Research Article

Choonkil Park, Mohammad Amin Tareeghee, Abbas Najati*, Yavar Khedmati Yengejeh, and Siriluk Paokanta*

Asymptotic behavior of Fréchet functional equation and some characterizations of inner product spaces

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Abstract: This article presents the general solution $f:\mathcal{G}\to\mathcal{V}$ of the following functional equation:

$$f(x)-4f(x+y)+6f(x+2y)-4f(x+3y)+f(x+4y)=0,\quad x,y\in\mathcal{G},$$

where $(\mathcal{G},+)$ is an abelian group and $\mathcal{V}$ is a linear space. We also investigate its Hyers-Ulam stability on some restricted domains. We apply the obtained results to present some asymptotic behaviors of this functional equation in the framework of normed spaces. Finally, we provide some characterizations of inner product spaces associated with the mentioned functional equation.

Keywords: Hyers-Ulam stability, quadratic function, cubic function, Fréchet functional equation, asymptotic behavior

MSC 2020: 39B82, 39B52, 39B62, 46C15

1 Introduction and preliminaries

The functional equation

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}g(x+jy)=0,\quad x,y\in\mathbb{R},\tag{1}$$

where $g:\mathbb{R}\to\mathbb{R}$, is known as the Fréchet functional equation. For $n=1$, the Fréchet functional equation becomes $g(x+y)=g(x)$. Then, g is a constant function. In the case of $n=2$, the Fréchet functional equation appears as $g(x+2y)+g(x)=2g(x+y)$, which is the Jensen functional equation. In [1, Theorem 7.20], the general solution of (1) was obtained for $n=3$ without assuming any regularity condition on g, where g is a function between two linear spaces. A result from Fréchet [2] states that if a continuous function $g:\mathbb{R}\to\mathbb{R}$ satisfies (1), then g is a polynomial of degree $<n$. Johnson [3] proved that a normed linear space X is an inner product space if and only if, for some integer $n\geqslant 3$,

*** Corresponding author: Abbas Najati,** Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran, e-mail: a.nejati@yahoo.com
*** Corresponding author: Siriluk Paokanta,** School of Science, University of Phayao, Phayao 56000, Thailand, e-mail: siriluk.pa@up.ac.th
Choonkil Park: Department of Mathematics, Hanyang University, Seoul, Korea, e-mail: baak@hanyang.ac.kr
Mohammad Amin Tareeghee: Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran, e-mail: amintareeghee30@gmail.com
Yavar Khedmati Yengejeh: Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran, e-mail: khedmati.y@uma.ac.ir

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\|x+jy\|^2 = 0, \quad x, y \in X.$$

It will be interesting to determine the general solutions of (1) without additional assumptions on g. In this article, we obtain the general solution of (1) for $n = 4$ without assuming any regularity condition on $g : \mathcal{G} \to \mathcal{V}$, where $(\mathcal{G}, +)$ is an abelian group and $\mathcal{V}$ is a linear space. Indeed, we deal with the following functional equation:

$$f(x) - 4f(x+y) + 6f(x+2y) - 4f(x+3y) + f(x+4y) = 0, \quad x, y \in \mathcal{G}. \tag{2}$$

We investigate the Hyers-Ulam stability of (2) on some restricted domains. The obtained results are used to present some asymptotic behaviors of the functional equation (2) in the framework of normed spaces. We also provide some characterizations of inner product spaces related to (2).

In 1940, Ulam [4] proposed the following question regarding the stability of homomorphisms between groups:

> Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with a metric d. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(x * y), f(x) \diamond f(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $h : G_1 \to G_2$, with $d(f(x), h(x)) < \varepsilon$ for all $x \in G_1$?

One year later, Hyers [5] answered Ulam's question for the case where G_1 and G_2 are assumed to be Banach spaces. It will also be interesting to study the stability problems of functional equations on restricted domains.

Skof [6] was the first author to study the Hyers-Ulam stability for additive functions on a restricted domain and applied the result to the study of an asymptotic behavior of additive functions. Jung [7] and Rassias [8] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains. The bounds and thus the stability results obtained in [7,8] were improved in [9].

Let us recall that a function $f : \mathcal{G} \to \mathcal{V}$, where $(\mathcal{G}, +)$ is an abelian group and $\mathcal{V}$ is a linear space, is called

- additive, if $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathcal{G}$;
- quadratic, if $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in \mathcal{G}$;
- cubic, if $f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)$ for all $x, y \in \mathcal{G}$.

For more information on functional equations and the concept of Hyers-Ulam stability and its applications, we refer the reader to [1,10–21].

2 General solution of (2)

In this section, $(\mathcal{G}, +)$ is an abelian group and $\mathcal{V}$ denotes a linear space.

Lemma 2.1. *Let $f : \mathcal{G} \to \mathcal{V}$ be a function satisfying*

$$f(x-y) + 2f(x+y) - f(y) - 3f(x) = 0, \quad x, y \in \mathcal{G}. \tag{3}$$

Then, f is additive.

