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Stability of Some Advanced Functional Equations in Various Spaces

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Preface

The investigation of Hyers-Ulam stability of mathematical equations is a fascinating and developing research area in mathematical analysis. This theory has emerged from the well-known question raised by S. M. Ulam [30] in the year 1941. His question is concerned with the existence of approximate solution near to the exact solution of homomorphisms in group theory. In the subsequent year 1941, D. H. Hyers [40] provided a partial response to the question of Ulam by considering Cauchy functional equation in Banach spaces and the result obtained by Hyers-Ulam stability of functional equations and the method adopted in proving this result is called as the direct method. The approximate solution is obtained directly from the given inequality in direct method.

In the year 1978, T. M. Rassias [83] further generalized Hyer's result by considering the upper bound as a sum of the power of norms. Later, in the year 2002, C. Park [68] proved the stability of linear mappings in the spirit of Hyers, Ulam and Rassias in Banach modules. There are several different versions in the stability results obtained by many mathematicians, viz, Aoki [5], Gavruta [34] J. M. Rassias [84]. There are various techniques applied to prove the stabilities of functional equations. Some of them are fixed point technique [12–18], stability on restricted domains [10], stability in dislocated metric spaces [74]. The direct method has influenced many mathematicians to obtain stabilities of mathematical equations as it is simple and easy when compared to other methods.

The foremost reply presented by Hyers [40] considered Banach spaces to achieve the stability results. Later, this research work is carried out by many mathematicians in different normed spaces. Further, the results of this research field from the beginning to the latest results motivated us to deal with a few different functional equations to determine their stabilities in modern normed spaces.

The fundamental aim of this book is to present several new results concerning solution and various stabilities of some functional equations in different spaces such as Banach space, Banach algebra, fuzzy normed space, intuitionistic fuzzy normed space, random normed space, quasi-Banach algebra, random 2-normed space, and generalized 2-normed space.

In particular, fuzzy type normed spaces are considered in this book, as these normed spaces find many significant roles in solving problems with uncertainties. These stability results obtained in various fuzzy settings could be implemented to solve problems with inexactness and vagueness. We do hope these results would pave a different direction to analyze approximate solutions to a given equation whenever uncertainty occurs. The results gained in this book can help readers learn about proving stabilities in advanced normed spaces.

We dealt with functional equations derived from additive, quadratic, and other higher-order functions in this book. We achieved their general solutions and established their stabilities in various spaces using direct and fixed point approaches. It has been attempted to offer this book in such a way that postgraduate students and research scholars can gain some useful ideas for studying novel functional equations. However, readers may want to go into additional sources to learn the fundamentals of functional equations and how they develop in different directions.

The contributions of numerous researchers in the field of functional equations have benefited the authors, and also sincerely acknowledge the suggestions and ideas received from many colleagues, including P. Narasimman, B. V. Senthil Kumar, and others. The first author gratefully acknowledges the SERB, India grant (F. No.: MTR/2020/000534).

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Chapter 1
Functional Equations

1.1 Fundamentals of Functional Equation

Functional equations are one of the most interesting topics in the field of mathematics. A functional equation is much like a regular algebraic equation, though instead of unknown elements in some set, we are interested in finding a function satisfying our equation.

Functional equations arose concurrently with the definition of function and flourished at all levels of mathematics. The most fundamental functional equations include the definition of an even function $f(x) = f(-x)$, for all x, the definition of an odd function $f(x) = -f(-x)$, for all x, and the definition of a periodic function $f(x+T) = f(x)$, for all x, where $T > 0$.

Despite the fact that functional equations had been known since the 16th century, it took another two centuries for mathematicians to attempt to organize the notations and theory. The work of Grégoire de Saint-Vincent (1584–1667) is a good place to start. He discovered that the area under hyperbolic graphs can be described by a function f(x) with the property $f(x) + f(y) = f(xy)$. This property is possessed by logarithmic functions, which are easily shown by modern calculus, but at that time, Grégoire used a geometric argument to obtain a functional equation, which would ultimately have a logarithmic solution [62, 91].

1.2 Definition of Functional Equations

The following definition is due to Aczél [1, 2], a renowned specialist in the field of functional equations.

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Definition 1.1 Functional equations are equations in which both sides are terms constructed from a finite number of unknown functions and a finite number of independent variables.

That is, functional equations are equations in which the unknown functions of their variables are range over functions.

The following is an useful example (see also, [35, 45, 46]).

Example 1.2 (i) $A(x + y) = A(x) + A(y)$; (Cauchy's Additive Functional Equation)
(ii) $E(x + y) = E(x)E(y)$; (Cauchy's Exponential Functional Equation)
(iii) $M(xy) = M(x)M(y)$; (Cauchy's Multiplicative Functional Equation)
(iv) $L(xy) = L(x) + L(y)$; (Cauchy's Logarithmic Functional Equation)

The field of functional equations includes differential equations, difference equations, and integral equations.

Definition 1.3 An algebroid functional equation for the unknown function $f(x)$ is an equation of the form

$$F(x, f(x)) = 0, \tag{1.1}$$

for all $x \in D$, where F is a known function of two variables and D is a given set. Unfortunately, Eq. (1.1) hides some important aspects of functional equations. The function F may contain parameters. These parameters are frequently assumed in the form of variables in the functional equation with values in D. For example,

$$F(x, y, f(x), f(y)) = 0, \tag{1.2}$$

for all $x, y \in D$ is a functional equation of $f(x)$ with y and $f(y)$ appearing as parameters. The fact that we permit parameters to enter in ways similar to this allows a functional equation to have an endless number of possibilities. For convenience's sake, we shall assume that Eq. (1.1) is a shorthand notation for all such possibilities. Furthermore, $f(x)$ is a one-variable function, but it can satisfy the two-variable functional equation, $f(x + y) = f(x) + f(y)$.

1.3 Solution of Functional Equations

A solution of a functional equation is a function which satisfies the given condition of the functional equation.

Example 1.4 The functions $A(x) = k\,x$, $E(x) = e^x$, $M(x) = x^c$ and $L(x) = logx$ are solutions of the Cauchy additive, exponential, multiplicative and logarithmic functional equations respectively.

Example 1.5 $f(x) = cx + a$ is the solution of the functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}.$$

Particular Solution of Functional Equation: A function (or) set of functions is said to be a particular solution of a functional equation (or) system of functional equations if it satisfies the functional equation.

Example 1.6 The functional equation $f(x + y) = g(x) + h(y)$ has solution $f(x) = 2x + 3$; $g(x) = 2x + 1$; $h(x) = 2x + 2$ respectively.

General Solution of Functional Equation: The general solution of a functional equation (or) system is the totality of particular solutions.

Example 1.7 $f(x) = cx + a + b$; $g(x) = cx + a$; $h(x) = cx + b$ are solutions of the functional equation $f(x + y) = g(x) + h(y)$.

1.4 Applications of Functional Equations

Functional equations arise not only in mathematics but also in many other areas of science, engineering, and social sciences. On the research front, the functional equations have been at the heart of many different areas of mathematics and theoretical physics, such as lattice integrable systems, factorized scattering in quantum field theory, Braid and Knot theory, and quantum groups.

The sum of the kth powers of the first n natural numbers is calculated using the additive Cauchy functional equation for $k = 1, 2, 3, \ldots$ We can show that the number of possible pairs among n things can be determined using the additive Cauchy functional equation. Furthermore, the additive Cauchy functional equation can be used to find the sum of certain finite series.

Using the properties of the integrals and the nth logarithmic function, we can derive $\int_1^x \frac{1}{t} dt$, for all $x \in (0, \infty)$. In elementary calculus, the Cauchy functional equation is used to define natural logarithm. For example, they may be used in the continuation of functions into the complex plane. The Gamma function's functional equation, $\Gamma(z + 1) = z\Gamma(z)$, for example, allows us to understand the function's continuation to the left of the line $Re(z) = 0$.

1.5 Stability of Functional Equations

One can ask the following question in general for functional equations. Whether it is true, that the solution of an equation differing slightly from a given one, should be close to the solution of the given equation? or, if we replace a functional equation with a functional inequality, when can one assert that the solutions of the inequality lie near to the solutions of the equation?

The stability of the functional equation has been a hot topic over the last seven decades. Initially, it sustained on the question of Ulam [92] and several unsolved problems proposed by him. This was the starting point of the stability theory of functional equations [10].

Ulam Problem: Suppose G is a group, $H(d)$ is a metric group and $f : G \to H$. For any $\epsilon > 0$, does there exist $\delta > 0$ such that

$$d\big(f(xy), f(x)f(y)\big) < \delta,$$

holds for all $x, y \in G$ and implies there is a unique homomorphism $M : G \to H$ such that

$$d\big(f(x), M(x)\big) < \epsilon,$$

for all $x \in G$.

If the answer is affirmative, we can say that the Cauchy equation

$$f(xy) = f(x)f(y)$$

is stable. This question forms the basis of stability theory.

Hyers [40] in 1941 answered S.M. Ulam's question affirmatively by considering Banach spaces in places of the group G and metric group $H(d)$ in the form of the following theorem.

Theorem 1.8 ([40]) *Let X, Y be Banach spaces and let $f : X \to Y$ be a mapping satisfying*

$$\| f(x+y) - f(x) - f(y) \| \leq \epsilon, \tag{1.3}$$

for some $\epsilon > 0$ and for all $x, y \in X$. Then the limit

$$a(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n},$$

exists for all $x \in X$ and $a : X \to Y$ is the unique additive mapping satisfying

$$\| f(x) - a(x) \| \leq \epsilon,$$

for all $x \in X$.

The above stability is known as **Hyers-Ulam stability** of the additive functional equation for any pair of Banach spaces. In the preceding theorem, the method for constructing the additive function $a(x)$ is direct. This method is an important and powerful tool for studying the stability of various functional equations [80].

In 1950, Aoki [4] generalized the Hyers theorem for additive mappings involving normed spaces X and Y as follows.

Theorem 1.9 ([4]) *Let the transformation $f(x)$ from X to Y be approximately linear, i.e., there exist $k \geq 0$ and $0 \leq p < 1$ such that*

$$\| f(x+y) - f(x) - f(y) \| \leq k\Big(\| x \|^p + \| y \|^p \Big), \tag{1.4}$$

for all x, y in X.

Then there is a linear transformation $\pi(x)$ from X to Y near $f(x)$, i.e., there exist $k \geq 0$ and $0 \leq p < 1$ such that

$$\| f(x) - \pi(x) \| \leq k \| x \|^p, \tag{1.5}$$

for all x in X.

Such $\pi(x)$ is unique.

The above stability is called **Hyers-Ulam-Aoki Stability** of the additive functional equation. Rassias [83], re-modified Aoki's theorem, and proved the following theorem.

Theorem 1.10 (Hyers-Ulam-Rassias Stability) *Let X and Y be two Banach spaces. Let $\theta \geq 0$ and $p \in [0, 1)$. If a function $f : X \to Y$ satisfies the inequality*

$$\| f(x+y) - f(x) - f(y) \| \leq \theta\Big(\| x \|^p + \| y \|^p \Big), \tag{1.6}$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\| f(x) - T(x) \| \leq \frac{2\theta}{2-2^p} \| x \|^p, \tag{1.7}$$

for all x in X. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then the function T is linear.

Th.M. Rassias during the 27th international symposium on functional equations asked the question of whether such a Hyers-Ulam-Rassias theorem can also be proved for a value of p greater or equal to 1 after noticing that the above theorem also works for $p < 0$. In 1991, Gajda [32] provided an affirmative solution to Th.M. Rassiass's question for p strictly greater than one. In fact, he established the following result.

Theorem 1.11 ([32]) *Let X and Y be two (real) normed linear spaces and assume that X is complete. Let $f : X \to Y$ be a mapping for which there exists two constants $\epsilon > 0$ and $p \in R/\{1\}$ such that*

$$\| f(x+y) - f(x) - f(y) \| \leq \epsilon\Big(\| x \| + \| y \| \Big), \tag{1.8}$$

for all $x, y \in X$*, then there exists a unique additive mapping* $T : X \to Y$ *such that*

$$\| f(x) - T(x) \| \leq \delta, \tag{1.9}$$

for all x *in* X*, where*

$$\delta = \begin{cases} \frac{2\epsilon}{(2-2^p)} & for\ p < 1 \\ \frac{2\epsilon}{(2^p-2)} & for\ p > 1, \end{cases}$$

moreover, for each $x \in X$*, the transformation* $t \to f(tx)$ *is continuous, then the mapping* T *is linear.*

It turns out that 1 is the only critical value of p to which the Theorem 1.10 cannot be extended. Z. Gajda showed that this theorem is false for $p = 1$ by constructing a counterexample (see, [32]).

Rassias [78] replaced the factor (**sum of norms**) $\| x \|^p + \| y \|^q$ by (**product norms**) $\| x \|^p \cdot \| y \|^q$ for $p, q \in \mathbb{R}$, $p + q \neq 1$ and proved the following theorem.

Theorem 1.12 *Let* $(X, \|.\|_1)$ *be a normed linear space and* $(Y, \|.\|_2)$ *be a Banach space. Assume in addition that* $f : X \to Y$ *is a mapping such that* $f(t.x)$ *is continuous in* t *for each fixed* x*. If there exists* p, q*,* $0 \leq p + q < 1$*, and* $\theta \geq 0$ *such that*

$$\| f(x + y) - [f(x) + f(y)] \|_2 \leq \theta \| x \|_1^p . \| y \|_1^q, \tag{1.10}$$

for all $x, y \in X$*, then there exists a unique linear mapping* $T : X \to Y$ *such that*

$$\| f(x) - T(x) \|_2 \leq \frac{\theta}{2 - 2^{p+q}} \| x \|_1^{p+q}, \tag{1.11}$$

for all x *in* X*.*

Theorem 1.13 *Let* $(E, +)$ *be an abelian group,* F *be a Banach space and let* $\phi : E \times E \to [0, \infty)$ *be a function satisfying*

$$\phi(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \pi(2^k x, 2^k y) < \infty, \tag{1.12}$$

for all $x, y \in E$*. Assume that a mapping* $f : E \to F$ *satisfies the functional inequality*

$$\| f(x + y) - f(x) - f(y) \| \leq \phi(x, y),$$

for all $x, y \in E$*. Then there exists a unique additive mapping* $T : E \to F$ *which satisfies*

$$\| f(x) - T(x) \| \leq \frac{\phi(x,x)}{2},$$

for all $x \in E$. If moreover $f(tx)$ is continuous in t for fixed $x \in E$, then T is linear.

This stability result, proved by Gavruta's [34] is a generalisation of the **Hyers-Ulam-Rassias stability** of approximately additive mappings.

K. Ravi obtained a special case of Gavruta's theorem for the unbounded Cauchy difference in 2008 by considering the summation

$$\| x \|^p \| y \|^q + \| x \|^{(p+q)} + \| y \|^{(p+q)}$$

introduced by J. M. Rassias.

Theorem 1.14 *Let $(E, \perp)$ denote an orthogonality normed space with norm $\| \, . \, \|_E$ and $(F, \| \, . \, \|_F)$ is a Banach space and $f : E \to F$ be a mapping which satisfying the inequality*

$$\begin{aligned} &\| f(mx+y) + f(mx-y) - 2f(x+y) - 2f(x-y) - 2(m^2-2)f(x) + 2f(y) \|_F \\ &\leq \epsilon \Big\{ \| x \|_E^p \| y \|_E^p + \big(\| x \|_E^{2p} \| y \|_E^{2p} \big) \Big\}, \end{aligned} \tag{1.13}$$

for all $x, y \in E$ with $x \perp y$, where ϵ, $p > 0$ and either $m > 1$; $p < 1$ or $m < 1$; $p > 1$ with $m \neq 0$; $m \neq \pm 1$; $m \neq \pm\sqrt{2}$ and $-1 \neq |m|^{p-1} < 1$. Then the limit

$$Q(x) = \lim_{n \to \infty} \frac{f(m^n x)}{m^{2n}},$$

exists for all $x \in E$ and $q : E \to F$ is the unique orthogonality Euler-Lagrange quadratic mapping, such that

$$\| f(x) - Q(x) \|_F \leq \frac{\epsilon}{2|m^1 - m^{2p}|} \| x \|_E^{2p},$$

for all $x \in E$.

1.6 Fixed Point Theory on Stability of Functional Equations

In Theorem 1.8, the additive function a is explicitly constructed by Hyers [40], directly from the given function f as

$$a(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}, \tag{1.14}$$

This is considered a very powerful method to study the stability of functional equations of several types.

Cuadariu and Radu [17, 18] used the fixed point approach for studying the stability of Jensen's functional equation and the Cauchy functional equation.

Now, we recall some basic definitions and theorems on fixed point theory that will be required in the sequel [22, 76].

Definition 1.15 (*Metric*) Let X be a set, A function $d : X \times X \to [0, \infty)$ is called a metric on X if d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$
(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$
(iii) $d(x, z) \le d(x, y) + d(y, z)$, for all $x, y, z \in X$.

Definition 1.16 (*Generalized Metric*) Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$
(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$
(iii) $d(x, z) \le d(x, y) + d(y, z)$, for all $x, y, z \in X$.

The distinction between the generalized metric and the usual metric is that the range of the former includes infinity.

Theorem 1.17 *Let (X, d) be a complete metric space, and consider a mapping $A : X \to X$, which is strictly contractive, that is*

$$d(Ax, Ay) \le Ld(x, y), \ \forall x, y \in X$$

for some (Lipschitz constant) $L < 1$. Then

(i) The mapping A has one and only one, fixed point $x^* = A(x^*)$
(ii) The fixed point x^* is globally attractive, that is $\lim_{n\to\infty} A^n x = x^*$, for any starting point $x \in X$
(iii) One has the following estimation inequalities:

$$d(A^n x, x^*) \le L^n d(x, x^*), \forall n \ge 0 \text{ and } x \in X$$
$$d(A^n x, x^*) \le \frac{1}{1-L} d(A^n, A^{n+1}), \ \forall n \ge 0 \text{ and } x \in X$$
$$d(x, x^*) \le \frac{1}{1-L} d(x, Ax), \ \forall x \in X.$$

Theorem 1.18 *Let (X, d) be a complete generalized metric space and a strictly contractive mapping $A : X \to X$, with the Lipschitz constant L. Then, for each fixed element $x \in X$, either*

(a1) $d(A^n x, A^{n+1} x) = +\infty, \ \forall n \ge 0$ *or*
(a2) There exists a natural number n_0 such that
(b1) $d(A^n x, A^{n+1} x) < +\infty, \forall n \ge n_0$
(b2) the sequence $\{A^n x\}$ is convergent to a fixed point y^* of A
(b3) y^* is the unique fixed point of A in the set

$$Y = \left\{y \in X, d(A^{n_0} x, y)\right\}$$

(b4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Ay)$, *for all* $x, y \in Y$.

The fixed point y^*, if it exists, is not necessarily unique in the whole space X. In fact, if $(a2)$ is true, then (Y, d) is a complete metric space and $A(Y) \subset Y$. As a result, the properties $(b1) - (b3)$ are easily deduced from Theorem 1.17 (see also [17, 18, 21, 64]).

Example 1.19 Let $A = \{0, 2\}$, $B = \{\frac{1}{n} : n \in N\}$, $X = A \cup B$. Define d on $X \times X$ as follows:

1. $d(x, y) = 0$ if $x = y$
2. $d(x, y) = 1$ if $x \neq y$
3. $\{x, y\} \subseteq A$ or $\{x, y\} \subseteq B$, $d(x, y) = d(y, x) = y$ if $x \in A$ and $y \in B$.

Then it is easy to show (X, d) is a generalized metric space but (X, d) is not a standard metric space because it lacks the triangular property:

$$1 = d\left(\frac{1}{2}, \frac{1}{3}\right) > d\left(\frac{1}{2}, 0\right) + d\left(0, \frac{1}{3}\right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

In this g.m.s, the sequence $\left\{\frac{1}{n}\right\}$ converges to both 0 and 2 and it is not a Cauchy sequence. (X, d) is not a Hausdorff space and d is not continuous distance in a sense

$$\lim_{n\to\infty} d\left(\frac{1}{n}, \frac{1}{2}\right) \neq d\left(0, \frac{1}{2}\right).$$

Chapter 2
Additive Functional Equations

2.1 Additive Functional Equation

One of the most famous additive functional equation is

$$f(x + y) = f(x) + f(y). \tag{2.1}$$

It was first solved by A. L. Cauchy in the class of continuous real-valued functions. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of the natural and social sciences. Every solution of the additive functional equation (2.1) is called an additive function.

