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Local regularity for nonlocal equations with variable exponents

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Abstract
In this paper, we study local regularity properties of minimizers of nonlocal variational functionals with variable exponents and weak solutions to the corresponding Euler–Lagrange equations. We show that weak solutions are locally bounded when the variable exponent p is only assumed to be continuous and bounded. Furthermore, we prove that bounded weak solutions are locally Hölder continuous under some additional assumptions on p. On the one hand, the class of admissible exponents is assumed to satisfy a log-Hölder–type condition inside the domain, which is essential even in the case of local equations. On the other hand, since we are concerned with nonlocal problems, we need an additional assumption on p outside the domain.

KEYWORDS
Caccioppoli estimate, De Giorgi iteration, fractional Sobolev space, Hölder regularity, local boundedness, nonlocal equation, variable exponent, weak solution

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1 | INTRODUCTION

The aim of this paper is to study the regularity theory for minimizers of the nonlocal variational functional

$$\mathcal{F}(u) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{n+sp(x,y)}} \, \mathrm{d}y \, \mathrm{d}x \tag{1.1}$$

and for weak solutions to the corresponding Euler–Lagrange equation, where $n \in \mathbb{N}$, $s \in (0,1)$, and $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that

$$p(x,y) = p(y,x) \tag{1.2}$$

and

$$1 < \inf_{x,y \in \mathbb{R}^n} p(x,y) \le \sup_{x,y \in \mathbb{R}^n} p(x,y) < +\infty. \tag{1.3}$$

This functional is a nonlocal analog of a local variational functional

$$\mathcal{F}_{\mathrm{loc}}(u) = \int_\Omega \frac{1}{p(x)} |Du(x)|^{p(x)} \, \mathrm{d}x, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $p : \Omega \to \mathbb{R}$ is a measurable function such that $1 < \inf_{x\in\Omega} p(x) \le \sup_{x\in\Omega} p(x) < +\infty$. The functional in (1.4) was first considered by Zhikov [51, 52]. The regularity properties for minimizers of (1.4) or more general local variational functionals have been established in several works. See, for instance, [2, 3, 5, 17, 20, 21, 32, 34, 37, 38, 46–48, 53, 54] and the references therein.

A function $u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^n)$ is said to be a *minimizer of $\mathcal{F}$* in Ω if

$$\mathcal{F}(u) \le \mathcal{F}(u + \varphi)$$

for any measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$ supported inside Ω. See Section 2.2 for the definition of the function space $W^{s,p(\cdot,\cdot)}(\mathbb{R}^n)$. It is standard to show that minimizers of $\mathcal{F}$ in Ω are weak solutions to the Euler–Lagrange equation

$$(-\Delta)^s_{p(\cdot,\cdot)} u = 0 \tag{1.5}$$

in Ω, where $(-\Delta)^s_{p(\cdot,\cdot)}$ is the fractional $p(\cdot,\cdot)$-Laplacian defined by

$$(-\Delta)^s_{p(\cdot,\cdot)} u(x) = \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x-y|^{n+sp(x,y)}} \, \mathrm{d}y, \quad x \in \mathbb{R}^n.$$

See Section 3 for the precise definition of weak solution.

Before we formulate the assumptions on p and the main results of this paper, let us recall the regularity results for local variational functionals and the corresponding local operators. It is known [34] that minimizers of (1.4) in Ω and weak solutions to the corresponding Euler–Lagrange equation $-\Delta_{p(\cdot)} u = 0$ in Ω are locally bounded in Ω, provided that $p : \Omega \to \mathbb{R}$ is continuous on Ω. Moreover, if the modulus of continuity ω of p satisfies

$$\limsup_{R\to 0} \omega(R) \log\left(\frac{1}{R}\right) < +\infty, \tag{1.6}$$

then the minimizers and weak solutions are locally Hölder continuous. The log-Hölder continuity (1.6) is sharp in the sense that regularity properties such as Hölder continuity and even higher integrability fail to hold if the condition (1.6) is violated (see [54]). Moreover, it is proved [54] that the functional (1.4) exhibits the Lavrentiev phenomenon if and only if the condition (1.6) is dropped. Furthermore, the singular part of the measure representation of relaxed integrals with variable exponent disappears if and only if (1.6) holds (see [1]).

The log-Hölder–type condition (1.6) is equivalent to the condition that there exists a constant $L > 0$ such that

$$R^{p_-(B_R(x_0)) - p_+(B_R(x_0))} \le L \quad \text{for all } \overline{B_R(x_0)} \subset \Omega,$$

where

$$p_+(E) := \sup_{x\in E} p(x) \quad \text{and} \quad p_-(E) := \inf_{x\in E} p(x),$$

see [29]. It is natural to expect that a similar condition on p is required to obtain Hölder regularity results for the nonlocal variational functional (1.1) and the nonlocal equation (1.5). We introduce the following condition on p.

Definition 1.1. We say that a function $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies the condition (P1) in Ω if there exists a constant $L > 0$ such that

$$R^{p_-(B\times B) - p_+(B\times B)} \le L \quad \text{for all } \overline{B} = \overline{B_R(x_0)} \subset \Omega, \tag{P1}$$

where

$$p_+(E \times F) = \sup_{x \in E, y \in F} p(x,y) \quad \text{and} \quad p_-(E \times F) = \inf_{x \in E, y \in F} p(x,y).$$

Since we are concerned with nonlocal problems, we also need the information of p outside the domain.

Definition 1.2. We say that a function $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies the condition (P2) in Ω if

$$p_+(B \times B^c) \leq p_+(B \times B) \quad \text{and} \quad p_-(B \times B^c) \leq p_-(B \times B) \quad \text{for all } \overline{B} = \overline{B_R(x_0)} \subset \Omega. \tag{P2}$$

Let us make some comments on the conditions (P1) and (P2).

Remark 1.3.

(i) Note that condition (P1) does not imply that p is log-Hölder continuous as a $2n$-variable function, since $B \times B$ in (P1) is not a ball with respect to the Euclidean metric in $\mathbb{R}^{2n}$. The condition (P1) is actually weaker than the log-Hölder continuity of p. Let us first prove that the log-Hölder continuity of p implies (P1). If p is log-Hölder continuous, that is,

$$|p(x_1, y_1) - p(x_2, y_2)| \leq \frac{C}{-\log \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}}$$

for all $(x_1, y_1), (x_2, y_2) \in \Omega \times \Omega$ with $\sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2} \leq 1/2$, then

$$|p_-(B \times B) - p_+(B \times B)| \leq \frac{C}{-\log(2\sqrt{2}R)}$$

for any $\overline{B} = \overline{B_R(x_0)} \subset \Omega$ with $R \leq 1/8$. Thus,

$$R^{p_-(B \times B) - p_+(B \times B)} \leq R^{C/\log(2\sqrt{2}R)} = \exp\left(C \frac{\log R}{\log(2\sqrt{2}R)}\right) \leq e^{2C}.$$

If $R > 1/8$, then $R^{p_-(B \times B) - p_+(B \times B)} \leq 8^{p_+(B \times B) - p_-(B \times B)} \leq 8^{\|p\|_\infty}$. Therefore, (P1) is proved for any $R > 0$.

Let us next provide an example of p that is not log-Hölder continuous, but satisfies the condition (P1). The example will be given in $\mathbb{R} \times \mathbb{R}$, but it can be easily extended to $\mathbb{R}^n \times \mathbb{R}^n$. Let ω be a modulus of continuity that is smooth, bounded, concave, increasing, and satisfies

$$\lim_{R \to 0} \frac{1}{-\log R} \frac{1}{\omega(R)} = 0. \tag{1.7}$$

Define $p(x,y) = |x|\omega(|y|)$, then p is clearly not log-Hölder continuous by (1.7). To show that p satisfies (P1) in $(-1,1)$, let $x, y \in B := (x_0 - R, x_0 + R)$ with $R < 1$. Then,

$$p_+ := p_+(B \times B) = (|x_0| + R)\omega(|x_0| + R) \quad \text{and}$$

$$p_- := p_-(B \times B) = \begin{cases} 0 & \text{if } |x_0| < R, \\ (x_0 - R)\omega(x_0 - R) & \text{if } x_0 \geq R, \\ (x_0 + R)\omega(x_0 + R) & \text{if } x_0 \leq -R. \end{cases}$$

When $|x_0| < R$, we have

$$R^{p_- - p_+} = R^{-(|x_0| + R)\omega(|x_0| + R)} \leq R^{-2R\omega(2R)} \leq R^{-2\|\omega\|_\infty R} \leq 2\|\omega\|_\infty.$$

If $x_0 \geq R$, then by the mean value theorem,
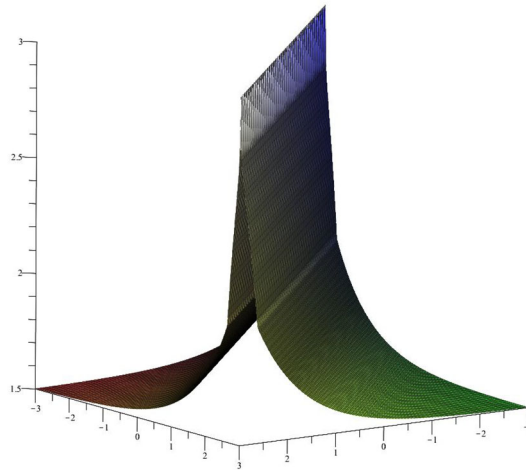
$$p_+ - p_- = 2Rf'(x_*)$$

FIGURE 1 Visualization of an example.

for some $x_* \in B$, where $f(t) = t\omega(t)$. Since f is concave and bounded, we have $f'(t) = \omega(t) + t\omega'(t) \leq 2\omega(t) \leq 2\|\omega\|_\infty$. Thus,

$$R^{p_- - p_+} \leq R^{-2Rf'(x_*)} \leq R^{-4\|\omega\|_\infty R} \leq 4\|\omega\|_\infty.$$

The case $x_0 \leq -R$ can be treated in the same way. Therefore, p satisfies (P1) in $(-1, 1)$.

For an explicit example of such p, one can consider a modulus of continuity ω that behaves like $1/\log(-\log R)$ or $\log(\log(1/R))/(-\log R)$ near zero.

(ii) Let us provide a nontrivial example of a function $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying (P1) and (P2) in B_1. Let ω be given by

$$\omega(r) = \begin{cases} 3 - \dfrac{1}{\log(1/r)} & \text{if } r < \dfrac{1}{e}, \\ \omega_0(r) & \text{if } r \geq \dfrac{1}{e}, \end{cases}$$

where ω_0 is any nonincreasing function such that $\omega_0(1/e) = 2$ and $\lim_{r\to\infty} \omega_0(r) > 1$, and define $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $p(x, y) = \omega(|x - y|)$ (see Figure 1). Then, p satisfies (P1) because p is log-Hölder continuous as a $2n$-variable function. Moreover, for any $\overline{B} = \overline{B_R(x_0)} \subset B_1$, we have

$$p_+(B \times B^c) = 3 = p_+(B \times B) \quad \text{and} \quad p_-(B \times B^c) = 2 \leq \omega(2R) = p_-(B \times B).$$

Therefore, p also satisfies (P2).

(iii) The conditions (P1) and (P2) do not restrict p on $\Omega^c \times \Omega^c$. In fact, for the local regularity results, we do not need any information about p on $\Omega^c \times \Omega^c$ except for the global bound (1.3). This is because the double integral over $\Omega^c \times \Omega^c$ vanishes whenever we use a cutoff function.

Let us now present the main results of this paper. The first result is the local boundedness of weak solutions to (1.5). Throughout the paper, we always assume that $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function satisfying (1.2) and (1.3). Note that the following theorem does not require the conditions (P1) and (P2).

Theorem 1.4. *Let Ω be a bounded domain in $\mathbb{R}^n$. If $u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^n)$ satisfies*

$$\sup_{x \in \Omega} \int_{\mathbb{R}^n} \frac{u_+(y)^{p(x,y)-1}}{1 + |y|^{n+sp(x,y)}} \, \mathrm{d}y < +\infty \tag{1.8}$$

and is a weak subsolution to (1.5) *in* Ω, *then* u *is locally bounded from above in* Ω. *Furthermore, for each* $x_0 \in \Omega$ *with* $p(x_0, x_0) \leq n/s$, *there is a radius* $R \in (0, 1)$ *such that* $B_R = B_R(x_0) \subset \Omega$, $p_+ < p_*^* := \frac{np_-}{n - \sigma p_-}$, *and*

$$\sup_{B_{R/2}} u \leq C \left(\int_{B_R} u_+^{p_+}(x) \, \mathrm{d}x \right)^{\frac{1}{p_+}} + \left(\sup_{x \in B_R} \int_{\mathbb{R}^n \setminus B_{R/2}} \frac{u_+(y)^{p(x,y)-1}}{|y - x_0|^{n+sp(x,y)}} \, \mathrm{d}y \right)^{1/(p_+ - 1)} + 1 \tag{1.9}$$

for any $\sigma \in (0, s)$ *and* $q \in (\max\{p_+, \frac{n}{n-\sigma}\}, p_-^*)$, *where* $p_\pm = p_\pm(B_R \times B_R)$. *The constant* C *depends on* n, s, σ, $p_+(B_R \times \mathbb{R}^n)$, $p_-(B_R \times B_R)$, q, *and* R.