Proof. The functional equation (3) gives us $f(0) = 0$ by letting $x = y = 0$. Putting $x = 0$ in (3), we infer that f is odd. Replacing y by $-y$ in (3), we obtain

$$f(x+y) + 2f(x-y) + f(y) - 3f(x) = 0, \quad x, y \in \mathcal{G}. \tag{4}$$

Multiplying (3) by -2 and adding the resultant to (4), we obtain $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathcal{G}$. This completes the proof. □

Lemma 2.2. *Let $f : \mathcal{G} \to \mathcal{V}$ be an odd function satisfying* (2) *for all $x, y \in \mathcal{G}$. Then, the function $g : \mathcal{G} \to \mathcal{V}$ defined by $g(x) = f(2x) - 8f(x)$ is additive and the function $h : \mathcal{G} \to \mathcal{V}$ defined by $h(x) = f(2x) - 2f(x)$ is cubic.*

Proof. First, we prove g is additive. Since f is odd, replacing x by $3x - y$ and y by $y - x$ in (2), we have

$$f(3x - y) - 4f(2x) + 6f(x + y) - 4f(2y) - f(x - 3y) = 0, \quad x, y \in \mathcal{G}. \tag{5}$$

Also, replacing x by $-4x$ and y by $x + y$ in (2), we have

$$-f(4x) + 4f(3x - y) - 6f(2x - 2y) + 4f(x - 3y) + f(4y) = 0, \quad x, y \in \mathcal{G}. \tag{6}$$

Multiplying (5) by -4 and adding the obtained equation to (6), we have

$$8f(x - 3y) - 6f(2x - 2y) - 24f(x + y) - f(4x) + 16f(2x) + f(4y) + 16f(2y) = 0, \tag{7}$$

for all $x, y \in \mathcal{G}$. For $x = -y$, equation (2) obtains us

$$f(3y) = 4f(2y) - 5f(y), \quad y \in \mathcal{G}.$$

So, by considering $x = 0$ in (2), we obtain

$$f(4y) = 10f(2y) - 16f(y), \quad y \in \mathcal{G}. \tag{8}$$

By (8), equation (7) is equivalent to

$$8f(x - 3y) - 6f(2x - 2y) - 24f(x + y) + 6f(2x) + 16f(x) + 26f(2y) - 16f(y) = 0, \tag{9}$$

for all $x, y \in \mathcal{G}$. On the other hand, replacing x by $x - 3y$ in (2), we have

$$f(x - 3y) = 4f(x - 2y) - 6f(x - y) + 4f(x) - f(x + y), \quad x, y \in \mathcal{G}. \tag{10}$$

By (9) and (10), we obtain

$$\begin{aligned} &32f(x - 2y) - 48f(x - y) - 32f(x + y) - 6f(2x - 2y) + 6f(2x) + 48f(x) + 26f(2y) - 16f(y) \\ &\quad = 0, \ \ x, y \in \mathcal{G}. \end{aligned} \tag{11}$$

Also, replacing x by $x - 2y$ in (2), we have

$$f(x - 2y) = 4f(x - y) - 6f(x) + 4f(x + y) - f(x + 2y), \quad x, y \in \mathcal{G}. \tag{12}$$

By (11) and (12), we obtain

$$\begin{aligned} &80f(x - y) - 6f(2x - 2y) + 96f(x + y) - 32f(x + 2y) + 6f(2x) - 144f(x) + 26f(2y) - 16f(y) \\ &\quad = 0, \ \ x, y \in \mathcal{G}. \end{aligned} \tag{13}$$

Replacing x by $2x$ and y by $2y$ in (13) and applying (8), we have

$$8f(2x + 4y) = 5f(2x - 2y) + 24f(x - y) + 24f(2x + 2y) - 21f(2x) - 24f(x) + 61f(2y) - 104f(y), \ \ x, y \in \mathcal{G}. \tag{14}$$

Since f is odd, replacing x by $x - y$ and y by $x + y$ in (9) obtains us

$$8f(2x + 4y) = 6f(2x - 2y) + 26f(2x + 2y) + 16f(x - y) - 16f(x + y) - 24f(2x) + 6f(4y), \quad x, y \in \mathcal{G}. \tag{15}$$

By (14), (15), and (8), it is concluded

$$[f(2x - 2y) - 8f(x - y)] + 2[f(2x + 2y) - 8f(x + y)] = [f(2y) - 8f(y)] + 3[f(2x) - 8f(x)], \quad x, y \in \mathcal{G}.$$

This means

$$g(x - y) + 2g(x + y) - g(y) - 3g(x) = 0, \quad x, y \in \mathcal{G}.$$

So, by Lemma 2.1, the function g is additive.

Now, we show h is a cubic function. By interchanging x and y in (11), we have

$$-32f(2x - y) = -48f(x - y) + 32f(x + y) - 6f(2x - 2y) - 6f(2y) - 48f(y) - 26f(2x) + 16f(x), \ \ x, y \in \mathcal{G}. \tag{16}$$

Replacing x by $2x$ and y by $2y$ in (16) and multiplying the obtained equation by $-\frac{1}{2}$, we obtain

$$16f(4x-2y) = 24f(2x-2y) - 16f(2x+2y) + 3f(4x-4y) + 3f(4y) + 24f(2y) + 13f(4x) - 8f(2x), \quad x,y \in \mathcal{G}. \tag{17}$$

Replacing y by $-y$ in (16), we have

$$-32f(2x+y) = -48f(x+y) + 32f(x-y) - 6f(2x+2y) + 6f(2y) + 48f(y) - 26f(2x) + 16f(x), \quad x,y \in \mathcal{G}. \tag{18}$$

Also replacing y by $-y$ in (17), we obtain

$$16f(4x+2y) = 24f(2x+2y) - 16f(2x-2y) + 3f(4x+4y) - 3f(4y) - 24f(2y) + 13f(4x) - 8f(2x), \quad x,y \in \mathcal{G}. \tag{19}$$

Adding (16), (17), (18), and (19) yields

$$\begin{aligned} &16f(4x+2y) + 16f(4x-2y) - 32f(2x+y) - 32f(2x-y) \\ &\quad = 3f(4x+4y) + 3f(4x-4y) + 2f(2x+2y) + 2f(2x-2y) - 16f(x+y) - 16f(x-y) \\ &\qquad + 26f(4x) - 68f(2x) + 32f(x), \end{aligned} \tag{20}$$

for all $x,y \in \mathcal{G}$. So, by (20) and (8), we obtain

$$\begin{aligned} &[f(4x+2y) - 2f(2x+y)] + [f(4x-2y) - 2f(2x-y)] \\ &\quad = [2f(2x+2y) - 4f(x+y)] + [2f(2x-2y) - 4f(x-y)] + [12f(2x) - 24f(x)], \quad x,y \in \mathcal{G}. \end{aligned}$$

