]$ and $B_1 \in \mathcal{B}_{2i_\ell}$ such that

$$(C \cap D) \cap (V(P_{2i_\ell}) \cup V(B_1)) = \emptyset.$$

Similarly, since $C \cap D$ does not meet one of the sets in $\{V(P_{2i_t - 1}) \cup \{a_{i_t}\} : t \in [q]\}$, there exist $\ell' \in [q]$ and $B_2 \in \mathcal{B}_{2j_{\ell'}-1}$ such that

$$(C \cap D) \cap (V(P_{2j_{\ell'}-1}) \cup \{a_{j_{\ell'}}\} \cup V(B_2)) = \emptyset.$$

On the other hand, by the construction of $\mathcal{B}$, B_1 and B_2 intersect. Since each of B_1 and B_2 is strongly connected, $B_1 \cup B_2$ is also strongly connected. This implies that there is a path from a_{i_ℓ} to $a_{j_{\ell'}}$ in

$$B_1 \cup B_2 \cup P_{2i_\ell} \cup G[V(P_{2j_{\ell'}-1}) \cup \{a_{j_{\ell'}}\}],$$

which avoids $C \cap D$, a contradiction. We conclude that A is well-linked.

Lastly, we verify the second bullet. Let $Z \subseteq T$ with $|Z| \geq |T|/2$. Suppose that there is no linkage of order p from A to Z in G. Then, by Menger's theorem, there is a separation (C, D) of order less than p with $A \subseteq C$ and $Z \subseteq D$.

As $|C \cap D| < p$, there exist $\ell \in [p]$ and $B \in \mathcal{B}_{2i_\ell}$ such that $(C \cap D) \cap (V(P_{2i_\ell}) \cup V(B)) = \emptyset$. Since $a_{i_\ell} \in C \setminus D$, we have $V(B) \subseteq C \setminus D$ and B does not intersect $Z \subseteq D$. It contradicts the fact that every set of $\mathcal{B}$ contains more than half of the vertices in T.

We conclude that there is a linkage of order p from A to Z, and in the same way, we can show that there is a linkage of order p from Z to A. ∎

5 A Half-integral Erdős-Pósa theorem for odd cycles

In this section, we prove Theorem 1.2.

We verify that if $\alpha_{k-1} \neq \infty$ and a directed graph G with $\nu_2(G) < k$ has a hitting set T of directed odd cycles with $|T| = \tau(G)$, then T is $2\alpha_{k-1}$-externally-well-linked.

LEMMA 5.1. *Let $k \geq 2$ be an integer such that α_{k-1} exists. Let G be a directed graph with $\nu_2(G) < k$ and let $T \subseteq V(G)$ with $|T| = \tau(G)$ meeting all odd cycles in G. Then T is $(2\alpha_{k-1})$-externally-well-linked.*

Proof. Let $A, B \subseteq T$ be disjoint sets with $|A| = |B| = r \geq 2\alpha_{k-1}$. We claim that there is a linkage in G from A to B of order r containing no vertex in $T \setminus (A \cup B)$. Suppose that there is no such a linkage.

Let $Z = T \setminus (A \cup B)$. By Menger's theorem applied to $G - Z$, there is a separation (X, Y) of G with $A \subseteq X$, $B \subseteq Y$ such that $Z \subseteq X \cap Y$ and $|(X \cap Y) \setminus Z| < r$. Let $W := (X \cap Y) \setminus Z$.

Let $T_A := (T \setminus A) \cup W$ and $T_B := (T \setminus B) \cup W$. Note that T_A is a set obtained from T by removing $A \setminus B$ and adding $W \setminus T$, because $A \cap B \subseteq W$. On the other hand, we have

$$|W \setminus T| < r - |A \cap B| = |A| - |A \cap B| = |A \setminus B|.$$

Therefore, $|T_A| < |T| = \tau(G)$ and by a similar reason, $|T_B| < |T| = \tau(G)$. Thus, none of T_A and T_B is a hitting set for odd cycles.

It means that there are an odd cycle C_A in $G - T_A$, and an odd cycle C_B in $G - T_B$. Since T is a hitting set for odd cycles, C_A must contain a vertex of A and C_B must contain a vertex of B. So, $G - Y$ contains C_A and $G - X$ contains C_B while $V(G - Y) \cap V(G - X) = \emptyset$.

By the definition of α_{k-1}, $G - Y$ has a hitting set M_Y of size at most α_{k-1}, and $G - X$ has a hitting set M_X of size at most α_{k-1}. Since A and B are disjoint, $|T| - |Z| = 2r$. It implies that $M_X \cup M_Y \cup (X \cap Y)$ is a hitting set for odd cycles in G of size at most

$$2\alpha_{k-1} + ((r-1) + |Z|) = 2\alpha_{k-1} + (|T| - r) - 1.$$

So, $\tau(G) \leq 2\alpha_{k-1} + \tau(G) - r - 1$ and $r < 2\alpha_{k-1}$, which contradicts the choice of r. ∎

As we discussed in the introduction, we will consider a set N of nails in a bipartite cylindrical wall W', and apply Lemma 3.1 for odd N-walks. When Lemma 3.1 outputs a hitting set for odd N-walks, the following proposition will imply that there is a small hitting set for odd cycles.

PROPOSITION 5.1. *Let r, t, and w be positive integers with $w \geq 2t$ and $t \geq r$. Let G be a directed graph, and let T be a set of at least $6t - 4$ vertices in G such that T is a hitting set of odd cycles, and it is r-externally-well-linked. Let W be a cylindrical wall of order w in G satisfying that for every subset Z of T of size at least $|T|/2$ and every set F of w nails in W, there is a linkage of order at least $w/2$ from Z to F, and there is a linkage of order at least $w/2$ from F to Z. Let N be a set of nails of W with $|N| \geq w^2$.*

If X is a set of less than t vertices in G hitting all odd N-walks, then G has a set of at most $3(t-1)$ vertices hitting all odd cycles.

Proof. Let $\{H_1, H_2, \ldots, H_m\}$ be the set of all strong components of $G - X$, and assume that it is ordered in an acyclic ordering, that is, for distinct $i, j \in [m]$, there can be an edge from H_i to H_j only if $i < j$.

As $t \geq r$ and T is an r-externally-well-linked set of size at least $6t - 4$, by Lemma 2.2, T is t-linked. Since T is t-linked and X has size less than t, $G - X$ has a unique strong component, say H_x, having more than half of the vertices in T. Note that H_x contains at least t vertices of T, as $3t - 2 \geq t$. If $\bigcup_{i \in [x-1]} V(H_i)$ contains at least t vertices of T, then since T is r-externally-well-linked and $t \geq r$, there is a linkage of order t from $T \cap V(H_x)$ to $T \cap (\bigcup_{i \in [x-1]} V(H_i))$. But every path in the linkage must contain a vertex of X, and it contradicts the assumption that $|X| < t$. Therefore, $\bigcup_{i\in[x-1]} V(H_i)$ contains less than t vertices of T, and similarly, $\bigcup_{i\in[m]\setminus[x]} V(H_i)$ contains less than t vertices of T.