It is well known that if an additive function $f : R \to R$ satisfies one of the following conditions.

(a) f is continuous at a point
(b) f is monotonic on an interval of positive length
(c) f is bounded on an interval of positive length
(d) f is integrable
(e) f is measurable.

Then f is of the form $f(x) = cx$ with a real constant c. That is to say f has the linearity. That is, if a solution of the additive Eq. (2.1) satisfies one of the very weak conditions (a) to (e), then it does have the linearity. But every additive functional which is not linear displays a very strange behavior.

The stability of the additive functional equation (2.1) was discussed by Hyers [40], Aoki [4], Rassias [83], Gajda [32], Rassias [77], Gavruta [34].

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The solution and stability of the following additive functional equations

$$f(2x-y)+f(x-2y)=3f(x)-3f(y) \tag{2.2}$$
$$f(x+y-2z)+f(2x+2y-z)=3f(x)+3f(y)-3f(z) \tag{2.3}$$
$$f(x+ky)+f(x-ky)=k^2f(x+y)+k^2f(x-y)+2(1-k^2)f(x) \tag{2.4}$$
$$f\Big(\sum_{i=1}^{n}kx_i\Big)+\sum_{j=1}^{n}f\Big(-kx_j+\sum_{\substack{i=1\\ i\neq j}}^{n}kx_i\Big)=(n-1)\Big[\sum_{i=1}^{n}(2i-1)f(x_i)\Big] \tag{2.5}$$
$$\begin{aligned}
&f(nx_1+n^2x_2+n^3x_3+n^4x_4+n^5x_5)+f(-nx_1+n^2x_2+n^3x_3+n^4x_4+n^5x_5)\\
&+f(nx_1-n^2x_2+n^3x_3+n^4x_4+n^5x_5)+f(nx_1+n^2x_2-n^3x_3+n^4x_4+n^5x_5)\\
&+f(nx_1+n^2x_2+n^3x_3-n^4x_4+n^5x_5)+f(nx_1+n^2x_2+n^3x_3+n^4x_4-n^5x_5)\\
&=2[f(nx_1+n^2x_2)+f(nx_1+n^3x_3)+f(nx_1+n^4x_4)+f(nx_1+n^5x_5)\\
&+f(n^2x_2+n^3x_3)+f(n^2x_2+n^4x_4)+f(n^2x_2+n^5x_5)+f(n^3x_3+n^4x_4)\\
&+f(n^3x_3+n^5x_5)+f(n^4x_x+n^5x_5)]-2n[f(x_1)-f(-x_1)]\\
&-2n^2[f(x_2)-f(-x_2)]-2n^3[f(x_3)-f(-x_3)]-2n^4[f(x_4)-f(-x_4)]\\
&-2n^5[f(x_5)-f(-x_5)]-n^2[f(x_1)+f(-x_1)]-n^4[f(x_2)+f(-x_2)\\
&-n^6[f(x_3)+f(-x_3)]-n^8[f(x_4)+f(-x_4)]-n^{10}[f(x_5)+f(-x_5)].
\end{aligned} \tag{2.6}$$

were discussed by Lee [59], Gordji and Parviz [23].

In this chapter, we discuss the solution and stability of the additive functional equation in Banach space with the help of direct and fixed point methods (see [3, 11, 33, 55]).

2.2 General Solution of Additive Functional Equation

Theorem 2.1 *An odd mapping $f : X \to Y$ satisfies the functional equation (2.1) for all $x, y \in X$, if and only if $f : X \to Y$ satisfies the functional equation (2.6) for all $x_1, x_2, x_3, x_4, x_5 \in X$.*

Proof Let $f : X \to Y$ satisfies the functional equation (2.1). Setting (x, y) by $(0, 0)$ in (2.1), we get $f(0) = 0$. Replacing (x, y) by (x, x) and $(x, 2x)$ respectively in (2.1), we obtain

$$f(2x)=2f(x) \text{ and } f(3x)=3f(x), \tag{2.7}$$

for all $x \in X$. In general for any positive integer a, we have

$$f(ax)=af(x), \tag{2.8}$$

for all $x \in X$. It is easy to verify from (2.8) that

$$f(a^2x) = a^2 f(x) \; and \; f(a^3x) = a^3 f(x), \tag{2.9}$$

for all $x \in X$. Replacing (x, y) by (nx_1, n^2x_2) in (2.1), we get

$$f(nx_1 + n^2x_2) = f(nx_1) + f(n^2x_2), \tag{2.10}$$

for all $x_1, x_2 \in X$. Replacing (x, y) *by* $(nx_1 + n^2x_2, n^3x_3)$ in (2.1) and using (2.10), we obtain

$$f(nx_1 + n^2x_2 + n^3x_3) = f(nx_1) + f(n^2x_2) + f(n^3x_3), \tag{2.11}$$

for all $x_1, x_2, x_3 \in X$. Replacing (x, y) *by* $(nx_1 + n^2x_2 + n^3x_3, n^4x_4)$ in (2.1) and using (2.11), we get

$$f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4) = f(nx_1) + f(n^2x_2) + f(n^3x_3) + f(n^4x_4), \tag{2.12}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) *by* $(nx_1 + n^2x_2 + n^3x_3 + n^4x_4, n^5x_5)$ in (2.1) and using (2.12), we attain

$$f(nx_1 + n^2x_2 + n^3x_3 + n^4x_x + n^5x_5) = f(nx_1) + f(n^2x_2) + f(n^3x_3) + f(n^4x_4) + f(n^5x_5), \tag{2.13}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Replacing x_1 *by* $-x_1$ in (2.13) and using oddness of f, we obtain

$$f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_x + n^5x_5) = -f(nx_1) + f(n^2x_2) + f(n^3x_3) + f(n^4x_4), \tag{2.14}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Again replacing x_2 *by* $-x_2$ in (2.13) and using oddness of f, we obtain

$$f(nx_1 - n^2x_2 + n^3x_3 + n^4x_x + n^5x_5) = f(nx_1) - f(n^2x_2) + f(n^3x_3) + f(n^4x_4) + f(n^5x_5), \tag{2.15}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Letting x_3 *by* $-x_3$ in (2.13) and using oddness of f, we have

$$f(nx_1 + n^2x_2 - n^3x_3 + n^4x_x + n^5x_5) = f(nx_1) + f(n^2x_2) - f(n^3x_3) + f(n^4x_4) + f(n^5x_5), \tag{2.16}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Replacing x_4 *by* $-x_4$ in (2.13) and using oddness of f, we get

$$f(nx_1 + n^2x_2 + n^3x_3 - n^4x_x + n^5x_5) = f(nx_1) + f(n^2x_2) + f(n^3x_3) - f(n^4x_4) + f(n^5x_5), \quad (2.17)$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Replacing x_5 *by* $-x_5$ in (2.13) and using oddness of f, we have

$$f(nx_1 + n^2x_2 + n^3x_3 + n^4x_x - n^5x_5) = f(nx_1) + f(n^2x_2) + f(n^3x_3) + f(n^4x_4) - f(n^5x_5), \quad (2.18)$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Adding the Eqs. (2.13), (2.14), (2.15), (2.16), (2.17) and (2.18), we have

$$\begin{aligned}
&f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5)\\
&+f(nx_1 - n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4 + n^5x_5)\\
&+f(nx_1 + n^2x_2 + n^3x_3 - n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 - n^5x_5)\\
&= 4f(nx_1) + 4f(n^2x_2) + 4f(n^3x_3) + 4f(n^4x_4) + 4f(n^5x_5), \qquad (2.19)
\end{aligned}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Using oddness of f in (2.19) and rearranging, we obtain

$$\begin{aligned}
&f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5)\\
&+f(nx_1 - n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4 + n^5x_5)\\
&+f(nx_1 + n^2x_2 + n^3x_3 - n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 - n^5x_5)\\
&= 2[f(nx_1 + n^2x_2) + f(nx_1 + n^3x_3) + f(nx_1 + n^4x_4) + f(nx_1 + n^5x_5)\\
&+f(n^2x_2 + n^3x_3) + f(n^2x_2 + n^4x_4) + f(n^2x_2 + n^5x_5) + f(n^3x_3 + n^4x_4)\\
&+f(n^3x_3 + n^5x_5) + f(n^4x_4 + n^5x_5)] - 2n[f(x_1) - f(-x_1)]\\
&-2n^2[f(x_2) - f(-x_2)] - 2n^3[f(x_3) - f(-x_3)]\\
&-2n^4[f(x_4) - f(-x_4)] - 2n^5[f(x_5) - f(-x_5)], \qquad (2.20)
\end{aligned}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$.

Adding $-n^2f(x_1) - n^4f(x_2) - n^6f(x_3) - n^8f(x_4) - n^{10}f(x_5)$ on both sides, rearranging and using oddness of f, we reach (2.6) as desired.

Conversely, assume that $f : X \to Y$ satisfies the functional equation (2.2). Using oddness of f in (2.10), we have

$$\begin{aligned}
&f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5)\\
&+f(nx_1 - n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4 + n^5x_5)\\
&+f(nx_1 + n^2x_2 + n^3x_3 - n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 - n^5x_5)\\
&= 4nf(x_1) + 4n^2f(x_2) + 4n^3f(x_3) + 4n^4f(x_4) + 4n^5f(x_5), \qquad (2.21)
\end{aligned}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$.

Replacing $(x_1, x_2, x_3, x_4, x_5)$ by $(x, 0, 0, 0, 0)$, $(0, x, 0, 0, 0)$, $(0, 0, x, 0, 0)$, $(0, 0, 0, x, 0)$ and $(0, 0, 0, 0, x)$ respectively in (2.21), we obtain

$$f(nx) = nf(x),\ \ f(n^2x) = n^2 f(x),\ \ f(n^3x) = n^3 f(x),\ \ f(n^4x) = n^4 f(x),$$
$$and\ \ f(n^5x) = n^5 f(x), \tag{2.22}$$

for all $x \in X$. One can easy to verify from (2.22) that

$$f\left(\frac{x}{n^i}\right) = \frac{1}{n^i} f(x);\ \ i = 1, 2, 3, 4, 5 \tag{2.23}$$

for all $x \in X$.

Replacing $(x_1, x_2, x_3, x_4, x_5)$ by $(\frac{x}{n}, \frac{y}{n^2}, 0, 0, 0)$ in (2.21) and using oddness of f and (2.23), we obtain our result.

2.3 Additive Functional Equation: Odd Case–Direct Method

Theorem 2.2 *Let $j \in \{-1, 1\}$. Let $\alpha : X^5 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{kj}x_1, n^{kj}x_2, n^{kj}x_3, n^{kj}x_4, n^{kj}x_5)}{n^{kj}} \tag{2.24}$$

converges in $\mathbb{R}$, for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f_a : X \to Y$ be an odd mapping satisfying the inequality

$$\| Df_a(x_1, x_2, x_3, x_4, x_5) \| \leq \alpha(x_1, x_2, x_3, x_4, x_5), \tag{2.25}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$ (Df—Domain of a function). Then there exists a unique additive mapping $A : X \to Y$ which satisfies the functional equation (2.6) and

$$\| f_a(x) - sA(x) \| \leq \frac{1}{4n} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(n^{kj}x, 0, 0, 0, 0)}{n^{kj}}, \tag{2.26}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k\to\infty} \frac{f_a(n^{kj}x)}{n^{kj}}, \tag{2.27}$$

for all $x \in X$.

Proof Assume that $j = 1$. Setting $(x_1, x_2, x_3, x_4, x_5)$ by $(x, 0, 0, 0, 0)$ in (2.25) and using oddness of f_a, we get

$$\| 4nf_a(x) - 4f_a(nx) \| \leq \alpha(x, 0, 0, 0, 0), \tag{2.28}$$

for all $x \in X$. It follows from (2.28) that

$$\left\|\frac{f_a(nx)}{n} - f_a(x)\right\| \leq \frac{1}{4n}\alpha(x, 0, 0, 0, 0), \tag{2.29}$$

for all $x \in X$. Now replacing x by nx and dividing by n in (2.29), we get

$$\left\|\frac{f_a(n^2x)}{n^2} - \frac{f_a(nx)}{n}\right\| \leq \frac{1}{4n^2}\alpha(nx, 0, 0, 0, 0), \tag{2.30}$$

for all $x \in X$. Adding (2.29) and (2.30), we have

$$\left\|\frac{f_a(n^2x)}{n^2} - f_a(x)\right\| \leq \frac{1}{4n}\left[\alpha(x, 0, 0, 0, 0) + \frac{\alpha(nx, 0, 0, 0, 0)}{n}\right],$$

for all $x \in X$. In general for any positive integer i one can easy to verify that

$$\left\|\frac{f_a(n^ix)}{n^i} - f_a(x)\right\| \leq \frac{1}{4n}\sum_{k=0}^{i-1}\frac{\alpha(n^kx, 0, 0, 0, 0)}{n^k}$$

$$\left\|\frac{f_a(n^ix)}{n^i} - f_a(x)\right\| \leq \frac{1}{4n}\sum_{k=0}^{\infty}\frac{\alpha(n^kx, 0, 0, 0, 0)}{n^k}, \tag{2.31}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{\frac{f_a(n^ix)}{n^i}\right\}$, replacing x by n^ix and dividing n^i in (2.31), for $i, l > 0$, we get

$$\left\|\frac{f_a(n^{i+l}x)}{n^{i+l}} - \frac{f_a(n^lx)}{n^l}\right\| \leq \frac{1}{4n}\sum_{k=0}^{i-1}\frac{\alpha(n^{k+l}x, 0, 0, 0, 0)}{n^{k+l}} \quad \to 0 \; as \; l \to \infty, \tag{2.32}$$

for all $x \in X$. Hence the sequence $\left\{\frac{f_a(n^ix)}{n^i}\right\}$, is Cauchy sequence. Since Y is complete, there exists a mapping $A : X \to Y$ such that $A(x) = \lim_{i\to\infty} \frac{f_a(n^ix)}{n^i}$, for all $x \in X$. Letting $i \to \infty$ in (2.31) we see that (2.26) holds for $x \in X$. To prove that A satisfies (2.6), replacing $(x_1, x_2, x_3, x_4, x_5)$ by $(n^lx, n^lx, n^lx, n^lx, n^lx)$ and dividing n^l in (2.25), we get

$$\frac{1}{n^l} \parallel Df_a(n^l x, n^l x, n^l x, n^l x, n^l x) \parallel \leq \frac{1}{n^l} \parallel \alpha(n^l x, n^l x, n^l x, n^l x, n^l x) \parallel,$$

for all $x_1, x_2, x_3, ..., x_n \in X$. Letting $l \to \infty$ in above inequality and using the definition of $A(x)$, we see that $A(x_1, x_2, x_3, x_4, x_5) = 0$. Hence A satisfies (2.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$. To show that A is unique. Let B be another additive mapping satisfying (2.6) and (2.26), then

$$\begin{aligned} \parallel A(x) - B(x) \parallel \ &\leq \frac{1}{n^l}\left\{\left\|\left(A(n^l x) - f_a(n^l x)\right)\right\| + \left\|\left(f_a(n^l x) - B(n^l x)\right)\right\|\right\} \\ &\leq \frac{1}{4n}\sum_{k=0}^{\infty} \frac{\alpha(n^{k+l}x, 0, 0, 0, 0)}{n^{k+l}} \ \to 0 \ \ as \ l \to \infty, \end{aligned}$$

for all $x \in X$. Hence A is unique. For $j = -1$, we can prove a similar stability result. This completes the proof of Theorem 2.2.

Proposition 2.3 *Let $j \in \{-1, 1\}$. Let $\alpha : X^5 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{2kj}x_1, n^{2kj}x_2, n^{2kj}x_3, n^{2kj}x_4, n^{2kj}x_5)}{n^{2kj}},$$

converges in $\mathbb{R}$, for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f_a : X \to Y$ be a mapping satisfying the inequality (2.25) for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique mapping $A : X \to Y$ which satisfies the functional equation (2.6) and

$$\parallel f_a(x) - A(x) \parallel \leq \frac{1}{4n^2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, n^{2kj}x, 0, 0, 0)}{n^{2kj}},$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k \to \infty} \frac{f_a(n^{2kj}x)}{n^{2kj}},$$

for all $x \in X$.

Remark: Let $j \in \{-1, 1\}$. Let $\alpha : X^5 \to [0, \infty)$ be a function such that

$$\sum_{k=0}^{\infty} \frac{\alpha(2^{2kj}x_1, 2^{2kj}x_2, 2^{2kj}x_3, 2^{2kj}x_4, 2^{2kj}x_5)}{2^{2kj}},$$

converges in $\mathbb{R}$, for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f_a : X \to Y$ be a mapping satisfying the inequality (2.25) for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique mapping $A : X \to Y$ which satisfies the functional equation (2.6) and

$$\| f_a(x) - A(x) \| \leq \frac{1}{16} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 2^{2kj}x, 0, 0, 0)}{2^{2kj}},$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k\to\infty} \frac{f_a(2^{2kj}x)}{2^{2kj}},$$

for all $x \in X$.

Proposition 2.4 *Let $j \in \{-1, 1\}$. Let $\alpha : X^5 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{3kj}x_1, n^{3kj}x_2, n^{3kj}x_3, n^{3kj}x_4, n^{3kj}x_5)}{n^{3kj}},$$

converges in $\mathbb{R}$, for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f_a : X \to Y$ be an odd mapping satisfying the inequality (2.25) for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive mapping $A : X \to Y$ which satisfies the functional equation (2.6) and

$$\| f_a(x) - A(x) \| \leq \frac{1}{4n^3} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 0, n^{3kj}x, 0, 0)}{n^{3kj}},$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k\to\infty} \frac{f_a(n^{3kj}x)}{n^{3kj}},$$

for all $x \in X$.

Remark: Let $j \in \{-1, 1\}$. Let $\alpha : X^5 \to [0, \infty)$ be a function such that

$$\sum_{k=0}^{\infty} \frac{\alpha(2^{3kj}x_1, 2^{3kj}x_2, 2^{3kj}x_3, 2^{3kj}x_4, 2^{3kj}x_5)}{2^{3kj}},$$

converges in $\mathbb{R}$, for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f_a : X \to Y$ be an odd mapping satisfying the inequality (2.25) for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive mapping $A : X \to Y$ which satisfies the functional equation (2.6) and

$$\| f_a(x) - A(x) \| \leq \frac{1}{32} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 0, 2^{3kj}x, 0, 0)}{2^{3kj}},$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k\to\infty} \frac{f_a(2^{3kj}x)}{2^{3kj}},$$

for all $x \in X$.

Proposition 2.5 *Let $j \in \{-1, 1\}$. Let $\alpha : X^5 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{4kj}x_1, n^{4kj}x_2, n^{4kj}x_3, n^{4kj}x_4, n^{4kj}x_5)}{n^{4kj}},$$

converges in $\mathbb{R}$, for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f_a : X \to Y$ be a mapping satisfying the inequality (2.25) for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive mapping $A : X \to Y$ which satisfies the functional equation (2.6) and

$$\| f_a(x) - A(x) \| \leq \frac{1}{4n^4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 0, 0, n^{4kj}x, 0)}{n^{4kj}},$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k\to\infty} \frac{f_a(n^{4kj}x)}{n^{4kj}}, \ \forall\, x \in X$$

Proposition 2.6 *Let $j \in \{-1, 1\}$. Let $\alpha : X^5 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{5kj}x_1, n^{5kj}x_2, n^{5kj}x_3, n^{5kj}x_4, n^{5kj}x_5)}{n^{5kj}},$$

converges in $\mathbb{R}$ for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f_a : X \to Y$ be an odd mapping satisfying the inequality (2.25) for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive mapping $A : X \to Y$ which satisfies the functional equation (2.6) and

$$\| f_a(x) - A(x) \| \leq \frac{1}{4n^5} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 0, 0, 0, n^{5kj})}{n^{5kj}},$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{k\to\infty} \frac{f_a(n^{5kj}x)}{n^{5kj}},$$

for all $x \in X$.