This means

$$h(2x+y) + h(2x-y) = 2[h(x+y) + h(x-y)] + 12h(x), \quad x,y \in \mathcal{G}.$$

Therefore, h is a cubic function. □

Lemma 2.3. *Let $f : \mathcal{G} \to \mathcal{V}$ be an even function satisfying* (2) *for all $x,y \in \mathcal{G}$. If $f(0) = 0$, then f is a quadratic function.*

Proof. Since f is even and $f(0) = 0$, replacing x by $-2y$ in equation (2), we obtain

$$f(2y) = 4f(y), \quad y \in \mathcal{G}. \tag{21}$$

Replacing x by $x-2y$ in (2), we have

$$f(x-2y) - 4f(x-y) + 6f(x) - 4f(x+y) + f(x+2y) = 0, \quad x,y \in \mathcal{G}. \tag{22}$$

Since f is even, replacing x by $2y$ and y by x in (22) and applying (21), we obtain

$$f(x-y) + 6f(y) + f(x+y) = f(x-2y) + f(x+2y), \quad x,y \in \mathcal{G}. \tag{23}$$

Adding (22) and (23), we infer that $f(x+y) + f(x-y) = 2f(x) + 2f(y)$, i.e., f is quadratic. □

Lemma 2.4. *Each additive, quadratic, and cubic function satisfies equation* (2).

Proof. For the case that a function is additive, the proof is obvious. Let $Q : \mathcal{G} \to \mathcal{V}$ be quadratic. We have

$$2Q(x) + 2Q(y) = Q(x+y) + Q(x-y), \quad x,y \in \mathcal{G}. \tag{24}$$

and also

$$Q(x+2y) + Q(x-2y) = 2Q(x) + 8Q(y), \quad x,y \in \mathcal{G}. \tag{25}$$

Multiplying (24) by 4 and adding the resultant to (25), we have

$$Q(x-2y) + 6Q(x) + Q(x+2y) = 4Q(x-y) + 4Q(x+y), \quad x,y \in \mathcal{G}. \tag{26}$$

Replacing x by $x+2y$ in (26), we obtain Q that satisfies (2). Now, let $C : \mathcal{G} \to \mathcal{V}$ be cubic. Then, we have

$$2C(x+y) + 2C(x-y) + 12C(x) = C(2x+y) + C(2x-y), \quad x,y \in \mathcal{G}. \tag{27}$$

It is obvious that $C(2x) = 8C(x)$ for all $x \in \mathcal{G}$. Replacing y by $2y$ in (27), we obtain

$$C(2x + 2y) + C(2x - 2y) = 2C(x + 2y) + 2C(x - 2y) + 12C(x), \quad x, y \in \mathcal{G}. \tag{28}$$

Using $C(2x) = 8C(x)$ in (28), we obtain

$$C(x - 2y) - 4C(x - y) + 6C(x) - 4C(x + y) + C(x + 2y) = 0, \quad x, y \in \mathcal{G}. \tag{29}$$

Replacing x by $x + 2y$ in (29), the function C satisfies (2). □

Theorem 2.5. *A function $f : \mathcal{G} \to \mathcal{V}$ satisfies (2) if and only if f has the form $f = A + Q + C + f(0)$, where $A, Q, C : \mathcal{G} \to \mathcal{V}$ are additive, quadratic, and cubic, respectively.*

Proof. Let f_o and f_e be the odd and even parts of f, i.e.,

$$f_o(x) = \frac{f(x) - f(-x)}{2}, \quad f_e(x) = \frac{f(x) + f(-x)}{2}, \quad x \in \mathcal{G}.$$

First, we assume that f satisfies (2). Then, f_o and f_e fulfill (2). Consider the functions $g, h : \mathcal{G} \to \mathcal{V}$ defined by:

$$g(x) = f_o(2x) - 8f_o(x), \quad h(x) = f_o(2x) - 2f_o(x), \quad x \in \mathcal{G}.$$

Then, by Lemma 2.2, g is additive and h is cubic. Moreover, we have

$$f_o(x) = \frac{1}{6}[h(x) - g(x)], \quad x \in \mathcal{G}.$$

Also, $f_e - f(0)$ is quadratic by Lemma 2.3. Set

$$A = -\frac{1}{6}g, \quad Q = f_e - f(0), \quad \text{and} \quad C = \frac{1}{6}h.$$

Hence, $f = f_o + f_e = A + Q + C + f(0)$.

Conversely, let $f = A + Q + C + f(0)$. Hence, by Lemma 2.4, f satisfies (2). □

3 Stability of (2) on some restricted domains

In this section, $\mathcal{X}$ and $\mathcal{W}$ denote linear normed spaces and $\mathcal{Y}$ is a Banach space. For convenience, we let

$$Df(x, y) := f(x) - 4f(x + y) + 6f(x + 2y) - 4f(x + 3y) + f(x + 4y),$$

where f is a function between two linear spaces.