As $w \geq 2t$, there is a set $\mathcal{C}$ of at least $w-(t-1) \geq w/2+1$ columns of W containing no vertex of X. We claim that for each $C \in \mathcal{C}$, C is contained in H_x. Let F be a set of w nails of W that are contained in C. Note that $V(H_x) \cap T$ is a subset of T of size at least $|T|/2$. So, by the assumption, there is a linkage of order at least $w/2 \geq t$ from $V(H_x) \cap T$ to F. Since C does not contain a vertex of X and C is strongly connected, C is contained in one of the strong components in $\{H_i : i \in [m]\}$. But if C is contained in a strong component other than H_x, then either there is no linkage of order t from $V(H_x) \cap T$ to F, or there is no linkage of order t from F to $V(H_x) \cap T$. This is a contradiction. Therefore, the claim holds.

In particular, it implies that H_x contains $w/2+1$ columns of W. Since N contains at least half of the nails of W, H_x contains at least two nails of W in N, say v and z.

We claim that H_x contains no odd cycle. Suppose for contradiction that H_x contains an odd cycle H. Since H_x is strongly connected, there is a path P_v from v to H in H_x, and there is a path P_z from H to z in H_x. In $H \cup P_v \cup P_z$, there are two walks from v to z, namely, one is obtained by using the shortest path in H from the endvertex of P_v in H to the endvertex of P_z in H, and the other one is obtained by traversing H one more time. As H is an odd cycle, the two walks have different parities. So G contains an odd walk between two nails of W that is contained in H_x, which contradicts the assumption that X hits all odd N-walks. Thus, H_x has no odd cycle.

For other strong components $H_y \neq H_x$, if $T \cap V(H_y)$ intersects all odd cycles in H_y. Therefore, $(T \cap (V(G) \setminus V(H_x))) \cup X$ hits all odd cycles. We remind that $\bigcup_{i\in[x-1]} V(H_i)$ contains less than t vertices of T, and similarly, $\bigcup_{i\in[m]\setminus[x]} V(H_i)$ contains less than t vertices of T. Thus, $(T \cap (V(G) \setminus V(H_x))) \cup X$ has size at most $3(t-1)$. ∎

By Proposition 5.1, we may assume that Lemma 3.1 outputs a large half-integral packing of odd paths whose endvertices are distinct nails of W'. We will give a formal proof of this in the proof of Theorem 1.2. The rest of this section devotes to find a half-integral packing of k odd cycles from it.

Proposition 5.2. *There is a function $g_{path} : \mathbb{N} \to \mathbb{R}$ satisfying the following. Let k be a positive integer, and let W be a bipartite cylindrical wall of order at least $(2k+3)(6g_{path}(k)+1)$ in a directed graph G. Let N be a set of nails of W that are contained in the same part of the bipartition of W. Let $\mathcal{U}$ be a half-integral packing of $12(g_{path}(k)-1)+1$ odd N-paths in G such that the endvertices of paths in $\mathcal{U}$ are disjoint. Then G contains a half-integral packing of k odd cycles.*

We prove two auxiliary lemmas, and then prove Proposition 5.2. Let W be a bipartite cylindrical wall in a directed graph G. For $v, w \in V(W)$, a walk P in G from v to w is *parity-breaking* if the parity of the length of P is different from the parity of a path from v to w in W. If the parities are the same, then we say that P is *parity-preserving.*

Lemma 5.2. *Let G be a directed graph, and let W be a bipartite cylindrical wall in G. If P is a parity-breaking walk for W from a to b, then either $G[V(P)]$ contains an odd cycle, or it contains a parity-breaking path from a to b.*

Proof. Let $Q = (q_1, q_2, \dots, q_m)$ be a shortest parity-breaking walk from a to b contained in $G[V(P)]$. If Q has no repeated vertices, then Q is a parity-breaking path. Assume that there is a pair of repeated vertices. We choose such a pair (q_i, q_j) with $|j-i|$ is minimum. If the length from q_i to q_j is odd, then $G[V(P)]$ contains an odd cycle. Otherwise, it has even length, and by removing this part, we can find a shorter parity-breaking walk with same endvertices. It contradicts the minimality of Q. ∎

Lemma 5.3. *Let k and m be positive integers. Let G be a directed graph, W be a bipartite cylindrical wall in G, and let A, B, C, D be disjoint subsets of $V(W)$ of size m. Let $\mathcal{Q}$ be a linkage of order m from A to B in W, and let $\mathcal{R}$ be a linkage of order m from C to D in W. Let $\mathcal{U}$ be a half-integral packing of m parity-breaking paths from B to C in G. If $m \geq 8k$, then there is either*

- *a half-integral packing of k odd cycles, or*

- *a half-integral packing of k parity-breaking paths from A to D in $(\bigcup \mathcal{Q}) \cup (\bigcup \mathcal{R}) \cup (\bigcup \mathcal{U})$ such that the first vertices of the paths are all distinct and the last vertices of the paths are all distinct.*

Proof. We construct a graph F_1 starting from the vertex set $V(W)$ and the empty edge set as follows.

- For every edge (u, v) in $E(W)$, we add a new vertex x_{uv} and two edges (u, x_{uv}) and (x_{uv}, v).
- For every W-path P from a vertex u to a vertex v that is a subpath of some path in $\mathcal{U}$, if P is parity-breaking, then we add an edge (u, v), and otherwise, we add a vertex z_{uv} and two edges (u, z_{uv}) and (z_{uv}, v).
- For every $q \in A$, we add two new vertices q^1 and q^2 and add edges (q^1, q^2) and (q^2, q).
- For every $r \in D$, we add two new vertices r^1 and r^2 and add edges (r, r^2) and (r^2, r^1).

We assign $A_1 := \{q^1 : q \in A\}$ and $D_1 := \{r^1 : r \in D\}$. Observe that a walk between two vertices of W in $(\bigcup \mathcal{Q}) \cup (\bigcup \mathcal{R}) \cup (\bigcup \mathcal{U})$ is parity-breaking if and only if the corresponding walk in F_1 is odd. Note that two paths in $\mathcal{U}$ may share a vertex on a path in $\mathcal{Q} \cup \mathcal{R}$. Thus, there is a set of m odd walks from A_1 to D_1 in F_1 such that

- every vertex of G is used at most 4 times, and
- for each vertex $w \in A_1 \cup D_1$, there is exactly one walk containing w in the m odd walks.

Now, we obtain a bipartite directed graph F_2 with bipartition (X, Y) from F_1 such that

- $X = \{v_1 : v \in V(F_1)\}$ and $Y = \{v_2 : v \in V(F_1)\}$,
- $E(F_2) = \{(v_1, w_2), (v_2, w_1) : (v, w) \in E(F_1)\}$.

Let $A_2 := \{q^1_1 : q \in A\}$ and $D_2 := \{r^1_2 : r \in D\}$.

In F_2, there is a set of m walks from A_2 to D_2 in F_2 such that every vertex is used at most 4 times, because each walk from A_2 to D_2 in F_2 corresponds to an odd walk from A_1 to D_1 in F_1. So, there is a 1/4-integral packing of m paths from A_2 to D_2 in F_2. Since $m \geq 8k$, by Lemma 2.1, there is a linkage of order $2k$ from A_2 to D_2 in F_2.

It implies that there is a set $\mathcal{L}_1$ of $2k$ odd walks from A_1 to D_1 in F_1 such that

- every vertex of F_1 is used at most twice,
- the first vertices of paths in $\mathcal{L}_1$ are all distinct, and
- the last vertices of paths in $\mathcal{L}_1$ are all distinct.

Furthermore, there is a set $\mathcal{L}_2$ of $2k$ parity-breaking walks from A to D such that

- every vertex of G is used at most twice,
- the first vertices of paths in $\mathcal{L}_2$ are all distinct, and
- the last vertices of paths in $\mathcal{L}_2$ are all distinct.