Remark 1.5. If $u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^n)$ is a weak supersolution to (1.5) in Ω satisfying (1.8) with u_+ replaced by u_-, then u is locally bounded from below in Ω and (1.9) holds with u replaced by $-u$.

As a consequence of Theorem 1.4, we know that every minimizer of (1.1) in Ω is locally bounded in Ω since it is a weak solution to (1.5) in Ω.

The strategy for the proof of Theorem 1.4 is to develop the De Giorgi theory for the nonlocal functional $\mathcal{F}$ with variable exponent. This approach for nonlocal functionals with constant exponent has been studied extensively in the last few years. See, for instance, [18, 33, 36, 41, 42] for the case $p = 2$, and [16, 22, 26, 27, 45] for $p > 1$. For a deeper discussion on fractional De Giorgi classes and their applications for the regularity of nonlocal problems, we refer the reader to [23] and the references therein. Analogously, we obtain the Caccioppoli-type estimate that contains terms with variable exponents, and then use the De Giorgi iteration technique to establish Theorem 1.4. Due to the variable exponent in the Caccioppoli-type estimate, an additional difficulty arises in the De Giorgi iteration that does not occur in the case of the constant exponent. That is, different exponents involving p_+ and p_- come into play in the iteration. Thus, the supremum of u is controlled by a maximum of two L^{p_+}-norms of u with different powers, and the nonlocal tail term having the variable exponent.

Let us mention that local boundedness of weak solutions to more general problems involving subcritical nonlinearity has been recently settled by Ho and Kim [39]. However, their result requires an additional log-Hölder–type assumption on p to cover the subcritical nonlinearity with a variable exponent. For Equation (1.5), this type of additional regularity on p is not necessary.

The second main result is the Hölder continuity of bounded weak solutions to (1.5).

Theorem 1.6. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$. *Assume that* p *satisfies* (P1) *and* (P2) *in* Ω. *If* $u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^n)$ *is a weak solution to* (1.5) *in* Ω *satisfying* (1.8), *then* u *is locally Hölder continuous in* Ω. *Furthermore, for each* $x_0 \in \Omega$ *with* $p(x_0, x_0) \leq n/s$, *there exists* $R \in (0, 1)$ *such that* $B_R = B_R(x_0) \Subset \Omega$ *and* $p_+(B_R \times B_R) < p_-^*(B_R \times B_R)$, *and*

$$[u]_{C^\alpha(\overline{B_{R/2}})} \leq C \|u\|_{L^\infty(B_R)} + R^s + 1 + \left(R^{sp_+(\{x_0\} \times B_R^c)} \sup_{x \in B_{3R/4}} \int_{\mathbb{R}^n \setminus B_R} \frac{|u(y)|^{p(x,y)-1}}{|y - x_0|^{n+sp(x,y)}} \, \mathrm{d}y \right)^{\frac{1}{p_+(\{x_0\} \times B_R^c)-1}} \tag{1.10}$$

for any $\sigma \in (0, s)$, *where the constants* α *and* C *depend on* n, s, σ, $p_+(\Omega \times \mathbb{R}^n)$, $p_-(\Omega \times \mathbb{R}^n)$, R, *and* L.

The Hölder estimate for the fractional p-Laplacian–type equations was first established by Di Castro–Kuusi–Palatucci [27]. Theorem 1.6 generalizes their result to the case of variable exponents.

Theorem 1.6 follows from the so-called growth lemma (Lemma 5.2), which provides the control of oscillation of supersolutions. In order to prove the growth lemma, we need two ingredients: an improved Caccioppoli-type estimate and a fractional De Giorgi isoperimetric-type inequality. When p is constant, the Caccioppoli-type estimate was first established in [27] and improved in [22] (see also [18]). The Caccioppoli-type estimate we establish to prove Theorem 1.4 is an improved version. Our Caccioppoli-type estimate not only makes it possible to use the fractional De Giorgi isoperimetric-type inequality as in [22], but also takes variable exponents into account.

The proof of Theorem 1.6 is significantly different in the De Giorgi iteration from the one for the case of the constant exponent. As in the proof of Theorem 1.4, we also encounter different exponents involving p_+ and p_- in the De Giorgi iteration. However, this mismatch of exponents causes a more serious problem when we investigate the modulus of continuity of weak solutions. We will see that the assumption (P1) on p solves this problem.

Another difference is that the variable exponent in the nonlocal tail term affects the iteration as well. This difficulty does not exist in the local variational problems with variable exponent as well as the nonlocal problem with constant exponent. The variable exponent in mixed regions, which appears in the nonlocal tail, interacts with the variable exponent in local terms. With this regard, the assumption (P2) on p is required.

After finishing our work, we have learned from [49] that similar results to ours can be obtained by using a different iteration method. By introducing a nonstandard nonlocal tail term, which involves L^∞-norm of u inside, he was able to obtain the Hölder estimate without assuming (P2) on p. However, in order to get a standard nonlocal tail term in the Hölder estimates as in (1.10), it is inevitable to impose an additional assumption on p in mixed regions.

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1.1 | Outline

The paper is organized as follows. In Section 2, we recall the variable exponents Lebesgue spaces, fractional Sobolev spaces with variable exponents, and fractional Sobolev embedding theorems. Section 3 is devoted to the proof of the improved Caccioppoli-type estimate with variable exponent, which will be used in the proofs of local boundedness and Hölder regularity for weak solutions. In Section 4, we prove Theorem 1.4, which provides a quantitative local estimate on the supremum of weak subsolutions. Finally, we prove Theorem 1.6 in Section 5 by establishing a growth lemma. This is proved by using the improved Caccioppoli-type estimate and the isoperimetric-type inequality.

2 | PRELIMINARIES

In this section, we briefly review the variable exponent Lebesgue spaces and fractional Sobolev spaces with variable exponents. Furthermore, we recall the fractional Sobolev embedding theorems for the constant exponent case.

2.1 | Variable exponents Lebesgue spaces

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $p : \Omega \to \mathbb{R}$ be a measurable function satisfying

$$1 < \inf_{x \in \Omega} p(x) \le \sup_{x \in \Omega} p(x) < +\infty.$$

We define the variable exponent Lebesgue spaces

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \varrho_{L^{p(\cdot)}(\Omega)}(u/\lambda) < +\infty \text{ for some } \lambda > 0 \right\}$$

endowed with the norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \varrho_{L^{p(\cdot)}(\Omega)}(u/\lambda) \le 1 \right\},$$

where

$$\varrho_{L^{p(\cdot)}(\Omega)}(u) = \int_\Omega |u(x)|^{p(x)} \, dx.$$

It is well known that $L^{p(\cdot)}(\Omega)$ is a Banach space, see [31, 35, 44] for instance. Let us collect some useful inequalities for later use.

Lemma 2.1. *[35, Theorem 1.3] Let $u \in L^{p(\cdot)}(\Omega)$ and $p_\pm = p_\pm(\Omega)$, then*

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \, (= 1; < 1)$ *if and only if* $\varrho_{L^{p(\cdot)}(\Omega)} > 1 \, (= 1; < 1)$;

(ii) *if* $\|u\|_{L^{p(\cdot)}(\Omega)} \geq 1$, *then* $\|u\|_{L^{p(\cdot)}(\Omega)}^{p_-} \leq \varrho_{L^{p(\cdot)}(\Omega)}(u) \leq \|u\|_{L^{p(\cdot)}(\Omega)}^{p_+}$;

(iii) *if* $\|u\|_{L^{p(\cdot)}(\Omega)} \leq 1$, *then* $\|u\|_{L^{p(\cdot)}(\Omega)}^{p_+} \leq \varrho_{L^{p(\cdot)}(\Omega)}(u) \leq \|u\|_{L^{p(\cdot)}(\Omega)}^{p_-}$.

Lemma 2.2. *[44, Theorem 2.1] For every $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, it holds that*

$$\int_{\Omega} |u(x)v(x)| \, dx \leq 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)},$$

where $1/p(x) + 1/p'(x) = 1$.

See [24, 31, 35, 44] for more properties of the variable exponent Lebesgue spaces.

2.2 | Fractional Sobolev spaces with variable exponents

The fractional Sobolev spaces with variable exponents were first introduced recently by Kaufmann, Rossi, and Vidal [43], and have been studied in different contexts. See [4, 6–15, 19, 25, 39, 40, 50, 55] and references therein. Note that the Triebel–Lizorkin spaces with variable smoothness and integrability have been introduced in [30], which are isomorphic to $W^{k,p(\cdot)}(\mathbb{R}^n)$ if $k \in \mathbb{N} \cup \{0\}$, respectively, the variable exponent Bessel potential space $\mathcal{L}^{\alpha,p(\cdot)}(\mathbb{R}^n)$ for $\alpha > 0$ under suitable assumptions on p. In the scope of this paper, we will focus on the fractional Sobolev spaces with variable exponents introduced in [43].

In this section, let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ or $\Omega = \mathbb{R}^n$. Let $p \in C(\overline{\Omega} \times \overline{\Omega})$ be such that $p(x,y) = p(y,x)$ and

$$1 < p_-(\Omega \times \Omega) \leq p_+(\Omega \times \Omega) < +\infty,$$

and define $\bar{p}(x) = p(x,x)$. For $s \in (0,1)$, the fractional Sobolev space with variable exponents is defined as

$$W^{s,p(\cdot,\cdot)}(\Omega) := \left\{ u \in L^{\bar{p}(\cdot)}(\Omega) : \varrho_{W^{s,p(\cdot,\cdot)}(\Omega)}(u/\lambda) < +\infty \text{ for some } \lambda > 0 \right\},$$

where

$$\varrho_{W^{s,p(\cdot,\cdot)}(\Omega)}(u) = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{n+sp(x,y)}} \, dy \, dx.$$

We define a seminorm

$$[u]_{W^{s,p(\cdot,\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \varrho_{W^{s,p(\cdot,\cdot)}(\Omega)}(u/\lambda) \leq 1 \right\}.$$

It is well known [43] that $W^{s,p(\cdot,\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{s,p(\cdot,\cdot)}(\Omega)} = \|u\|_{L^{\bar{p}(\cdot)}(\Omega)} + [u]_{W^{s,p(\cdot,\cdot)}(\Omega)}.$$

Let us also define

$$\tilde{\varrho}_{W^{s,p(\cdot,\cdot)}(\Omega)}(u) = \varrho_{L^{\bar{p}(\cdot)}(\Omega)}(u) + \varrho_{W^{s,p(\cdot,\cdot)}(\Omega)}(u)$$

and a norm

$$|u|_{W^{s,p(\cdot,\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \tilde{\varrho}_{W^{s,p(\cdot,\cdot)}(\Omega)}(u/\lambda) \leq 1 \right\}.$$

Then, it is clear that two norms $\|u\|_{W^{s,p(\cdot,\cdot)}(\Omega)}$ and $|u|_{W^{s,p(\cdot,\cdot)}(\Omega)}$ are comparable, see [39]. It is also easy to obtain the following lemma from the definitions of the norms.

Lemma 2.3. *Let $u \in W^{s,p(\cdot,\cdot)}(\Omega)$ and $p_\pm = p_\pm(\Omega \times \Omega)$, then*

(i) *if $[u]_{W^{s,p(\cdot,\cdot)}(\Omega)} \geq 1$, then $[u]^{p_-}_{W^{s,p(\cdot,\cdot)}(\Omega)} \leq \varrho_{W^{s,p(\cdot,\cdot)}(\Omega)}(u) \leq [u]^{p_+}_{W^{s,p(\cdot,\cdot)}(\Omega)}$;*

(ii) *if $[u]_{W^{s,p(\cdot,\cdot)}(\Omega)} \leq 1$, then $[u]^{p_+}_{W^{s,p(\cdot,\cdot)}(\Omega)} \leq \varrho_{W^{s,p(\cdot,\cdot)}(\Omega)}(u) \leq [u]^{p_-}_{W^{s,p(\cdot,\cdot)}(\Omega)}$;*

(iii) *if $|u|_{W^{s,p(\cdot,\cdot)}(\Omega)} \geq 1$, then $|u|^{p_-}_{W^{s,p(\cdot,\cdot)}(\Omega)} \leq \tilde{\varrho}_{W^{s,p(\cdot,\cdot)}(\Omega)}(u) \leq |u|^{p_+}_{W^{s,p(\cdot,\cdot)}(\Omega)}$;*

(iv) *if $|u|_{W^{s,p(\cdot,\cdot)}(\Omega)} \leq 1$, then $|u|^{p_+}_{W^{s,p(\cdot,\cdot)}(\Omega)} \leq \tilde{\varrho}_{W^{s,p(\cdot,\cdot)}(\Omega)}(u) \leq |u|^{p_-}_{W^{s,p(\cdot,\cdot)}(\Omega)}$.*

Recently, the fractional Sobolev embeddings with variable exponents have been studied in [39, 40, 43]. However, the fractional Sobolev embeddings with constant exponents are sufficient for the local regularity theory with variable exponents. Let us recall the following embedding theorems for constant exponent fractional Sobolev spaces.