The following corollaries are immediate consequences of Theorem 2.2, Propositions 2.3–2.6 respectively.

Corollary 2.7 *Let ϵ and s be non-negative real numbers. If a mapping $f_a : X \to Y$ satisfies the inequality*

$$\| Df_a(x_1, x_2, x_3, x_4, x_5) \| \leq \begin{cases} \epsilon, \\ \epsilon\left\{ \sum_{i=1}^{5} \| x_i \|^s \right\}, \\ \epsilon\left\{ \prod_{i=1}^{5} \| x_i \|^s + \sum_{i=1}^{5} \| x_i \|^{5s} \right\} \end{cases} \tag{2.33}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{4|n-1|}, \\ \frac{\epsilon \|x\|^s}{4|n-n^s|}, & s \neq 1 \\ \frac{\epsilon \|x\|^{5s}}{4|n-n^{5s}|}, & s \neq \frac{1}{5} \end{cases} \tag{2.34}$$

for all $x \in X$.

Example 2.8 Let ϵ and s be non-negative real numbers. If a mapping $f_a : X \to Y$ satisfies the inequality (2.33), for all $x_1, x_2, x_3, x_4, x_5 \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{|4|} \\ \frac{\epsilon \|x\|^s}{4|2-2^s|} & ; s \neq 1 \\ \frac{\epsilon \|x\|^{5s}}{4|2-2^{5s}|} & ; s \neq \frac{1}{5}, \end{cases} \tag{2.35}$$

for all $x \in X$.

Corollary 2.9 *Let ϵ and s be non-negative real numbers. If a mapping $f_a : X \to Y$ satisfies the inequality (2.33), then there exists a unique additive mapping $A : X \to Y$ such that*

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{4|n^2-1|} \\ \frac{\epsilon \|x\|^s}{4|n^2-n^{2s}|} & ; s \neq 1 \\ \frac{\epsilon \|x\|^{5s}}{4|n^2-n^{10s}|} & ; s \neq \frac{1}{5}, \end{cases} \tag{2.36}$$

for all $x \in X$.

Example 2.10 Let ϵ and s be non-negative real numbers. If a mapping $f_a : X \to Y$ satisfies the inequality (2.33), then there exists a unique additive mapping $A : X \to Y$ such that

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{4|12|} \\ \frac{\epsilon \|x\|^s}{4|2^2 - 2^{2s}|} & ; s \neq 1 \\ \frac{\epsilon \|x\|^{5s}}{4|2^2 - 2^{10s}|} & ; s \neq \frac{1}{5}, \end{cases} \tag{2.37}$$

for all $x \in X$.

Corollary 2.11 *Let ϵ and s be non-negative real numbers. If a mapping $f_a : X \to Y$ satisfies the inequality (2.33), then there exists a unique additive mapping $A : X \to Y$ such that*

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{4|n^3 - 1|} \\ \frac{\epsilon \|x\|^s}{4|n^3 - n^{3s}|} & ; s \neq 1 \\ \frac{\epsilon \|x\|^{5s}}{4|n^3 - n^{15s}|} & ; s \neq \frac{1}{5}, \end{cases} \tag{2.38}$$

for all $x \in X$.

Corollary 2.12 *Let ϵ and s be non-negative real numbers. If a mapping $f_a : X \to Y$ satisfies the inequality (2.33), then there exists a unique additive mapping $A : X \to Y$ such that*

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{4|n^4 - 1|} \\ \frac{\epsilon \|x\|^s}{4|n^4 - n^{4s}|} & ; s \neq 1 \\ \frac{\epsilon \|x\|^{5s}}{4|n^4 - n^{20s}|} & ; s \neq \frac{1}{5}, \end{cases} \tag{2.39}$$

for all $x \in X$.

Corollary 2.13 *Let ϵ and s be non-negative real numbers. If a mapping $f_a : X \to Y$ satisfies the inequality (2.33), then there exists a unique additive mapping $A : X \to Y$ such that*

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{4|n^5 - 1|} \\ \frac{\epsilon \|x\|^s}{4|n^5 - n^{5s}|} & ; s \neq 1 \\ \frac{\epsilon \|x\|^{5s}}{4|n^5 - n^{25s}|} & ; s \neq \frac{1}{5}, \end{cases} \tag{2.40}$$

for all $x \in X$.

2.4 Functional Equation: Odd Case—Fixed Point Method

Theorem 2.14 *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{\alpha(\eta_i^k x_1, \eta_i^k x_2, \eta_i^k x_3, \eta_i^k x_4, \eta_i^k x_5)}{\eta_i^k} = 0, \tag{2.41}$$

where $\eta_i = \begin{cases} n & i = 0; \\ \frac{1}{n} & i = 1; \end{cases}$ *satisfying the functional inequality*

$$\| Df_a(x_1, x_2, x_3, x_4, x_5) \| \leq \alpha(x_1, x_2, x_3, x_4, x_5), \tag{2.42}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function

$$x \to \gamma(x) = \frac{1}{4}\alpha\left(\frac{x}{n}, 0, 0, 0, 0\right),$$

has the property

$$\frac{\gamma(\eta_i x)}{\eta_i} = L\gamma(x), \tag{2.43}$$

for all $x \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying the functional equation (2.6) and

$$\| f_a(x) - A(x) \| \leq \frac{L^{1-i}}{1-L}\gamma(x), \tag{2.44}$$

for all $x \in X$.

Proof Let us consider the set $\Omega = \{p | p : U^2 \to V,\ p(0) = 0\}$ and d be a generalized metric on Ω, such that

$$d(p, q) = inf\left\{k \in (0, \infty) :\| p(x) - q(x) \| \leq k\gamma(x), x \in X\right\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tg(x) = \frac{1}{\eta_i} g(\eta_i x)$, for all $x \in X$. For $p, q \in \Omega$ and $x \in X$, we have

$$\begin{aligned}
& d(p, q) = k \Rightarrow \| p(x) - q(x) \| \leq k\gamma(x), \\
& \Rightarrow \left\|\frac{p(\eta_i x)}{\eta_i} - \frac{q(\eta_i x)}{\eta_i}\right\| \leq \frac{1}{\eta_i} k\gamma(\eta_i x), \\
& \Rightarrow \left\|Tp(x) - Tq(x)\right\| \leq \frac{1}{\eta_i} k\gamma(\eta_i x), \\
& \Rightarrow \left\|Tp(x) - Tq(x)\right\| \leq Lk\gamma(x) \Rightarrow d(Tp(x), Tq(x) \leq KL.
\end{aligned}$$

That is, $d(Tp(x), Tq(x) \leq Ld(p, q)$. Therefore T is strictly contractive mapping on Ω with Lipschitz constant L. It follows from (2.28) that

$$\| 4nf_a(x) - 4f_a(nx) \| \leq \alpha(x, 0, 0, 0, 0), \tag{2.45}$$

for all $x \in X$. It follows from (2.45) that

$$\| nf_a(x) - f_a(nx) \| \leq \frac{\alpha(x, 0, 0, 0, 0)}{4}, \tag{2.46}$$

for all $x \in X$. Using the definition of $\gamma(x)$ in the above equation for $i = 0$, we get

$$\left\| f_a(x) - \frac{f_a(nx)}{n} \right\| \leq \frac{1}{n} L\gamma(x) \Rightarrow \| f_a(x) - Tf_a(x) \| \leq L\gamma(x),$$

for all $x \in X$. Hence, we obtain

$$d(Tf_a(x) - f_a(x) \leq L = L^{1-i}, \tag{2.47}$$

for all $x \in X$. Replacing x by $\frac{x}{n}$ in (2.46), we have

$$\left\| nf_a\left(\frac{x}{n}\right) - f_a(x) \right\| \leq \frac{1}{4}\alpha\left(\frac{x}{n}, 0, 0, 0, 0\right), \tag{2.48}$$

for all $x \in X$. Using the definition of $\gamma(x)$ in the above equation for $i = 0$, we have

$$\left\| \frac{nf_a(x)}{n} - f_a(x) \right\| \leq \gamma(x) \Rightarrow \| Tf_a(x) - f_a(x) \| \leq \gamma(x),$$

for all $x \in X$. Hence we get

$$d(f_a(x) - Tf_a(x)) \leq n = L^{1-i}, \tag{2.49}$$

for all $x \in X$. From (2.47) and (2.49), we can conclude

$$d(f_a(x) - Tf_a(x)) \leq n = L^{1-i} < \infty, \tag{2.50}$$

for all $x \in X$. Now from the fixed point alternative in both cases, it follows that there exists a fixed point A of T in Ω such that

$$A(x) = \lim_{k\to\infty} \frac{f_a(\eta_i^k x)}{\eta_i^k}, \tag{2.51}$$

for all $x \in X$. In order to prove $A : X \to Y$ satisfies the functional equation (2.6), the proof is similar to that of Theorem 2.2. Since A is unique fixed point of T in the set $\Delta = \{f_a \in \Omega / d(f_a, A) < \infty\}$. Therefore, A is an unique mapping such that

$$d(f_a, A) \leq \frac{1}{1-L} d(f_a, Tf_a) \Rightarrow d(f_a, A) \leq \frac{L^{1-i}}{1-L},$$
$$i.e., \| f_a(x) - A(x) \| \leq \frac{L^{1-i}}{1-L} \gamma(x),$$

for all $x \in X$. This completes the proof of the theorem.

Example 2.15 Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition (2.41), where $\eta_i = \begin{cases} 2 & i = 0; \\ \frac{1}{2} & i = 1; \end{cases}$ satisfying the functional inequality (2.42), for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function

$$x \to \gamma(x) = \frac{1}{4} \alpha\left(\frac{x}{2}, 0, 0, 0, 0\right)$$

has the property (2.43), for all $x \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying the functional equations (2.6) and (2.44), for all $x \in X$.

Proposition 2.16 *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition (2.41), where $\eta_i = \begin{cases} n^2 & i = 0; \\ \frac{1}{n^2} & i = 1; \end{cases}$ satisfying the functional inequality (2.42), for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function*

$$x \to \gamma(x) = \frac{1}{4} \alpha\left(0, \frac{x}{n^2}, 0, 0, 0\right)$$

has the property (2.43), for all $x \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying the functional equations (2.6) and (2.44).

Example 2.17 Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition (2.41), where $\eta_i = \begin{cases} 2^3 & i = 0; \\ \frac{1}{2^3} & i = 1; \end{cases}$ satisfying the functional inequality (2.42), for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function

$$x \to \gamma(x) = \frac{1}{4} \alpha\left(0, 0, \frac{x}{2^3}, 0, 0\right)$$

has the property (2.43), for all $x \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying the functional equations (2.6) and (2.44), for all $x \in X$.

Proposition 2.18 *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition (2.41), where $\eta_i = \begin{cases} n^3 & i = 0; \\ \frac{1}{n^3} & i = 1; \end{cases}$ satisfying the functional inequality (2.42), for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function*

$$x \to \gamma(x) = \frac{1}{4}\alpha\left(0, 0, \frac{x}{n^3}, 0, 0\right)$$

has the property (2.43), for all $x \in X$. Then there exists a unique mapping $A : X \to Y$ satisfying the functional equations (2.6) and (2.44), for all $x \in X$.

Proposition 2.19 *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition (2.41), where $\eta_i = \begin{cases} n^4 & i = 0; \\ \frac{1}{n^4} & i = 1; \end{cases}$ satisfying the functional inequality (2.42), for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function*

$$x \to \gamma(x) = \frac{1}{4}\alpha\left(0, 0, 0, \frac{x}{n^4}, 0\right)$$

has the property (2.43), for all $x \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying the functional equations (2.6) and (2.44).

Proposition 2.20 *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition (2.41), where $\eta_i = \begin{cases} n^5 & i = 0; \\ \frac{1}{n^5} & i = 1; \end{cases}$ satisfying the functional inequality (2.42), for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function*

$$x \to \gamma(x) = \frac{1}{4}\alpha\left(0, 0, 0, 0, \frac{x}{n^5}\right)$$

has the property (2.43), for all $x \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying the functional equations (2.6) and (2.44), for all $x \in X$.

The following corollary is an immediate consequence of Theorem 2.14, Proposition 2.16, and Propositions 2.18–2.20 respectively.

Corollary 2.21 *Let ϵ and s be a non-negative real numbers. If a mapping $f_a : X \to Y$ satisfies the inequality (2.33), then there exists an unique additive mapping $A : X \to Y$ satisfying (2.34) for all $x \in X$.*

Proof Set

$$\alpha(x_1, x_2, x_3, x_4, x_5) \leq \begin{cases} \epsilon, \\ \epsilon\left\{ \sum_{i=1}^{5} \| x_i \|^s \right\}, \\ \epsilon\left\{ \prod_{i=1}^{5} \| x_i \|^s + \sum_{i=1}^{5} \| x_i \|^{5s} \right\}, \end{cases}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Now

$$\begin{aligned} \frac{\alpha(\eta_i^k x_1, \eta_i^k x_2, \eta_i^k x_3, \eta_i^k x_4, \eta_i^k x_5)}{\eta_i^k} &= \begin{cases} \frac{\epsilon}{\eta_i^k}, \\ \frac{\epsilon}{\eta_i^k}\left\{ \sum_{i=1}^{5} \| \eta_i x_i \|^s \right\}, \\ \frac{\epsilon}{\eta_i^k}\left\{ \prod_{i=1}^{5} \| \eta_i x_i \|^s + \sum_{i=1}^{5} \| \eta_i x_i \|^{5s} \right\}, \end{cases} \\ &= \begin{cases} \to 0 \ \ as \ k \to \infty \\ \to 0 \ \ as \ k \to \infty \\ \to 0 \ \ as \ k \to \infty, \end{cases} \end{aligned}$$

i.e., (2.45) holds. So we have

$$\gamma(x) = \frac{1}{4}\alpha\left(\frac{x}{n}, 0, 0, 0, 0\right)$$

$$\gamma(x) = \begin{cases} \frac{\epsilon}{4} \\ \frac{\epsilon \|x\|^s}{4n^s} \\ \frac{\epsilon \|x\|^{5s}}{4n^{5s}}. \end{cases}$$

Also,

$$\frac{1}{\eta_i}\gamma(\eta_i x) = \begin{cases} \frac{1}{\eta_i}\frac{\epsilon}{4} \\ \frac{1}{\eta_i}\frac{\epsilon \|x\|^s \eta_i^s}{4n^s} \\ \frac{1}{\eta_i}\frac{\epsilon \|x\|^{5s} \eta_i^{5s}}{4n^{5s}} \end{cases} = \begin{cases} \eta_i^{-1}\gamma(x) \\ \eta_i^{s-1}\gamma(x) \\ \eta_i^{ns-1}\gamma(x) \end{cases}$$

for all $x \in X$. Hence the inequality (2.6) holds for following cases:
$L = n^{-1} \ \ if \ \ i = 0 \ \ and \ \ L = n \ \ if \ \ i = 1.$

$L = n^{s-1} \ \ for \ \ s < 1 \ \ if \ \ i = 0 \ \ and \ \ L = n^{1-s} \ \ for s > 1 \ \ if \ \ i = 1.$
$L = n^{5s-1} \ \ for \ \ s < 1 \ \ if \ \ i = 0 \ \ and \ \ L = n^{1-5s} \ \ for s > 1 \ \ if \ \ i = 1.$

Now, from (2.45), we prove the following cases.

Case 1. $L = n^{-1} \ \ if \ \ i = 0$

$$f_a(x) - A(x) \ \| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{n^{-1}}{1-n^{-1}}\frac{\epsilon}{4} = \frac{\epsilon}{4(n-1)}.$$

Case 2. $L = n \;\; if \;\; i = 1$

$$f_a(x) - A(x) \parallel \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{1}{1-n}\frac{\epsilon}{4} = \frac{\epsilon}{4(1-n)}.$$

Case 3. $L = n^{s-1} \;\; for \;\; s < 1 \;\; if \; i = 0$

$$f_a(x) - A(x) \parallel \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{n^{s-1}}{1-n^{s-1}}\frac{\epsilon \parallel x \parallel^s}{4n^s} = \frac{\epsilon \parallel x \parallel^s}{4(n-n^s)}.$$

Case 4. $L = n^{1-s} \;\; for \;\; s > 1 \;\; if \; i = 1$

$$f_a(x) - A(x) \parallel \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{1}{1-n^{1-s}}\frac{\epsilon \parallel x \parallel^s}{4n^s} = \frac{\epsilon \parallel x \parallel^s}{4(n^s-n)}.$$

Case 5. $L = n^{5s-1} \;\; for \;\; s < 1 \;\; if \; i = 0$

$$f_a(x) - A(x) \parallel \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{n^{5s-1}}{1-n^{5s-1}}\frac{\epsilon \parallel x \parallel^{5s}}{4n^{5s}} = \frac{\epsilon \parallel x \parallel^{5s}}{4(n-n^{5s})}.$$

Case 6. $L = n^{1-5s} \;\; for \;\; s > 1 \;\; if \; i = 1$

$$f_a(x) - A(x) \parallel \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{1}{1-n^{1-5s}}\frac{\epsilon \parallel x \parallel^{5s}}{4n^{5s}} = \frac{\epsilon \parallel x \parallel^{5s}}{4(n^{5s}-n)}.$$

Hence the proof is complete.

Proposition 2.22 *Let ϵ and s be non-negative real numbers. If a mapping $f_a : X \to Y$ satisfies the inequality (2.33), for all $x_1, x_2, x_3, x_4, x_5 \in X$, then there exists an unique mapping satisfying the inequality*

$$\| f_\alpha(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{|4|} \\ \frac{\epsilon \|x\|^s}{4|2-2^s|}; & s \neq 1 \\ \frac{\epsilon \|x\|^{5s}}{4|2-2^{5s}|}; & s \neq \frac{1}{5} \end{cases}$$

for all $x \in X$.

Corollary 2.23 *Let ϵ and s be non-negative real numbers. If a mapping $f_a : X \to Y$ satisfies the inequality (2.33), for all $x_1, x_2, x_3, x_4, x_5 \in X$, then there exists an unique mapping satisfying the inequality (2.36) for all $x \in X$.*

Proposition 2.24 *Let ϵ and s be non-negative real numbers. If a mapping $f_a : X \to Y$ satisfies the inequality (2.33), for all $x_1, x_2, x_3, x_4, x_5 \in X$, then there exists an unique mapping satisfying the inequality*

$$\|f_\alpha(x) - A(x)\| \leq \begin{cases} \frac{\epsilon}{|12|} \\ \frac{\epsilon\|x\|^s}{4|2^2-2^s|}; & s \neq 1 \\ \frac{\epsilon\|x\|^{5s}}{4|2^2-2^{5s}|}; & s \neq \frac{1}{5} \end{cases}$$

for all $x \in X$.

Corollary 2.25 *Let ϵ and s be non-negative real numbers. If a mapping $f_a : X \to Y$ satisfies the inequality (2.33), for all $x_1, x_2, x_3, x_4, x_5 \in X$, then there exists an unique additive mapping satisfying the inequality (2.38) for all $x \in X$.*

Corollary 2.26 *Let ϵ and s be non-negative real numbers. If a mapping $f_a : X \to Y$ satisfies the inequality (2.33), for all $x_1, x_2, x_3, x_4, x_5 \in X$, then there exists an unique mapping satisfying the inequality (2.39) for all $x \in X$.*

Corollary 2.27 *Let ϵ and s be non-negative real numbers. If a mapping $f_a : X \to Y$ satisfies the inequality (2.33), for all $x_1, x_2, x_3, x_4, x_5 \in X$, then there exists an unique additive mapping satisfying the inequality (2.40) for all $x \in X$.*

2.5 Examples

In this section, the counterexample for non stable cases is discussed. Now, we will provide an example to illustrate that the functional equation (2.6) is not stable for $x = 1$.