Now, by Lemma 5.2, either there is a half-integral packing of k odd cycles, or there is a half-integral packing of k parity-breaking paths from A to D in $(\bigcup \mathcal{Q}) \cup (\bigcup \mathcal{R}) \cup (\bigcup \mathcal{U})$, where the first vertices are all distinct, and the last vertices are all distinct. ∎

Proof. [Proof of Proposition 5.2] Let w be the order of W. We set

- $g_3(k) = 8k$
- $g_2(k) = 4g_3(k)$,
- $g_1(k) = (2g_2(k) - 1)^2 + 1$,
- $g_{path}(k) = g(k) = (2g_1(k) - 1)^2 + 1$.

Since every path in $\mathcal{U}$ is an odd path between two nails in the same part of the bipartition of W, every path in $\mathcal{U}$ is parity-breaking. We start with finding subpaths of some paths in $\mathcal{U}$ so that they are still parity-breaking and do not intersect many N^W-paths in W.

CLAIM 1. *For every $t \in [g(k)]$, there is a half-integral packing of parity-breaking paths $U_1, U_2, \dots, U_t$ for W such that for each $i \in [t]$,*

(i) $\bigcup_{j\in[i]} U_j$ intersects at most $6i$ N^W-paths in W, and

(ii) $\bigcup_{j\in[i-1]} U_j$ does not intersect any N^W-path containing an endvertex of U_i.

PROOF OF THE CLAIM: We prove the statement by induction on $1 \le t \le g(k)$. Assume that such a set of paths $U_1, \dots, U_{t-1}$ has been constructed for some $t \le g(k)$. By Property (i), $\bigcup_{j\in[t-1]} U_j$ intersects at most $6(t-1) \le 6(g(k)-1)$ N^W-paths in W. Let $\mathcal{A}$ be the set of all N^W-paths in W that contain a vertex of $\bigcup_{j\in[t-1]} U_j$, and let $B := \bigcup_{Q\in\mathcal{A}} V(Q)$. Note that B contains at most $12(g(k)-1)$ nails. Since $|\mathcal{U}| = 12(g(k)-1)+1$ and the endvertices of paths in $\mathcal{U}$ are disjoint, there is a path $U \in \mathcal{U}$ such that the endvertices of U are not contained in B.

Let $U = u_1u_2\cdots u_m$. Let $\mathcal{Q}$ be the set of all subpaths U^* of U where its endvertices are in $V(W) \setminus B$ and all internal vertices are not in $V(W) \setminus B$.

Note that the paths in $\mathcal{Q}$ are pairwise edge-disjoint, and $\bigcup_{Q\in\mathcal{Q}} E(Q) = E(U)$. Since U is parity-breaking, $\mathcal{Q}$ contains at least one parity-breaking path. Let U' be a parity-breaking path in $\mathcal{Q}$. Note that every vertex of W is contained in at most three N^W-paths. Since all the internal vertices of U' are not contained in $V(W) \setminus B$, $U_1 \cup U_2 \cup \cdots \cup U_{t-1} \cup U'$ intersects at most $6(t-1)+6 \le 6t$ N^W-paths in W, and the N^W-paths containing the endvertices of U' are not used by paths in $U_1, \dots, U_{t-1}$. Thus, the claim holds. ◊

By the claim, there is a half-integral packing of parity-breaking paths $U_1, U_2, \dots, U_{g(k)}$ that intersect at most $6g(k)$ N^W-paths.

Recall that the order of W is at least $(2k+3)(6g(k)+1)$, and each N^W-path may intersect at most two columns and at most two rows. As

$$(2k+3)(6g(k)+1) - 12g(k) \ge (2k+1)(6g(k)+1)+1,$$

there is a set of $2k+2$ consecutive columns, say $C_{z+1}, C_{z+2}, \dots, C_{z+2k+2}$, containing no vertices of B. Also, since

$$(4k+6)(6g(k+1)+1) - 12g(k) \ge (4k+4)(6g(k+1)+1)+1,$$

there is a set of $4k+5$ consecutive rows containing no vertices of B. Among these $4k+5$ rows, we choose $4k+4$ consecutive rows $P_{y+1}, P_{y+2}, \dots, P_{y+4k+4}$ such that P_{y+1} is a row traversing from C_1 to C_w. We define

$$W^* = P_{y+1} \cup P_{y+2} \cup \cdots \cup P_{y+4k+4} \cup C_{z+1} \cup C_{z+2} \cup \cdots \cup C_{z+2k+2}.$$

See Figure 4 for an illustration of W^*. Observe that $V(W^*) \cap B = \emptyset$.

Let L be the bijection from the set of all nails of W to $[w] \times [2w] \times [2]$ satisfying the following.

- Let $i \in [w]$ and $j \in [2w]$. When we traverse C_i from P_1 to P_{2w}, C_i contains two nails of each P_j, and for the first vertex v, $L(v) = (i, j, 1)$ and for the second vertex v, $L(v) = (i, j, 2)$.

For each $i \in [g(k)]$, we define the following.

- Let p_i and q_i be the endvertices of U_i such that U_i is a path from p_i to q_i.
- If p_i is a nail, then let $p_i^* := p_i$, $A_i := G[\{p_i\}]$, and $(a_i, b_i, c_i) := L(p_i^*)$. Otherwise, let p_i^* be the first vertex of the N^W-path in W containing p_i, and let A_i be the subpath from p_i^* to p_i in the N^W-path, and $(a_i, b_i, c_i) := L(p_i^*)$.
- If q_i is a nail, then let $q_i^* := q_i$, $D_i := G[\{q_i\}]$, and $(d_i, e_i, f_i) := L(q_i^*)$. Otherwise, let q_i^* be the last vertex of the N^W-path in W containing q_i, and let D_i be the subpath from q_i to q_i^* in the N^W-path, and $(d_i, e_i, f_i) := L(q_i^*)$.

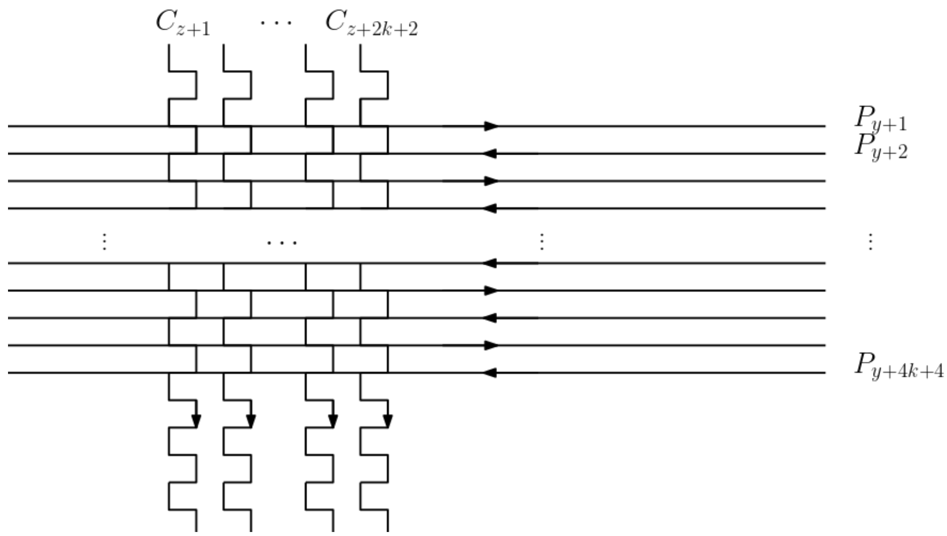


Figure 4: Selected consecutive columns and rows in the proof of Proposition 5.2.