Theorem 2.4. *[28, Theorem 6.7] Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $s \in (0,1)$ and $p \in [1, n/s)$. Then, there exists a constant $C = C(n, s, p, \Omega) > 0$ such that, for any $u \in W^{s,p}(\Omega)$, we have*

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)}$$

for any $q \in [1, np/(n - sp)]$.

Theorem 2.5. *[22, Corollary 4.9] Let $s \in (0,1)$, $p \in [1, n/s)$, and $R > 0$. Let $u \in W_0^{s,p}(B_R)$ and suppose that $u = 0$ on a set $\Omega_0 \subset B_R$ with $|\Omega_0| \geq \gamma|B_R|$ for some $\gamma \in (0,1]$. Then,*

$$\|u\|_{L^{\frac{np}{n-sp}}(B_R)} \leq C[u]_{W^{s,p}(B_R)}$$

for some $C = C(n, s, p, \gamma) > 0$.

Theorem 2.6. *[28, Theorem 8.2] Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $s \in (0,1)$ and $p > n/s$. Then there exists a constant $C = C(n, s, p, \Omega) > 0$ such that, for any $u \in L^p(\Omega)$, we have*

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C\|u\|_{W^{s,p}(\Omega)},$$

where $\alpha = (sp - n)/p$.

3 | CACCIOPPOLI-TYPE ESTIMATE

This section is devoted to the Caccioppoli-type estimate for weak subsolutions and supersolutions to (1.5). Let us first provide the definitions of weak subsolutions and supersolutions.

Definition 3.1. A function $u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^n)$ is a weak subsolution (weak supersolution, respectively) to (1.5) in Ω if

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+sp(x,y)}} \, dy \, dx \leq 0 \quad (\geq 0, \text{ respectively})$$

for every nonnegative $\varphi \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^n)$ such that $\varphi = 0$ a.e. outside Ω. A function $u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^n)$ is a weak solution to (1.5) in Ω if it is a weak subsolution and supersolution.

The Caccioppoli-type estimate is a key ingredient for the local regularity results. This type of estimate has been established by many authors (see, for instance, [16, 22, 27, 45]) for the case of the fractional p-Laplacian with a constant $p > 1$. The main difference between Caccioppoli-type estimates for the local and nonlocal operators is that the estimate for the nonlocal operator involves a nonlocal tail term. Moreover, in [22], Cozzi improved the estimate to take an isoperimetric-type inequality into account. In this section, we generalize Cozzi's estimate to the fractional $p(\cdot, \cdot)$-Laplacian.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^n)$ be a weak subsolution to* (1.5) *in Ω. Then, for any* $B_r(x_0) \Subset B_R(x_0) \subset \Omega$ *and any $k \in \mathbb{R}$,*

$$\varrho_{W^{s,p(\cdot,\cdot)}(B_r(x_0))}(w_+) + \int_{B_r(x_0)} w_+(x) \int_{B_R(x_0)} \frac{w_-(y)^{p(x,y)-1}}{|x-y|^{n+sp(x,y)}} \, dy \, dx$$

$$\leq C \int_{B_R(x_0)} \int_{B_R(x_0)} \left| \frac{w_+(x)}{R-r} \right|^{p(x,y)} \frac{dy \, dx}{|x-y|^{n-(1-s)p(x,y)}} \tag{3.1}$$

$$+ C \left(\sup_{x \in B_{\frac{R+r}{2}}(x_0)} \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{w_+(y)^{p(x,y)-1}}{|y-x_0|^{n+sp(x,y)}} \left(\frac{2R}{R-r} \right)^{n+sp(x,y)} dy \right) \int_{B_R(x_0)} w_+(x) \, dx,$$

where $w_{\pm} := (u-k)_{\pm}$. The constant C depends only on $p_{\pm}(B_R(x_0) \times B_R(x_0))$.

Remark 3.3. If $u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^n)$ is a weak supersolution to (1.5) in Ω, then (3.1) holds with w_+ and w_- replaced by w_- and w_+, respectively.

In order to prove Theorem 3.2, we need an algebraic inequality. Recall that, in the case of $p(\cdot)$-Laplacian with $1 < p_- \leq p(x) \leq p_+ < \infty$, the inequalities

$$|Dw|^{p(x)-2} Dw \cdot D(w\eta^{p_+}) \geq |Dw|^{p(x)} \eta^{p_+} - p_+ |Dw|^{p(x)-1} \eta^{p_+-1} w |D\eta|$$

$$\geq |Dw|^{p(x)} \eta^{p_+} - \frac{1}{2} |Dw|^{p(x)} \eta^{(p_+-1)\frac{p(x)}{p(x)-1}} - C w^{p(x)} |D\eta|^{p(x)} \tag{3.2}$$

$$\geq \frac{1}{2} |Dw|^{p(x)} \eta^{p_+} - C w^{p(x)} |D\eta|^{p(x)}$$

for some $C > 0$, play a crucial role for establishing Caccioppoli-type estimates (see, e.g., [34]). The following lemma is a discrete version of (3.2).

Lemma 3.4. *Let $a, b \geq 0$, $\tau_1, \tau_2 \in [0,1]$, and $1 < p_- \leq p(x,y) \leq p_+ < \infty$. Then,*

$$|a-b|^{p(x,y)-2}(a-b)\left(a\tau_1^{p_+} - b\tau_2^{p_+}\right) \geq \frac{1}{2}|a-b|^{p(x,y)}(\max\{\tau_1,\tau_2\})^{p_+} - C(\max\{a,b\})^{p(x,y)}|\tau_1-\tau_2|^{p(x,y)}, \tag{3.3}$$

for some $C = C(p_+, p_-) > 0$.

Proof. Since

$$(a-b)\left(a\tau_1^{p_+} - b\tau_2^{p_+}\right) \geq (a-b)^2\tau_1^{p_+} - b|a-b||\tau_1^{p_+} - \tau_2^{p_+}| \quad \text{and}$$
$$(a-b)\left(a\tau_1^{p_+} - b\tau_2^{p_+}\right) \geq (a-b)^2\tau_2^{p_+} - a|a-b||\tau_1^{p_+} - \tau_2^{p_+}|,$$

we have

$$(a-b)\left(a\tau_1^{p_+} - b\tau_2^{p_+}\right) \geq (a-b)^2(\max\{\tau_1,\tau_2\})^{p_+} - \max\{a,b\}|a-b||\tau_1^{p_+} - \tau_2^{p_+}|. \tag{3.4}$$

By convexity of the function $f(\tau) = \tau^{p_+}$,

$$|\tau_1^{p_+} - \tau_2^{p_+}| \leq \max\{f'(\tau_1), f'(\tau_2)\}|\tau_1 - \tau_2| \leq p_+|\tau_1 - \tau_2|(\max\{\tau_1,\tau_2\})^{p_+-1}. \tag{3.5}$$

Thus, it follows from (3.4), (3.5), and Young's inequality that

$$|a - b|^{p(x,y)-2}(a - b)(a\tau_1^{p_+} - b\tau_2^{p_+})$$

$$\geq |a - b|^{p(x,y)}(\max\{\tau_1, \tau_2\})^{p_+} - p_+\frac{p(x,y) - 1}{p(x,y)}\varepsilon^{\frac{p(x,y)}{p(x,y)-1}}|a - b|^{p(x,y)}(\max\{\tau_1, \tau_2\})^{(p_+-1)\frac{p(x,y)}{p(x,y)-1}}$$

$$- \frac{p_+}{p(x,y)}\varepsilon^{-p(x,y)}(\max\{a, b\})^{p(x,y)}|\tau_1 - \tau_2|^{p(x,y)}$$

$$\geq \left(1 - p_+\varepsilon^{p_+/(p_+-1)}\right)|a - b|^{p(x,y)}(\max\{\tau_1, \tau_2\})^{p_+} - \frac{p_+}{p_-}\varepsilon^{-p_+}(\max\{a, b\})^{p(x,y)}|\tau_1 - \tau_2|^{p(x,y)}.$$

Taking $\varepsilon = (1/(2p_+))^{(p_+-1)/p_+}$, we obtain (3.3) with $C = \frac{p_+}{p_-}(2p_+)^{p_+-1}$. □

Proof of Theorem 3.2. In this proof, every ball is centered at x_0. Let η be a cut-off function satisfying $\eta \in [0, 1]$, supp$\eta \subset B_{\frac{R+r}{2}} \subset B_R$, $\eta \equiv 1$ in B_r, and $|D\eta| \leq 4/(R - r)$). Let $p_\pm = p_\pm(B_R \times B_R)$. We first assume that $u \in L^\infty(B_{2R})$, then $\varphi(x) = w_+(x)\eta(x)^{p_+} \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^n)$ by [49, Lemma 4.1]. Applying the definition of weak subsolutions with the test function φ, we have

$$0 \geq \int_{B_R}\int_{B_R} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+sp(x,y)}}\,dy\,dx$$

$$+ 2\int_{B_R}\int_{\mathbb{R}^n\setminus B_R} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))w_+(x)\eta(x)^{p_+}}{|x - y|^{n+sp(x,y)}}\,dy\,dx =: I_1 + I_2. \tag{3.6}$$

It is easy to see that

$$|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))$$

$$\geq |w_+(x) - w_+(y)|^{p(x,y)-2}(w_+(x) - w_+(y))(w_+(x)\eta(x)^{p_+} - w_+(y)\eta(y)^{p_+}), \tag{3.7}$$

as in the proof of [27, Lemma 1.4]. Moreover, by Lemma 3.4 with $a = w_+(x)$, $b = w_+(y)$, $\tau_1 = \eta(x)$, and $\tau_2 = \eta(y)$, we obtain

$$|w_+(x) - w_+(y)|^{p(x,y)-2}(w_+(x) - w_+(y))(w_+(x)\eta(x)^{p_+} - w_+(y)\eta(y)^{p_+})$$

$$\geq \frac{1}{2}|w_+(x) - w_+(y)|^{p(x,y)}(\max\{\eta(x), \eta(y)\})^{p_+} - C(\max\{w_+(x), w_+(y)\})^{p(x,y)}|\eta(x) - \eta(y)|^{p(x,y)} \tag{3.8}$$

for all $x, y \in B_R$, where $C = C(p_+, p_-) > 0$. On the other hand, it is obvious that

$$|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y)) = 0 \tag{3.9}$$

when $u(x), u(y) \leq k$. Furthermore, if $u(x) > k$ and $u(y) \leq k$, then

$$|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y)) \geq c\left((w_+(x) - w_+(y))^{p(x,y)} + w_+(x)w_-(y)^{p(x,y)-1}\right)\eta^{p_+}(x) \tag{3.10}$$

by a similar argument as in [22, Proposition 8.5], where $c = 2^{p_- - 2} \wedge 1$. Therefore, combining (3.7)–(3.10) and using the symmetry of $p(x, y)$, we estimate I_1 by

$$I_1 = \left(\int_{A_{k,R}^+} \int_{A_{k,R}^+} + 2 \int_{A_{k,R}^+} \int_{B_R \setminus A_{k,R}^+} \right) \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+sp(x,y)}} \, dy \, dx$$

$$\geq c \int_{B_r} \int_{B_r} \frac{|w_+(x) - w_+(y)|^{p(x,y)}}{|x - y|^{n+sp(x,y)}} \, dy \, dx + 2c \int_{B_r} w_+(x) \int_{B_R} \frac{w_-(y)^{p(x,y)-1}}{|x - y|^{n+sp(x,y)}} \, dy \, dx$$

$$- C \int_{B_R} \int_{B_R} \frac{w_+(x)^{p(x,y)} |\eta(x) - \eta(y)|^{p(x,y)}}{|x - y|^{n+sp(x,y)}} \, dy \, dx, \tag{3.11}$$

where $A_{k,R}^+ = B_R \cap \{u > k\}$.