Example 2.28 Let $\Upsilon : R \to R$ be a function defined by

$$\Upsilon(x) = \begin{cases} \nu x & \text{if } |x| < 1 \\ \nu & \text{otherwise,} \end{cases}$$

where $\nu > 0$ is a constant and define a function $f : R \to R$ by

$$N(f(x), u) = \sum_{m=0}^{\infty} \frac{\Upsilon(n^m x)}{n^m}, \quad \text{for all } x \in R.$$

Then f satisfies the functional inequality

$$N\left(f\left(nx_0 \pm \hat{\text{A}}_{\text{¸}} \sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_0 \pm x_i), u\right) \leq \frac{n \times (n+1)\nu}{n-1} \sum_{i=0}^{n} \|x_i\|, \tag{2.52}$$

for all $x_0, x_1, ..., x_n \in R$ and $u \in R$. But there does not exist any additive mapping $A : R \to R$ and any constant $\delta > 0$ such that

$$N(f(x) - A(x), u) \leq \delta\|x\|, \quad \text{for all } x \in R. \tag{2.53}$$

Proof Now

$$\begin{aligned} N(f(x), u) &\leq \sum_{m=0}^{\infty} \frac{\Upsilon(n^m x)}{n^m} \\ &= \sum_{n=0}^{\infty} \frac{\nu}{n^m} = \frac{n}{n-1}\nu. \end{aligned}$$

Therefore, we see that f is bounded. We are going to prove that f satisfies (2.52).

If $x_i = 0$, $i = 0, 1, 2, ..., n$, then (2.52) is trivial.

If $\sum_{i=0} \|x_i\| \geq 1$, then the left hand side of (2.52) is less than $\frac{n(n+1)}{n-1}\nu$.

Now suppose that $0 < \sum_{i=0} \|x_i\| < 1$. Then there exists a positive integer k such that

$$\frac{1}{n^k} \leq \sum_{i=0} \|x_i\| < \frac{1}{n^k - 1}. \tag{2.54}$$

So $n^{k-1}\|x_i\| < 1,\ i = 0, 1, 2, ..., n$ and consequently

$$n^{k-1}\left(nx_0 \pm \sum_{i=1}^{n} x_i\right), \quad n^{k-1}\sum_{i=1}^{n}(x_0 \pm x_i) \in (-1, 1).$$

Therefore, for each $m = 0, 1, ..., k-1$, we have

$$n^m\left(nx_0 \pm \sum_{i=1}^{n} x_i\right), \quad n^m\sum_{i=1}^{n}(x_0 \pm x_i) \in (-1, 1)$$

and

$$N\left(\Upsilon\left(n^m\left(nx_0 \pm \sum_{i=1}^{n} x_i\right)\right) - \sum_{i=1}^{n} \Upsilon(n^m(x_0 \pm x_i)), u\right) = 0$$

for $m = 0, 1, ..., k-1$. From the definition of f and (2.54), we obtain that

$$\begin{aligned}
N&\left(f\left(n^m\left(nx_0 \pm \sum_{i=1}^{n} x_i\right)\right) - \sum_{i=1}^{n} f(n^m(x_0 \pm x_i)), u\right) \\
&\leq \sum_{m=0}^{\infty} \frac{1}{n^m} N\left(\Upsilon\left(n^m\left(nx_0 \pm \sum_{i=1}^{n} x_i\right)\right) - \sum_{i=1}^{n} \Upsilon(n^m(x_0 \pm x_i)), u\right) \\
&\leq \sum_{m=k}^{\infty} \frac{1}{n^m} N\left(\Upsilon\left(n^m\left(nx_0 \pm \sum_{i=1}^{n} x_i\right)\right) - \sum_{i=1}^{n} \Upsilon(n^m(x_0 \pm x_i)), u\right) \\
&\leq \sum_{m=k}^{\infty} \frac{1}{n^m} (n+1)\nu \\
&= \frac{n \times (n+1)\nu}{(n-1)n^k} \\
&\leq \frac{n \times (n+1)\nu}{n-1} \sum_{i=0}^{n} \|x_i\|.
\end{aligned}$$

Thus f satisfies (2.52), for all $x_i \in R,\ i = 0, 1, 2, ..., n$ with $0 < \sum_{i=0}^{n} \|x_i\| < 1$.

We claim that the additive functional equation (2.6) is not stable for $x = 1$.

Suppose on the contrary that there exist an additive mapping $A : R \to R$ and a constant $\delta > 0$ satisfying (2.53). Since f is bounded and continuous for all $x \in R$, A is bounded on any open interval containing the origin and continuous at the origin. Then, A must have the form $A(x) = cx$, for any x in R. Thus we obtain that

$$N(f(x), u) \leq (\delta + \|c\|)\|x\|. \tag{2.55}$$

But we can choose a positive integer l with $l\nu > \beta + \|c\|$.

If $x \in \left(0, \frac{1}{n^{l-1}}\right)$, then $n^m x \in (0, 1)$, for all $m = 0, 1, ..., l-1$. For this x, we get

$$\begin{aligned}
N(f(x), u) &= \sum_{m=0}^{\infty} \frac{\Upsilon(n^m x)}{n^m} \\
&\geq \sum_{m=0}^{l-1} \frac{\nu n^m x}{n^m} \\
&= l\nu x \\
&> (\delta + \|c\|)\|x\|
\end{aligned}$$

which contradicts to (2.55). Therefore, the additive functional equation (2.6) is not stable in sense of Hyers-Ulam if s = 1.

Now we will provide an example to illustrate that the functional equation (2.6) is not stable for $x = \frac{1}{n+1}$.

Example 2.29 Let $\Upsilon : R \to R$ be a function defined by

$$\Upsilon(x) = \begin{cases} \nu x & \text{if } |x| < \frac{1}{n+1} \\ \frac{\nu}{n+1} & \text{otherwise,} \end{cases}$$

where $\nu > 0$ is a constant and define a function $f : R \to R$ by

$$N(f(x), u) = \sum_{m=0}^{\infty} \frac{\Upsilon(n^m x)}{n^m}, \quad \text{for all } x \in R.$$

Then f satisfies the functional inequality

$$N\left(f\left(nx_0 \pm \text{Â¸}\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_0 \pm x_i), u\right) \leq \frac{(n+1)\nu}{(n-1)}\left\{\prod_{i=0}^{n} \|x_i\|^{\frac{n^2}{n+1}} + \sum_{i=0}^{n} \|x_i\|\right\}, \tag{2.56}$$

for all $x_0, x_1, ..., x_n \in R$ and $u \in R$. Then there do not exist any additive mapping $A : R \to R$ and any constant $\delta > 0$ such that

$$N(f(x) - A(x), u) \leq \delta\|x\|, \quad \text{for all } x \in R. \tag{2.57}$$

Chapter 3
Additive Functional Equations in Banach Algebras

3.1 Banach Algebra—Additive Functional Equation

In 1982, Rassias [77] replaced the factor $\| x \|^p + \| y \|^q \quad by \quad \| x \|^p \| y \|^q$ for $p, q \in \mathbb{R}$. Gavruta [34] obtained a generalisation of stability results in 1994 by replacing the unbounded Cauchy difference with a general control function $\varphi(x, y)$. In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi, considering the summation of both the sum and the product of two p-norms. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [52]) and the references cited therein. In 1951, Bourgin [9] treated the Ulam stability problem for additive mappings. In 1978, Rassias [83] proved a generalised stability theorem for the linear mapping by applying Hyers' direct method.

In this chapter, we will show the stability of the additive functional equation (2.6) by using the direct and fixed point methods in Banach algebra.

3.2 Additive Functional Equation: Odd Case—Direct Method

In this section, we discussed the Hyers-Ulam stability of additive functional equation in Banach Algebra using direct and fixed point methods.

Definition 3.1 Let X be a Banach algebra. An additive mapping $A : X \to X$ is said to be an additive derivation if the additive mapping A satisfies,

$$A(x_1x_2) = A(x_1)x_2 + x_1A(x_2), \tag{3.1}$$

for all $x_1, x_2 \in X$. Also the additive derivation for five variables satisfies

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https://doi.org/10.1007/978-3-031-33704-8_3

$$A(x_1x_2x_3x_4x_5) = A(x_1)x_2x_3x_4x_5 + x_1A(x_2)x_3x_4x_5 + x_1x_2A(x_3)x_4x_5 \\ +x_1x_2x_3A(x_4)x_5 + x_1x_2x_3x_4A(x_5), \tag{3.2}$$

for five variables in the benefits of spectra operator theory in additive functional equation.

Theorem 3.2 *Let $j = \pm 1$. Let $f_a : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{kj}x_1, n^{kj}x_2, n^{kj}x_3, n^{kj}x_4, n^{kj}x_5)}{n^{kj}} \tag{3.3}$$

converges in $\mathbb{R}$ and

$$\sum_{k=0}^{\infty} \frac{\beta(n^{kj}x_1, n^{kj}x_2, n^{kj}x_3, n^{kj}x_4, n^{kj}x_5)}{n^{5kj}} \tag{3.4}$$

converges in $\mathbb{R}$ and the functional inequalities

$$\| Df_a(x_1, x_2, x_3, x_4, x_5) \| \leq \alpha(x_1, x_2, x_3, x_4, x_5), \tag{3.5}$$

and

$$\| f_a(x_1x_2x_3x_4x_5) - f_a(x_1)x_2x_3x_4x_5 - x_1f_a(x_2)x_3x_4x_5 - x_1x_2f_a(x_3)x_4x_5 \\ -x_1x_2x_3f_a(x_4)x_5 - x_1x_2x_3x_4f_a(x_5) \| \leq \beta(x_1, x_2, x_3, x_4, x_5), \tag{3.6}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique derivation $A : X \to X$ satisfying the functional equation (2.6) and

$$\| f_a(x) - A(x) \| \leq \frac{1}{4n} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(n^{kj}x, 0, 0, 0, 0)}{n^{kj}}, \tag{3.7}$$

for all $x \in X$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{n\to\infty} \frac{f_a(n^{kj}x)}{n^{kj}}, \tag{3.8}$$

for all $x \in X$.

Proof It follows from Theorem 2.2 that A is a unique additive mapping satisfying (2.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$. It follows from (3.6) that

$$
\begin{aligned}
&\| f_a(x_1x_2x_3x_4x_5) - f_a(x_1)x_2x_3x_4x_5 - x_1 f_a(x_2)x_3x_4x_5 - x_1x_2 f_a(x_3)x_4x_5 \\
&-x_1x_2x_3 f_a(x_4)x_5 - x_1x_2x_3x_4 f_a(x_5) \| \\
&\leq \frac{1}{n^{5k}} \left|\left| f_a(n^k(x_1x_2x_3x_4x_5)) \right.\right. \\
&\quad - f_a(n^k x_1)(n^k x_2 n^k x_3 n^k x_4 n^k x_5) - n^k x_1 f_a(n^k x_2) n^k x_3 n^k x_4 n^k x_5 \\
&\quad -n^k x_1 n^k x_2 f_a(n^k x_3) n^k x_4 n^k x_5 - n^k x_1 n^k x_2 n^k x_3 n^k f_a(n^k x_4) n^k x_5 \\
&\quad \left.\left. -n^k x_1 n^k x_2 n^k x_3 n^k x_4 f_a(n^k x_5) \right|\right| \\
&\leq \beta(n^k x_1, n^k x_2, n^k x_3, n^k x_4, n^k x_5) \quad \to 0 \ \ as \quad k \to \infty,
\end{aligned}
$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Hence, the mapping $A : X \to X$ is a unique additive derivation satisfying (3.7). □

Example 3.3 Let $j = \pm 1$. Let $f_a : X \to Y$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with condition

$$\sum_{k=0}^{\infty} \frac{\alpha(2^{kj}x_1, 2^{kj}x_2, 2^{2kj}x_3, 2^{kj}x_4, 2^{kj}x_5)}{2^{kj}}$$

converges in $\mathbb{R}$ and

$$\sum_{k=0}^{\infty} \frac{\beta(2^{kj}x_1, 2^{kj}x_2, 2^{kj}x_3, 2^{kj}x_4, 2^{kj}x_5)}{n^{5kj}},$$

converges in $\mathbb{R}$ such that the functional inequalities (3.5) and (3.6) for all $x_1, x_2, x_3, x_4, x_5 \in X$.

Then there exists a unique additive derivation $A : X \to X$ satisfying the functional equation (2.6) and for all $x \in X$,

$$\| f_a(x) - A(x) \| \leq \frac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(2^{kj}x, 0, 0, 0, 0)}{2^{kj}}.$$

The mapping $A(x)$ is defined by $A(x) = \lim\limits_{n\to\infty} \frac{f_a(2^{kj}x)}{n^{kj}}$, for all $x \in X$.

Proposition 3.4 *Let $j = \pm 1$. Let $f_a : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with condition*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{2kj}x_1, n^{2kj}x_2, n^{2kj}x_3, n^{2kj}x_4, n^{2kj}x_5)}{n^{2kj}}$$

converges in $\mathbb{R}$ and

$$\sum_{k=0}^{\infty} \frac{\beta(n^{2kj}x_1, n^{2kj}x_2, n^{2kj}x_3, n^{2kj}x_4, n^{2kj}x_5)}{n^{10kj}}$$

converges in $\mathbb{R}$ *such that the functional inequalities (3.5) and (3.6), for all* $x_1, x_2, x_3, x_4, x_5 \in X$.

Then there exists a unique additive derivation $A : X \to X$ *satisfying the functional equation (2.6) and for all* $x \in X$

$$\| f_a(x) - A(x) \| \leq \frac{1}{4n^2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, n^{2kj}x, 0, 0, 0)}{n^{2kj}}.$$

The mapping $A(x)$ *is defined by* $A(x) = \lim_{n\to\infty} \frac{f_a(n^{2kj}x)}{n^{2kj}}$, *for all* $x \in X$.

Example 3.5 Let $j = \pm 1$. Let $f_a : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with condition

$$\sum_{k=0}^{\infty} \frac{\alpha(2^{2kj}x_1, 2^{2kj}x_2, 2^{2kj}x_3, 2^{2kj}x_4, 2^{2kj}x_5)}{2^{2kj}},$$

converges in $\mathbb{R}$ and

$$\sum_{k=0}^{\infty} \frac{\beta(2^{2kj}x_1, 2^{2kj}x_2, 2^{2kj}x_3, 2^{2kj}x_4, 2^{2kj}x_5)}{2^{10kj}},$$

converges in $\mathbb{R}$ such that the inequalities (3.5) and (3.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$.

Then there exists a unique additive derivation $A : X \to X$ satisfying the functional equation (2.6) and for all $x \in X$

$$\| f_a(x) - A(x) \| \leq \frac{1}{16} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 2^{2kj}x, 0, 0, 0)}{n^{2kj}}.$$

The mapping $A(x)$ is defined by $A(x) = \lim_{n\to\infty} \frac{f_a(2^{2kj}x)}{2^{2kj}}$, for all $x \in X$.

Proposition 3.6 *Let* $j = \pm 1$. *Let* $f_a : X \to X$ *be a mapping for which there exist functions* $\alpha, \beta : X^5 \to [0, \infty)$ *with condition*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{3kj}x_1, n^{3kj}x_2, n^{3kj}x_3, n^{3kj}x_4, n^{3kj}x_5)}{n^{3kj}}$$

converges in $\mathbb{R}$ *and*

$$\sum_{k=0}^{\infty} \frac{\beta(n^{3kj}x_1, n^{3kj}x_2, n^{3kj}x_3, n^{3kj}x_4, n^{3kj}x_5)}{n^{15kj}}$$

converges in $\mathbb{R}$ *such that the inequalities (3.5) and (3.6), for all* $x_1, x_2, x_3, x_4, x_5 \in X$.

Then there exists a unique additive derivation $A : X \to X$ *satisfying the functional equation (2.6) and for all* $x \in X$

$$\| f_a(x) - A(x) \| \leq \frac{1}{4n^3} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 0, n^{3kj}x, 0, 0)}{n^{3kj}}.$$

The mapping $A(x)$ *is defined by* $A(x) = \lim_{n\to\infty} \frac{f_a(n^{3kj}x)}{n^{3kj}}$, *for all* $x \in X$.

Proposition 3.7 *Let* $j = \pm 1$. *Let* $f_a : X \to X$ *be a mapping for which there exist functions* $\alpha, \beta : X^5 \to [0, \infty)$ *with condition*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{4kj}x_1, n^{4kj}x_2, n^{4kj}x_3, n^{4kj}x_4, n^{4kj}x_5)}{n^{4kj}}$$

converges in $\mathbb{R}$ *and*

$$\sum_{k=0}^{\infty} \frac{\beta(n^{4kj}x_1, n^{4kj}x_2, n^{4kj}x_3, n^{4kj}x_4, n^{4kj}x_5)}{n^{20kj}}$$

converges in $\mathbb{R}$ *such that the functional inequalities (3.5) and (3.6), for all* $x_1, x_2, x_3, x_4, x_5 \in X$.

Then there exists a unique additive derivation mapping $A : X \to X$ *satisfying the functional equation (2.6) and for all* $x \in X$

$$\| f_a(x) - A(x) \| \leq \frac{1}{4n^4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 0, 0, n^{4kj}x, 0)}{n^{4kj}}.$$

The mapping $A(x)$ *is defined by* $A(x) = \lim_{n\to\infty} \frac{f_a(n^{4kj}x)}{n^{4kj}}$, *for all* $x \in X$.

Proposition 3.8 *Let* $j = \pm 1$. *Let* $f_a : X \to X$ *be a mapping for which there exist functions* $\alpha, \beta : X^5 \to [0, \infty)$ *with condition*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{5kj}x_1, n^{5kj}x_2, n^{5kj}x_3, n^{5kj}x_4, n^{5kj}x_5)}{n^{5kj}},$$

converges in $\mathbb{R}$ *and*

$$\sum_{k=0}^{\infty} \frac{\beta(n^{5kj}x_1, n^{5kj}x_2, n^{5kj}x_3, n^{5kj}x_4, n^{5kj}x_5)}{n^{25kj}},$$

converges in $\mathbb{R}$ *such that the functional inequalities (3.5) and (3.6), for all* $x_1, x_2, x_3, x_4, x_5 \in X$.

Then there exists a unique additive derivation $A : X \to X$ *satisfying the functional equation (2.6) and for all* $x \in X$

$$\| f_a(x) - A(x) \| \leq \frac{1}{4n^5} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 0, 0, 0, n^{5kj}x)}{n^{5kj}}.$$

The mapping $A(x)$ *is defined by* $A(x) = \lim_{n\to\infty} \frac{f_a(n^{5kj}x)}{n^{5kj}}$, *for all* $x \in X$.

The following corollaries are the immediate consequence of Theorem 3.2, Proposition 3.4–3.8 respectively, concerning the stability of (2.6).

Corollary 3.9 *Let* $f_a : X \to X$ *be a mapping and assume that there exist real numbers* ϵ *and* s *such that (2.33) and*

$$\begin{aligned} &\| f_a(x_1x_2x_3x_4x_5) - f_a(x_1)x_2x_3x_4x_5 - x_1 f_a(x_2)x_3x_4x_5 - x_1x_2 f_a(x_3)x_4x_5 \\ &-x_1x_2x_3 f_a(x_4)x_5 - x_1x_2x_3x_4 f_a(x_5) \| \\ &\leq \begin{cases} \epsilon, \\ \epsilon\left\{ \sum_{i=1}^{5} \| x_i \|^s \right\}, \quad s \neq 1 \\ \epsilon\left\{ \prod_{i=1}^{5} \| x_i \|^s + \sum_{i=1}^{5} \| x_i \|^{5s} \right\}, \quad s \neq \frac{1}{5}, \end{cases} \end{aligned} \tag{3.9}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. *Then there exists a unique additive derivation* $A : X \to Y$ *such that*

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{4|n-1|} \\ \frac{\epsilon\|x\|^s}{4|n-n^s|} \\ \frac{\epsilon\|x\|^{5s}}{4|n-n^{5s}|}, \end{cases} \tag{3.10}$$

for all $x \in X$.

Remark: Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s such that (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ such that

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{|4|} \\ \frac{\epsilon \|x\|^s}{4|2-2^s|} \\ \frac{\epsilon \|x\|^{5s}}{4|2-2^{5s}|}, \end{cases}$$

for all $x \in X$.

Corollary 3.10 *Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s such that (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ such that*

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{4|n^2-1|} \\ \frac{\epsilon \|x\|^s}{4|n^2-n^{2s}|} \\ \frac{\epsilon \|x\|^{5s}}{4|n^2-n^{10s}|}, \end{cases} \tag{3.11}$$

for all $x \in X$.

Remark: Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s such that (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ such that

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{|12|} \\ \frac{\epsilon \|x\|^s}{4|2^2-2^{2s}|} \\ \frac{\epsilon \|x\|^{5s}}{4|2^2-n^{10s}|}, \end{cases}$$

for all $x \in X$.