Since $g(k) = (2g_1(k) - 1)^2 + 1$, there is a subset $I_1 \subseteq [g(k)]$ of size $2g_1(k)$ such that either

- all integers in $(a_i : i \in I_1)$ are distinct, or
- all integers in $(a_i : i \in I_1)$ are the same.

There is a subset $I_2 \subseteq I_1$ with $|I_2| \geq g_1(k)$ such that all integers in $(c_i : i \in I_2)$ are the same. Since all integers in $(c_i : i \in I_2)$ are the same, all integers in $(b_i : i \in I_2)$ are distinct. Furthermore, as $g_1(k) = (2g_2(k) - 1)^2 + 1$, there is a subset $I_3 \subseteq I_2$ of size $2g_2(k)$ such that either

- all integers in $(d_i : i \in I_3)$ are distinct, or
- all integers in $(d_i : i \in I_3)$ are the same.

There is a subset $I_4 \subseteq I_3$ of size $g_2(k)$ such that all integers in $(f_i : i \in I_4)$ are the same. Since all integers in $(f_i : i \in I_4)$ are the same, all integers in $(e_i : i \in I_3)$ are distinct.

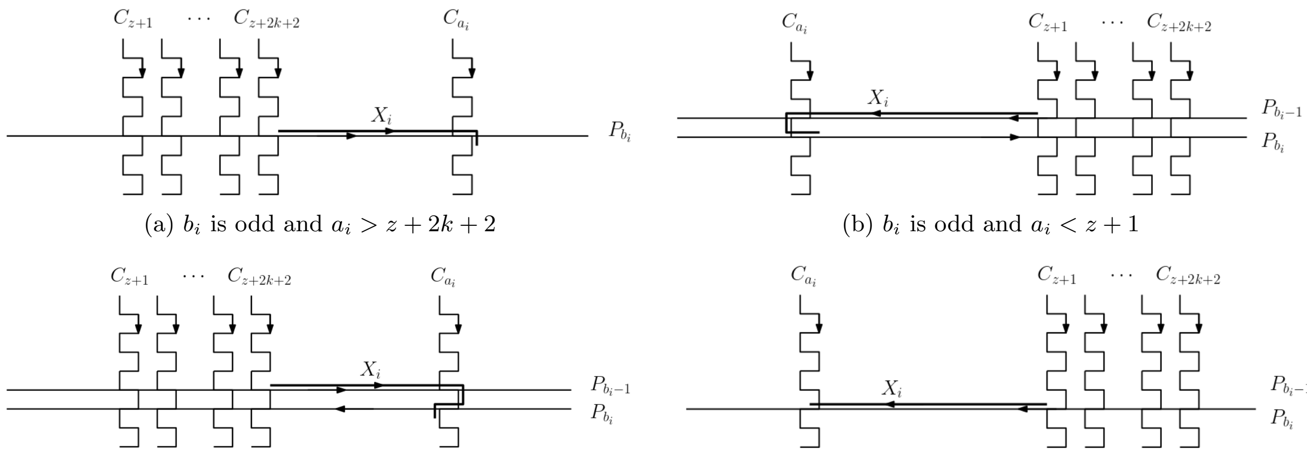
Lastly, we take a subset $I_5 \subseteq I_4$ of size $g_2(k)/4 = g_3(k)$ such that

- if all integers in $(a_i : i \in I_4)$ are the same, then $y + 4k + 5 \notin (b_i : i \in I_5)$ (as modulo $2w$) and $|b_{i_1} - b_{i_2}| \geq 2 \pmod{2w}$ for all distinct $i_1, i_2 \in I_5$, and
- if all integers in $(d_i : i \in I_3)$ are the same, then $y \notin (e_i : i \in I_5)$ (as modulo $2w$) and $|e_{i_1} - e_{i_2}| \geq 2 \pmod{2w}$ for all distinct $i_1, i_2 \in I_5$.

We can greedily choose elements of I_5 from I_4.

Now, we construct a linkage $\{X_i : i \in I_5\}$ from $V(W^*)$ to $\{p_i : i \in I_5\}$ in W, and a linkage $\{Y_i : i \in I_5\}$ from $\{q_i : i \in I_5\}$ to $V(W^*)$ in W. We will apply Lemma 5.3, together with the half-integral linkage $\{U_1, \ldots, U_{g(k)}\}$.

- Assume that all integers in $(a_i : i \in I_5)$ are distinct. Let X_i be the path starting at $L^{-1}(a_i, y + 4k + 4, 2)$, traversing to p_i^* in C_{a_i}, and traversing to p_i in A_i.
- Otherwise, all integers in $(a_i : i \in I_5)$ are the same and all integers in $(b_i : i \in I_5)$ are distinct. We divide into four cases. See Figure 5 for illustrations.
  - (b_i is odd and $a_i > z + 2k + 2$.) Let X_i be the path starting at $L^{-1}(z + 2k + 2, b_i, 2)$, traversing to p_i^* in P_{b_i}, and traversing to p_i in A_i.
  - (b_i is odd and $a_i < z + 1$.) Let X_i be the path starting at $L^{-1}(z + 1, b_i - 1, 2)$, traversing to $L^{-1}(a_i, b_i - 1, 2)$ in $P_{b_i - 1}$, traversing to p_i^* in C_{a_i}, and traversing to p_i in A_i.



(a) b_i is odd and $a_i > z + 2k + 2$ (b) b_i is odd and $a_i < z + 1$

(c) b_i is even and $a_i > z + 2k + 2$ (d) b_i is even and $a_i < z + 1$

Figure 5: The construction of X_i when all integers in $(a_i : i \in I_5)$ are the same.

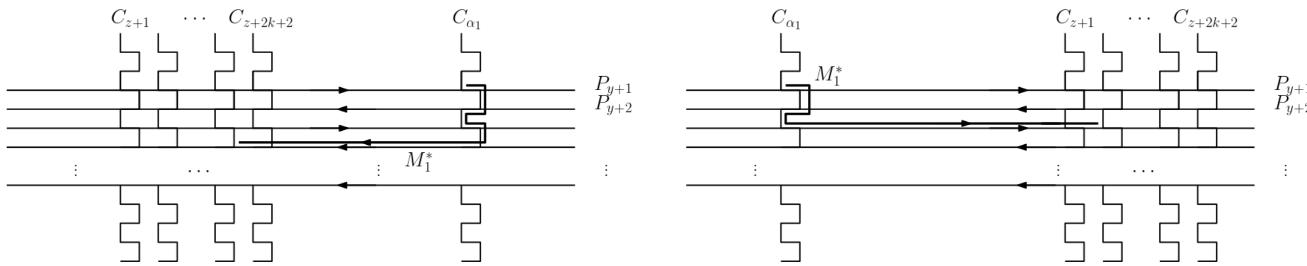
- (b_i is even and $a_i > z + 2k + 2$.) Let X_i be the path starting at $L^{-1}(z + 2k + 2, b_i - 1, 2)$, traversing to $L^{-1}(a_i, b_i - 1, 2)$ in P_{b_i-1}, traversing to p_i^* in C_{a_i}, and traversing to p_i in A_i.
- (b_i is even and $a_i < z + 1$.) Let X_i be the path starting at $L^{-1}(z + 1, b_i, 2)$, traversing to p_i^* in P_{b_i}, and traversing to p_i in A_i.