For I_2, we use the inequalities

$$|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))w_+(x) \geq -(u(y) - u(x))_+^{p(x,y)-1} w_+(x) \geq -w_+(y)^{p(x,y)-1} w_+(x)$$

and

$$\frac{|y - x_0|}{|x - y|} \leq 1 + \frac{|x - x_0|}{|x - y|} \leq 1 + \frac{R + r}{R - r} = \frac{2R}{R - r}, \quad x \in B_{\frac{R+r}{2}}, y \in \mathbb{R}^n \setminus B_R,$$

to obtain

$$I_2 \geq -2 \int_{B_R} \int_{\mathbb{R}^n \setminus B_R} \frac{w_+(y)^{p(x,y)-1} w_+(x) \eta(x)^{p_+}}{|x - y|^{n+sp(x,y)}} \, dy \, dx$$

$$\geq -2 \left(\sup_{x \in \text{supp} \, \eta} \int_{\mathbb{R}^n \setminus B_R} \frac{w_+(y)^{p(x,y)-1}}{|x - y|^{n+sp(x,y)}} \, dy \right) \int_{B_R} w_+(x) \eta(x)^{p_+} \, dx \tag{3.12}$$

$$\geq -2 \left(\sup_{x \in B_{\frac{R+r}{2}}} \int_{\mathbb{R}^n \setminus B_R} \frac{w_+(y)^{p(x,y)-1}}{|y - x_0|^{n+sp(x,y)}} \left(\frac{2R}{R - r} \right)^{n+sp(x,y)} \, dy \right) \int_{B_R} w_+(x) \, dx.$$

Therefore, (3.1) follows from (3.6), (3.11), (3.12), and $|D\eta| \leq 4/(R - r)$.

The general case $u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^n)$ follows by using a test function $\varphi(x) = (\min\{u, M\} - k)_+ \eta(x)^{p_+}$ instead of $\varphi(x) = w_+(x)\eta(x)^{p_+}$ and then taking a limit $M \to \infty$ in the resulting inequality. □

4 | LOCAL BOUNDEDNESS

In this section, we prove Theorem 1.4. The idea of the proof of the local boundedness is to fix a point $x_0 \in \Omega$ and find a small ball $B_{R/2}(x_0) \subset \Omega$ on which u is bounded. We do not only prove the local boundedness of weak subsolutions in $B_{R/2}(x_0)$ but also provide a quantitative estimate of their supremum. The proof of Theorem 1.4 is based on the De Giorgi iteration technique.

The Caccioppoli-type inequality and the fractional Sobolev inequality are crucial tools for the De Giorgi iteration. One can make use of the fractional Sobolev inequality with variable exponent developed in [39], but it requires the assumption $p_+(B_R(x_0) \times B_R(x_0)) < n/s$, which is stronger than the assumption made in Theorem 1.4, namely, $p(x_0, x_0) \leq n/s$. Thus, we will use the fractional Sobolev inequality with a constant exponent (Theorem 2.4).

For local variational problems, we have the continuous embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,p_-}(\Omega)$ by a simple inequality $\int |Du|^{p_-} \, dx \leq \int (|Du|^{p(x)} + 1) \, dx$. However, a similar continuous embedding $W^{s,p(\cdot,\cdot)}(\Omega) \hookrightarrow W^{s,p_-}(\Omega)$ is not available. Instead, we prove the following lemma, which shows a continuous embedding into a larger space with smaller orders of differentiability $\sigma < s$ and integrability $q < p_-$. This lemma is a generalization of [22, Lemma 4.6].

Lemma 4.1. *Let $\Omega' \subset \Omega \subset \mathbb{R}^n$ be two bounded measurable sets with $d := \operatorname{diam}(\Omega) \leq 1$. Let $1 \leq q < p_- \leq p(x, y) \leq p_+$ and $0 < \sigma < s < 1$, where $p_\pm = p_\pm(\Omega \times \Omega)$, then $W^{s, p(\cdot, \cdot)}(\Omega) \hookrightarrow W^{\sigma, q}(\Omega)$. In particular, for any $u \in W^{s, p(\cdot, \cdot)}(\Omega)$,*

$$\left(\int_\Omega \int_{\Omega'} \frac{|u(x) - u(y)|^q}{|x - y|^{n + \sigma q}} \, dy \, dx \right)^{1/q} \leq C \max \left\{ \left(|\Omega'| d^{(s-\sigma)\frac{p_+ + q}{p_+ - q}} \right)^{\frac{p_+ - q}{p_+ q}}, \left(|\Omega'| d^{(s-\sigma)\frac{p_+ + q}{p_+ - q}} \right)^{\frac{p_- - q}{p_- q}} \right\} [u]_{W^{s, p(\cdot, \cdot)}(\Omega)},$$

where $C = C(n, s, \sigma, p_+, p_-, q) > 0$.

Proof. We define

$$U(x, y) := \frac{|u(x) - u(y)|^q}{|x - y|^{n + \sigma q}},$$

then

$$U(x, y) = \frac{|u(x) - u(y)|^q}{|x - y|^{(n + sp(x,y))\frac{q}{p(x,y)}}} \frac{1}{|x - y|^{n\frac{p(x,y) - q}{p(x,y)} - (s - \sigma)q}} =: V(x, y) W(x, y).$$

Thus, by Lemma 2.2, we obtain

$$\int_\Omega \int_{\Omega'} \frac{|u(x) - u(y)|^q}{|x - y|^{n + \sigma q}} \, dy \, dx = \|U\|_{L^1(\Omega \times \Omega')} \leq 2 \|V\|_{L^{\frac{p(\cdot,\cdot)}{q}}(\Omega \times \Omega')} \|W\|_{L^{\frac{p(\cdot,\cdot)}{p(\cdot,\cdot) - q}}(\Omega \times \Omega')} = 2 [u]^q_{W^{s, p(\cdot,\cdot)}(\Omega)} \|W\|_{L^{\frac{p(\cdot,\cdot)}{p(\cdot,\cdot) - q}}(\Omega \times \Omega')}.$$

Using Lemma 2.1, we have

$$\|W\|_{L^{\frac{p(\cdot,\cdot)}{p(\cdot,\cdot) - q}}(\Omega \times \Omega')} \leq \max \left\{ \varrho_{L^{\frac{p(\cdot,\cdot)}{p(\cdot,\cdot) - q}}(\Omega \times \Omega')}(W)^{\frac{p_+ - q}{p_+}}, \varrho_{L^{\frac{p(\cdot,\cdot)}{p(\cdot,\cdot) - q}}(\Omega \times \Omega')}(W)^{\frac{p_- - q}{p_-}} \right\}.$$

Since $d \leq 1$,

$$\varrho_{L^{\frac{p(\cdot,\cdot)}{p(\cdot,\cdot) - q}}(\Omega \times \Omega')}(W) = \int_{\Omega'} \int_\Omega \frac{|x - y|^{(s-\sigma)\frac{p(x,y)q}{p(x,y) - q}}}{|x - y|^n} \, dy \, dx$$

$$\leq \int_{\Omega'} \int_\Omega \frac{|x - y|^{(s-\sigma)\frac{p_+ q}{p_+ - q}}}{|x - y|^n} \, dy \, dx$$

$$\leq \frac{p_+ - q}{(s - \sigma) p_+ q} |\mathbb{S}^{n-1}| |\Omega'| d^{(s-\sigma)\frac{p_+ q}{p_+ - q}}.$$

Therefore, combining the previous estimates finishes the proof. $\square$

As mentioned before, we will prove Theorem 1.4 by using the De Giorgi iteration technique. For this purpose, we need the following lemma.

Lemma 4.2. *Suppose that a sequence $\{Y_j\}_{j=0}^\infty$ of nonnegative numbers satisfies the recursion relation*

$$Y_{j+1} \leq C b^j \max \left\{ Y_j^{1 + \beta_1}, Y_j^{1 + \beta_2}, \dots, Y_j^{1 + \beta_N} \right\}$$

for some constants $C \geq 1$, $b > 1$, $N \in \mathbb{N}$, and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_N > 0$. If

$$Y_0 \leq C^{-\frac{1}{\beta_N}} b^{-\frac{1}{\beta_N^2}}, \tag{4.1}$$

then

$$Y_j \leq C^{-\frac{1}{\beta_N}} b^{-\frac{1}{\beta_N^2} - \frac{j}{\beta_N}} \quad \text{for all } j \geq 0, \tag{4.2}$$

and, consequently, $Y_j \to 0$ as $j \to \infty$.

Proof. Since $Y_0 \leq 1$, one can easily prove by induction that

$$Y_j \leq C^{\frac{(1+\beta_N)^j - 1}{\beta_N}} b^{\frac{(1+\beta_N)^j - 1}{\beta_N^2} - \frac{j}{\beta_N}} Y_0^{(1+\beta_N)^j} \leq 1, \quad \text{for all } j \geq 0,$$

under the assumption (4.1). This yields (4.2). □

We are now in a position to prove Theorem 1.4 by using Theorem 3.2, Lemma 4.1, and Lemma 4.2.

Proof of Theorem 1.4. Suppose that $u \in W^{s,p(\cdot,\cdot)}(\mathbb{R}^n)$ is a weak subsolution to (1.5) in Ω satisfying (1.8). Let us fix $x_0 \in \Omega$. We distinguish two different cases.

If $p(x_0, x_0) > n/s$, then the fact that $u \in W^{s,p(\cdot,\cdot)}(\Omega)$ implies that u is bounded in a neighborhood of x_0. Indeed, by the continuity of p, we can take $R > 0$ such that $\overline{B_R(x_0)} \subset \Omega$ and

$$p_- := p_-(B_R(x_0) \times B_R(x_0)) > \frac{n}{\sigma}$$

for $\sigma \in (0, s)$ sufficiently close to s. Let $q \in (n/\sigma, p_-)$, then by Theorem 2.6,

$$\|u\|_{C^\alpha(\overline{B_R(x_0)})} \leq \|u\|_{W^{\sigma,q}(B_R(x_0))}.$$

Moreover, by Lemma 4.1 and Lemma 2.2, we obtain

$$\|u\|_{W^{\sigma,q}(B_R(x_0))} \leq C\|u\|_{W^{s,p(\cdot,\cdot)}(B_R(x_0))} < +\infty.$$

Therefore,

$$\|u\|_{L^\infty(B_R(x_0))} \leq \|u\|_{C^\alpha(\overline{B_R(x_0)})} < +\infty.$$

It remains to study the case $p(x_0, x_0) \leq n/s$. By the continuity of p, we can take $R \in (0, 1/2)$ sufficiently small such that $B_R(x_0) \subset \Omega$ and

$$p_+ < p_-^* := \frac{np_-}{n - \sigma p_-}, \tag{4.3}$$

where $p_\pm = p_\pm(B_R(x_0) \times B_R(x_0))$. Note that $\sigma p_- < sp_- \leq n$.

We fix $k \in \mathbb{R}$ and $\tilde{k} \in \mathbb{R}^+$. In order to use the De Giorgi iteration, we set for each $j \in \mathbb{N} \cup \{0\}$

$$r_j = \frac{1}{2}(1 + 2^{-j})R, \quad \tilde{r}_j = \frac{r_j + r_{j+1}}{2}, \quad k_j = k + (1 - 2^{-j})\tilde{k}, \quad \text{and} \quad \tilde{k}_j = \frac{k_{j+1} + k_j}{2},$$

and define

$$w_j = (u - k_j)_+ \quad \text{and} \quad \tilde{w}_j = (u - \tilde{k}_j)_+.$$

For simplicity, we write $B_j = B(x_0, r_j)$ and $\tilde{B}_j = B(x_0, \tilde{r}_j)$.

By (4.3), we can choose a constant q such that

$$\max \left\{ p_+, \frac{n}{n - \sigma} \right\} < q < p_-^*. \tag{4.4}$$

Then, $q = t^* = \frac{nt}{n - \sigma t}$ for some $1 < t < p_- \leq n/\sigma$. By applying Theorem 2.4 to $\tilde{w}_j$ in $\tilde{B}_j$, we have

$$\|\tilde{w}_j\|_{L^q(\tilde{B}_j)} \leq C \|\tilde{w}_j\|_{W^{\sigma,t}(\tilde{B}_j)}$$

for some $C > 0$ depending on $\tilde{r}_j$. Since $\tilde{r}_j \in [R/2, R]$, we may assume that C depends on R, but not on j, with a possibly larger constant C. Since the quantities $n, s, \sigma, p_+(B_R \times \mathbb{R}^n), p_-(B_R \times B_R), q$, and R are not important for the iteration, we will absorb these quantities into constants C. Moreover, using Lemma 4.1 and Lemma 2.2, we have

$$\|\tilde{w}_j\|_{W^{\sigma,t}(\tilde{B}_j)} \leq C \|\tilde{w}_j\|_{W^{s,p(\cdot,\cdot)}(\tilde{B}_j)}$$

for some $C = C(n, s, \sigma, p_+, p_-, q, R) > 0$. By Lemma 2.3,

$$\|\tilde{w}_j\|_{W^{s,p(\cdot,\cdot)}(\tilde{B}_j)} \leq 2|\tilde{w}_j|_{W^{s,p(\cdot,\cdot)}(\tilde{B}_j)} \leq 2 \max \left\{ \tilde{\varrho}_{W^{s,p(\cdot,\cdot)}(\tilde{B}_j)}(\tilde{w}_j)^{1/p_-}, \tilde{\varrho}_{W^{s,p(\cdot,\cdot)}(\tilde{B}_j)}(\tilde{w}_j)^{1/p_+} \right\}. \tag{4.5}$$

We set $A_{h,r}^+ = B_r \cap \{u > h\}$ and

$$Y_j = \int_{B_j} w_j^{p_+}(x) \, dx,$$

and estimate $\tilde{\varrho}_{W^{s,p(\cdot,\cdot)}(\tilde{B}_j)}(\tilde{w}_j)$ in terms of j and Y_j.