Corollary 3.11 *Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s such that (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ such that*

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{4|n^3-1|} \\ \frac{\epsilon \|x\|^s}{4|n^3-n^{3s}|} \\ \frac{\epsilon \|x\|^{5s}}{4|n^3-n^{15s}|}, \end{cases} \tag{3.12}$$

for all $x \in X$.

Remark: Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s such that (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ such that

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{|28|} \\ \frac{\epsilon \|x\|^s}{4|2^3-2^{3s}|} \\ \frac{\epsilon \|x\|^{5s}}{4|2^3-n^{15s}|}, \end{cases}$$

for all $x \in X$.

Corollary 3.12 *Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s such that (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ such that*

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{4|n^4-1|} \\ \frac{\epsilon \|x\|^s}{4|n^4-n^{4s}|} \\ \frac{\epsilon \|x\|^{5s}}{4|n^4-n^{20s}|}, \end{cases} \tag{3.13}$$

for all $x \in X$.

Corollary 3.13 *Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s such that (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ such that*

$$\| f_a(x) - A(x) \| \leq \begin{cases} \frac{\epsilon}{4|n^5-1|} \\ \frac{\epsilon \|x\|^s}{4|n^5-n^{5s}|} foll \\ \frac{\epsilon \|x\|^{5s}}{4|n^5-n^{25s}|}, \end{cases} \tag{3.14}$$

for all $x \in X$.

3.3 Additive Functional Equation: Odd Case—Fixed Point Method

Theorem 3.14 *Let $j = \pm 1$. Let $f_a : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with condition*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{kj}x_1, n^{kj}x_2, n^{kj}x_3, n^{kj}x_4, n^{kj}x_5)}{n^{kj}} \tag{3.15}$$

converges in $\mathbb{R}$ and

$$\sum_{k=0}^{\infty} \frac{\beta(n^{kj}x_1, n^{kj}x_2, n^{kj}x_3, n^{kj}x_4, n^{kj}x_5)}{n^{5kj}} \tag{3.16}$$

converges in $\mathbb{R}$, where $\eta_i = \begin{cases} n & i = 0; \\ \frac{1}{n} & i = 1; \end{cases}$ satisfying the functional inequalities

$$\| Df_a(x_1, x_2, x_3, x_4, x_5) \| \leq \alpha(x_1, x_2, x_3, x_4, x_5) \tag{3.17}$$

and

$$\| f_a(x_1x_2x_3x_4x_5) - f_a(x_1)x_2x_3x_4x_5 - x_1 f_a(x_2)x_3x_4x_5 - x_1x_2 f_a(x_3)x_4x_5$$
$$-x_1x_2x_3 f_a(x_4)x_5 - x_1x_2x_3x_4 f_a(x_5) \| \leq \beta(x_1, x_2, x_3, x_4, x_5), \tag{3.18}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. *Assume that there exists* $L = L(i) < 1$ *such that the function* $x \to \beta(x) = \frac{1}{4}\alpha\left(\frac{x}{n}, 0, 0, 0, 0\right)$ *has the property*

$$\frac{1}{\eta_i}\beta(\eta_i x) = L\beta(x). \tag{3.19}$$

Then there exists a unique additive derivative $A : X \to X$ *satisfying the functional equation (2.6) and*

$$\| f_a(x) - A(x) \| \leq \frac{L^{1-i}}{1-L}\beta(x), \tag{3.20}$$

for all $x \in X$.

Proof It follows from Theorem 2.14 that A is a unique additive mapping and satisfies (2.6), for all $x \in X$. It follows from (3.15), (3.16) and (3.18) that

$$\begin{aligned}
&\| A(x_1x_2x_3x_4x_5) - A(x_1)x_2x_3x_4x_5 - x_1 A(x_2)x_3x_4x_5 - x_1x_2 A(x_3)x_4x_5 \\
&-x_1x_2x_3 A(x_4)x_5 - x_1x_2x_3x_4 A(x_5) \| \\
&\leq \frac{1}{\eta_i^k}\Big|\Big| f_a(\eta_i^k(x_1x_2x_3x_4x_5)) - f_a(\eta_i^k x_1)(\eta_i^k x_2\eta_i^k x_3\eta_i^k x_4\eta_i^k x_5) \\
&\quad -\eta_i^k x_1 f_a(\eta_i^k x_2)\eta_i^k x_3\eta_i^k x_4\eta_i^k x_5 - \eta_i^k x_1\eta_i^k x_2 f_a(\eta_i^k x_3)\eta_i^k x_4\eta_i^k x_5 \\
&\quad -\eta_i^k x_1\eta_i^k x_2\eta_i^k x_3 f_a(\eta_i^k x_4)\eta_i^k x_5 - \eta_i^k x_1\eta_i^k x_2\eta_i^k x_3\eta_i^k x_4 f_a(\eta_i^k x_5)\Big|\Big|, \\
&\leq \frac{1}{\eta_i^k}\beta(\eta_i^k x_1, \eta_i^k x_2, \eta_i^k x_3, \eta_i^k x_4, \eta_i^k x_5) \ \to 0 \ \ as \ \ k \to \infty,
\end{aligned} \tag{3.21}$$

Thus the mapping $A : X \to X$ unique additive derivation satisfying (2.6). □

Example 3.15 Let $j = \pm 1$. Let $f_a : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with conditions (3.15) and (3.16), where $\eta_i = \begin{cases} 2\ if\ i = 0; \\ \frac{1}{2}\ if\ i = 1; \end{cases}$ satisfying the functional inequalities (3.17) and (3.18) for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists $L = L(i) < 1$ such that the function $x \to \beta(x) = \frac{1}{4}\alpha\left(\frac{x}{2}, 0, 0, 0, 0\right)$ has the property (3.19), for all $x \in X$. Then there exists a unique additive derivation $A : X \to X$ satisfying the functional equations (2.6) and (3.20), for all $x \in X$.

Proposition 3.16 *Let* $j = \pm 1$. *Let* $f_a : X \to X$ *be a mapping for which there exist functions* $\alpha, \beta : X^5 \to [0, \infty)$ *with conditions (3.15) and (3.16), where* $\eta_i =$

$\begin{cases} n^2 & i = 0; \\ \frac{1}{n^2} & i = 1; \end{cases}$ *satisfying the functional inequalities (3.17) and (3.18) for all* $x_1, x_2, x_3, x_4, x_5 \in X$. *Assume that there exists* $L = L(i) < 1$ *such that the function* $x \to \beta(x) = \frac{1}{4}\alpha\left(0, \frac{x}{n^2}, 0, 0, 0\right)$ *has the property (3.19), for all* $x \in X$. *Then there exists a unique additive derivation* $A : X \to X$ *satisfying the functional equations (2.6) and (3.20), for all* $x \in X$.

Example 3.17 Let $j = \pm 1$. Let $f_a : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with conditions (3.15) and (3.16), where $\eta_i = \begin{cases} 2^2 \; if \; i = 0; \\ \frac{1}{2^2} \; if \; i = 1; \end{cases}$ satisfying the functional inequalities (3.17) and (3.18) for all x_1, x_2, x_3,
$x_4, x_5 \in X$. Assume that there exists $L = L(i) < 1$ such that the function $x \to \beta(x) = \frac{1}{4}\alpha\left(0, \frac{x}{2^2}, 0, 0, 0\right)$ has the property (3.19), for all $x \in X$. Then there exists a unique additive derivation $A : X \to X$ satisfying the functional equations (2.6) and (3.20), for all $x \in X$.

Proposition 3.18 *Let* $j = \pm 1$. *Let* $f_a : X \to X$ *be a mapping for which there exist functions* $\alpha, \beta : X^5 \to [0, \infty)$ *with conditions (3.15) and (3.16), where* $\eta_i = \begin{cases} n^3 & i = 0; \\ \frac{1}{n^3} & i = 1; \end{cases}$ *satisfying the functional inequalities (3.17) and (3.18) for all* $x_1, x_2, x_3, x_4, x_5 \in X$. *Assume that there exists* $L = L(i) < 1$ *such that the function* $x \to \beta(x) = \frac{1}{4}\alpha\left(0, 0, \frac{x}{n^3}, 0, 0\right)$ *has the property (3.19), for all* $x \in X$. *Then there exists a unique additive derivation* $A : X \to X$ *satisfying the functional equations (2.6) and (3.20), for all* $x \in X$.

Proposition 3.19 *Let* $j = \pm 1$. *Let* $f_a : X \to X$ *be a mapping for which there exist functions* $\alpha, \beta : X^5 \to [0, \infty)$ *with conditions (3.15) and (3.16), where* $\eta_i = \begin{cases} n^4 & i = 0; \\ \frac{1}{n^4} & i = 1; \end{cases}$ *satisfying the functional inequalities (3.17) and (3.18), for all* $x_1, x_2, x_3, x_4, x_5 \in X$. *Assume that there exists* $L = L(i) < 1$ *such that the function* $x \to \beta(x) = \frac{1}{4}\alpha\left(0, 0, 0\frac{x}{n^4}, 0, \right)$ *has the property (3.19), for all* $x \in X$. *Then there exists a unique additive derivation* $A : X \to X$ *satisfying the functional equations (2.6) and (3.20), for all* $x \in X$.

Proposition 3.20 *Let* $j = \pm 1$. *Let* $f_a : X \to X$ *be a mapping for which there exist functions* $\alpha, \beta : X^5 \to [0, \infty)$ *with conditions (3.15) and (3.16), where* $\eta_i = \begin{cases} n^5 & i = 0; \\ \frac{1}{n^5} & i = 1; \end{cases}$ *satisfying the functional inequalities (3.17) and (3.18), for all* $x_1, x_2, x_3, x_4, x_5 \in X$. *Assume that there exists* $L = L(i) < 1$ *such that the function* $x \to \beta(x) = \frac{1}{4}\alpha\left(0, 0, 0, 0, \frac{x}{n^5}\right)$ *has the property (3.19), for all* $x \in X$. *Then there exists a unique additive derivation* $A : X \to X$ *satisfying the functional equations (2.6) and (3.20), for all* $x \in X$.

This following corollaries are the immediate consequences of Theorems 3.14–3.20 respectively, concerning the stability of (2.6).

Corollary 3.21 *Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s, such that the inequalities (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ satisfying the functional inequality (3.10), for all $x \in X$.*

Remark: Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s such that the inequalities (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ satisfying the functional inequality

$$\|f_a(x) - A(x)\| \leq \begin{cases} \frac{\epsilon}{4} \\ \frac{\epsilon\|x\|^s}{4|2-2^s|} \\ \frac{\epsilon\|x\|^{5s}}{4|2-2^{5s}|}, \end{cases}$$

for all $x \in X$.

Corollary 3.22 *Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s such that the inequalities (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ satisfying the functional inequality (3.11), for all $x \in X$.*

Remark: Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s such that the inequalities (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ satisfying the functional inequality

$$\|f_a(x) - A(x)\| \leq \begin{cases} \frac{\epsilon}{12} \\ \frac{\epsilon\|x\|^s}{4|2-2^{2s}|} \\ \frac{\epsilon\|x\|^{5s}}{4|2^2-2^{10s}|}, \end{cases}$$

for all $x \in X$.

Corollary 3.23 *Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s such that the inequalities (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ satisfying the functional inequality (3.12), for all $x \in X$.*

Remark: Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s such that the inequalities (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ satisfying the functional inequality

$$\|f_a(x) - A(x)\| \leq \begin{cases} \frac{\epsilon}{28} \\ \frac{\epsilon\|x\|^s}{4|2^3-2^{3s}|} \\ \frac{\epsilon\|x\|^{5s}}{4|2^3-2^{15s}|}, \end{cases}$$

for all $x \in X$.

Corollary 3.24 *Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s such that the inequalities (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ satisfying the functional inequality (3.13), for all $x \in X$.*

Corollary 3.25 *Let $f_a : X \to X$ be a mapping and assume that there exist real numbers ϵ and s such that the inequalities (2.33) and (3.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique additive derivation $A : X \to X$ satisfying the functional inequality (3.14), for all $x \in X$.*

Chapter 4
n-Dimensional Additive Functional Equations in Random 2-Normed Spaces

4.1 n-Dimensional Additive Functional Equation

Let V be a commutative group, W be a linear space, n be a positive integer and $k \in 0, 1, 2, \ldots, n$. A mapping $f : V^n \to W$ is called k-additive and $n - k$-quadratic (briefly, multi-additive quadratic) if f is a additive in each of some work k variables and is quadratic in each of the other variables, but one can obtain analog our results without this assumption. Let us note that for $k = n$ the above definition leads to the so called multi-additive mappings (some basic facts on such mappings can be found for instance in [57], where their application of polynomial functions is also presented), for $k = 0$ we obtain the notion of multi-quadratic function (see [8]), 1-additive and 1-quadratic mapping is just an additive-quadratic mapping defined by Park et al. [67].

A function $f : \mathbb{R} \to \mathbb{R}$ is called a biadditive function if it is additive in each variable, that is,

$$f(x + y, z) = f(x, z) + f(y, z),$$

and

$$f(x, y + z) = f(x, y) + f(x, z),$$

for all $x, y, z \in \mathbb{R}$. It is well known that every continuous biadditive function $f : \mathbb{R} \to \mathbb{R}$ is of the form

$$f(x, y) = mxy,$$

for all $x, y, z \in \mathbb{R}$, where m is a constant.

$$f\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} f\left(-x_j + \sum_{i=1; i \neq j}^{n} x_i\right) = (n-1) \sum_{i=1}^{n} f(x_i), \tag{4.1}$$

where n is a positive integer with $n \geq 3$.

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In this chapter, we discuss about the general solution and stability of the n-dimensional additive functional equation (4.1) in random 2-normed space with the help of direct and fixed point method.

4.2 Solutions of n-Dimensional Additive Functional Equation

Theorem 4.1 *Let X and Y be real vector spaces. Then a mapping $f : X \to Y$ satisfies the functional equation (2.1), for all $x, y \in X$ if and only if $f : X \to Y$ satisfies the functional equation (4.1), for all $x_1,\ x_2, \ldots, x_n \in X$.*

Proof Assume that $f : X \to Y$ satisfies the functional equation (2.1). Letting $x = y = 0$ in (2.1), we get $f(0) = 0$. Replacing y by $-x$ in (2.1), we get $f(-x) = -f(x)$, for all $x \in X$. Hence f is an odd mapping. Replacing (x, y) by (x, x) and $(2x, x)$ in (2.1), we have

$$f(2x) = 2f(x) \ \text{ and } \ f(3x) = 3f(x), \tag{4.2}$$

for all $x \in X$. In general in any positive integer a, we obtain

$$f(ax) = af(x), \tag{4.3}$$

for all $x \in X$. Replacing x by $\frac{x}{a}$ in (4.3), we get $f(\frac{x}{a}) = \frac{1}{a}$, for all $x \in X$. One can easy to verify from (4.2) that

$$f(x_1 + x_2 + \cdots + x_n) = f(x_1) + f(x_2) + \cdots + f(x_n), \tag{4.4}$$

for all $x_1,\ x_2, \ldots, x_n \in X$. Replacing x_1 by $-x_1$, and using oddness of f in (4.4), we obtain

$$f(-x_1 + x_2 + \cdots + x_n) = -f(x_1) + f(x_2) + \cdots + f(x_n), \tag{4.5}$$

for all $x_1,\ x_2, \ldots, x_n \in X$,. Again setting x_2 by $-x_2$, and using oddness of f in (4.4), we obtain

$$f(x_1 - x_2 + \cdots + x_n) = f(x_1) - f(x_2) + \cdots + f(x_n), \tag{4.6}$$

for all $x_1,\ x_2, \ldots, x_n \in X$. Replacing x_n by $-x_n$, and using oddness of f in (4.4), we obtain

$$f(x_1 + x_2 + \cdots - x_n) = f(x_1) + f(x_2) + \cdots - f(x_n), \tag{4.7}$$

for all $x_1,\ x_2, \ldots, x_n \in X$. Adding the above n equations, we obtain

$$f(x_1 + x_2 + \cdots + x_n) + f(-x_1 + x_2 + \cdots + x_n) + \ldots f(x_1 + x_2 + \cdots - x_n) \\ = (n-1)\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right].$$

Conversely, let $f : X \to Y$ satisfies the functional equation (4.1). Replacing $(x_1, x_2, \ldots, x_n)$ by $(0, 0, \ldots, 0)$ in (4.1), we obtain $f(0) = 0$. Now replacing $(x_1, x_2, \ldots, x_n)$ by $(x, x, \ldots, 0)$ in (4.1), we get the desired result. □

4.3 n-Dimensional Additive Functional Equation—Direct Method

Definition 4.2 A distribution function is an element of Δ^+, where $\Delta^+ = \{f : \mathbb{R} \to [0, 1];\ f$ is left-continuous, non decreasing, $f(0) = 0$ and $f(+\infty)\}$ and the subset $D^+ \subset \Delta^+$ is the set

$$D^+ = \left\{f \in \Delta^+; \tau f(+\infty) = 1\right\}.$$

Here, $\tau f(+\infty)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, i.e., $f \le g$ if and only if $f(x) \le g(x)$, for all $x \in \mathbb{R}$. For any $a \in \mathbb{R}$, H_a is a distribution function defined by

$$H_a(x) = \begin{cases} 0 \; if \;\; x \le a \\ 1 \; if \;\; x > a. \end{cases}$$

The set Δ, as well as its subsets, can be partially ordered by the usual pointwise order: in this order, H_0 is the maximal element in Δ^+. A triangle function is a binary operation on Δ^+, namely, a function $\mu : \Delta^+ \times \Delta^+ \to \Delta^+$ is associative, commutative, non decreasing and which has ϵ as unit, that is, for all $f, g, h \in \Delta^+$, we obtain:

(i) $\mu(\mu(f, g), h) = \mu(f, \mu(g, h))$,
(ii) $\mu(f, g) = \mu(g, f)$
(iii) $\mu(f, g) = \mu(g, f)$ whenever $f \le g$,
(iv) $\mu(f, H_0) = f$.

A $t-$norm is continuous mapping $* : [0, 1] \times [0, 1] \to [0, 1]$ such that $([0, 1], a)$ is abelian monoid with unit one and $c * d \ge a * b$ if $c \ge a$ and $d \ge b$, for all $a, b, c, d \in [0, 1]$.

Example 4.3 Let $(X, \|., .\|)$ be a 2-normed space with $\|x, z\| = \|x_1 z_2 - x_2 z_1\|$, $x = (x_1, x_2)$, $z = (z_1, z_2)$ and $a * b = ab$ for $a, b \in [0, 1]$. For all $x \in X$, $t > 0$ and non zero $z \in X$, consider

$$F_{x,z}(t) = \begin{cases} \frac{t}{t+\|x,z\|} & if \quad t > 0 \\ 0 & if \quad t \leq 0. \end{cases}$$

Then $(X, F, *)$ is a random 2-normed space.

Definition 4.4 Let X be a linear space dimension greater than 1. Suppose $||., .||$ is a real-valued function on $X \times X$ satisfying the following conditions:

i $||x, y|| = 0$ if and only if x, y are linearly depended vectors,
ii $||x, y|| = ||y, x||$, for all $x, y \in X$.
iii $||\lambda x, y|| = |\lambda| \; ||y, x||$, for all $\lambda \in \mathbb{R}$, for all $x, y \in X$.
iv $||x + y, z|| = ||x, y|| + ||y + z||$, for all $x, y, z \in X$.

Then $||., .||$ is called a 2-norm linear space. Some of the basic properties of 2-norm are that they are non-negative and $||x, y + \lambda x|| = ||x, y||$, for all $\lambda \in \mathbb{R}$ and for all $x, y \in X$.

Remark: Every 2-normed space $(X, \|., .\|)$ can be made a random 2-normed space in natural way, by setting $F_{x,y}(t) = H_0(t - \|x, y\|)$, for every $x, y \in X, t > 0$ and $a * b = \min\{a, b\}, a * b = \min\{a, b\}, a, b \in [0, 1]$.

Definition 4.5 A sequence $x = (x_k)$ is convergent in $(X, F, *)$ or simply F-convergent to l if for every $\epsilon > 0$ and $\theta \in (0, 1)$ there exists $k_0 \in N$ such that $F_{x_k - l, z}(\epsilon) > 1 - \theta$ whenever $k \geq k_0$ and non zero $z \in X$. In this case, we write $F - \lim_{k \to \infty} x_k = l$ and l is called the F-limit of $x = (x_k)$.