We observe that all paths in $\{X_i : i \in I_5\}$ are pairwise vertex-disjoint. When all integers in $(a_i : i \in I_5)$ are distinct, each path A_i is starting from a vertex of C_{a_i}, but does not meet other column of W. So, all paths in $\{A_i : i \in I_5\}$ are pairwise vertex-disjoint and all paths in $\{X_i : i \in I_5\}$ are pairwise vertex-disjoint. The case when all integers in $(a_i : i \in I_5)$ are the same is similar, and for the second and third subcases of the second case, we additionally use the fact that $y + 4k + 5 \notin (b_i : i \in I_5)$ (as modulo $2w$) and $|b_{i_1} - b_{i_2}| \geq 2 \pmod{2w}$ for all distinct $i_1, i_2 \in I_5$.

We define paths T_i in a symmetric way.

- Assume that all integers in $(d_i : i \in I_5)$ are distinct. Let Y_i be the path starting at q_i, traversing to q_i^* in D_i, and traversing to $L^{-1}(d_i, y + 1, 1)$ in C_{d_i}.
- Otherwise, all integers in $(d_i : i \in I_5)$ are the same and all integers in $(e_i : i \in I_5)$ are distinct. We divide into four cases.
  - (e_i is odd and $d_i > z + 2k + 2$.) Let Y_i be the path starting at q_i, traversing to q_i^* in D_i, traversing to $L^{-1}(d_i, e_i + 1, 1)$ in C_{d_i}, and traversing to $L^{-1}(z + 2k + 2, e_i + 1, 1)$ in P_{e_i+1}.
  - (e_i is odd and $d_i < z + 1$.) Let Y_i be the path starting at q_i, traversing to q_i^* in D_i, and traversing to $L^{-1}(z + 1, e_i, 1)$ in P_{e_i}.
  - (e_i is even and $d_i > z+2k+2$.) Let Y_i be the path starting at q_i, traversing to q_i^* in D_i, and traversing to $L^{-1}(z + 2k + 2, e_i, 1)$ in P_{e_i}.
  - (e_i is even and $d_i < z + 1$.) Let Y_i be the path starting at q_i, traversing to q_i^* in D_i, traversing to $L^{-1}(d_i, e_i + 1, 1)$ in C_{d_i}, and traversing to $L^{-1}(z + 1, e_i + 1, 1)$ in P_{e_i+1}.

We observe that all paths in $\{Y_i : i \in I_5\}$ are pairwise vertex-disjoint. When all integers in $(d_i : i \in I_5)$ are distinct, each path D_i is ending at a vertex of C_{d_i}, but does not meet other column of W. So, all paths in $\{D_i : i \in I_5\}$ are pairwise vertex-disjoint and all paths in $\{Y_i : i \in I_5\}$ are pairwise vertex-disjoint. The case when all integers in $(d_i : i \in I_5)$ are the same is similar, and for the first and fourth subcases of the second case, we additionally use the fact that $y \notin (e_i : i \in I_5)$ (as modulo $2w$) and $|e_{i_1} - e_{i_2}| \geq 2 \pmod{2w}$ for all distinct $i_1, i_2 \in I_5$.



(a) $\alpha_i > z + 2k + 2$ (b) $\alpha_i < z + 1$

Figure 6: The construction of M_1^* when $r_1 \in V(P_{y+1})$.

Now, we apply Lemma 5.3 for linkages $\{X_i : i \in I_5\}$, $\{Y_i : i \in I_5\}$, and a half-integral packing of parity-breaking paths $\{U_i : i \in I_5\}$. Since $|I_5| = g_3(k) = 8k$, by Lemma 5.3, there is either a half-integral packing of k odd cycles, or a half-integral packing of parity-breaking paths $\mathcal{Z} = \{Z_i : i \in [k]\}$ such that the first vertices of paths in $\mathcal{Z}$ are all distinct, and the last vertices of paths in $\mathcal{Z}$ are all distinct. For each $i \in [k]$, let s_i and r_i be the first and last vertices of Z_i, respectively. Because paths in $\{X_i, Y_i, U_i : i \in I_5\}$ do not use any edge of W^*, $\{s_i : i \in [k]\}$ and $\{r_i : i \in [k]\}$ cannot share a vertex.

Let $\text{clos}(W^*)$ be the subwall of W that is the union of all N^W-paths whose both endvertices are in W^*. Now, we construct a path Z_i^* for each $i \in [k]$ in $\text{clos}(W^*)$ so that $Z_i \cup Z_i^*$ is an odd cycle. and Z_i^* does not intersect $\bigcup_{j\in[k]} Z_j$ except the vertices in $\{r_i, s_i\}$.

- Observe that r_i is contained in $P_{y+1} \cup C_{z+1} \cup C_{z+2k+2}$. Let C_{α_i} and P_{β_i} be the column and row containing r_i of W, respectively.
  - (Type 1. $r_i \in V(P_{y+1})$.) See Figure 6 for illustrations. If $\alpha_i > z + 2k + 2$, then let M_i^* be the path starting at r_i, traversing to $L^{-1}(\alpha_i, y+2i+2, 1)$ in C_{α_i}, and traversing to $L^{-1}(z+2k+2-i, y+2i+2, 1)$ in P_{y+2i+2}. If $\alpha_i < z + 1$, then let M_i^* be the path starting at r_i, traversing to $L^{-1}(\alpha_i, y + 2i + 2, 1)$ in C_{α_i}, and traversing to $L^{-1}(z + 1 + i, y + 2i + 2, 1)$ in P_{y+2i+2}.
  - (Type 2. $r_i \in V(C_{z+1})$.) Let M_i^* be the path starting at r_i and traversing to $L^{-1}(z + 1 + i, \beta_i, 1)$ in P_{β_i}.
  - (Type 3. $r_i \in V(C_{z+2k+2})$.) Let M_i^* be the path starting at r_i and traversing to $L^{-1}(z+2k+2-i, \beta_i, 1)$ in P_{β_i}.
- The vertex s_i is contained in $P_{y+4k+4} \cup C_{z+1} \cup C_{z+2k+2}$. Let C_{η_i} and P_{θ_i} be the column and row containing r_i of W, respectively.
  - (Type 1. $s_i \in V(P_{y+4k+4})$.) If $\eta_i > z + 2k + 2$, then let M_i^{**} be the path starting at $L^{-1}(z + 2k + 2 - i, y + 4k + 3 - 2i, 2)$, traversing to $L^{-1}(\eta_i, y + 4k + 3 - 2i, 2)$ in $P_{y+4k+3-2i}$, and traversing to $L^{-1}(\eta_i, y + 4k + 4, 2)$ in C_{η_i}. If $\eta_i < z + 1$, then let M_i^{**} be the path starting at $L^{-1}(z+1+i, y+4k+4-2i, 2)$, traversing to $L^{-1}(\eta_i, y+4k+4-2i, 2)$ in $P_{y+4k+4-2i}$, and traversing to $L^{-1}(\eta_i, y + 4k + 4, 2)$ in C_{η_i}.
  - (Type 2. $s_i \in V(C_{z+1})$.) Let M_i^{**} be the path starting at $L^{-1}(z + 1 + i, \theta_i, 2)$ and traversing to $L^{-1}(z + 1, \theta_i, 2)$ in P_{θ_i}.
  - (Type 3. $s_i \in V(C_{z+2k+2})$.) Let M_i^{**} be the path starting at $L^{-1}(z + 2k + 2 - i, \theta_i, 2)$ and traversing to $L^{-1}(z + 2k + 2, \theta_i, 2)$ in P_{θ_i}.
- Observe that the last vertex of M_i^* and the first vertex of M_i^{**} are contained in $C_{z+1+i} \cup C_{z+2k+2-i}$. Also, the subgraph H_i obtained from $C_{z+1+i} \cup C_{z+2k+2-i}$ by adding the subpath of P_{y+1+2i} from C_{z+1+i} to $C_{z+2k+2-i}$ and the subpath of P_{y+2+2i} from $C_{z+2k+2-i}$ to C_{z+1+i} is strongly connected. Let M_i^{***} be a shortest path from the last vertex of M^* to the first vertex of M^{**} in H_i, and let $Z_i^* := M_i^* \cup M_i^{**} \cup M_i^{***}$. Clearly, Z_i^* is a path from r_i to s_i in $\text{clos}(W^*)$.