By Theorem 3.2, we have

$$\tilde{\varrho}_{W^{s,p(\cdot,\cdot)}(\tilde{B}_j)}(\tilde{w}_j) \leq C 2^{jp_+} \int_{B_j} \int_{B_j} \frac{\tilde{w}_j(x)^{p(x,y)}}{|x-y|^{n+sp(x,y)}} |x-y|^{p(x,y)} \, dy \, dx$$

$$+ C \left(\sup_{x \in B(x_0, \frac{1}{2}(r_j + \tilde{r}_j))} \int_{\mathbb{R}^n \setminus B_j} \frac{\tilde{w}_j(y)^{p(x,y)-1}}{|y-x_0|^{n+sp(x,y)}} \left(\frac{4r_j}{r_j - r_{j+1}} \right)^{n+sp(x,y)} \, dy \right) \int_{B_j} \tilde{w}_j(x) \, dx$$

$$+ \int_{\tilde{B}_j} \tilde{w}_j^{\bar{p}(x)}(x) \, dx$$

$$=: I_1 + I_2 + I_3.$$

Since $\tilde{w}_j = 0$ on $B_j \setminus A_{\tilde{k}_j, r_j}^+$, $\tilde{w}_j \leq w_j$, and $R < 1/2$, we estimate I_1 as follows:

$$I_1 \leq C 2^{jp_+} \int_{A_{\tilde{k}_j, r_j}^+} \int_{B_j} (w_j(x)^{p_+} + 1) \frac{|x-y|^{(1-s)p_-}}{|x-y|^n} \, dy \, dx \leq C 2^{jp_+} \left(\int_{A_{\tilde{k}_j, r_j}^+} w_j(x)^{p_+} \, dx + |A_{\tilde{k}_j, r_j}^+| \right) \leq C 2^{jp_+} \left(Y_j + |A_{\tilde{k}_j, r_j}^+| \right).$$

Similarly, I_3 is estimated as

$$I_3 \leq C\left(Y_j + |A^+_{\tilde{k}_j, r_j}|\right).$$

For I_2, we use $p(x, y) \leq p_+(B_R \times \mathbb{R}^n)$ and $(\tilde{k}_j - k_j)^{p_+ - 1} \tilde{w}_j \leq w_j^{p_+}$. Then,

$$I_2 \leq C2^{j(n + sp_+(B_R \times \mathbb{R}^n))}\left(\sup_{x \in B_R} \int_{\mathbb{R}^n \setminus B_{R/2}} \frac{\tilde{w}_j(y)^{p(x,y) - 1}}{|y - x_0|^{n + sp(x,y)}}\, dy\right) \int_{B_j} \frac{w_j^{p_+}(x)}{(\tilde{k}_j - k_j)^{p_+ - 1}}\, dx$$

$$\leq C2^{j(n + sp_+(B_R \times \mathbb{R}^n))}\left(\sup_{x \in B_R} \int_{\mathbb{R}^n \setminus B_{R/2}} \frac{w_0(y)^{p(x,y) - 1}}{|y - x_0|^{n + sp(x,y)}}\, dy\right)\left(\frac{2^{j+2}}{\tilde{k}}\right)^{p_+ - 1} \int_{A^+_{\tilde{k}_j, r_j}} w_j^{p_+}\, dx$$

$$\leq C2^{j(n + 2p_+(B_R \times \mathbb{R}^n))} \frac{T}{\tilde{k}^{p_+ - 1}} Y_j,$$

where

$$T = \sup_{x \in B_R} \int_{\mathbb{R}^n \setminus B_{R/2}} \frac{u_+(y)^{p(x,y) - 1}}{|y - x_0|^{n + sp(x,y)}}\, dy.$$

Combining the estimates above and using

$$|A^+_{\tilde{k}_j, r_j}| \leq \frac{1}{(\tilde{k}_j - k_j)^{p_+}} \int_{A_{\tilde{k}_j, r_j}} w_j^{p_+}\, dx \leq C\left(\frac{2^j}{\tilde{k}}\right)^{p_+} Y_j,$$

yield that

$$\tilde{\varrho}_{W^{s, p(\cdot, \cdot)}(\tilde{B}_j)}(\tilde{w}_j) \leq C2^{j(n + 2p_+(B_R \times \mathbb{R}^n) - 1)}\left(1 + \frac{1}{\tilde{k}^{p_+}} + \frac{T}{\tilde{k}^{p_+ - 1}}\right) Y_j.$$

Assuming

$$\tilde{k} \geq T^{1/(p_+ - 1)} + 1, \tag{4.6}$$

we arrive at

$$\tilde{\varrho}_{W^{s, p(\cdot, \cdot)}(\tilde{B}_j)}(\tilde{w}_j) \leq C2^{j(n + 2p_+(B_R \times \mathbb{R}^n))} Y_j. \tag{4.7}$$

On the other hand, recalling that (4.4) holds, we have $\tilde{w}_j^q \geq (k_{j+1} - \tilde{k}_j)^{q - p_+} w_{j+1}^{p_+}$, and hence

$$\|\tilde{w}_j\|^q_{L^q(\tilde{B}_j)} \geq \|\tilde{w}_j\|^q_{L^q(B_{j+1})} \geq c(2^{-j}\tilde{k})^{q - p_+} Y_{j+1}. \tag{4.8}$$

Therefore, from (4.5), (4.7), and (4.8), we deduce

$$Y_{j+1} \leq C\tilde{k}^{p_+ - q} b^j \max\left\{Y_j^{1 + \beta_1}, Y_j^{1 + \beta_2}\right\},$$

where $\beta_1 = q/p_- - 1 > 0$, $\beta_2 = q/p_+ - 1 > 0$, $b = 2^{q - p_+ + (n + 2p_+(B_R \times \mathbb{R}^n))q/p_-}$, and $C > 0$ is a constant depending only on $n, s, \sigma, p_+, p_-, q,$ and R.

By Lemma 4.2, if

$$Y_0 \leq (C\tilde{k}^{p_+ - q})^{-\frac{1}{\beta_2}} b^{-\frac{1}{\beta_2^2}}, \tag{4.9}$$

then $Y_j \to 0$ as $j \to \infty$. Thus, if we take

$$\tilde{k} \geq \left(C^{\frac{1}{\beta_2}} b^{\frac{1}{\beta_2^2}} Y_0 \right)^{\frac{1}{p_+}}, \tag{4.10}$$

then (4.9) is satisfied, and hence

$$\sup_{B_{R/2}} u \leq k + \tilde{k}.$$

Note that the choice

$$\tilde{k} = \left(C^{\frac{1}{\beta_2}} b^{\frac{1}{\beta_2^2}} \right)^{\frac{1}{p_+}} \left(\int_{B_R} w_0^{p_+}(x)\,dx \right)^{\frac{1}{p_+}} + T^{1/(p_+ - 1)} + 1$$

is in accordance with (4.6) and (4.10). The constant C_0 depends on $n, s, \sigma, p_+(B_R \times \mathbb{R}^n), p_-(B_R \times B_R), q$, and R. We finish the proof by choosing $k = 0$. $\square$

5 | HÖLDER ESTIMATE

This section is devoted to the proof of the local Hölder regularity of weak solutions to (1.5). In this part of the paper, the assumptions (P1) and (P2) on p take an important role for our analysis. The key step in establishing the local Hölder regularity is a growth lemma, see Lemma 5.2. We start with an auxiliary result that is needed in the proof of the growth lemma.

Lemma 5.1. *Let $B_R = B_R(x_0) \subset \mathbb{R}^n$ with $R \in (0,1)$. Let $H > 0$, $\delta \in (0, 1/8]$ and $0 < \sigma < s < 1$. Assume that p satisfies (P1) and (P2) in B_R, $H^{p_+ - p_-} \leq 2$, and $p_+ < p_-^*$, where $p_\pm = p_\pm(B_R \times B_R)$ and $p_-^* = \frac{np_-}{n - \sigma p_-}$. Let $u \in W^{s, p(\cdot, \cdot)}(\mathbb{R}^n)$ be a weak supersolution to (1.5) in B_R such that*

$$0 \leq u \leq 2H \quad \text{in } B_R \quad \text{and} \quad |B_{R/2} \cap \{u \geq H\}| \geq \gamma |B_{R/2}| \tag{5.1}$$

for some $\gamma \in (0,1)$. Assume $R^s \leq \delta H$ and

$$\sup_{x \in B_{3R/4}} \int_{\mathbb{R}^n \setminus B_R} \frac{u_-(y)^{p(x,y)-1}}{|y - x_0|^{n + sp(x,y)}}\,dy \leq R^{-sp_+}(\delta H)^{p_+ - 1} + R^{-sp_-}(\delta H)^{p_- - 1}. \tag{5.2}$$

Let $1 \leq q < p_-$. Then, there is a constant $C = C(n, s, \sigma, p_+(B_R \times \mathbb{R}^n), p_-(B_R \times \mathbb{R}^n), q, L) > 0$ such that for any $\ell \in [2\delta H, H]$,

$$[(u - \ell)_-]^q_{W^{\sigma,q}(B_{R/2})} \leq C\ell^q R^{-\sigma q} \max\left\{ |A_{\ell,R}^-|, |A_{\ell,R}^-|^{1 + \frac{q}{p_-} - \frac{q}{p_+}}, |A_{\ell,R}^-|^{1 + \frac{q}{p_+} - \frac{q}{p_-}} \right\},$$

where $A_{\ell,R}^- = B_R \cap \{u < \ell\}$.

Proof. Let $\ell \in [2\delta H, H]$. The idea of the proof is to estimate $[(u - \ell)_-]^q_{W^{\sigma,q}(B_{R/2})}$ using Lemma 4.1 and then applying the Caccioppoli-type inequality to estimate $\varrho_{W^{s,p(\cdot,\cdot)}(B_{R/2})}((u - \ell)_-)$. In the following, $C > 0$ denotes a constant depending on $n, s, \sigma, p_+(B_R \times \mathbb{R}^n), p_-(B_R \times \mathbb{R}^n), q$, and L whose exact value is not important and might change from line to line.

Let $r = R/2$. First, by Lemma 4.1,

$$[(u - \ell)_-]^q_{W^{\sigma,q}(B_r)} \leq C \int_{B_r} \int_{B_r} \frac{|(u(x) - \ell)_- - (u(y) - \ell)_-|^q}{|x - y|^{n+\sigma q}} \, dx \, dy$$

$$\leq C \int_{A^-_{\ell,r}} \int_{B_r} \frac{|(u(x) - \ell)_- - (u(y) - \ell)_-|^q}{|x - y|^{n+\sigma q}} \, dx \, dy$$

$$\leq C \max \left\{ \left(|A^-_{\ell,r}| R^{(s-\sigma)\frac{p_++q}{p_+-q}} \right)^{\frac{p_+-q}{p_+}}, \left(|A^-_{\ell,r}| R^{(s-\sigma)\frac{p_++q}{p_+-q}} \right)^{\frac{p_--q}{p_-}} \right\} [(u - \ell)_-]^q_{W^{s,p(\cdot,\cdot)}(B_r)}$$

$$\leq C \max \left\{ |A^-_{\ell,r}|^{\frac{p_+-q}{p_+}} R^{q(s-\sigma)}, |A^-_{\ell,r}|^{\frac{p_--q}{p_-}} R^{(s-\sigma)\frac{p_++q(p_--q)}{(p_+-q)p_-}} \right\}$$

$$\times \max \left\{ \left(\varrho_{W^{s,p(\cdot,\cdot)}(B_r)}((u - \ell)_-) \right)^{\frac{q}{p_+}}, \left(\varrho_{W^{s,p(\cdot,\cdot)}(B_r)}((u - \ell)_-) \right)^{\frac{q}{p_-}} \right\}.$$