Theorem 3.1. *Take $\varepsilon \geqslant 0$ and $d > 0$. Consider an odd function $f : \mathcal{X} \to \mathcal{Y}$ satisfying one of the following conditions*:

$$\|Df(x, y)\| \leqslant \varepsilon, \quad x, y \in \mathcal{X} : \|x + y\| \geqslant d; \tag{30}$$

$$\|Df(x, y)\| \leqslant \varepsilon, \quad x, y \in \mathcal{X} : \min\{\|x\|, \|y\|\} \geqslant d; \tag{31}$$

$$\|Df(x, y)\| \leqslant \varepsilon, \quad x, y \in \mathcal{X} : \|x\| \geqslant d; \tag{32}$$

$$\|Df(x, y)\| \leqslant \varepsilon, \quad x, y \in \mathcal{X} : \|y\| \geqslant d. \tag{33}$$

Then, there is a unique additive function $A : \mathcal{X} \to \mathcal{Y}$ such that we have

$$\|f(2x) - 8f(x) - A(x)\| \leqslant \frac{59}{6}\varepsilon, \quad x \in \mathcal{X}. \tag{34}$$

Proof. We assume that f is satisfying (30) (we have a similar argument if f satisfies (31), (32), or (33)) . Since f is odd, replacing x by $-3y$ in (30), we have

$$\|f(3y) - 4f(2y) + 5f(y)\| \leqslant \varepsilon, \quad \|y\| \geqslant d. \tag{35}$$

Also, by putting $x = -4y$ in (30), we obtain

$$\|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\| \leqslant \varepsilon, \quad \|y\| \geqslant d. \tag{36}$$

Multiplying (35) by 4 and adding the obtained inequality to (36), we conclude

$$\|f(4y) - 10f(2y) + 16f(y)\| \leqslant 5\varepsilon, \quad \|y\| \geqslant d. \tag{37}$$

So, by considering $g(y) = f(2y) - 8f(y)$ for all $y \in \mathcal{X}$, (37) is equivalent to

$$\|g(2y) - 2g(y)\| \leqslant 5\varepsilon, \quad \|y\| \geqslant d. \tag{38}$$

Replacing y by $2^n y$, $n \in \mathbb{N} \cup \{0\}$, in (38), and multiplying the reached inequality by $2^{-(n+1)}$, we have

$$\left\| \frac{1}{2^{n+1}} g(2^{n+1}y) - \frac{1}{2^n} g(2^n y) \right\| \leqslant \frac{5\varepsilon}{2^{n+1}}, \quad \|y\| \geqslant d. \tag{39}$$

By (39), for integer numbers $n \geq m \geq 0$, we derive

$$\left\| \frac{1}{2^{n+1}} g(2^{n+1}y) - \frac{1}{2^m} g(2^m y) \right\| \leqslant \sum_{i=m}^{n} \frac{5\varepsilon}{2^{i+1}}, \quad \|y\| \geqslant d. \tag{40}$$

From (40), we can see $\{\frac{1}{2^n} g(2^n y)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for all $y \in \mathcal{Y}$. So, by the completeness of $\mathcal{Y}$, it is convergent. Now, the function $A : \mathcal{X} \to \mathcal{Y}$ with $A(x) = \lim_{n\to\infty} \frac{1}{2^n} g(2^n x)$ is well defined. By (30), we can obtain

$$\|g(x) - 4g(x+y) + 6g(x+2y) - 4g(x+3y) + g(x+4y)\| \leqslant 9\varepsilon, \quad \|x+y\| \geqslant d. \tag{41}$$

Replacing x and y by $2^n x$ and $2^n y$ in (41) and multiplying the obtained inequality by 2^{-n}, and then letting $n \to \infty$, we conclude that A satisfies (2) for all x, y with $x + y \neq 0$. It follows from (35) that

$$\|g(3y) - 4g(2y) + 5g(y)\| \leqslant 9\varepsilon, \quad \|y\| \geqslant d. \tag{42}$$

Because $A(0) = 0$, (42) yields $A(3y) - 4A(2y) + 5A(y) = 0$ for all $y \in \mathcal{X}$. Since A is an odd function, the last equation implies

$$A(x) - 4A(0) + 6A(-x) - 4A(-2x) + A(-3x) = 0, \quad x \in \mathcal{X}.$$

This means that A satisfies (2) for $x + y = 0$. Therefore, A satisfies (2) for all $x, y \in \mathcal{X}$. So, the mapping $x \mapsto A(2x) - 8A(x)$ is additive by Lemma 2.2. On the other hand, by the definition of A, we obtain $A(2x) = 2A(x)$. Hence, A is additive. Letting $m = 0$ and $n \to \infty$ in (40), we have

$$\|g(y) - A(y)\| \leqslant 5\varepsilon, \quad \|y\| \geqslant d.$$

Therefore,

$$\begin{aligned} &\|A(y-2x) - g(y-2x)\| \leqslant 5\varepsilon, \quad \|y-2x\| \geqslant d; \\ &\|4g(2y-x) - 4A(2y-x)\| \leqslant 20\varepsilon, \quad \|2y-x\| \geqslant d; \\ &\|4g(x+4y) - 4A(x+4y)\| \leqslant 20\varepsilon, \quad \|x+4y\| \geqslant d; \\ &\|A(2x+5y) - g(2x+5y)\| \leqslant 5\varepsilon, \quad \|2x+5y\| \geqslant d. \end{aligned} \tag{43}$$

Replacing x by $y - 2x$ and y by $x + y$ in (41), we obtain

$$\|g(y-2x) - 4g(2y-x) + 6g(3y) - 4g(x+4y) + g(2x+5y)\| \leqslant 9\varepsilon, \quad \|2y-x\| \geqslant d. \tag{44}$$

Let $y \in \mathcal{X}$ and choose $x \in \mathcal{X}$ such that $\|x\| \geqslant d + 4\|y\|$. Then,

$$\min\{\|y-2x\|, \|2y-x\|, \|x+4y\|, \|2x+5y\|\} \geqslant d.$$

Now, it follows from (43) and (44)

$$\|6g(3y) + A(y - 2x) - 4A(2y - x) - 4A(x + 4y) + A(2x + 5y)\| \leqslant 59\varepsilon. \tag{45}$$