We claim that $\{Z_i^* : i \in [k]\}$ is a half-integral packing. First observe that the set $\{M_i^* : i \in [k]\}$ is a half-integral packing. In fact, if M_i^* intersects M_j^* for some distinct $i, j \in [k]$, then they are both paths of type 1, and either $\alpha_i, \alpha_j > z + 2k + 2$ or $\alpha_i, \alpha_j < z + 1$. But since they traverse with pairwise distinct rows, no vertex can be shared by three paths in $\{M_i^* : i \in [k]\}$, and furthermore, the possible intersection is not contained in the columns $C_{z+1}, \ldots, C_{z+2k+2}$. Similarly, the set $\{M_i^{**} : i \in [k]\}$ is a half-integral packing. Moreover, $\bigcup_{i\in[k]} M_i^*$ and $\bigcup_{i\in[k]} M_i^{**}$ are vertex-disjoint, because we use rows $P_{y+3}, \ldots, P_{y+2k+2}$ for M_i^* of type 1, and rows $P_{y+2k+3}, \ldots, P_{y+4k+2}$ for M_i^{**} of type 1, and all paths of type 2 or 3 are pairwise vertex-disjoint (r_i cannot be same as s_j because of the directions).

Now, we observe that $\{Z_i^* : i \in [k]\}$ is a half-integral packing. It is sufficient to consider nails contained in $C_{z+1}, \ldots, C_{z+2k+2}$, as paths in $\{M_i^{***} : i \in [k]\}$ do not use nails not contained in $C_{z+1}, \ldots, C_{z+2k+2}$. Suppose for contradiction that there is a nail v in $C_{z+1}, \ldots, C_{z+2k+2}$ that is contained in some three paths in $\{M_i^* : i \in [k]\} \cup \{M_i^{**} : i \in [k]\} \cup \{M_i^{***} : i \in [k]\}$. Since paths in $\{M_i^* : i \in [k]\} \cup \{M_i^{**} : i \in [k]\}$ do not intersect on a nail in $C_{z+1}, \ldots, C_{z+2k+2}$, v is contained in two paths in $\{M_i^{***} : i \in [k]\}$, say $M_{i_1}^{***}$ and $M_{i_2}^{***}$. Since $\{H_i : i \in [k]\}$ is a half-integral packing, the other path should be a path in $\{M_i^* : i \in [k]\} \cup \{M_i^{**} : i \in [k]\}$.

By the construction of $\{H_i : i \in [k]\}$, v is contained in one of the rows used by $M_{i_1}^{***}$ and $M_{i_2}^{***}$. But by the construction of $\{M_i^* : i \in [k]\} \cup \{M_i^{**} : i \in [k]\}$, the other path should use the same row, and therefore, it has to have the same index as one of i_1 and i_2. Then the intersection vertex is contained in one path of $\{Z_i^* : i \in [k]\}$, contradicting the assumption that it is contained in three paths of $\{Z_i^* : i \in [k]\}$. We conclude that $\{Z_i^* : i \in [k]\}$ is a half-integral packing. ∎

We now prove Theorem 1.2. We recall that α_k is the minimum integer such that for every directed graph G with $\nu_2(G) < k$, we have $\tau(G) \le \alpha_k$, if such an integer exists, and otherwise α_k is defined to be ∞.

Proof. [Proof of Theorem 1.2] We prove by induction on k that $\alpha_k \neq \infty$. We know $\alpha_1 = 0$. So, we may assume that $k > 1$ and $\alpha_{k-1} \neq \infty$.

Let f_{wall} be the function defined in Theorem 2.2, and let g_{path} be the function defined in Proposition 5.2. Let $r = 2\alpha_{k-1}$. We set

- $f_3(k) = \max(k, r/4, 12(g_{path}(k) - 1) + 1)$,
- $f_2(k) = \max((2k + 3)(6g_{path}(k) + 1), 8f_3(k))$,
- $f_1(k) = \max(r, f_{wall}(kf_2(k)))$,
- $f(k) = \max(12f_1(k)(f_1(k) + 1) + 1, 24f_3(k) - 4)$.

For convenience, let $w := f_2(k)$. We show that for every directed graph G, if $\nu_2(G) < k$, then $\tau(G) \le f(k)$.

Suppose for contradiction that $\nu_2(G) < k$ and $\tau(G) > f(k)$ for some directed graph G. Let T be a minimum-size hitting set of odd cycles in G. By the assumption, $|T| = \tau(G) > f(k)$. Also, by Lemma 5.1, T is r-externally-well-linked.

Note that $2f_1(k)(f_1(k) + 1) \ge r$ as $f_1(k) \ge r$. Since $|T| > f(k) \ge 12f_1(k)(f_1(k) + 1) + 1$, by Lemma 4.2, G contains a well-linked set A of size $f_1(k)$ such that

(∗) for every subset Z of T of size at least $|T|/2$, there is a linkage of order $f_1(k)$ from A to Z, and there is a linkage of order $f_1(k)$ from Z to A.

Since A is a well-linked set of size $f_1(k) \ge f_{wall}(kf_2(k))$, by Theorem 2.2, G contains a cylindrical wall W of order $kf_2(k) = kw$ such that for every set F of kw nails of W, there is a linkage of order kw from F to A, and there is a linkage of order kw from A to F. Let $C_1, \ldots, C_{kw}$ be the columns of W and $P_1, \ldots, P_{2kw}$ be the rows of W. We consider the following k vertex-disjoint subwalls of W. For each $j \in [k]$, let W_j be the subwall of W consisting of columns $C_{w(j-1)+1}, \ldots, C_{wj}$ and the minimal subpaths of rows P_i with $i \in [2w]$ containing $C_{w(j-1)+1} \cap P_i$ and $C_{wj} \cap P_i$.

We claim that for each $j \in [k]$,

(∗∗) for every set F of w nails of W_j, there is a linkage of order w from F to A, and there is a linkage of order w from A to F.

Let F be a set of w nails of W_j. We choose a set F' of $(k-1)w$ nails of W that are not contained in W_j. We can choose such nails because there are $(k-1)w$ columns of W that are not contained in W_j. By the property of W, there is a linkage of order kw from $F \cup F'$ to A, and there is a linkage of order kw from A to $F \cup F'$. If we restrict paths whose endvertices are in F, then we obtain a linkage of order w from F to A, and a linkage of order w from A to F. Thus, the claim holds.

If each of $W_1, \ldots, W_k$ contains an odd cycle, then we have k vertex-disjoint odd cycles, contradicting the assumption that $\nu_2(G) < k$. Thus, one of $W_1, \ldots, W_k$, say W', does not contain an odd cycle.