By Theorem 3.2, we can estimate $\varrho_{W^{s,p(\cdot,\cdot)}(B_r)}((u - \ell)_-)$ as follows:

$$\varrho_{W^{s,p(\cdot,\cdot)}(B_r)}((u - \ell)_-) + \int_{B_r} (u - \ell)_-(x) \int_{B_r} \frac{(u(y) - \ell)_+^{p(x,y)-1}}{|x - y|^{n+sp(x,y)}} \, dy \, dx$$

$$\leq C \int_{B_R} \int_{B_R} \left| \frac{(u(x) - \ell)_-}{R - r} \right|^{p(x,y)} \frac{dy \, dx}{|x - y|^{n-(1-s)p(x,y)}}$$

$$+ C \left(\sup_{x \in B_{\frac{R+r}{2}}} \int_{\mathbb{R}^n \backslash B_R} \frac{(u(y) - \ell)_-^{p(x,y)-1}}{|y - x_0|^{n+sp(x,y)}} \left(\frac{2R}{R - r} \right)^{n+sp(x,y)} \, dy \right) \int_{B_R} (u(x) - \ell)_- \, dx$$

$$=: I_1 + I_2.$$

First, we consider I_1. By the nonnegativity of u in B_R,

$$I_1 = C \int_{A^-_{\ell,R}} \int_{B_R} \left| \frac{(u(x) - \ell)_-}{R - r} \right|^{p(x,y)} \frac{dy \, dx}{|x - y|^{n-(1-s)p(x,y)}}$$

$$\leq C |A^-_{\ell,R}| \left(\left| \frac{\ell}{R - r} \right|^{p_+} + \left| \frac{\ell}{R - r} \right|^{p_-} \right) \left(R^{(1-s)p_+} + R^{(1-s)p_-} \right)$$

$$\leq C |A^-_{\ell,R}| \left(R^{-sp_+} \ell^{p_+} + R^{-sp_-} \ell^{p_-} \right),$$

where we used (P1) in the last inequality. Next, we study I_2. Note that by the assumption $R^s \leq \delta H \leq \ell$ and (P2), we have

$$R^{-sp_\pm(B_R \times B_R^c)} \ell^{p_\pm(B_R \times B_R^c)} \leq R^{-sp_\pm} \ell^{p_\pm}. \tag{5.3}$$

Using the nonnegativity of u in B_R, the tail estimate (5.2), and (5.3):

$$I_2 \leq C \sup_{x \in B_{\frac{R+r}{2}}} \int_{\mathbb{R}^n \setminus B_R} \frac{(u(y) - \ell)_-^{p(x,y)-1}}{|y - x_0|^{n+sp(x,y)}} \, dy \int_{B_R} (u(x) - \ell)_- \, dx$$

$$\leq C \sup_{x \in B_{\frac{R+r}{2}}} \left(\int_{\mathbb{R}^n \setminus B_R} \frac{u(y)_-^{p(x,y)-1}}{|y - x_0|^{n+sp(x,y)}} \, dy + \int_{\mathbb{R}^n \setminus B_R} \left(\frac{\ell}{|y - x_0|^s} \right)^{p(x,y)} \frac{\ell^{-1}}{|y - x_0|^n} \, dy \right) \ell |A_{\ell,R}^-|$$

$$\leq C\ell |A_{\ell,R}^-| \left(R^{-sp_+}(\delta H)^{p_+-1} + R^{-sp_-}(\delta H)^{p_--1} + \ell^{p_+(B_R \times B_R^c)-1} R^{-sp_+(B_R \times B_R^c)} + \ell^{p_-(B_R \times B_R^c)-1} R^{-sp_-(B_R \times B_R^c)} \right)$$

$$\leq C|A_{\ell,R}^-|(R^{-sp_+}\ell^{p_+} + R^{-sp_-}\ell^{p_-}).$$

Hence, since $R < 1$,

$$\varrho_{W^{s,p(\cdot,\cdot)}(B_{R/2})}((u-\ell)_-) + \int_{B_r} (u-\ell)_-(x) \int_{B_r} \frac{(u(y)-\ell)_+^{p(x,y)-1}}{|x-y|^{n+sp(x,y)}} \, dy \, dx \leq C|A_{\ell,R}^-|(R^{-sp_+}\ell^{p_+} + R^{-sp_-}\ell^{p_-})$$

$$\leq C|A_{\ell,R}^-|R^{-sp_+}(\ell^{p_+} + \ell^{p_-}). \tag{5.4}$$

Combining the previous estimates, we get

$$R^{\sigma q}[(u-\ell)_-]_{W^{\sigma,q}(B_{R/2})}^q \leq CR^{\sigma q} \max \left\{ |A_{\ell,r}^-|^{\frac{p_+-q}{p_+}} R^{q(s-\sigma)}, |A_{\ell,r}^-|^{\frac{p_--q}{p_-}} R^{(s-\sigma)\frac{p_++q(p_--q)}{(p_++q)p_-}} \right\}$$

$$\times \max \left\{ \left(|A_{\ell,R}^-|R^{-sp_+}(\ell^{p_+} + \ell^{p_-}) \right)^{q/p_+}, \left(|A_{\ell,R}^-|R^{-sp_+}(\ell^{p_+} + \ell^{p_-}) \right)^{q/p_-} \right\}$$

$$=: CR^{\sigma q} \max\{Y_1, Y_2\} \max\{\Phi_1, \Phi_2\}.$$

We need to check the four possible cases for that inequality. Before doing that, note that since $\ell \in [2\delta H, H]$ and $R^s \leq \delta H$, there is a constant $C > 0$ such that

$$1 + \ell^{p_--p_+} \leq 1 + R^{s(p_--p_+)} \leq 1 + L^s \leq C, \tag{5.5}$$

where we used (P1).

Case 1: We have by (5.5),

$$R^{\sigma q} Y_1 \Phi_1 = |A_{\ell,R}^-|(\ell^{p_+} + \ell^{p_-})^{\frac{q}{p_+}} \leq C|A_{\ell,R}^-|\ell^q.$$

Case 2: By $\ell \in [2\delta H, H]$ and the assumptions $H^{p_+-p_-} \leq 2$ and (P1), we get

$$R^{\sigma q} Y_1 \Phi_2 = |A_{\ell,R}^-|^{1+\frac{q}{p_-}-\frac{q}{p_+}} R^{\frac{sq}{p_-}(p_--p_+)}(\ell^{p_+} + \ell^{p_-})^{\frac{q}{p_-}} \leq C|A_{\ell,R}^-|^{1+\frac{q}{p_-}-\frac{q}{p_+}}\ell^q.$$

Case 3: Using (5.5) together with (P1), we get

$$R^{\sigma q} Y_2 \Phi_1 = |A_{\ell,R}^-|^{1-\frac{q}{p_-}+\frac{q}{p_+}} R^{(s-\sigma)\frac{q^2(p_--p_+)}{(p_++q)p_-}}(\ell^{p_+} + \ell^{p_-})^{\frac{q}{p_+}} \leq C|A_{\ell,R}^-|^{1-\frac{q}{p_-}+\frac{q}{p_+}}\ell^q.$$

Case 4: By $H^{p_+-p_-} \leq 2$, and (P1), we get

$$R^{\sigma q} Y_2 \Phi_2 = |A_{\ell,R}^-|R^{\frac{(sp_+-\sigma q)q}{p_--(p_++q)}(p_--p_+)}(\ell^{p_+} + \ell^{p_-})^{\frac{q}{p_-}} \leq C|A_{\ell,R}^-|\ell^q.$$

Combining the estimates from the previous four cases proves the assertion of the lemma. □

We are now in a position to prove the growth lemma. It is the main ingredient for the proof of the local Hölder regularity estimate.

Lemma 5.2. *Let $B_R = B_R(x_0) \subset \mathbb{R}^n$ with $R \in (0, 1)$. Let $H > 0$, and $0 < \sigma < s < 1$. Assume that p satisfies (P1) and (P2) in B_R, $H^{p_+ - p_-} \leq 2$, and $p_+ < p_-^*$, where $p_\pm = p_\pm(B_R \times B_R)$ and $p_-^* = \frac{np_-}{n - \sigma p_-}$. Let $u \in W^{s, p(\cdot, \cdot)}(\mathbb{R}^n)$ be a weak supersolution to (1.5) in B_R such that (5.1) is satisfied for some $\gamma \in (0, 1)$. Then, there exists $\delta \in (0, 1/8]$, such that, if $R^s \leq \delta H$ and (5.2) is satisfied, then*

$$u \geq \delta H \qquad in \ B_{R/4}. \tag{5.6}$$

The constant δ depends on n, s, σ, $p_+(B_R \times \mathbb{R}^n)$, $p_-(B_R \times \mathbb{R}^n)$, and L.

Proof. The proof follows the ideas of [22, Proof of Lemma 6.3]. Let $0 < \delta < 1/8$ and $0 < \tau \leq 2^{-n-1}$ to be specified later. We first suppose

$$|B_{R/2} \cap \{u < 2\delta H\}| \leq \tau |B_{R/2}| \tag{5.7}$$

and prove the assertion of the lemma under this additional assumption. Afterwards we prove that this precondition (5.7) is indeed a consequence of the given assumptions of the lemma.

We use $C > 0$ for a constant depending on n, s, σ, $p_+(B_R \times \mathbb{R}^n)$, $p_-(B_R \times \mathbb{R}^n)$, q, and L whose exact value is not important and that might change from line to line.

The idea to prove the assertion of the lemma is by iteration and the use of Lemma 4.2. For this purpose, we need to establish some auxiliary results. Let $\delta H \leq h < k \leq 2\delta H$ and $\frac{R}{4} \leq \rho < r \leq \frac{R}{2}$. Note that by (5.7),

$$|B_\rho \cap \{(u - k)_- = 0\}| = |B_\rho \setminus \{u < k\}| \geq |B_\rho| - |B_{R/2} \cap \{u < 2\delta H\}|$$

$$\geq |B_\rho| - \tau |B_{R/2}| = \left(1 - \tau \left(\frac{R/2}{\rho}\right)^n\right)|B_\rho| \geq (1 - 2^n \tau)|B_\rho| \geq \frac{1}{2}|B_\rho|. \tag{5.8}$$

Using (5.8), Theorem 2.5, and Lemma 5.1, we have

$$(k - h)|A_{h,\rho}^-|^{\frac{n-\sigma}{n}} \leq \left(\int_{A_{h,\rho}^-} (k - u(x))^{\frac{n}{n-\sigma}} \, dx\right)^{\frac{n-\sigma}{n}} \leq \left(\int_{B_\rho} (u(x) - k)_-^{\frac{n}{n-\sigma}} \, dx\right)^{\frac{n-\sigma}{n}}$$

$$\leq C \int_{B_\rho} \int_{B_\rho} \frac{|(u(x) - k)_- - (u(y) - k)_-|}{|x - y|^{n+\sigma}} \, dx \, dy \tag{5.9}$$

$$\leq C k r^{-\sigma} \max \left\{ |A_{k,r}^-|, |A_{k,r}^-|^{1 + \frac{1}{p_-} - \frac{1}{p_+}}, |A_{k,r}^-|^{1 + \frac{1}{p_+} - \frac{1}{p_-}} \right\},$$

where $A_{k,r}^- = B_r \cap \{u < r\}$. In the proceeding, we use (5.9) to prove the assertion of the lemma by iteration. We define for $j \in \mathbb{N} \cup \{0\}$

$$r_j = \frac{1}{4}(1 + 2^{-j})R, \quad k_j = (1 + 2^{-j})\delta H, \quad \text{and} \quad y_j = \frac{|A_{k_j, r_j}^-|}{|B_{r_j}|}.$$

Then, $r_j \in (\frac{1}{4}R, \frac{1}{2}R]$ and $k_j \in (\delta H, 2\delta H]$. Choosing $k = k_j$, $h = k_{j+1}$, $\rho = r_{j+1}$, and $r = r_j$, we get from (5.9)

$$\frac{\delta H}{2^{j+1}} \left(y_{j+1}|B_{r_{j+1}}|\right)^{\frac{n-\sigma}{n}} \leq C(\delta H) r_j^{-\sigma} \max \left\{ r_j^n y_j, (r_j^n y_j)^{1 + \frac{p_+ - p_-}{p_+ p_-}}, (r_j^n y_j)^{1 + \frac{p_- - p_+}{p_+ p_-}} \right\},$$

which leads to

$$y_{j+1} \leq C\left(2^j r_j^{-n} \max\left\{ r_j^n y_j, (r_j^n y_j)^{1+\frac{p_+ - p_-}{p_+ p_-}}, (r_j^n y_j)^{1+\frac{p_- - p_+}{p_+ p_-}} \right\}\right)^{\frac{n}{n-\sigma}}. \tag{5.10}$$

If we prove that there are $\beta_1, \beta_2, \beta_3 > 0$ such that

$$y_{j+1} \leq C 2^{\frac{n}{n-\sigma} j} \max\left\{ y_j^{1+\beta_1}, y_j^{1+\beta_2}, y_j^{1+\beta_3} \right\} \tag{5.11}$$

and y_0 is sufficiently small, then we can apply Lemma 4.2, which would prove (5.6). We have three cases for the maximum in (5.10):

Case 1: In the first case, we have

$$y_{j+1} \leq C 2^{\frac{n}{n-\sigma} j} y_j^{\frac{n}{n-\sigma}}.$$

Since $\frac{n}{n-\sigma} > 1$, this proves the assertion in the first case.