Since A is additive, (45) yields

$$\|g(3y) - A(3y)\| \leqslant \frac{59}{6}\varepsilon.$$

Hence, we obtain (34). The uniqueness of A is a simple consequence of (34). □

Theorem 3.2. *Assume that an odd function $f : \mathcal{X} \to \mathcal{Y}$ satisfies one of the conditions* (30)–(33) *for some* $\varepsilon \geqslant 0$ *and* $d > 0$. *Then, there is a unique cubic function $C : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \leqslant \frac{71}{42}\varepsilon, \quad x \in \mathcal{X}. \tag{46}$$

Proof. Let f satisfy (30). As we shown in the proof of Theorem 3.1, f satisfies (37). Then,

$$\|h(2y) - 8h(y)\| \leqslant 5\varepsilon, \quad \|y\| \geqslant d, \tag{47}$$

where $h(y) = f(2y) - 2f(y)$. Replacing y by $2^n y$, $n \geqslant 0$, in (47) and multiplying the resultant inequality by $8^{-(n+1)}$, we have

$$\left\| \frac{1}{8^{n+1}} h(2^{n+1}y) - \frac{1}{8^n} h(2^n y) \right\| \leqslant \frac{5\varepsilon}{8^{n+1}}, \quad \|y\| \geqslant d.$$

So, we derive

$$\left\| \frac{1}{8^{n+1}} h(2^{n+1}y) - \frac{1}{8^m} h(2^m y) \right\| \leqslant \sum_{i=m}^{n} \frac{5\varepsilon}{8^{i+1}}, \quad \|y\| \geqslant d, \; n \geqslant m \geqslant 0. \tag{48}$$

From (48), we can infer that $\{\frac{1}{8^n} h(2^n y)\}_n$ is a Cauchy sequence for all $y \in \mathcal{Y}$, and then, it is convergent by completeness of $\mathcal{Y}$. Now, the function $C : \mathcal{X} \to \mathcal{Y}$ given by $C(x) = \lim_{n\to\infty} \frac{1}{8^n} h(2^n x)$ is well defined. By (30), we can obtain

$$\|h(x) - 4h(x + y) + 6h(x + 2y) - 4h(x + 3y) + h(x + 4y)\| \leqslant 3\varepsilon, \quad \|x + y\| \geqslant d. \tag{49}$$

Replacing x and y by $2^n x$ and $2^n y$ in (49) and multiplying the obtained inequality by 8^{-n}, and then taking $n \to \infty$, we conclude that C satisfies (2) for x, y with $x + y \neq 0$. It follows from (35) that

$$\|h(3y) - 4h(2y) + 5h(y)\| \leqslant 3\varepsilon, \quad \|y\| \geqslant d. \tag{50}$$

Because $C(0) = 0$, (50) yields $C(3y) - 4C(2y) + 5C(y) = 0$ for all $y \in \mathcal{X}$. Since C is an odd function, the last equation implies

$$C(x) - 4C(0) + 6C(-x) - 4C(-2x) + C(-3x) = 0, \quad x \in \mathcal{X}.$$

This means that C satisfies (2) for $x + y = 0$. Therefore, C satisfies (2) for all $x, y \in \mathcal{X}$. Since C is odd, we obtain the map $x \to C(2x) - 2C(x)$ is a cubic function by Lemma 2.2. On the other hand, we have $C(2x) = 8C(x)$ by the definition of C. So C is cubic. Letting $m = 0$ and $n \to \infty$ in (48), we obtain

$$\|h(y) - C(y)\| \leqslant \frac{5}{7}\varepsilon, \quad \|y\| \geqslant d.$$

Therefore,

$$\begin{aligned}
\|C(y - 2x) - h(y - 2x)\| &\leqslant \frac{5}{7}\varepsilon, \quad \|y - 2x\| \geqslant d; \\
\|4h(2y - x) - 4C(2y - x)\| &\leqslant \frac{20}{7}\varepsilon, \quad \|2y - x\| \geqslant d; \\
\|4h(x + 4y) - 4C(x + 4y)\| &\leqslant \frac{20}{7}\varepsilon, \quad \|x + 4y\| \geqslant d; \\
\|C(2x + 5y) - h(2x + 5y)\| &\leqslant \frac{5}{7}\varepsilon, \quad \|2x + 5y\| \geqslant d.
\end{aligned} \tag{51}$$

Replacing x by $y-2x$ and y by $x+y$ in (49), we obtain

$$\|h(y-2x)-4h(2y-x)+6h(3y)-4h(x+4y)+h(2x+5y)\| \leqslant 3\varepsilon, \quad \|2y-x\| \geqslant d. \tag{52}$$

Let $y \in \mathcal{X}$ and choose $x \in \mathcal{X}$ such that $\|x\| \geqslant d+4\|y\|$. Then,

$$\min\{\|y-2x\|, \|2y-x\|, \|x+4y\|, \|2x+5y\|\} \geqslant d.$$

Now, it follows from (51) and (52)

$$\|6h(3y)+C(y-2x)-4C(2y-x)-4C(x+4y)+C(2x+5y)\| \leqslant \frac{71}{7}\varepsilon. \tag{53}$$

Since C satisfies (2), it follows from (53) that

$$\|h(3y)-C(3y)\| \leqslant \frac{71}{42}\varepsilon.$$

Hence, we obtain (46). The uniqueness of A is a simple consequence of (46). □

Theorem 3.3. *Take $\varepsilon \geqslant 0, d>0$ and consider an even function $f:\mathcal{X}\to\mathcal{Y}$ satisfying one of the conditions* (30)–(33). *Then, there is a unique quadratic function $Q:\mathcal{X}\to\mathcal{Y}$, such that*

$$\|f(x)-Q(x)-f(0)\| \leqslant \frac{4}{9}\varepsilon, \quad x \in \mathcal{X}. \tag{54}$$

Proof. Let f satisfy (30). Since f is an even function, replacing x by $-2y$ in (30), we obtain

$$\|f(2y)-4f(y)+3f(0)\| \leqslant \frac{\varepsilon}{2}, \quad \|y\| \geqslant d. \tag{55}$$