Now, by $(*)$ and $(**)$ and Lemma 2.1, we have that

$(***)$ for every subset Z of T of size at least $|T|/2$ and every set F of w nails of W, there is a linkage of order at least $w/2$ from F to Z, and there is a linkage of order at least $w/2$ from Z to F.

Indeed, combining the linkage from A to Z and the linkage of order w from F to A, we obtain a half-integral linkage of order w from F to Z. Lemma 2.1 implies that there is a linkage of order at least $w/2$ from F to Z. The other direction is similar.

Since W' has order w, W' has $2w^2$ nails. Let N be a set of w^2 nails of W' such that they are contained in the same part of the bipartition of W'. Now, we apply Lemma 3.1 for a tuple $(G, N, f_3(k))$. As G has no half-integral packing of k odd cycles and $f_3(k) \geq k$, G contains either

- a half-integral packing $\mathcal{U}$ of $f_3(k)$ odd N-paths whose endvertices are pairwise disjoint, or
- a set Y of at most $4f_3(k) - 1$ vertices such that $G - Y$ has no odd N-walks.

Assume that the latter case happens. Observe that $f_2(k) \geq 8f_3(k)$, $4f_3(k) \geq r$, and $f(k) \geq 6f_3(k) - 4$. We apply Proposition 5.1 with $(r, t, w) = (r, 4f_3(k), f_2(k))$. We can apply the proposition because of the property $(***)$. By Proposition 5.1, G has a set of at most $3(4f_3(k) - 1)$ vertices hitting all odd cycles. It contradicts the fact that $\tau(G) > 24f_3(k) - 4 \geq 12f_3(k) - 3$.

Thus, we may assume that the former case happens. Observe that W' is a bipartite cylindrical wall of order

$$w = f_2(k) \geq (2k+3)(6g_{path}(k) + 1)$$

and $\mathcal{U}$ is a half-integral packing of

$$f_3(k) \geq 12(g_{path}(k) - 1) + 1$$

odd N-paths such that the endvertices of paths in $\mathcal{U}$ are disjoint. So, by Proposition 5.2, $\nu_2(G) \geq k$, a contradiction.

We conclude that $\tau(G) \leq f(k)$.

The algorithmic result will be presented in the next section. □

6 Algorithmic applications

We now discuss how to turn this combinatorial result into a polynomial-time algorithm to find a half-integral packing of k odd cycles or a hitting set of size at most $f(k)$, for fixed integer k.

First by considering all sets S of at most $f(k)$ vertices in G and testing whether $G - S$ has no odd cycles, we can detect a hitting set of size at most $f(k)$ if one exists. Note that we can test in polynomial time whether a given directed graph has an odd cycle, as it is sufficient to test whether the underlying undirected graph of each strong component is bipartite. Therefore, we may assume that G has no hitting set of odd cycles of size at most $f(k)$, that is, $\tau(G) > f(k)$. So, we want to find a half-integral packing of k odd cycles.

Note that we cannot guess the set T, as $\tau(G)$ may be much larger than k (and must contain a half-integral packing of k odd cycles). On the other hand, as $\tau(G) > f(k)$, there should be a well-linked set of size $f_1(k)$ as in the proof. We consider all sets A of size $f_1(k)$ and test whether it is well-linked. As k is a fixed integer, we can test in polynomial time whether A is well-linked, by repeatedly applying Menger's theorem. We construct the set

$$\mathcal{U} := \{A : |A| = f_1(k), A \text{ is well-linked}\}.$$

As $f_1(k) \geq f_{wall}(kf_2(k))$, for each $A \in \mathcal{U}$, by applying Theorem 2.2, we obtain a cylindrical wall W_A of order $kf_2(k) = kw$ such that

- for every set F of kw nails of W_A, there is a linkage of order kw from F to A, and there is a linkage of order kw from A to F.

Note that it runs in polynomial time for fixed k, and Campos et al. [4] recently discussed how to modify this into an FPT algorithm. By dividing W_A into k subwalls as in the proof, we find either a half-integral packing of k odd cycles or a bipartite cylindrical subwall W'_A of order w such that

- for every set F of w nails of W'_A, there is a linkage of order w from F to A, and there is a linkage of order w from A to F.

We choose a set N_A of w^2 nails in W'_A contained in the same part of the bipartition of W'_A. We apply Lemma 3.1 for the tuple $(G, N_A, f_3(k))$. Clearly, Lemma 3.1 can be simulated in polynomial time, as we only use Menger's theorem. If it outputs a half-integer packing of k odd cycles, then we are done. So, we may assume that it outputs either

- a half-integral packing $\mathcal{U}_A$ of $f_3(k)$ odd N_A-paths whose endvertices are pairwise disjoint, or
- a set Y_A of at most $4f_3(k) - 1$ vertices such that $G - Y_A$ has no odd N_A-walks.

If this output Y_A and the current A and the set T satisfy the property $(*)$, then there is a hitting set of size at most $12f_3(k) - 3 \leq f(k)$, which contradicts the assumption that $\tau(G) > f(k)$. So, if the second outcome occurs, then it means that the current A and T do not satisfy $(*)$, and we skip this A. If it outputs $\mathcal{U}_A$, then following the proof of Proposition 5.2 we can obtain a half-integral packing of k odd cycles in polynomial time.

But since there should exist a set $A \in \mathcal{U}$ satisfying $(*)$, by considering all sets A in $\mathcal{U}$, we will either output a hitting set of size at most $f(k)$ or a half-integral packing of k odd cycles. This concludes the algorithm.

We now turn to the proof of Theorem 1.3. Following the notion in [8], the k-Half-Or-No-Integral Disjoint Paths problem asks for given a directed graph G and pairs of source/sink vertices $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$, to either find a half-integral linkage $\{P_i : i \in [k]\}$ where each P_i connects s_i and t_i, or conclude that it has no linkage $\{P_i : i \in [k]\}$ where each P_i connects s_i and t_i. In [8], the following polynomial time algorithm is obtained.

Theorem 6.1. *For every fixed positive integer k, k-*Half-Or-No-Integral Disjoint Paths *can be solved in polynomial time.*

To prove Theorem 1.3, we need the following variation.

Theorem 6.2. *For every fixed positive integer k, there is a polynomial-time algorithm that given a directed graph G having no odd cycles, and given pairs of source/sink vertices $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ and $a_1, \ldots, a_k \in \{0, 1\}$, either*

- *finds a half-integral linkage $\{P_i : i \in [k]\}$ where each P_i connects s_i and t_i and the length of P_i is $a_i \pmod 2$, or*
- *concludes that it has no linkage $\{P_i : i \in [k]\}$ where each P_i connects s_i and t_i and the length of P_i is $a_i \pmod 2$.*

Proof. Let $\{H_1, H_2, \ldots, H_m\}$ be the set of strong components of G, and assume that it is ordered in an acyclic ordering. Let F be the set of all edges that are incident with two strong components of G.

For each $i \in [k]$, we choose a set U_i of edges in F so that

- $U_1, \ldots, U_k$ are pairwise disjoint, and
- for every $1 \leq x < y \leq m$, each U_i contains at most one edge incident with both H_x and H_y.

We will ask to find P_i for which $E(P_i) \cap F = U_i$. Note that if a path in G traverses from H_x to H_y with $y > x$, then it cannot come back to H_x. This means that it is sufficient for U_i to contain at most 1 edge between H_x and H_y for every pair of strong components (H_x, H_y). So, the number of possible tuples $(U_i : i \in [k])$ is at most $n^2 \cdot n^{2k}$, where n is the number of vertices in G.