Remark: Let $(X, \|.\|)$ be a normed linear space. Then

$$N(x, t) = \begin{cases} \frac{t}{t+\|x\|}, t > 0, & for \;\; all \;\; x \in X \\ 0, \quad\;\; t \leq 0, & for \;\; all \;\; x \in X, \end{cases}$$

is a fuzzy norm on X.

Definition 4.6 A sequence $x = x_k$ is said to be a Cauchy sequence in $(X, F, *)$ if for every $\epsilon > 0, \;\; \theta > 0$ and non-zero $z \in X$, there exist a number $N = N(\epsilon, z)$ such that $\lim F_{x_n - x_m, z}(\epsilon) > 1 - \theta$, for all $n, m \geq N$. RTN-space $(X, F, *)$ is said to be complete if every F-Cauchy is F-convergent. In this case, $(X, F, *)$ is called random 2-Banach space.

Note: All through this part, let X be a linear space Y, μ, T be a complete RN-space.
Let us define a mapping $Df : X \to Y$ by

$$Df(x_1, x_2, \ldots, x_n) = f\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} f\left(-x_j + \sum_{i=1; i \neq j}^{n} x_i\right) - (n-1) \sum_{i=1}^{n} f(x_i),$$

for all $x_1, x_2, \ldots, x_n \in X$.

Theorem 4.7 *Let $j = \pm 1$, $f : X \to Y$ be a mapping for which there exists a function with the condition $\eta : X^n \to D^+$*

$$\lim_{k\to\infty} T_{i=0}^{\infty}\left(\eta_{\left(3^{(k+i)x_1},3^{(k+i)x_2},3^{(k+i)x_3}\ldots,3^{(k+i)x_n}\right),z}\left(3^{(k+i+1)j}t\right)\right) = 1, \tag{4.8}$$

$$= \lim_{k\to\infty}\eta_{(3^{kj}x_1,3^{kj}x_2,3^{kj}x_3,\ldots,3^{kj}x_n),z}\left(3^{kj}t\right), \tag{4.9}$$

satisfying $f(0) = 0$ and

$$\mu_{Df(x_1,x_2,\ldots,x_n),z}(t) \geq \eta_{\left(x_1,x_2,\ldots,x_n\right),z}(t), \tag{4.10}$$

for all $x_1, x_2, \ldots, x_n \in X$ and all $t > 0$. Then there exists an additive mapping $C : X \to Y$ satisfying the functional equation (4.1) and

$$\mu_{C(x)-f(x),z}(t) \geq T_{i=0}^{\infty}\left(\eta_{\left(3^{(i+1)j}x,3^{(i+1)j}x,3^{(i+1)j}x,x0\ldots,0\right),z}\left(3^{(k+i+1)j}t\right)\right), \tag{4.11}$$

for all $x \in X$ and all $t > 0$. The mapping $C(x)$ is defined by

$$\mu_{C(x),z}(t) = \lim_{k\to\infty}\mu_{\frac{f\left(3^{kj}x\right)}{3^{kj}},z}(t), \tag{4.12}$$

for all $x \in X$ and all $t > 0$.

Proof Assume that $j = 1$. Replacing $(x_1, x_2, x_3, x_4 \ldots, x_n)$ by $(x, x, x, 0, \ldots, 0)$ in (4.8), we have

$$\mu_{(n-2)f(3x)-3(n-2)f(x),z}(t) \geq \eta_{(x,x,x,0,\ldots,0),z}(t), \tag{4.13}$$

for all $x \in X$ and all $t > 0$. It follows from (4.12) and (RN2), that

$$\mu_{\frac{f(3x)}{3}-f(x),z}(t) \geq \eta_{\left(x,x,x,0,\ldots,0\right),z}\left(3(n-2)3t\right), \tag{4.14}$$

for all $x \in X$ and all $t > 0$. Replacing x by $3^k x$ in (4.13), we obtain

$$\begin{aligned}\mu_{\frac{f(3^{k+1}x)}{3^{(k+1)}}-\frac{\mu_{f(3^kx)}}{3^k},z}(t) &\geq \eta_{(3^kx,3^kx,3^kx,x,0,\ldots,0),z}(3^k(n-2)3t)\\ &\geq \eta_{(x,x,x,0,\ldots,0),z}\left(\frac{3^k(n-2)3}{\alpha^k}t\right),\end{aligned} \tag{4.15}$$

for all $x \in X$ and all $t > 0$. It follows from

$$\frac{f(3^n x)}{3^n} - f(x) = \sum_{k=0}^{n-1}\frac{f(3^{k+1}x)}{3^{k+1}} - \frac{f(3^k x)}{3^k}$$

and (4.15) that

$$\mu_{\frac{f(3^n x)}{3^n}-f(x),z}\left(t\sum_{k=0}^{n-1}\frac{\alpha^k}{3^k(n-2)3}\right) \geq T_{k=0}^{n-1}\eta_{(x,x,x,0,\ldots,0),z}(t), \\ = \eta_{(x,x,x,0,\ldots,0),z}(t), \tag{4.16}$$

$$\mu_{\frac{f(3^n x)}{3^n}-f(x),z}(t) \geq \eta_{(x,x,x,0,\ldots,0),z}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{3^k(n-2)3}}\right), \tag{4.17}$$

for all $x \in X$ and all $t > 0$. Replacing x by $3^m x$ in (4.17), we obtain

$$\mu_{\frac{f(3^{n+m}x)}{3^{n+m}}-\frac{f(3^m x)}{3^m},z}(t) \geq \eta_{(x,x,x,0,\ldots,0),z}\left(\frac{t}{\sum_{k=m}^{m+n}\frac{\alpha^k}{3^k(n-2)3}}\right), \tag{4.18}$$

Since $\eta_{(x,x,x,0,\ldots,0),z}\left(\frac{t}{\sum_{k=m}^{m+n}\frac{\alpha^k}{3^k(n-2)3}}\right) \to 1$ as $m, n \to \infty$, $\left\{\frac{f(3^n x)}{3^n}\right\}$ is a Cauchy sequence in (Y, μ, T). Since (Y, μ, T) is a complete random 2-normed space, this sequence converges to some point $C(x) \in Y$. Fixing $x \in X$ and putting $m = 0$ in (4.18), we have

$$\mu_{\frac{f(3^n x)}{3^n}-f(x),z}(t) \geq \eta_{(x,x,x,0,\ldots,0),z}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{3^k(n-2)3}}\right), \tag{4.19}$$

and so, for every $\delta > 0$, we get

$$\mu_{C(x)-f(x),z}(t+\delta) \geq T\left(\mu_{C(x)-\frac{f(3^n x)}{3^n},z}(\delta), \mu_{\frac{f(3^n x)}{3^n}-f(x),z}(t)\right) \\ \geq T\left(\mu_{C(x)-\frac{f(3^n x)}{3^n},z}(\delta), \eta_{(x,x,x,0,\ldots,0),z}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{3^k(n-2)3}}\right)\right), \tag{4.20}$$

for all $x \in X$, $t > 0$. Taking the limit as $n \to \infty$ and using (4.20), we have

$$\mu_{C(x)-f(x),z}(t+\delta) \geq \eta_{(x,x,x,0,\ldots,0),z}((n-2)(3-\alpha)t), \tag{4.21}$$

for all $x \in X$, $t > 0$. Since δ was arbitrary, by taking $\delta \to 0$ in (4.21), we have

$$\mu_{C(x)-f(x),z}(t) \geq \eta_{(x,x,x,0,\ldots,0),z}((n-2)(3-\alpha)t), \tag{4.22}$$

for all $x \in X, t > 0$. Replacing $x_1, x_2, \ldots, x_n \in X$ by $(3^n x_1, 3^n x_2, \ldots, 3^n x_n)$ in (4.10) respectively, we acquire

$$\mu_{Df(3^n x_1, 3^n x_2, \ldots, 3^n x_n),z}\big(t\big) \geq \eta_{\left(3^n x_1, 3^n x_2, \ldots, 3^n x_n\right),z}\big(3^n t\big), \tag{4.23}$$

for all $x_1, x_2, \ldots, x_n \in X$ and for all $t > 0$. So

$$\lim_{k\to\infty} T_{i=0}^{\infty}\left(\eta_{\left(3^{(k+i)}x_1, 3^{(k+i)}x_2, \ldots, 3^{(k+i)}x_n\right),z}\big(3^{(k+i+1)j}t\big)\right) = 1.$$

We conclude that C fulfills (4.1). To prove the uniqueness of the additive mapping C, assume that there exists another additive mapping D from X to Y, which satisfies (4.22). Fix $x \in X$. Clearly, $C(3^n x) = 3^n C(x)$ and $D(3^n x) = 3^n D(x)$, for all $x \in X$. It follows from (4.22) that

$$\mu_{C(x)-D(x),z}\big(t\big) = \lim_{n\to\infty} \mu_{\frac{C(3^n x)}{3^n} - \frac{D(3^n x)}{3^n},z}\big(t\big)$$

$$\begin{aligned}
\mu_{\frac{C(3^n x)}{3^n} - \frac{D(3^n x)}{3^n},z}\big(t\big) &\geq \min\left\{\mu_{\frac{C(3^n x)}{3^n} - \frac{f(3^n x)}{3^n},z}\Big(\frac{t}{2}\Big), \mu_{\frac{D(3^n x)}{3^n} - \frac{f(3^n x)}{3^n},z}\Big(\frac{t}{2}\Big)\right\}, \\
&\geq \eta_{\left(3^n x, 3^n x, 3^n x, 0\ldots,0\right),z}\big(3^n(n-2)(3-\alpha)t\big) \\
&\geq \eta_{\left(x,x,x,0,\ldots,0\right),z}\left(\frac{3^n(n-2)(3-\alpha)t}{\alpha^n}\right).
\end{aligned}$$

Since $\lim_{n\to\infty}\left(\frac{3^n(n-2)(3-\alpha)t}{\alpha^n}\right) = \infty$, we get $\lim_{n\to\infty} \eta_{\left(x,x,x,0,\ldots,0\right),z}\left(\frac{3^n(n-2)(3-\alpha)t}{\alpha^n}\right) = 1$. Thus $\mu_{C(x)-f(x),z}\big(t\big) = 1$, for all $t > 0$ and so $C(x) = D(x)$. This completes the proof. □

Corollary 4.8 *Let ϵ and s be non-negative real numbers. Let an additive mapping $f : X \to Y$ satisfies the inequality*

$$\mu_{Df(x_1,x_2,x_3,\ldots,x_n),z^{(t)}} \geq \begin{cases} \eta_{\epsilon,z}(t) \\ \eta_{\epsilon}\sum_{i=1}^{n}||x_i||^s(t), \quad s \neq 1 \\ \eta_{\epsilon}(\prod_{i=1}^{n}||x_i||^s + \sum_{i=1}^{n}||x_i||^{ns})(t), \quad s \neq \frac{1}{n} \end{cases} \tag{4.24}$$

for all $x_1, x_2, x_3, \ldots, x_n \in X$ and all $t > 0$. Then there exists a unique additive mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \begin{cases} \eta_{\frac{\epsilon}{(n-2)|2|},z}(t) \\ \eta_{\frac{\epsilon||x||^s}{(n-2)|3-3^s|},z}(t) \\ \eta_{\frac{\epsilon||x||^{ns}}{(n-2)|3-3^{ns}|},z}(t) \end{cases} \tag{4.25}$$

for all $x \in X$ and all $t > 0$.

4.4 *n*-Dimensional Additive Functional Equation—Fixed Point Method

Theorem 4.9 *Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^n \to D^+$ with the condition*

$$\lim_{k\to\infty} \eta_{\left(\delta_i^k x_1, \delta_i^k x_2, \delta_i^k x_3, \ldots, \delta_i^k x_n\right), z}\left(\delta_i^k t\right) = 1, \tag{4.26}$$

for all $x_1, x_2, x_3, \ldots, x_n \in X$ and all $t > 0$, where

$$\delta_i = \begin{cases} 3, & i = 0; \\ \frac{1}{3} & i = 1; \end{cases}$$

satisfying the functional inequality

$$\mu_{Df}(x_1, x_2, x_3, \ldots, x_n)(t) \geq \eta_{(x_1,x_2,x_3,\ldots,x_n),z}\left(t\right), \tag{4.27}$$

for all $x_1, x_2, x_3, \ldots, x_n \in X$ and all $t > 0$. If there exists $L = L(i)$ such that the function

$$x \to \beta(x, t) = \eta_{\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}, 0, \ldots, 0\right), z}((n-2)t), \tag{4.28}$$

has the property

$$\beta(x, t) \leq L\frac{1}{\delta_i}\beta(\delta_i x, t), \tag{4.29}$$

for all $x \in X$ and $t > 0$, then there exists a unique additive mapping $C : X \to Y$ satisfying the functional equation (4.1) and

$$\mu_{C(x)-f(x),z}\left(\frac{L^{1-i}}{1-L}t\right) \geq \beta(x, t), \tag{4.30}$$

for all $x \in X$ and $t > 0$.

Proof Let us consider the set $\Omega = \{p | p : U^2 \to V,\ p(0) = 0\}$ and d be a generalized metric on Ω, such that

$$d(g, h) = \inf\{k \in (0, \infty)/\mu_{g(x)-h(x),z}(kt) \geq \beta(x, t), x \in X, t > 0\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tg(x), z = \frac{1}{\delta_i}(\delta_i x), z$, for all $x \in X$. Now for $g, h \in \Omega$, we have $d(g, h) \leq K$.

$$\mu_{g(x)-h(x),z}(kt) \geq \beta(x, t),$$

$$\Longrightarrow \mu_{(Tg(x)-Th(x)),z}\left(\frac{Kt}{\delta_i^3}\right) \geq \beta(x,t),$$

$$\Longrightarrow d(Tg(x),Th(x)) \leq KL,$$

$$\Longrightarrow d(Tg,Th) \leq Ld(g,h), \tag{4.31}$$

for all $g, h \in \Omega$. Therefore T is strictly contractive mapping on ω with Lipschitz constant L. It follows from (4.13) that

$$\mu_{(n-2)f(3x)-3(n-2)f(x),z}(t) \geq \eta_{(x,x,x,0,\ldots,0),z}(t), \tag{4.32}$$

for all $x \in X$. It follows from (4.32) that

$$\mu_{\frac{f(3x)}{3}-f(x),z}(t) \geq \eta_{(x,x,x,0,\ldots,0),z}\left(\big((n-2)\big)3t\right), \tag{4.33}$$

for all $x \in X$. Using (4.29) for the case $i = 0$, we get

$$\mu_{\frac{f(3x)}{3}-f(x),z}(t) \geq L\beta(x,t),$$

for all $x \in X$. Hence, we obtain

$$d\left(\mu_{Tf(x)-f(x),z}\right) \geq L = L^{l-i} < \infty, \tag{4.34}$$

for all $x \in X$. Replacing x by $\frac{x}{3}$ in (4.33), we get

$$\mu_{\frac{f(x)}{3}-f(\frac{x}{3},z)}(t) \geq \eta_{\left(\frac{x}{3},\frac{x}{3},\frac{x}{3},0,\ldots,0\right)}\big((n-2)3t\big), \tag{4.35}$$

for all $x \in X$. Using (4.29) for the case $i = 1$, we get

$$\mu_{3f(\frac{x}{3})-f(x),z}(t) \geq \beta(x,t) \implies \mu_{Tf(x)-f(x),z}(t) \geq \beta(x,t),$$

for all $x \in X$. Hence we get

$$d\big(\mu_{Tf(x)-f(x),z}\big) \geq L = L^{l-i} < \infty, \tag{4.36}$$

for all $x \in X$. From (4.34) and (4.36), we can conclude

$$d\big(\mu_{Tf(x)-f(x),z}\big) \geq L = L^{l-i} < \infty, \tag{4.37}$$

for all $x \in X$. In order to prove $C : X \to Y$ satisfies the functional equation (4.1), the remaining proof is similar to the proof of Theorem 12.6. Thus C is unique fixed point of T in the set $\Delta = f \in \Omega/d(f, C) < \infty$. Thus, C is a unique additive mapping such that

$$\mu_{f(x)-C(x),z}\left(\frac{L^{1-i}}{1-L}t\right) \geq \beta(x,t),$$

for all $x \in X$ and $t > 0$. This completes the proof of the theorem. □

Remark: Let $(X, \|.\|)$ be a normed linear space. We define a function N by

$$N(x,t) = \begin{cases} \frac{t^2-\|x\|^2}{t^2+\|x\|^2}, if & t > \|x\| \\ 0, \quad\quad if & t \leq \|x\|. \end{cases}$$

Then N defines a fuzzy norm on X.

Corollary 4.10 *If a mapping $f : X \to Y$ satisfies the inequality (4.24), for all $x_1, x_2, x_3, \ldots, x_n \in X$ and all $t > 0$, where ϵ, s are constants with $\epsilon > 0$, then there exists a unique cubic additive mapping $C : X \to Y$, such that (4.25), for all $x \in X$ and all $t > 0$.*

Proof Set

$$\mu_{Df(x_1,x_2,x_3,\ldots,x_n),z}(t) \geq \begin{cases} \eta_{\epsilon,z}(t) \\ \eta_{\epsilon \sum_{i=1}^n ||x_i||^s(t),z}(t) \\ \eta_{\epsilon\left(\prod_{i=1}^n ||x_i||^s + \sum_{i=1}^n ||x_i||^{ns}\right),z}(t), \end{cases}$$

for all $x_1, x_2, x_3, \ldots, x_n \in X$ and all $t > 0$. Then

$$\eta_{\left(\delta_i^k x_1, \delta_i^k x_2, \delta_i^k x_3, \ldots, \delta_i^k x_n\right),z}\left(\delta_i^k t\right) = \begin{cases} \eta_{\epsilon\delta_i^k,z}(t) \\ \eta_{\epsilon \sum_{i=1}^n ||x_i||^s \delta_i^{(1-s)k},z}(t), \\ \eta_{\epsilon\left(\prod_{i=1}^n ||x_i||^s \delta_i^{(1-s)k} + \sum_{i=1}^n ||x_i||^{ns} \delta_i^{(1-ns)k}\right),z}(t), \end{cases}$$

$$= \begin{cases} \to 1 \;\; as \;\; k \to \infty \\ \to 1 \;\; as \;\; k \to \infty \\ \to 1 \;\; as \;\; k \to \infty. \end{cases}$$

But we have that $\beta(x,t) = \eta_{(\frac{x}{3},\frac{x}{3},0,\ldots,0)}((n-2)t)$ has the property $L\frac{1}{\delta_i}\beta(\delta_i x,t)$, for all $x \in X$ and $t > 0$. Now

$$\beta(x,t) = \left\{\begin{matrix} \eta_{\frac{\epsilon}{3(n-2)},z}(t) \\ \eta_{\frac{\epsilon||x||^s}{3^s(n-2)},z}(t) \\ \eta_{\frac{\epsilon||x||^{ns}}{3^{ns}(n-2)},z}(t), \end{matrix}\right\},\ L\frac{1}{\delta_i}\beta(\delta_i x,t) = \left\{\begin{matrix} \eta_{\delta_i^{-1}\beta(x)}(t) \\ \eta_{\delta_i^{s-1}\beta(x)}(t) \\ \eta_{\delta_i^{ns-1}\beta(x)}(t) \end{matrix}\right\}.$$

By Proposition 12.8, we prove the following six cases:

$L = 3^{-1}$ if $i = 0$ and $L = 3$ if $i = 1$.
$L = 3^{s-1}$ for $s < 1$ if $i = 0$ and $L = 3^{1-s}$ for $s > 1$ if $i = 1$.
$L = 3^{ns-1}$ for $s < \frac{1}{n}$ if $i = 0$ and $L = 3^{1-ns}$ for $s > \frac{1}{n}$ if $i = 1$.