Similarly, replacing x by $-3y$ in (30), we obtain

$$\|f(3y)-4f(2y)+7f(y)-4f(0)\| \leqslant \varepsilon, \quad \|y\| \geqslant d. \tag{56}$$

Replacing y by $2^n y$, $n \in \mathbb{N}\cup\{0\}$, in (55) and multiplying the resultant inequality by $4^{-(n+1)}$, we have

$$\left\|\frac{1}{4^{n+1}}f(2^{n+1}y)-\frac{1}{4^n}f(2^n y)+\frac{3}{4^{n+1}}f(0)\right\| \leqslant \frac{\varepsilon}{2^{2n+3}}, \quad \|y\| \geqslant d.$$

Then, for integers $n \geqslant m \geqslant 0$, we obtain

$$\left\|\frac{1}{4^{n+1}}f(2^{n+1}y)-\frac{1}{4^m}f(2^m y)+\sum_{i=m}^{n}\frac{3}{4^{i+1}}f(0)\right\| \leqslant \sum_{i=m}^{n}\frac{\varepsilon}{2^{2i+3}}, \quad \|y\| \geqslant d. \tag{57}$$

It follows from (57) that the sequence $\{\frac{1}{4^n}f(2^n y)\}_{n\in\mathbb{N}}$ is Cauchy for all $y \in \mathcal{Y}$, and then it is convergent by the completeness of $\mathcal{Y}$. Now, the function $Q:\mathcal{X}\to\mathcal{Y}$ given by $Q(x)=\lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$ is well defined. Replacing x and y by $2^n x$ and $2^n y$ in (30) and multiplying the resultant inequality by 4^{-n}, and taking the limit as n tends to ∞, we infer that Q satisfies (2) for all $x, y \in \mathcal{X}$ with $x+y \neq 0$. It follows from (56) that $Q(3y)-4Q(2y)+7Q(y)=0$ for all $y \in \mathcal{X}$. Since $Q(0)=0$ and Q is an even function, the last equation implies

$$Q(x)-4Q(0)+6Q(-x)-4Q(-2x)+Q(-3x)=0, \quad x \in \mathcal{X}.$$

This means Q satisfies (2) for $x+y=0$. Therefore, Q satisfies (2) for all $x, y \in \mathcal{X}$. So, Q is quadratic by Lemma 2.3. Taking $m=0$ and $n\to\infty$ in (57), we obtain

$$\|f(y)-Q(y)-f(0)\| \leqslant \frac{\varepsilon}{6}, \quad \|y\| \geqslant d.$$

Therefore,

$$\begin{aligned}
\|Q(y-2x)-f(y-2x)+f(0)\| &\leqslant \frac{\varepsilon}{6}, \quad \|y-2x\| \geqslant d;\\
\|4f(2y-x)-4Q(2y-x)-4f(0)\| &\leqslant \frac{2}{3}\varepsilon, \quad \|2y-x\| \geqslant d;\\
\|4f(x+4y)-4Q(x+4y)-4f(0)\| &\leqslant \frac{2}{3}\varepsilon, \quad \|x+4y\| \geqslant d;\\
\|Q(2x+5y)-f(2x+5y)+f(0)\| &\leqslant \frac{\varepsilon}{6}, \quad \|2x+5y\| \geqslant d.
\end{aligned} \tag{58}$$

Replacing x by $y-2x$ and y by $x+y$ in (30), we obtain

$$\|f(y-2x)-4f(2y-x)+6f(3y)-4f(x+4y)+f(2x+5y)\| \leqslant \varepsilon, \quad \|2y-x\| \geqslant d. \tag{59}$$

Let $y \in \mathcal{X}$ and choose $x \in \mathcal{X}$ such that $\|x\| \geqslant d+4\|y\|$. Then,

$$\min\{\|y-2x\|, \|2y-x\|, \|x+4y\|, \|2x+5y\|\} \geqslant d.$$

Now, it follows from (58) and (59)

$$\|6f(3y)+Q(y-2x)-4Q(2y-x)-4Q(x+4y)+Q(2x+5y)-6f(0)\| \leqslant \frac{8}{3}\varepsilon. \tag{60}$$

Since Q satisfies (2), it follows from (60) that

$$\|f(3y)-Q(3y)-f(0)\| \leqslant \frac{4}{9}\varepsilon.$$

Hence, we obtain (54). The uniqueness of Q is a simple consequence of (54). □

Theorem 3.4. *Take $\varepsilon \geqslant 0, d > 0$ and consider a function $f : \mathcal{X} \to \mathcal{Y}$ satisfying one of the conditions* (30)–(33). *Then, there exist unique additive, quadratic, and cubic functions $A, Q, C : \mathcal{X} \to \mathcal{Y}$ such that*

$$\|f(x)-A(x)-Q(x)-C(x)-f(0)\| \leqslant \frac{149}{63}\varepsilon, \quad x \in \mathcal{X}. \tag{61}$$

Proof. We may assume that f satisfies (30). Let f_e and f_o be the even and odd parts of f. It is clear that f_e and f_o satisfy (30) for all $x, y \in \mathcal{X}$ with $\|x+y\| \geqslant d$. By Theorems 3.1, 3.2, and 3.3, we have additive, quadratic, and cubic functions $\widetilde{A}, Q, \widetilde{C} : \mathcal{X} \to \mathcal{Y}$ such that

$$\begin{aligned}
\|\widetilde{A}(x)-f_o(2x)+8f_o(x)\| &\leqslant \frac{59}{6}\varepsilon,\\
\|f_o(2x)-2f_o(x)-\widetilde{C}(x)\| &\leqslant \frac{71}{42}\varepsilon,\\
\|f_e(x)-Q(x)-f(0)\| &\leqslant \frac{4}{9}\varepsilon, \quad x \in \mathcal{X}.
\end{aligned}$$