Case 2: In the second case, using $r_j \leq 1$ and the fact that its exponent is positive,

$$y_{j+1} \leq C 2^{\frac{n}{n-\sigma} j} r_j^{\frac{p_+ - p_-}{p_+ p_-} \frac{n}{n-\sigma}} y_j^{\frac{n}{n-\sigma} + \frac{p_+ - p_-}{p_+ p_-} \frac{n}{n-\sigma}} \leq C 2^{\frac{n}{n-\sigma} j} y_j^{\frac{n}{n-\sigma} + \frac{p_+ - p_-}{p_+ p_-} \frac{n}{n-\sigma}}.$$

Since $\frac{n}{n-\sigma} + \frac{p_+ - p_-}{p_+ p_-} \frac{n}{n-\sigma} > \frac{n}{n-\sigma} > 1$, we have proven the assertion in the second case.

Case 3: In the third case, using (P1), we have

$$y_{j+1} \leq C 2^{\frac{n}{n-\sigma} j} r_j^{\frac{n}{(n-\sigma)p_+ p_-}(p_- - p_+)} y_j^{\frac{n}{n-\sigma}\left(1 + \frac{1}{p_+} - \frac{1}{p_-}\right)} \leq C 2^{\frac{n}{n-\sigma} j} y_j^{\frac{n}{n-\sigma}\left(1 + \frac{1}{p_+} - \frac{1}{p_-}\right)}.$$

Note that by assumption $p_+ < p_-^*$, where $p_-^* = \frac{np_-}{n - \sigma p_-}$, which is equivalent to

$$p_+ < p_-^* \iff \frac{\sigma}{n} > \frac{1}{p_-} - \frac{1}{p_+} \iff \frac{n}{n-\sigma}\left(1 + \frac{1}{p_+} - \frac{1}{p_-}\right) > 1.$$

This completes the proof in this case.

Hence, we have shown (5.11). Note that by (5.7)

$$y_0 = \frac{|A_{2\delta H, \frac{R}{2}}^-|}{|B_{\frac{R}{2}}|} \leq \tau.$$

Choosing τ sufficiently small, allows us to apply Lemma 4.2, which yields $y_j \to 0$ as $j \to \infty$ and proves (5.6).

In the remainder of the proof, we show (5.7). We prove this assertion by contradiction. Hence, suppose that (5.7) is not true, that is,

$$|B_{R/2} \cap \{u < 2\delta H\}| > \tau |B_{R/2}|. \tag{5.12}$$

We split the proof into two cases for $s \in (0, 1)$. First, when s is sufficiently large, we prove the assertion using an isoperimetric-type inequality by Cozzi [22, Proposition 5.1]. Second, in the case of small s, the assertion follows by a direct calculation.

Let $\bar{s}$ be the constant coming from the isoperimetric-type inequality [22, Proposition 5.1] (applied for the constant exponent case q and for σ) and let $s \in [\bar{s}, 1)$. For given δ, there is a unique $m \in \mathbb{N}$ such that

$$2^{-m-1} \leq \delta < 2^{-m}.$$

We define for $i = 0, \dots, m-1$, $k_i = 2^{-i}H$. Note that by definition $k_i \in (2\delta H, H]$. In the following, we check the conditions to apply [22, Proposition 5.1]. By (5.1) and (5.12), we get

$$|B_{R/2} \cap \{(u - k_{i-1})_- \leq 2^{-i}H\}| = |B_{R/2} \cap \{u \geq k_i\}| \geq |B_{R/2} \cap \{u \geq H\}| \geq \gamma |B_{R/2}|$$

and

$$|B_{R/2} \cap \{(u - k_{i-1})_- \geq 3 \cdot 2^{-i-1}H\}| = |B_{R/2} \cap \{u \leq k_{i+1}\}| \geq |B_{R/2} \cap \{u < 2\delta H\}| \geq \tau |B_{R/2}|$$

for $i = 1, \dots, m-2$. In order to apply [22, Proposition 5.1], it remains to prove that there is a constant $C > 0$ such that

$$\|(u - k_{i-1})_-\|_{L^q(B_{R/2})}^q + R^{\sigma q}[(u - k_{i-1})_-]_{W^{\sigma,q}(B_{R/2})}^q \leq C(k_i - k_{i+1})^q R^n. \tag{5.13}$$

Using the nonnegativity of u in B_R, we get

$$\|(u - k_{i-1})_-\|_{L^q(B_{R/2})}^q \leq C k_{i-1}^q R^n.$$

Combining this estimate together with Lemma 5.1 for $\ell = k_{i-1}$,

$$\|(u - k_{i-1})_-\|_{L^q(B_{R/2})}^q + R^{\sigma q}[(u - k_{i-1})_-]_{W^{\sigma,q}(B_{R/2})}^q \leq C k_{i-1}^q \max \left\{ |A_{k_{i-1},R}^-|, |A_{k_{i-1},R}^-|^{1 + \frac{q}{p_-} - \frac{q}{p_+}}, |A_{k_{i-1},R}^-|^{1 + \frac{q}{p_+} - \frac{q}{p_-}} \right\}$$
$$\leq C(k_i - k_{i+1})^q R^n \tag{5.14}$$

for some constant $C = C(n, s, \sigma, p_+(B_R \times \mathbb{R}^n), p_-(B_R \times \mathbb{R}^n), q, L) > 0$, where we used (P1) in the last inequality. This proves (5.13) and therefore, we can apply [22, Proposition 5.1] with $h = k_{i-1} - k_i$, $k = k_{i-1} - k_{i+1}$, and the function $(u - k_{i-1})_-$, that yields

$$(k_i - k_{i+1}) \big[|B_{R/2} \cap \{u \geq k_i\}| |B_{R/2} \cap \{u \leq k_{i+1}\}| \big]^{\frac{n-1}{n}}$$
$$\leq C R^{n-2+\sigma}[(u - k_{i-1})_-]_{W^{\sigma,q}(B_{R/2})} \big| B_{R/2} \cap \{k_{i+1} < u \leq k_i\} \big|^{\frac{q-1}{q}}. \tag{5.15}$$

In the following, we show that this inequality leads to a contradiction. On the one hand, the left-hand side can be estimated with (5.1) by

$$(k_i - k_{i+1}) \big[|B_{R/2} \cap \{u \geq k_i\}| |B_{R/2} \cap \{u \leq k_{i+1}\}| \big]^{\frac{n-1}{n}} \geq C k_{i+1} \big[R^n |B_{R/2} \cap \{u < 2\delta H\}| \big]^{\frac{n-1}{n}}.$$

On the other hand, we can estimate the right-hand side, using (5.14), by

$$R^{n-2+\sigma}[(u - k_{i-1})_-]_{W^{\sigma,q}(B_{R/2})} \big| B_{R/2} \cap \{k_{i+1} < u \leq k_i\} \big|^{\frac{q-1}{q}} \leq C R^{n-2+\frac{n}{q}} k_{i+1} \big| B_{R/2} \cap \{k_{i+1} < u \leq k_i\} \big|^{\frac{q-1}{q}}.$$

Hence, we get from (5.15)

$$|B_{R/2} \cap \{u < 2\delta H\}|^{\frac{q(n-1)}{(q-1)n}} \leq C R^{\frac{n-q}{q-1}} \big| B_{R/2} \cap \{k_{i+1} < u \leq k_i\} \big|.$$

Summing up this inequality over $i = 1, \dots, m-2$ gives us

$$(m-2)\left[|B_{R/2} \cap \{u < 2\delta H\}|\right]^{\frac{q(n-1)}{(q-1)n}} \leq CR^{\frac{n-q}{q-1}}\left|B_{R/2}\right| = CR^{\frac{q(n-1)}{q-1}},$$

which leads to

$$|B_{R/2} \cap \{u < 2\delta H\}| \leq CR^n m^{-\frac{n(q-1)}{(n-1)q}} \leq C|B_{R/2}||\log\delta|^{-\frac{n(q-1)}{(n-1)q}}.$$

Estimating the left-hand side by (5.12), we get

$$|\log\delta|^{-\frac{n(q-1)}{(n-1)q}} \geq C.$$

Hence, choosing δ sufficiently small results in a contradiction and finishes the proof for the case $s \in [\bar{s}, 1)$.

Now let $s \in (0, \bar{s})$. In this case, we get by (5.4), (5.1), and (5.12)

$$((4\delta H)^{p_+} + (4\delta H)^{p_-})R^{n-sp_+} \geq C \int_{B_{R/2}} \int_{B_{R/2}} \frac{(u(x)-4\delta H)_+^{p(x,y)-1}(u(y)-4\delta H)_-}{|x-y|^{n+sp(x,y)}} \, dy \, dx$$

$$\geq \frac{C}{R^{n+sp_-}} \int_{B_{R/2} \cap \{u \geq H\}} (u(x)-4\delta H)^{p(x,y)-1} \, dx \int_{B_{R/2} \cap \{u < 2\delta H\}} (4\delta H - u(y)) dy$$

$$\geq \frac{C}{R^{n+sp_-}} |B_{R/2} \cap \{u \geq H\}| \min\left\{\left(\frac{H}{2}\right)^{p_+-1}, \left(\frac{H}{2}\right)^{p_--1}\right\} 2\delta H |B_{R/2} \cap \{u < 2\delta H\}|$$

$$\geq CR^{n-sp_-} \delta \min\{H^{p_+}, H^{p_-}\}.$$

Hence, since by $R^{sp_- - sp_+} \leq L^s$ by (P1), we get

$$((4\delta H)^{p_+} + (4\delta H)^{p_-}) \geq C\delta \min\{H^{p_+}, H^{p_-}\}.$$

Choosing δ sufficiently small leads to a contradiction in this inequality and finishes the proof of the lemma. □

We would like to emphasize that we made use of Lemma 4.1 in the foregoing proof. For this reason, we were able to prove the growth lemma without using the Sobolev inequality for variable exponents. It was sufficient to make use the fractional Sobolev inequality for constant exponents.

Proof of Theorem 1.6. Let $x_0 \in \mathbb{R}^n$. If $p(x_0, x_0) > n/s$, then we can find $R > 0$ and $\alpha \in (0, 1)$ such that $B_R(x_0) \Subset \Omega$ and $u \in C^\alpha(\overline{B_R(x_0)})$ as in the proof of Theorem 1.4. Thus, let us assume $p(x_0, x_0) \leq n/s$ in the rest of the proof. In this case, for given $\sigma \in (0, s)$, we can find $R \in (0, 1)$ such that $B_R(x_0) \Subset \Omega$ and $p_+(B_R(x_0) \times B_R(x_0)) < p_-^*(B_R(x_0) \times B_R(x_0))$, where $p_-^*(B_R(x_0) \times B_R(x_0)) = \frac{np_-(B_R(x_0) \times B_R(x_0))}{n - \sigma p_-(B_R(x_0) \times B_R(x_0))}$. By Theorem 1.4, $u \in L^\infty(B_R(x_0))$.

Let $\delta \in (0, 1)$ be the constant from Lemma 5.2 and let

$$0 < \alpha < \min\left\{s, \log_4\left(\frac{2}{2-\delta}\right), \frac{sp_+(\Omega \times \mathbb{R}^n)}{2(p_+(\Omega \times \mathbb{R}^n) - 1)}\right\} \tag{5.16}$$

be chosen such that the following is satisfied:

$$\int_1^\infty \frac{((4t)^\alpha - 1)^{p_+(\Omega \times \mathbb{R}^n)-1}}{t^{1+sp_+(\Omega \times \mathbb{R}^n)}} \, dt + \int_1^\infty \frac{((4t)^\alpha - 1)^{p_-(\Omega \times \mathbb{R}^n)-1}}{t^{1+sp_-(\Omega \times \mathbb{R}^n)}} \, dt \leq \frac{\delta^{p_+(\Omega \times \mathbb{R}^n)-1}}{2^{p_+(\Omega \times \mathbb{R}^n)}n\omega_n}, \tag{5.17}$$

where ω_n denotes the volume the n-dimensional Euclidean unit ball. We define $j_0 \in \mathbb{N}$ to be the smallest natural number satisfying

$$
j_0 \geq \max \left\{ \frac{sp_+(\{x_0\} \times B_R^c)}{2} \left| \log_4 \left(\frac{\delta^{p_+(\{x_0\} \times B_R^c)-1}}{2C_0} \right) \right|, \frac{sp_-(\Omega \times \mathbb{R}^n)}{2} \left| \log_4 \left(\frac{\delta^{p_-(\Omega \times \mathbb{R}^n)-1}}{2C_0} \right) \right|, \frac{|\log_4(\frac{\delta}{2})|}{s-\alpha} \right\}, \tag{5.18}
$$

where $C_0 := \max\{1, 2^{p_+(\Omega \times \mathbb{R}^n)}\} \left(\frac{n\omega_n}{sp_-(\Omega \times \mathbb{R}^n)} + 1 \right)$.