Now, we fix a tuple $(U_i : i \in [k])$. To test whether there is a set $\{P_i : i \in [k]\}$ of paths where $E(P_i) \cap F = U_i$, it is sufficient to test for each strong component. If some strong component does not contain exactly two vertices among terminals $\{s_i, t_i\}$ or endvertices of U_i, then this tuple is not realizable, and so we skip it. Otherwise, the problem is reduced to k-Half-Or-No-Integral Disjoint Paths for each strong component. We may assume that in each strong component, we get a half-integral linkage between terminals restricted to the strong component.

An important point is that since G has no odd cycle, the underlying undirected graph of each strong component is bipartite. Therefore, paths inside a strong component between two specific vertices have the same parity. So, the parity of the resulting path only depends on the set U_i. If this parity is the same as what we require for P_i, then we accept the output. Otherwise, we skip the tuple.

This concludes the algorithm. □

Now, we prove Theorem 1.3.

By the polynomial-time algorithm in Theorem 1.2, we may assume that we obtain a set X of at most $f(k)$ vertices such that $G - X$ has no odd cycles.

If there are k vertex-disjoint odd cycles in G, then each odd cycle must go through at least one vertex in X. As $|X| \leq f(k)$, we can enumerate all the ways for k vertex-disjoint odd cycles to go through vertices of X, and they are bounded by $n^{f'(k)}$ for some function f' of k. So, we can guess all possible intersections on X in polynomial time. For each guess, we get a problem with $k' \leq 2f(k)$ pairs of terminals in $G - X$ where for each pair (s, t), we want to find a path from s to t with specific parity (as at the end, we need to test whether the cycle is odd). To test it, we apply Theorem 6.2. If we obtain a half-integral packing of k' paths with required parities at some moment, we obtain a half-integral packing of k odd cycles. Otherwise (i.e., for all of them, we conclude that there are no desired k' vertex-disjoint paths in Theorem 6.1), we conclude that there are no k vertex-disjoint odd cycles, as required.

In order to replace the second conclusion of Theorem 1.3 by "it concludes that there is no half-integral packing of k directed odd cycles.", we need to improve Theorem 6.1 to decide the k-half integral disjoint paths problem. This is indeed conjectured in [8].

References

[1] S. A. Amiri, K. Kawarabayashi, S. Kreutzer, and P. Wollan. The Erdős-Pósa property for directed graphs. *preprint*, arxiv.org/abs/1603.02504, 2016.

[2] E. Birmelé, J. A. Bondy, and B. A. Reed. The Erdős-Pósa property for long circuits. *Combinatorica*, 27(2):135–145, 2007.

[3] H. Bruhn, F. Joos, and O. Schaudt. Long cycles through prescribed vertices have the Erdős-Pósa property. *J. Graph Theory*, 87(3):275–284, 2018.

[4] V. Campos, R. Lopes, A. K. Maia, and I. Sau. Adapting the Directed Grid Theorem into an FPT algorithm. *SIAM J. Discret. Math.*, 36(3):1887–1917, 2022.

[5] P. Erdős and L. Pósa. On the independent circuits contained in a graph. *Canad. J. Math.*, 17:347–352, 1965.

[6] S. Fiorini, N. Hardy, B. Reed, and A. Vetta. Approximate min-max relations for odd cycles in planar graphs. *Math. Program.*, 110(1, Ser. B):71–91, 2007.

[7] S. Fiorini and A. Herinckx. A tighter Erdős-Pósa function for long cycles. *J. Graph Theory*, 77(2):111–116, 2014.

[8] A. C. Giannopoulou, K. Kawarabayashi, S. Kreutzer, and O. Kwon. Directed tangle tree-decompositions and applications. In *Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022*, pages 377–405. SIAM, 2022.

[9] P. Gollin, K. Hendrey, K. Kawarabayashi, O. Kwon, and S. Oum. A unified half-integral Erdős-Pósa theorem for cycles in graphs labelled by multiple abelian groups. *preprint*, arxiv.org/abs/2102.01986, 2021.

[10] P. Gollin, K. Hendrey, O. Kwon, S. Oum, and Y. Yoo. A unified Erdős-Pósa theorem for cycles in graphs labelled by multiple abelian groups. *preprint*, arxiv.org/abs/2209.09488, 2022.

[11] T. Huynh, F. Joos, and P. Wollan. A unified Erdős-Pósa theorem for constrained cycles. *Combinatorica*, 39(1):91–133, 2019.

[12] N. Kakimura and K. Kawarabayashi. Packing directed circuits through prescribed vertices bounded fractionally. *SIAM J. Discret. Math.*, 26(3):1121–1133, 2012.

[13] N. Kakimura, K. Kawarabayashi, and D. Marx. Packing cycles through prescribed vertices. *J. Combin. Theory Ser. B*, 101(5):378–381, 2011.

[14] K. Kawarabayashi, D. Král', M. Krcál, and S. Kreutzer. Packing directed cycles through a specified vertex set. In *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2013*, pages 365–377. SIAM, 2013.

[15] K. Kawarabayashi and S. Kreutzer. The directed grid theorem. *preprint*, arxiv.org/abs/1411.5681, 2014.

[16] K. Kawarabayashi and S. Kreutzer. The directed grid theorem. In *STOC'15—Proceedings of the 2015 ACM Symposium on Theory of Computing*, pages 655–664. ACM, New York, 2015.

[17] E. J. Kim and O. Kwon. Erdős-Pósa property of chordless cycles and its applications. *J. Combin. Theory Ser. B*, 145:65–112, 2020.

[18] D. Král̆, J.-S. Sereni, and L. Stacho. Min-max relations for odd cycles in planar graphs. *SIAM J. Discrete Math.*, 26(3):884–895, 2012.

[19] K. Menger. Zur allgemeinen Kurventheorie. *Fund. Math.*, 10:96–115, 1927.

[20] F. Mousset, A. Noever, N. Škorić, and F. Weissenberger. A tight Erdős-Pósa function for long cycles. *J. Combin. Theory Ser. B*, 125:21–32, 2017.

[21] M. Pontecorvi and P. Wollan. Disjoint cycles intersecting a set of vertices. *J. Combin. Theory Ser. B*, 102(5):1134–1141, 2012.

[22] J.-F. Raymond and D. M. Thilikos. Recent techniques and results on the Erdős-Pósa property. *Discrete Appl. Math.*, 231:25–43, 2017.

[23] B. Reed. Mangoes and blueberries. *Combinatorica*, 19(2):267–296, 1999.

[24] B. Reed, N. Robertson, P. Seymour, and R. Thomas. Packing directed circuits. *Combinatorica*, 16(4):535–554, 1996.

[25] N. Robertson and P. D. Seymour. Graph minors. V. Excluding a planar graph. *J. Combin. Theory Ser. B*, 41(1):92–114, 1986.

[26] C. Thomassen. On the presence of disjoint subgraphs of a specified type. *J. Graph Theory*, 12(1):101–111, 1988.

[27] W. C. van Batenburg, T. Huynh, G. Joret, and J.-F. Raymond. A tight Erdős-Pósa function for planar minors. *Advances in Combinatorics*, 2:33 pages, 2019.

[28] D. H. Younger. Graphs with interlinked directed circuits. *Proceedings of the Midwest Symposium on Circuit Theory*, 2:XVI 2.1 – XVI 2.7, 1973.