Case: 1 $L = 3^{-1}$ if $i = 0$

$$\mu_{f(x)-C(x),z}(t) \geq L\frac{1}{\delta_i}\beta(\delta_i x,t)(t) \geq \eta_{\left(\frac{\epsilon}{(n-2)(2)}\right),z}(t).$$

Case: 2 $L = 3$ if $i = 1$

$$\mu_{f(x)-C(x),z}(t) \geq L\frac{1}{\delta_i}\beta(\delta_i x,t)(t) \geq \eta_{\left(\frac{\epsilon}{(n-2)(2)}\right),z}(t).$$

Case: 3 $L = 3^{s-1}$ for $s < 1$ if $i = 0$

$$\mu_{f(x)-C(x),z}(t) \geq L\frac{1}{\delta_i}\beta(\delta_i x,t)(t) \geq \eta_{\left(\frac{\epsilon||x||^s}{(n-2)(3-3^s)}\right),z}(t).$$

Case: 4 $L = 3^{1-s}$ for $s > 1$ if $i = 0$

$$\mu_{f(x)-C(x),z}(t) \geq L\frac{1}{\delta_i}\beta(\delta_i x,t)(t) \geq \eta_{\left(\frac{\epsilon||x||^s}{(n-2)(3^s-3)}\right),z}(t).$$

Case: 5 $L = 3^{ns-1}$ for $s < \frac{1}{n}$ if $i = 0$

$$\mu_{f(x)-C(x),z}(t) \geq L\frac{1}{\delta_i}\beta(\delta_i x,t)(t) \geq \eta_{\left(\frac{\epsilon||x||^{ns}}{(n-2)(3-3^{ns})}\right),z}(t).$$

Case: 6 $L = 3^{1-ns}$ for $s > \frac{1}{n}$ if $i = 1$

$$\mu_{f(x)-C(x),z}(t) \geq L\frac{1}{\delta_i}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\epsilon||x||^{ns}}{(n-2)(3^{ns}-3)}\right),z}(t).$$

Hence the proof is complete. □

Chapter 5
Quadratic Functional Equations

5.1 Quadratic Functional Equation

The next famous functional equation in the field of stability of functional equations is the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{5.1}$$

The solution $f(x) = x^2$ of the functional equation (5.1) is the function. Hence, it is called the quadratic functional equation (or) "Euler-Lagrange functional equation", introduced by Rassias [79].

Note that this equation is sometimes called the "Euler-Lagrange-Rassias functional equation". Every solution of the quadratic functional equation (5.1) is called a quadratic function [9, 20, 48, 49].

A mapping $f : E_1 \to E_2$ between two vector spaces is quadratic.

The solution and stability of following quadratic functional equations

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(x+z) \tag{5.2}$$

$$f(x-y-z) + f(x) + f(y) + f(z) = f(x-y) + f(y+z) + f(z-x) \tag{5.3}$$

$$f(x+y+z) + f(x-y) + f(y-z) + f(z-x) = 3f(x) + 3f(y) + 3f(z) \tag{5.4}$$

$$f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1 \le i < j \le n} f(x_i - x_j) = n \sum_{i=1}^{n} f(x_i), \ (n \ge 2). \tag{5.5}$$

was investigated by Kannappan [52], Jung [48, 49]. The Eq. (2.6) is an quadratic functional equation.

In this chapter, we discuss about the general solution and stability of the quadratic functional equation (2.6) for even case in Banach space with the help of direct and fixed point methods (see [3, 14, 19, 54, 81, 87]).

H. Dutta et al., *Stability of Some Advanced Functional Equations in Various Spaces*, Studies in Systems, Decision and Control 479,
https://doi.org/10.1007/978-3-031-33704-8_5

5.2 General Solution of Quadratic Functional Equation

Theorem 5.1 *An even mapping $f : X \to Y$ satisfies the functional equation (5.1) for all $x, y \in X$ if and only if $f : X \to Y$ satisfies the functional equation (2.6), for all x_1, x_2, x_3, x_4, x_5.*

Proof Let $f : X \to Y$ satisfies the functional equation (5.1). Setting $x = y = 0$ in (5.1), we get $f(0) = 0$. Replacing y *by* x and y *by* $2x$ in (5.1), we obtain

$$f(2x) = 4f(x) \textit{ and } f(3x) = 9f(x), \tag{5.6}$$

for all $x \in X$. In general for any positive integer b, we get

$$f(bx) = b^2 f(x), \tag{5.7}$$

for all $x \in X$. It easy to verify from (5.7) that

$$f(b^2x) = b^4 f(x) \textit{ and } f(b^3x) = b^6 f(x), \tag{5.8}$$

for all $x \in X$. Replacing (x, y) by $(n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5, nx_1)$ in (5.1), we get

$$\begin{aligned} &f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(-nx_1 + n^2x_2 + n^3x_3 \\ &+n^4x_4 + n^5x_5) = 2f(n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + 2f(nx_1). \end{aligned} \tag{5.9}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Replacing (x, y) by $(nx_1 + n^4x_4 + n^5x_5, n^2x_2 - n^3x_3)$ in (5.1), we obtain

$$\begin{aligned} &f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 - n^2x_2 + n^3x_3 \\ &+n^4x_4 + n^5x_5) = 2f(nx_1 + n^4x_4 + n^5x_5) + 2f(n^2x_2 - n^3x_3), \end{aligned} \tag{5.10}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Replacing (x, y) by $(nx_1 + n^2x_2 + n^3x_3, n^4x_4 - n^5x_5)$ in (5.1), we obtain

$$\begin{aligned} &f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 - n^5x_5) + f(nx_1 + n^2x_2 \\ &+n^3x_3 - n^4x_4 + n^5x_5) = 2f(nx_1 + n^2x_2 + n^3x_3) \\ &+2f(n^4x_4 - n^5x_5), \end{aligned} \tag{5.11}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Adding Eqs. (5.9), (5.10) and (5.11), we get

$$\begin{aligned} f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(-nx_1 + n^2x_2 + n^3x_3 \\ +n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 \\ -n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 - n^5x_5) \\ + f(nx_1 + n^2x_2 + n^3x_3 - n^4x_4 + n^5x_5) = 2f(nx_1) + 2f(n^2x_2 + n^3x_3 \\ +n^4x_4 + n^5x_5) + 2f(nx_1 + n^4x_4 + n^5x_5) + 2f(n^2x_2 - n^3x_3) + 2f(nx_1 \\ +n^2x_2 + n^3x_3) + 2f(n^4x_4 - n^5x_5), \end{aligned} \tag{5.12}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Replacing (x, y) by $(n^2x_2 - n^3x_3, n^4x_4 - n^5x_5)$ in (5.1), we get

$$\begin{aligned} f(n^2x_2 - n^3x_3 + n^4x_4 - n^5x_5) + f(n^2x_2 - n^3x_3 - n^4x_4 + n^5x_5) \\ = 2f(n^2x_2 - n^3x_3) + 2f(n^4x_4 - n^5x_5), \end{aligned} \tag{5.13}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Using (5.13) in (5.12), we have

$$\begin{aligned} f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(-nx_1 + n^2x_2 + n^3x_3 \\ +n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 \\ -n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 - n^5x_5) \\ + f(nx_1 + n^2x_2 + n^3x_3 - n^4x_4 + n^5x_5) = 2f(nx_1) + 2f(n^2x_2 + n^3x_3 \\ +n^4x_4 + n^5x_5) + 2f(nx_1 + n^4x_4 + n^5x_5) + 2f(nx_1 + n^2x_2 + n^3x_3) \\ + f(n^2x_2 - n^3x_3 + n^4x_4 - n^5x_5) + f(n^2x_2 - n^3x_3 \\ -n^4x_4 + n^5x_5), \end{aligned} \tag{5.14}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Adding $2f(n^2x_2)$ on both sides of (5.14), we obtain

$$\begin{aligned} f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4 \\ +n^5x_5) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 - n^2x_2 + n^3x_3 \\ + + n^4x_4 + n^5x_5) f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 - n^5x_5) + f(nx_1 + n^2x_2 \\ +n^3x_3 - n^4x_4 + n^5x_5) = 2f(nx_1) + 2f(n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) \\ +2f(nx_1 + n^4x_4 + n^5x_5) + 2f(n^2x_2) + 2f(nx_1 + n^2x_2 + n^3x_3) \\ + f(n^2x_2 - n^3x_3 + n^4x_4 - n^5x_5) + f(n^2x_2 - n^3x_3 - n^4x_4 + n^5x_5) \\ -2f(n^2x_2), \end{aligned} \tag{5.15}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Replacing (x, y) by $(nx_1 + n^4x_4 + n^5x_5, n^2x_2)$ in (5.1), we obtain

$$\begin{aligned} f(nx_1 + n^4x_4 + n^5x_5 + n^2x_2) + f(nx_1 - n^2x_2 + n^4x_4 + n^5x_5) \\ = 2f(nx_1 + n^4x_4 + n^5x_5) + 2f(n^2x_2), \end{aligned} \tag{5.16}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Using (5.16) in (5.15), we obtain

$$f(nx_1+n^2x_2+n^3x_3+n^4x_4+n^5x_5)+f(-nx_1+n^2x_2+n^3x_3+n^4x_4 +n^5x_5)+f(nx_1+n^2x_2-n^3x_3+n^4x_4+n^5x_5)+f(nx_1-n^2x_2+n^3x_3 +n^4x_4+n^5x_5)+f(nx_1+n^2x_2+n^3x_3+n^4x_4-n^5x_5)+f(nx_1+n^2x_2 +n^3x_3-n^4x_4+n^5x_5)=2f(nx_1)+2f(n^2x_2+n^3x_3+n^4x_4+n^5x_5) +f(nx_1+n^2x_2+n^4x_4+n^5x_5)+f(nx_1-n^2x_2+n^4x_4+n^5x_5) +2f(nx_1+n^2x_2+n^3x_3)+f(n^2x_2-n^3x_3+n^4x_4-n^5x_5)+f(n^2x_2 -n^3x_3-n^4x_4+n^5x_5)-2f(n^2x_2), \tag{5.17}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Adding $2f(n^4x_4)$ on both sides of (5.17), we get

$$f(nx_1+n^2x_2+n^3x_3+n^4x_4+n^5x_5)+f(-nx_1+n^2x_2+n^3x_3+n^4x_4 +n^5x_5)+f(nx_1+n^2x_2-n^3x_3+n^4x_4+n^5x_5)+f(nx_1-n^2x_2+n^3x_3 +n^4x_4+n^5x_5)+f(nx_1+n^2x_2+n^3x_3+n^4x_4-n^5x_5)+f(nx_1+n^2x_2 +n^3x_3-n^4x_4+n^5x_5)=2f(nx_1)+2f(n^2x_2+n^3x_3+n^4x_4+n^5x_5) +f(nx_1+n^2x_2+n^4x_4+n^5x_5)+f(nx_1-n^2x_2+n^4x_4+n^5x_5)+2f(nx_1 +n^2x_2+n^3x_3)+2f(n^4x_4)+f(n^2x_2-n^3x_3+n^4x_4-n^5x_5)+f(n^2x_2 -n^3x_3-n^4x_4+n^5x_5)-2f(n^2x_2)-2f(n^4x_4), \tag{5.18}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Replacing (x, y) by $(nx_1+n^2x_2+n^3x_3, n^4x_4)$ in (5.1), we have

$$f(nx_1+n^2x_2+n^3x_3+n^4x_4)+f(nx_1+n^2x_2+n^3x_3-n^4x_4) =2f(nx_1+n^2x_2+n^3x_3)+2f(n^4x_4), \tag{5.19}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Using (5.19) in (5.18), we get

$$f(nx_1+n^2x_2+n^3x_3+n^4x_4+n^5x_5)+f(-nx_1+n^2x_2+n^3x_3+n^4x_4 +n^5x_5)+f(nx_1+n^2x_2-n^3x_3+n^4x_4+n^5x_5)+f(nx_1-n^2x_2+n^3x_3 +n^4x_4+n^5x_5)+f(nx_1+n^2x_2+n^3x_3+n^4x_4-n^5x_5)+f(nx_1+n^2x_2 +n^3x_3-n^4x_4+n^5x_5)=2f(nx_1)+2f(n^2x_2+n^3x_3+n^4x_4+n^5x_5) +f(nx_1+n^2x_2+n^4x_4+n^5x_5)+f(nx_1-n^2x_2+n^4x_4+n^5x_5)+f(nx_1 +n^2x_2+n^3x_3+n^4x_4)+f(nx_1+n^2x_2+n^3x_3-n^4x_4)+f(n^2x_2-n^3x_3 +n^4x_4-n^5x_5)++f(n^2x_2-n^3x_3-n^4x_4+n^5x_5)-2f(n^2x_2) -2f(n^4x_4), \tag{5.20}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Replacing (x, y) by $(n^2x_2+n^4x_4, n^3x_3+n^5x_5)$ in (5.1), we obtain

$$f(n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(n^2x_2 - n^3x_3 + n^4x_4 - n^5x_5)$$
$$= 2f(n^2x_2 + n^4x_4) + 2f(n^3x_3 + n^5x_5), \quad (5.21)$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Using (5.21) in (5.20), we obtain

$$f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4$$
$$+n^5x_5) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 - n^2x_2 + n^3x_3$$
$$+n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 - n^5x_5) + f(nx_1 + n^2x_2$$
$$+n^3x_3 - n^4x_4 + n^5x_5) = 2f(nx_1) + f(nx_1 + n^2x_2 + n^4x_4 + n^5x_5)$$
$$+f(nx_1 - n^2x_2 + n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4)$$
$$+f(nx_1 + n^2x_2 + n^3x_3 - n^4x_4) + 2f(n^2x_2 + n^4x_4) + 2f(n^3x_3 + n^5x_5)$$
$$+2f(n^2x_2 + n^5x_5) + 2f(n^3x_3 + n^4x_4) - 2f(n^2x_2) - 2f(n^4x_4), \quad (5.22)$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Replacing (x, y) by $(nx_1 + n^4x_4 + n^5x_5, n^2x_2)$ in (5.1), we get

$$f(nx_1 + n^2x_2 + n^4x_4 + n^5x_5) + f(nx_1 - n^2x_2 + n^4x_4 + n^5x_5)$$
$$= 2f(nx_1 + n^4x_4 + n^5x_5) + 2f(n^2x_2), \quad (5.23)$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Replacing (x, y) by $(nx_1 + n^2x_2 + n^3x_3, n^4x_4)$ in (5.1), we have

$$f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4) + f(nx_1 - n^2x_2 + n^3x_3 - n^4x_4)$$
$$= 2f(nx_1 + n^2x_2 + n^3x_3) + 2f(n^4x_4), \quad (5.24)$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Using (5.23) and (5.24) in (5.22), we have

$$f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4$$
$$+n^5x_5) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 - n^2x_2 + n^3x_3$$
$$+n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 - n^5x_5) + f(nx_1 + n^2x_2$$
$$+n^3x_3 - n^4x_4 + n^5x_5) = 2f(nx_1) + 2f(nx_1 + n^4x_4 + n^5x_5) + 2f(n^2x_2)$$
$$+2f(nx_1 + n^2x_2 + n^3x_3) + 2f(n^4x_4) + 2f(n^2x_2 + n^4x_4) + 2f(n^3x_+n^5x_5)$$
$$+2f(n^2x_2 + n^5x_5) + 2f(n^3x_3 + n^4x_4) - 2f(n^2x_2) - 2f(n^4x_4), \quad (5.25)$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Replacing (x, y) by $(nx_1 + n^4x_4 + n^5x_5, nx_1)$ in (5.1), we get

$$f(nx_1 + nx_1 + n^4x_4 + n^5x_5) + f(n^4x_4 + n^5x_5) = 2f(nx_1 + n^4x_4 + n^5x_5)$$
$$+2f(nx_1), \quad (5.26)$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Replacing (x, y) by $(nx_1 + n^2x_2 + n^3x_3, n^2x_2)$ in (5.1), we obtain

$$\begin{aligned} & f(nx_1 + n^2x_2 + n^2x_2 + n^3x_3) + f(nx_1 + n^3x_3) \\ & \quad = 2f(nx_1 + n^2x_2 + n^3x_3) + 2f(n^2x_2), \end{aligned} \tag{5.27}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Using (5.26) and (5.27) in (5.25), we have

$$\begin{aligned} & f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4 \\ & +n^5x_5) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 - n^2x_2 + n^3x_3 \\ & +n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 - n^5x_5) + f(nx_1 + n^2x_2 \\ & +n^3x_3 - n^4x_4 + n^5x_5) = f(nx_1 + nx_1 + n^4x_4 + n^5x_5) + f(n^4x_4 + n^5x_5) \\ & + f(nx_1 + n^2x_2 + n^2x_2 + n^3x_3) + f(nx_1 + n^3x_3) + 2f(n^4x_4) + 2f(n^2x_2 \\ & +n^4x_4) + 2f(n^3x_3 + n^5x_5) + 2f(n^2x_2 + n^5x_5) + 2f(n^3x_3 + n^4x_4) \\ & -2f(n^2x_2) - 2f(n^4x_4), \ \forall\, x_1, x_2, x_3, x_4, x_5 \in X \end{aligned} \tag{5.28}$$

Replacing (x, y) by $(nx_1 + n^4x_4, nx_1 + n^5x_5)$ in (5.1), we get

$$\begin{aligned} & f(nx_1 + nx_1 + n^4x_4 + n^5x_5) = 2f(nx_1 + n^4x_4) + 2f(nx_1 + n^5x_5) \\ & \quad -f(n^4x_4 - n^5x_5), \ \forall\, x_1, x_4, x_5 \in X \end{aligned} \tag{5.29}$$

Replacing (x, y) by $(nx_1 + n^2x_2, n^2x_2 + n^3x_3)$ in (5.1), we have

$$\begin{aligned} & f(nx_1 + n^2x_2 + n^2x_2 + n^3x_3) = 2f(nx_1 + n^2x_2) \\ & \quad +2f(n^2x_2 + n^3x_3) - f(nx_1 - n^3x_3), \end{aligned} \tag{5.30}$$

for all $x_1, x_2, x_3 \in X$. Using (5.29) and (5.30) in (5.28), we get

$$\begin{aligned} & f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4 \\ & + n^5x_5) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 - n^2x_2 + n^3x_3 \\ & + n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 - n^5x_5) + f(nx_1 + n^2x_2 \\ & + n^3x_3 - n^4x_4 + n^5x_5) = 2f(nx_1 + n^4x_4) + 2f(nx_1 + n^5x_5) - f(n^4x_4 \\ & - n^5x_5) + f(n^4x_4 + n^5x_5) + 2f(nx_1 + n^2x_2) + 2f(n^2x_2 + n^3x_3) - f(nx_1 \\ & - n^3x_3) + f(nx_1 + n^3x_3) + 2f(n^4x_4) + 2f(n^2x_2 + n^4x_4) + 2f(n^3x^3 \\ & + n^5x_5) + 2f(n^2x_2 + n^5x_5) + 2f(n^3x_3 + n^4x_4) - 2f(n^2x_2) \\ & - 2f(n^4x_4), \ \forall\, x_1, x_2, x_3, x_4, x_5 \in X \end{aligned} \tag{5.31}$$