In the following, we show that there is a nonincreasing sequence (M_j) and a nondecreasing sequence (m_j) in $\mathbb{R}$, such that for all $j \in \mathbb{N} \cup \{0\}$

$$
m_j \leq u \leq M_j \quad \text{in } B_{4^{-j}R}(x_0) \qquad \text{and} \qquad M_j - m_j = Z4^{-\alpha j}, \tag{5.19}
$$

where

$$
Z := 2 \cdot 4^{\alpha j_0} \|u\|_{L^\infty(B_R(x_0))} + R^s + 1 + \left(R^{sp_+(\{x_0\} \times B_R(x_0)^c)} \sup_{x \in B_{3R/4}(x_0)} \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{|u(y)|^{p(x,y)-1}}{|y - x_0|^{n+sp(x,y)}} \, dy \right)^{\frac{1}{p_+(\{x_0\} \times B_R(x_0)^c)-1}}.
$$

For $j \in \{0, \dots, j_0\}$, we define $M_j := 4^{-\alpha j}Z/2$ and $m_j := -4^{-\alpha j}Z/2$. Then, (5.19) is clearly satisfied for all $j \in \{0, \dots, j_0\}$. It remains to prove the assertion (5.19) for $j > j_0$. The proof of the assertion follows by induction. Let us fix $j \geq j_0$ and assume that (5.19) is true for all $i \in \{0, \dots, j\}$. We now construct the elements m_{j+1} and M_{j+1} of the sequences. We distinguish between two cases.

First, we assume

$$
\left| B_{\frac{4^{-j}R}{2}}(x_0) \cap \left\{ u \geq m_j + \frac{M_j - m_j}{2} \right\} \right| \geq \frac{1}{2} \left| B_{\frac{4^{-j}R}{2}}(x_0) \right|. \tag{5.20}
$$

In this case, we define $v := u - m_j$, $H := \frac{M_j - m_j}{2}$, and $\widetilde{R} := 4^{-j}R$. The main idea for constructing m_{j+1} and M_{j+1} is to apply Lemma 5.2 for the function v and the radius $\widetilde{R}$. Hence, we need to verify the requirements of the lemma. Note that by assumption we have $0 \leq v \leq 2H$ in $B_{\widetilde{R}}(x_0)$ and $|B_{\widetilde{R}/2}(x_0) \cap \{v \geq H\}| \geq \frac{1}{2}|B_{\widetilde{R}/2}(x_0)|$. It remains to prove $\widetilde{R}^s \leq \delta H$ and (5.2). First we show $\widetilde{R}^s \leq \delta H$. Note that $2H = M_j - m_j = Z4^{-\alpha j} \geq R^s4^{-\alpha j}$. Since $j \geq j_0$, we can use (5.18), which leads to

$$
\widetilde{R}^s = 4^{-js}R^s \leq 4^{j(\alpha-s)}2H \leq \delta H.
$$

It remains to prove (5.2). We split $\mathbb{R}^n \setminus B_{\widetilde{R}}(x_0)$ as follows:

$$
\mathbb{R}^n \setminus B_{\widetilde{R}}(x_0) = (\mathbb{R}^n \setminus B_R(x_0)) \cup \left(\bigcup_{l=0}^{j-1} B_{4^{-l}R}(x_0) \setminus B_{4^{-(l+1)}R}(x_0) \right).
$$

If $x \in B_{4^{-l}R}(x_0) \setminus B_{4^{-(l+1)}R}(x_0)$, then $|x - x_0| \geq 4^{-l-1}R$ and therefore

$$
v(x) = u(x) - m_j \geq m_l - M_l + 2H = 2H(-4^{(-l+j)\alpha} + 1) \geq -2H\left(\left(\frac{4|x - x_0|}{\widetilde{R}} \right)^\alpha - 1 \right).
$$

On the other hand, if $x \in \mathbb{R}^n \setminus B_R(x_0)$, we have $v(x) \geq -|u(x)| - Z/2$. Now we are in a position to finalize the verification of (5.2). By the previous estimates on v in $\mathbb{R}^n \setminus B_{\widetilde{R}}(x_0)$, we have

$$\sup_{x\in B_{3\widetilde{R}/4}(x_0)}\int_{\mathbb{R}^n\setminus B_{\widetilde{R}}(x_0)}\frac{v_-(y)^{p(x,y)-1}}{|y-x_0|^{n+sp(x,y)}}\,dy \le \sup_{x\in B_{3\widetilde{R}/4}(x_0)}\int_{\mathbb{R}^n\setminus B_{\widetilde{R}}(x_0)}\frac{\left(2H\left(\left(\frac{4|y-x_0|}{\widetilde{R}}\right)^\alpha-1\right)\right)^{p(x,y)-1}}{|y-x_0|^{n+sp(x,y)}}\,dy$$

$$+\max\{1,2^{p_+(B_R(x_0)\times\mathbb{R}^n)-1}\}\sup_{x\in B_{3\widetilde{R}/4}(x_0)}\int_{\mathbb{R}^n\setminus B_R(x_0)}\frac{|u(y)|^{p(x,y)-1}+Z^{p(x,y)-1}}{|y-x_0|^{n+sp(x,y)}}\,dy$$

$$=:J_1+J_2.$$

First note, that we can estimate J_1 as follows:

$$J_1\le\int_{\mathbb{R}^n\setminus B_{\widetilde{R}}(x_0)}\frac{\left(2H\left(\left(\frac{4|y-x_0|}{\widetilde{R}}\right)^\alpha-1\right)\right)^{p_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}}{|y-x_0|^{n+sp_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}}\,dy+\int_{\mathbb{R}^n\setminus B_{\widetilde{R}}(x_0)}\frac{\left(2H\left(\left(\frac{4|y-x_0|}{\widetilde{R}}\right)^\alpha-1\right)\right)^{p_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}}{|y-x_0|^{n+sp_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}}\,dy$$

$$=n\omega_n(2H)^{p_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}\widetilde{R}^{-sp_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}\int_1^\infty\frac{((4t)^\alpha-1)^{p_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}}{t^{1+sp_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}}\,dt$$

$$+n\omega_n(2H)^{p_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}\widetilde{R}^{-sp_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}\int_1^\infty\frac{((4t)^\alpha-1)^{p_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}}{t^{1+sp_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}}\,dt.$$

Using $p_-(\Omega\times\mathbb{R}^n)\le p_\pm(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)\le p_+(\Omega\times\mathbb{R}^n)$ and (5.17), we get

$$\int_1^\infty\frac{((4t)^\alpha-1)^{p_\pm(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}}{t^{1+sp_\pm(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}}\,dt\le\int_1^\infty\frac{((4t)^\alpha-1)^{p_+(\Omega\times\mathbb{R}^n)-1}}{t^{1+sp_+(\Omega\times\mathbb{R}^n)}}\,dt+\int_1^\infty\frac{((4t)^\alpha-1)^{p_-(\Omega\times\mathbb{R}^n)-1}}{t^{1+sp_-(\Omega\times\mathbb{R}^n)}}\,dt\le\frac{\delta^{p_+(\Omega\times\mathbb{R}^n)-1}}{2^{p_+(\Omega\times\mathbb{R}^n)}n\omega_n}.$$

Combing the previous two estimates, we arrive at

$$J_1\le\frac{1}{2}H^{p_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}\widetilde{R}^{-sp_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}\delta^{p_+(\Omega\times\mathbb{R}^n)-1}+\frac{1}{2}H^{p_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}\widetilde{R}^{-sp_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}\delta^{p_+(\Omega\times\mathbb{R}^n)-1}$$

$$\le\frac{1}{2}(\delta H)^{p_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}\widetilde{R}^{-sp_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}+\frac{1}{2}(\delta H)^{p_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}\widetilde{R}^{-sp_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}$$

$$\le\frac{1}{2}(\delta H)^{p_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0))-1}\widetilde{R}^{-sp_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0))}+\frac{1}{2}(\delta H)^{p_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0))-1}\widetilde{R}^{-sp_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0))}.$$

In the last inequality, we used that by (P2) and $\widetilde{R}^s\le\delta H$,

$$R^{-sp_\pm(B_R(x_0)\times B_R(x_0)^c)}(\delta H)^{p_\pm(B_R(x_0)\times B_R(x_0)^c)}\le R^{-sp_\pm(B_R(x_0)\times B_R(x_0))}(\delta H)^{p_\pm(B_R(x_0)\times B_R(x_0))}.$$

Next, we estimate J_2 as follows:

$$J_2\le\max\{1,2^{p_+(\Omega\times\mathbb{R}^n)-1}\}R^{-sp_+(\{x_0\}\times B_R(x_0)^c)}Z^{p_+(\{x_0\}\times B_R(x_0)^c)-1}$$

$$+\max\{1,2^{p_+(\Omega\times\mathbb{R}^n)-1}\}\int_{\mathbb{R}^n\setminus B_R(x_0)}\left(\frac{Z^{p_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}}{|y-x_0|^{n+sp_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}}+\frac{Z^{p_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}}{|y-x_0|^{n+sp_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}}\right)dy$$

$$\le C_0R^{-sp_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}Z^{p_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}+C_0R^{-sp_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}Z^{p_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}$$

$$=C_0(4^j\widetilde{R})^{-sp_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}(4^{\alpha j}2H)^{p_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}+C_0(4^j\widetilde{R})^{-sp_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}(4^{\alpha j}2H)^{p_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}$$

$$\le C_0 4^{\frac{-sp_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}{2}j_0}H^{p_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}\widetilde{R}^{-sp_+(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}$$

$$+C_0 4^{\frac{-sp_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}{2}j_0}H^{p_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)-1}\widetilde{R}^{-sp_-(B_{\widetilde{R}}(x_0)\times B_{\widetilde{R}}(x_0)^c)}$$

$$\leq \frac{1}{2}(\delta H)^{p_+(B_{\widetilde{R}}(x_0) \times B_{\widetilde{R}}(x_0)^c) - 1} \widetilde{R}^{-sp_+(B_{\widetilde{R}}(x_0) \times B_{\widetilde{R}}(x_0)^c)} + \frac{1}{2}(\delta H)^{p_-(B_{\widetilde{R}}(x_0) \times B_{\widetilde{R}}(x_0)^c) - 1} \widetilde{R}^{-sp_-(B_{\widetilde{R}}(x_0) \times B_{\widetilde{R}}(x_0)^c)}$$

$$\leq \frac{1}{2}(\delta H)^{p_+(B_{\widetilde{R}}(x_0) \times B_{\widetilde{R}}(x_0)) - 1} \widetilde{R}^{-sp_+(B_{\widetilde{R}}(x_0) \times B_{\widetilde{R}}(x_0))} + \frac{1}{2}(\delta H)^{p_-(B_{\widetilde{R}}(x_0) \times B_{\widetilde{R}}(x_0)) - 1} \widetilde{R}^{-sp_-(B_{\widetilde{R}}(x_0) \times B_{\widetilde{R}}(x_0))}, \tag{5.21}$$

where we used the definition of Z, (P2), (5.16), and (5.18). Note that the constant C_0 comes from (5.18). Combining the estimates of J_1 and J_2, proves (5.2).

Hence, we can apply Lemma 5.2, which leads to

$$u \geq m_j + \delta H = m_j + \delta \frac{M_j - m_j}{2} = m_j + \frac{\delta 4^{-\alpha j} Z}{2} > m_j + 4^{-\alpha j}(1 - 4^{-\alpha})Z \quad \text{in } B_{\widetilde{R}/4}(x_0),$$

where we used (5.16) in the last inequality. Hence, choosing $M_{j+1} = M_j$ and $m_{j+1} = m_j + 4^{-\alpha j}(1 - 4^{-\alpha})Z$ proves (5.19) for the case (5.20).

In the second case

$$\left| B_{\frac{4^{-j}R}{2}}(x_0) \cap \left\{ u \geq m_j + \frac{M_j - m_j}{2} \right\} \right| < \frac{1}{2} \left| B_{\frac{4^{-j}R}{2}}(x_0) \right|,$$

we can proceed similarly and consider the function $v := M_j - u$. In this case, we can choose the members of the sequences to be of the form $M_{j+1} = M_j - 4^{-\alpha j}(1 - 4^{-\alpha})Z$ and $m_{j+1} = m_j$. This completes the construction of the sequences (M_j) and (m_j) and completes the proof of (5.19). Now the local Hölder regularity follows in a standard way. □

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