Then,

$$\left\|f(x)+\frac{1}{6}\widetilde{A}(x)-\frac{1}{6}\widetilde{C}(x)-Q(x)-f(0)\right\| \leqslant \frac{149}{63}\varepsilon, \quad x \in \mathcal{X}.$$

So we obtain (61) by letting $A(x) = -\frac{1}{6}\widetilde{A}(x)$ and $C(x) = \frac{1}{6}\widetilde{C}(x)$ for all $x \in \mathcal{X}$. To prove the uniqueness of A, Q, and C, let $A', Q', C' : \mathcal{X} \to \mathcal{Y}$ be additive, quadratic, and cubic functions, respectively, satisfying (61). Let $\varphi = A-A', \theta = Q-Q'$, and $\psi = C-C'$. We show $\varphi = \theta = \psi = 0$. By (61), we have

$$\begin{aligned}
\|\varphi(x)+\theta(x)+\psi(x)\| &\leqslant \|f(x)-A'(x)-Q'(x)-C'(x)-f(0)\| + \|A(x)+Q(x)+C(x)+f(0)-f(x)\|\\
&\leqslant \frac{298}{63}\varepsilon,
\end{aligned} \tag{62}$$

for all $x \in \mathcal{X}$. Therefore,

$$\lim_{n\to\infty}\frac{1}{8^n}\|\varphi(2^n x)+\theta(2^n x)+\psi(2^n x)\|=0,\quad x\in\mathcal{X}. \tag{63}$$

Since φ, ψ, and θ are additive, quadratic, and cubic, respectively, we have

$$\varphi(2^n x)+\theta(2^n x)+\psi(2^n x)=2^n\varphi(x)+4^n\theta(x)+8^n\psi(x),\quad x\in\mathcal{X}.$$

So (63) obtains $\psi(x)=0$ for all $x\in\mathcal{X}$, and (62) yields

$$\|\varphi(x)+\theta(x)\|\leqslant\frac{298}{63}\varepsilon,\quad x\in\mathcal{X}. \tag{64}$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{4^n}\|\varphi(2^n x)+\theta(2^n x)\|=0,\quad x\in\mathcal{X}.$$

This implies that $\theta(x)=0$ for all $x\in\mathcal{X}$. Hence, by (64), we infer that the additive function φ is bounded, and this yields $\varphi=0$. □

Remark 3.5. Since

$$\begin{aligned}\{(x,y)\in\mathcal{X}\times\mathcal{X}:\|x+y\|\geqslant 2d\}&\subseteq\{(x,y)\in\mathcal{X}\times\mathcal{X}:\|x\|+\|y\|\geqslant 2d\}\\&\subseteq\{(x,y)\in\mathcal{X}\times\mathcal{X}:\max\{\|x\|,\|y\|\}\geqslant d\},\end{aligned}$$

the aforementioned results remain valid if the condition $\|x+y\|\geqslant d$ in (30) is replaced by $\|x\|+\|y\|\geqslant d$ or $\max\{\|x\|,\|y\|\}\geqslant d$.

Now, we can prove the following corollary concerning an asymptotic property of the functional equation (2).

Corollary 3.6. *Let $f:\mathcal{X}\to\mathcal{W}$ be a function. Then, the following statements are equivalent*:

(1) $\lim_{\|x\|+\|y\|\to\infty}Df(x,y)=0$;

(2) $\lim_{\|x+y\|\to\infty}Df(x,y)=0$;

(3) $\lim_{\min\{\|x\|,\|y\|\}\to\infty}Df(x,y)=0$;

(4) $\lim_{\max\{\|x\|,\|y\|\}\to\infty}Df(x,y)=0$;

(5) $\lim_{\|x\|\to\infty}Df(x,y)=0$;

(6) $\lim_{\|y\|\to\infty}Df(x,y)=0$;

(7) $f=A+Q+C+f(0)$, *where $A,Q,C:\mathcal{X}\to\mathcal{W}$ are additive, quadratic, and cubic, respectively.*

Corollary 3.7. *Let $\varphi:\mathcal{X}\times\mathcal{X}\to[0,+\infty)$. A function $f:\mathcal{X}\to\mathcal{W}$ satisfies*

$$Df(x,y)=0,\quad x,y\in\mathcal{X}$$

if one of the following conditions holds:

(1) $\lim_{\min\{\|x\|,\|y\|\}\to\infty}\varphi(x,y)=+\infty,\quad \limsup_{\min\{\|x\|,\|y\|\}\to\infty}\varphi(x,y)\|Df(x,y)\|<\infty$;

(2) $\lim_{\max\{\|x\|,\|y\|\}\to\infty}\varphi(x,y)=+\infty,\quad \limsup_{\max\{\|x\|,\|y\|\}\to\infty}\varphi(x,y)\|Df(x,y)\|<\infty$;

(3) $\lim_{\|x\|+\|y\|\to\infty}\varphi(x,y)=+\infty,\quad \limsup_{\|x\|+\|y\|\to\infty}\varphi(x,y)\|Df(x,y)\|<\infty$;

(4) $\lim_{\|x+y\|\to\infty}\varphi(x,y)=+\infty,\quad \limsup_{\|x+y\|\to\infty}\varphi(x,y)\|Df(x,y)\|<\infty$;

(5) $\lim_{\|x\|\to\infty}\varphi(x,y)=+\infty,\quad \limsup_{\|x\|\to\infty}\varphi(x,y)\|Df(x,y)\|<\infty$;

(6) $\lim_{\|y\|\to\infty}\varphi(x,y)=+\infty,\quad \limsup_{\|y\|\to\infty}\varphi(x,y)\|Df(x,y)\|<\infty$.

Corollary 3.8. *Let $\varepsilon\geqslant 0$, $p<0$, and $f:\mathcal{X}\to\mathcal{Y}$ be a function satisfying*

$$\|Df(x,y)\|\leqslant\varepsilon(\|x\|^p+\|y\|^p),\quad x,y\in\mathcal{X}\backslash\{0\}.$$

Then, $Df(x,y)=0$ for all $x,y\in\mathcal{X}$.

Corollary 3.9. *Let $\varepsilon \geqslant 0$, $\min\{p, q\} < 0$ and $f : \mathcal{X} \to \mathcal{Y}$ be a function satisfying*

$$\|Df(x, y)\| \leqslant \varepsilon\|x\|^p\|y\|^q, \quad x, y \in \mathcal{X}\backslash\{0\}.$$

Then, $Df(x, y) = 0$ for all $x, y \in \mathcal{X}$.

4 Some characterizations of inner product spaces

Jordan and von Neumann [22] established that in order to a normed linear space $\mathcal{X}$ to be an inner product space, it is necessary and sufficient that the following condition be satisfied:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x, y \in \mathcal{X}. \tag{65}$$

Some characterizations of inner product spaces could be found in [3,23–26].

Theorem 4.1. *Let $\mathcal{X} \neq \{0\}$ be a normed linear space such that*

$$\|x\|^p + 6\|x + 2y\|^q + \|x + 4y\|^r = 4\|x + y\|^\alpha + 4\|x + 3y\|^\beta, \quad x, y \in \mathcal{X}, \tag{66}$$

for some real numbers $p, q, r, \alpha, \beta \in (0, +\infty)$. Then, $\mathcal{X}$ is an inner product space.

Proof. Letting $x = -2y$ in (66), we obtain

$$2^p\|y\|^p + 2^r\|y\|^r = 4\|y\|^\alpha + 4\|y\|^\beta, \quad y \in \mathcal{X}. \tag{67}$$

Choosing $\|y\| = 1, 2, \frac{1}{2}, \frac{1}{4}$ in (67), we obtain

$$2^p + 2^r = 8; \tag{68}$$

$$4^p + 4^r = 4(2^\alpha + 2^\beta); \tag{69}$$

$$2^{1-\alpha} + 2^{1-\beta} = 1; \tag{70}$$

$$2^{-p} + 2^{-r} = 4^{1-\alpha} + 4^{1-\beta}. \tag{71}$$

Let

$$t = 2^p, \quad s = 2^r, \quad 2^{-\alpha} = z, \quad 2^{-\beta} = w.$$

Then by (68), (69) and (70), we obtain

$$32zw - tszw = 1. \tag{72}$$

On the other hand, (68), (70), and (71) yield

$$ts - 8tszw = 8. \tag{73}$$

By (73) and (72), one obtains $ts = 256zw$. Then, (72) yields $16zw = 1$, and so $ts = 16$. Since $t + s = 8$ and $2(z + w) = 1$, we conclude that $t = s = 4$ and $z = w = \frac{1}{4}$. Therefore, $p = r = \alpha = \beta = 2$.

Now, letting $y = 0$ and choosing $\|x\| = 2$ in (66), one obtains $q = 2$. Define $f : \mathcal{X} \to \mathbb{R}$ by $f(x) = \|x\|^2$. Then, (66) means that f satisfies (2) for all $x, y \in \mathcal{X}$, and we conclude f is quadratic by Lemma 2.3. Thus, $\|.\|$ fulfills (65). Hence, $\mathcal{X}$ is an inner product space. □

Corollary 4.2. *A normed linear space $\mathcal{X} \neq \{0\}$ is an inner product space if and only if*

$$\|x\|^2 + 6\|x + 2y\|^2 + \|x + 4y\|^2 = 4\|x + y\|^2 + 4\|x + 3y\|^2, \quad x, y \in \mathcal{X}.$$

Theorem 4.3. *Let $\mathcal{X}$ be a normed linear space and $\phi : [0, +\infty) \to \mathbb{R}$ be continuous with $\phi(0) = 0, \phi(1) \neq 0$ and satisfy*

$$\phi(\|x\|) + 6\phi(\|x + 2y\|) + \phi(\|x + 4y\|) = 4\phi(\|x + y\|) + 4\phi(\|x + 3y\|), \quad x, y \in \mathcal{X}. \tag{74}$$

Then, $\mathcal{X}$ is an inner product space.

Proof. Define $f : \mathcal{X} \to \mathbb{R}$ by $f(x) \coloneqq \phi(\|x\|)$. It is clear that f is even and fulfills (2). By Lemma 2.3, f is quadratic. Then,

$$\phi(r\|x\|) = \phi(\|rx\|) = f(rx) = r^2 f(x) = r^2\phi(\|x\|),$$

for all nonnegative rational r and all $x \in \mathcal{X}$. Choosing x with $\|x\| = 1$, we obtain $\phi(r) = r^2\phi(1)$ for all nonnegative rational r. Since ϕ is continuous, we infer that $\phi(t) = t^2\phi(1)$ for all $t \geqslant 0$. So, (74) becomes

$$\|x\|^2 + 6\|x + 2y\|^2 + \|x + 4y\|^2 = 4\|x + y\|^2 + 4\|x + 3y\|^2, \quad x, y \in \mathcal{X}.$$

Hence, $\mathcal{X}$ is an inner product space by Corollary 4.2. □

5 Conclusion

We presented the general solutions of the Fréchet functional equation:

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} g(x + jy) = 0,$$

for functions $g : \mathcal{G} \to \mathcal{V}$ in the case of $n = 4$, where $(\mathcal{G}, +)$ is an abelian group and $\mathcal{V}$ is a linear space. We also investigated its Hyers-Ulam stability on some restricted domains. The obtained results have been used to present some asymptotic behaviors of this functional equation in the framework of normed spaces. Finally, we provided some characterizations of inner product spaces associated with the mentioned functional equation.

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