Replacing (x, y) by (nx_1, n^3x_3) in (5.1), we get

$$f(nx_1 + n^3x_3) - 2f(nx_1) - 2f(n^3x_3) = -f(nx_1 - n^3x_3), \tag{5.32}$$

for all $x_1, x_3 \in X$. Replacing (x, y) by (n^4x_4, n^5x_5) in (5.1), we obtain

$$f(n^4x_4 + n^5x_5) - 2f(n^4x_4) - 2f(n^5x_5) = -f(n^4x_4 - n^5x_5), \tag{5.33}$$

for all $x_4, x_5 \in X$. Using (5.32) and (5.33) in (5.31), we obtain

$$\begin{aligned}
&f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4\\
&+ n^5x_5) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 - n^2x_2 + n^3x_3\\
&+ n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 - n^5x_5) + f(nx_1 + n^2x_2\\
&+ n^3x_3 - n^4x_4 + n^5x_5) = 2f(nx_1 + n^4x_4) + 2f(nx_1 + n^5x_5) + f(n^4x_4\\
&+ n^5x_5) - 2f(n^4x_4) - 2f(n^5x_5) + f(n^4x_4 + n^5x_5) + 2f(nx_1 + n^2x_2)\\
&+ 2f(n^2x_2 + n^3x_3) + f(nx_1 + n^3x_3) - 2f(nx_1) - 2f(n^3x_3) + f(nx_1 + n^3x_3)\\
&+ 2f(n^4x_4) + 2f(n^2x_2 + n^4x_4) + 2f(n^3x_3 + n^5x_5) + 2f(n^2x_2 + n^5x_5)\\
&+ 2f(n^3x_3 + n^4x_4) - 2f(n^2x_2) - 2f(n^4x_4),
\end{aligned} \tag{5.34}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. It follows from (5.34) that

$$\begin{aligned}
&f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4\\
&+ n^5x_5) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 - n^2x_2 + n^3x_3\\
&+ n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 - n^5x_5) + f(nx_1 + n^2x_2\\
&+ n^3x_3 - n^4x_4 + n^5x_5) = 2f(nx_1 + n^2x_2) + 2f(nx_1 + n^3x_3) + 2f(nx_1\\
&+ n^4x_4) + 2f(nx_1 + n^5x_5) + 2f(n^2x_2 + n^3x_3) + 2f(n^2x_2 + n^4x_4)\\
&+ 2f(n^2x_2 + n^5x_5) + 2f(n^3x_3 + n^4x_4) + 2f(n^3x_3 + n^5x_5)\\
&+ 2f(n^4x_4 + n^5x_5) - 2f(nx_1) - 2f(n^2x_2) - 2f(n^3x_3)\\
&- 2f(n^4x_4) - 2f(n^5x_5),
\end{aligned} \tag{5.35}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Using evenness of f in (5.35), we have

$$\begin{aligned}
&f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 + n^5x_5) + f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4\\
&+n^5x_5) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4 + n^5x_5) + f(nx_1 - n^2x_2 + n^3x_3\\
&+n^4x_4 + n^5x_5) + f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4 - n^5x_5) + f(nx_1 + n^2x_2\\
&+n^3x_3 - n^4x_4 + n^5x_5) = 2[f(nx_1 + n^2x_2) + f(nx_1 + n^3x_3) + f(nx_1 + n^4x_4)\\
&+f(nx_1 + n^5x_5) + f(n^2x_2 + n^3x_3) + f(n^2x_2 + n^4x_4) + f(n^2x_2 + n^5x_5)\\
&+f(n^3x_3 + n^4x_4) + f(n^3x_3 + n^5x_5) + f(n^4x_4 + n^5x_5)] - n^2[f(x_1) + f(-x_1)]\\
&-n^4[f(x_2) + f(-x_2)] - n^6[f(x_3) + f(-x_3)] - n^8[f(x_4) + f(-x_4)]\\
&-n^{10}[f(x_5) + f(-x_5)],
\end{aligned} \tag{5.36}$$

for all $x_1, x - 2, x_3, x_4, x_5 \in X$. Using evenness of f in (5.36) we get

$$\begin{aligned}
&f(nx_1+n^2x_2+n^3x_3+n^4x_4+n^5x_5)+f(-nx_1+n^2x_2+n^3x_3+n^4x_4\\
&+n^5x_5)+f(nx_1+n^2x_2-n^3x_3+n^4x_4+n^5x_5)+f(nx_1-n^2x_2+n^3x_3\\
&+n^4x_4+n^5x_5)+f(nx_1+n^2x_2+n^3x_3+n^4x_4-n^5x_5)+f(nx_1+n^2x_2\\
&+n^3x_3-n^4x_4+n^5x_5)=2[f(nx_1+n^2x_2)+f(nx_1+n^3x_3)+f(nx_1+n^4x_4)\\
&+f(nx_1+n^5x_5)+f(n^2x_2+n^3x_3)+f(n^2x_2+n^4x_4)+f(n^2x_2+n^5x_5)\\
&+f(n^3x_3+n^4x_4)+f(n^3x_3+n^5x_5)+f(n^4x_4+n^5x_5)]-2n[f(x_1)-f(-x_1)]\\
&-2n^2[f(x_2)-f(-x_2)]-2n^3[f(x_3)-f(-x_3)]-2n^4[f(x_4)-f(-x_4)]\\
&-2n^5[f(x_5)-f(-x_5)]-n^2[f(x_1)+f(-x_1)]-n^4[f(x_2)+f(-x_2)]\\
&-n^6[f(x_3)+f(-x_3)]-n^8[f(x_4)+f(-x_4)]-n^{10}[f(x_5)+f(-x_5)]. \quad (5.37)
\end{aligned}$$

Conversely, $f : X \to Y$ satisfies the functional equation (2.6). Using evenness of f in (5.2), we get (2.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Replacing $(x_1, x_2, x_3, x_4, x_5)$ by $(x, 0, 0, 0, 0)$, $(0, x, 0, 0, 0)$, $(0, 0, x, 0, 0)$, $(0, 0, 0, x, 0)$ and $(0, 0, 0, 0, x)$ in (2.6), we obtain

$$\begin{aligned}
&f(nx)=n^2f(x),\ \ f(n^2x)=n^4f(x)\ \ f(n^3x)=n^6f(x),\\
&f(n^4x)=n^8f(x)\ \textit{and}\ \ f(n^5x)=n^{10}f(x), \quad (5.38)
\end{aligned}$$

for all $x \in X$. It is easy to verify from (5.38), we have

$$f\left(\frac{x}{n^i}\right)=\left(\frac{1}{n^i}\right)^2;\ i=1,\ldots,5,\ \forall x\in X. \quad (5.39)$$

Replacing $(x_1, x_2, x_3, x_4, x_5)$ by $(\frac{x}{n}, \frac{y}{n^2}, 0, 0, 0)$ and using evenness of f and (5.39), we conclude our result. □

5.3 Quadratic Functional Equation: Even Case—Direct Method

Theorem 5.2 *Let $j \in \{-1, 1\}$. Let $\alpha : X^5 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty}\frac{\alpha(n^{kj}x_1, n^{kj}x_2, n^{kj}x_3, n^{kj}x_4, n^{kj}x_5)}{n^{2kj}} \quad (5.40)$$

converges in $\mathbb{R}$, for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f_q : X \to Y$ be an even mapping satisfying the inequality

$$\| Df_q(x_1, x_2, x_3, x_4, x_5) \| \le \alpha(x_1, x_2, x_3, x_4, x_5), \quad (5.41)$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$ (Df—Domain of a function). Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (2.6) and

$$\| f_q(x) - Q(x) \| \leq \frac{1}{2n^2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(n^{kj}x, 0, 0, 0, 0)}{n^{2kj}}, \tag{5.42}$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty} \frac{f_q(n^{kj}x)}{n^{2kj}}, \tag{5.43}$$

for all $x \in X$.

Proof Assume that $j = 1$. Replacing $(x_1, x_2, x_3, x_4, x_5)$ by $(x, 0, 0, 0, 0)$ in (5.41) and using evenness of f_q, we get

$$\| 2n^2 f_q(x) - 2 f_q(nx) \| \leq \alpha(x, 0, 0, 0, 0), \tag{5.44}$$

for all $x \in X$. It follows from (5.44) that

$$\left\| \frac{f_q(nx)}{n^2} - f_q(x) \right\| \leq \frac{1}{2n^2} \alpha(x, 0, 0, 0, 0), \tag{5.45}$$

for all $x \in X$. Now replacing x by nx and dividing by n^2 in (5.45), we get

$$\left\| \frac{f_q(n^2x)}{n^4} - \frac{f_q(nx)}{n^2} \right\| \leq \frac{1}{2n^4} \alpha(nx, 0, 0, 0, 0), \tag{5.46}$$

for all $x \in X$. Adding (5.45) and (5.46), we have

$$\left\| \frac{f_q(n^2x)}{n^4} - f_q(x) \right\| \leq \frac{1}{2n^2} \left[\alpha(x, 0, 0, 0, 0) + \frac{\alpha(nx, 0, 0, 0, 0)}{n^2} \right],$$

for all $x \in X$. In general for any positive integer i one can easily verify that

$$\left\| \frac{f_q(n^ix)}{n^{2i}} - f_q(x) \right\| \leq \frac{1}{2n^2} \sum_{k=0}^{i-1} \frac{\alpha(n^kx, 0, 0, 0, 0)}{n^{2k}}$$

$$\left\| \frac{f_q(n^ix)}{n^{2i}} - f_q(x) \right\| \leq \frac{1}{2n^2} \sum_{k=0}^{\infty} \frac{\alpha(n^kx, 0, 0, 0, 0)}{n^{2k}}, \tag{5.47}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{ \frac{f_q(n^ix)}{n^{2i}} \right\}$, replacing x by n^lx and dividing n^{2l} in (5.46), for $i, l > 0$, we get

$$\left\|\frac{f_q(n^{i+l}x)}{n^{2(i+l)}} - \frac{f_q(n^l x)}{n^{2l}}\right\| \leq \frac{1}{2n^2}\sum_{k=0}^{i-1}\frac{\alpha(n^{k+l}x, 0, 0, 0, 0)}{n^{2(k+l)}} \quad \to 0 \; as \; l \to \infty \quad (5.48)$$

for all $x \in X$. Hence the sequence $\left\{\frac{f_q(n^i x)}{n^{2i}}\right\}$, is Cauchy sequence. Since Y is complete, there exists a mapping $Q : X \to Y$ such that $Q(x) = \lim_{i\to\infty}\frac{f_q(n^i x)}{n^{2i}}$, for all $x \in X$. Letting $i \to \infty$ in (5.46), we see that (5.42) holds for $x \in X$. To prove that Q satisfies (2.6), replacing $(x_1, x_2, x_3, x_4, x_5)$ by $(n^l x, n^l x, n^l x, n^l x, n^l x)$ and dividing n^{2l} in (2.25), we get

$$\frac{1}{n^{2l}} \parallel Df_q(n^l x, n^l x, n^l x, n^l x, n^l x) \parallel \leq \frac{1}{n^{2l}} \parallel \alpha(n^l x, n^l x, n^l x, n^l x, n^l x) \parallel$$

for all $x_1, x_2, x_3, ..., x_n \in X$. Letting the limit as $l \to \infty$ in above inequality and using the definition of $Q(x)$, we see that $Q(x_1, x_2, x_3, x_4, x_5) = 0$. Hence Q satisfies (2.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$. To show that Q is unique, let B be another quadratic mapping satisfying (2.6) and (5.42). Then

$$\begin{aligned}\parallel Q(x) - B(x) \parallel \; &\leq \frac{1}{n^{2l}}\left\{\left\|\left(Q(n^l x) - f_q(n^l x)\right)\right\| + \left\|\left(f_q(n^l x) - B(n^l x)\right)\right\|\right\} \\ &\leq \frac{1}{2n^2}\sum_{k=0}^{\infty}\frac{\alpha(n^{k+l}x, 0, 0, 0, 0)}{n^{2(k+l)}} \quad \to 0 \;\; asl \to \infty,\end{aligned}$$

for all $x \in X$. Hence Q is unique. For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

Example 5.3 Let $j \in \{-1, 1\}$. Let $\alpha : X^5 \to [0, \infty)$ be a function such that

$$\sum_{k=0}^{\infty}\frac{\alpha(2^{kj}x_1, 2^{kj}x_2, 2^{kj}x_3, 2^{kj}x_4, 2^{kj}x_5)}{2^{2kj}},$$

converges in $\mathbb{R}$, for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f_q : X \to Y$ be an even mapping satisfying the inequality (5.41) for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$, which satisfies the functional equation (2.6) and

$$\parallel f_q(x) - Q(x) \parallel \leq \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\alpha(2^{kj}x, 0, 0, 0, 0)}{2^{2kj}},$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty}\frac{f_a(2^{kj}x)}{2^{2kj}},$$

for all $x \in X$.

Proposition 5.4 *Let $j \in \{-1, 1\}$. Let $\alpha : X^5 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{2kj}x_1, n^{2kj}x_2, n^{2kj}x_3, n^{2kj}x_4, n^{2kj}x_5)}{n^{4kj}}$$

converges in $\mathbb{R}$, for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f_q : X \to Y$ be an even mapping satisfying the inequality (5.41) for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique mapping $Q : X \to Y$ which satisfies the functional equation (2.6) and

$$\| f_q(x) - Q(x) \| \leq \frac{1}{2n^4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, n^{2kj}x, 0, 0, 0)}{n^{4kj}},$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty} \frac{f_q(n^{2kj}x)}{n^{4kj}}, \ \forall\, x \in X$$

Example 5.5 Let $j \in \{-1, 1\}$. Let $\alpha : X^5 \to [0, \infty)$ be a function such that

$$\sum_{k=0}^{\infty} \frac{\alpha(2^{2kj}x_1, 2^{2kj}x_2, 2^{2kj}x_3, 2^{2kj}x_4, 2^{2kj}x_5)}{2^{4kj}},$$

converges in $\mathbb{R}$, for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f_q : X \to Y$ be an even mapping satisfying the inequality (5.41) for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique mapping $Q : X \to Y$ which satisfies the functional equation (2.6) and

$$\| f_q(x) - Q(x) \| \leq \frac{1}{32} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 2^{2kj}x, 0, 0, 0)}{2^{4kj}},$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty} \frac{f_q(2^{2kj}x)}{2^{4kj}},$$

for all $x \in X$.

Proposition 5.6 *Let $j \in \{-1, 1\}$. Let $\alpha : X^5 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{3kj}x_1, n^{3kj}x_2, n^{3kj}x_3, n^{3kj}x_4, n^{3kj}x_5)}{n^{6kj}}$$

converges in $\mathbb{R}$*, for all* $x_1, x_2, x_3, x_4, x_5 \in X$*. Let* $f_q : X \to Y$ *be an even mapping satisfying the inequality (5.41) for all* $x_1, x_2, x_3, x_4, x_5 \in X$*. Then there exists a unique quadratic mapping* $Q : X \to Y$ *which satisfies the functional equation (2.6) and*

$$\| f_q(x) - Q(x) \| \leq \frac{1}{2n^6} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 0, n^{3kj} x, 0, 0)}{n^{6kj}},$$

for all $x \in X$*. The mapping* $Q(x)$ *is defined by*

$$Q(x) = \lim_{k\to\infty} \frac{f_a(n^{3kj} x)}{n^{6kj}},$$

for all $x \in X$*.*

Example 5.7 Let $j \in \{-1, 1\}$. Let $\alpha : X^5 \to [0, \infty)$ be a function such that

$$\sum_{k=0}^{\infty} \frac{\alpha(2^{3kj} x_1, 2^{3kj} x_2, 2^{3kj} x_3, 2^{3kj} x_4, 2^{3kj} x_5)}{2^{6kj}},$$

converges in $\mathbb{R}$, for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f_q : X \to Y$ be an even mapping satisfying the inequality (5.41) for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (2.6) and

$$\| f_q(x) - Q(x) \| \leq \frac{1}{128} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 0, 2^{3kj} x, 0, 0)}{2^{6kj}},$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty} \frac{f_a(2^{3kj} x)}{2^{6kj}}, \ \forall\, x \in X$$

Proposition 5.8 *Let* $j \in \{-1, 1\}$*. Let* $\alpha : X^5 \to [0, \infty)$ *be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{4kj} x_1, n^{4kj} x_2, n^{4kj} x_3, n^{4kj} x_4, n^{4kj} x_5)}{n^{8kj}}$$

converges in $\mathbb{R}$*, for all* $x_1, x_2, x_3, x_4, x_5 \in X$*. Let* $f_q : X \to Y$ *be an even mapping satisfying the inequality (5.41) for all* $x_1, x_2, x_3, x_4, x_5 \in X$*. Then there exists a unique mapping* $Q : X \to Y$ *which satisfies the functional equation (2.6) and*

$$\| f_q(x) - Q(x) \| \leq \frac{1}{2n^8} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 0, 0, n^{4kj}x, 0)}{n^{8kj}},$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty} \frac{f_a(n^{4kj}x)}{n^{8kj}},$$

for all $x \in X$.

Proposition 5.9 *Let $j \in \{-1, 1\}$. Let $\alpha X^5 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{5kj}x_1, n^{5kj}x_2, n^{5kj}x_3, n^{5kj}x_4, n^{5kj}x_5)}{n^{10kj}},$$

converges in $\mathbb{R}$, for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f_q : X \to Y$ be an even mapping satisfying the inequality (5.41) for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (2.6) and

$$\| f_q(x) - Q(x) \| \leq \frac{1}{2n^{10}} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 0, 0, 0, n^{5kj})}{n^{10kj}},$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty} \frac{f_a(n^{5kj}x)}{n^{10kj}},$$

for all $x \in X$.

The following corollaries are the immediate consequence of Theorem 5.2, and Propositions 5.4–5.9 respectively, concerning the stability of (2.6).

Corollary 5.10 *Let λ and p be non-negative real numbers. If a mapping $f_q : X \to Y$ satisfies the inequality*

$$\| Df_q(x_1, x_2, x_3, x_4, x_5) \| \leq \begin{cases} \lambda, \\ \lambda\left\{ \sum_{i=1}^{5} \| x_i \|^p \right\}, \\ \lambda\left\{ \prod_{i=1}^{5} \| x_i \|^p + \sum_{i=1}^{5} \| x_i \|^{5p} \right\}, \end{cases} \tag{5.49}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{6|n^2-1|} \\ \frac{\lambda\|x\|^p}{8|n-n^p|} & ; p \neq 1 \\ \frac{\lambda\|x\|^{5p}}{8|n-n^{5p}|} & ; p \neq \frac{1}{5}, \end{cases} \tag{5.50}$$

for all $x \in X$.

Remark: Let λ and p be non-negative real numbers and $f_q : X \to Y$ satisfy the inequality (5.49), for all x_1, x_2, x_3, x_4, x_5. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{|6|} \\ \frac{\lambda\|x\|^p}{2|2^2-2^{2p}|} & ; p \neq 1 \\ \frac{\lambda\|x\|^{5p}}{2|2^2-2^{10s}|} & ; p \neq \frac{1}{5}, \end{cases}$$

for all $x \in X$.

Corollary 5.11 *Let λ and p be non-negative real numbers. If a mapping $f_q : X \to Y$ satisfies the inequality (5.49), for all x_1, x_2, x_3, x_4, x_5, then there exists a unique mapping $Q : X \to Y$ such that*

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{8|n^4-1|} \\ \frac{\lambda\|x\|^p}{8|n^2-n^{2p}|} & ; p \neq 1 \\ \frac{\lambda\|x\|^{5p}}{8|n^2-n^{10p}|} & ; p \neq \frac{1}{5}, \end{cases} \tag{5.51}$$

for all $x \in X$.

Remark: Let λ and p be non-negative real numbers and $f_q : X \to Y$ satisfy the inequality (5.49), for all x_1, x_2, x_3, x_4, x_5. Then there exists a unique quadratic mapping $Q : X \to Y$ such that, for all $x \in X$

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{|30|} \\ \frac{\lambda\|x\|^p}{2|2^4-2^{2p}|} & ; p \neq 1 \\ \frac{\lambda\|x\|^{5p}}{2|2^4-2^{10s}|} & ; p \neq \frac{1}{5}. \end{cases}$$

Corollary 5.12 *Let λ and p be non-negative real numbers. If a mapping $f_q : X \to Y$ satisfies the inequality (5.49), then there exists a unique quadratic mapping $Q : X \to Y$ such that, for all $x \in X$*

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{4|n^3-1|} \\ \frac{\lambda \|x\|^p}{4|n^3-n^{3p}|} & ; p \neq 1 \\ \frac{\lambda \|x\|^{5p}}{4|n^3-n^{15p}|} & ; p \neq \frac{1}{5}. \end{cases} \tag{5.52}$$

Corollary 5.13 *Let λ and p be non-negative real numbers. If a mapping $f_q : X \to Y$ satisfying the inequality (5.49), then there exists a unique quadratic mapping $Q : X \to Y$ such that, for all $x \in X$*

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{4|n^4-1|} \\ \frac{\lambda \|x\|^p}{4|n^4-n^{4p}|}; & p \neq 1 \\ \frac{\epsilon \|x\|^{5p}}{4|n^4-n^{20p}|}; & p \neq \frac{1}{5}. \end{cases} \tag{5.53}$$

Corollary 5.14 *Let λ and p be non-negative real numbers. If a mapping $f_q : X \to Y$ satisfies the inequality (2.33), then there exists a unique quadratic mapping $Q : X \to Y$ such that, for all $x \in X$*

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{4|n^5-1|} \\ \frac{\lambda \|x\|^p}{4|n^5-n^{5p}|} & ; p \neq 1 \\ \frac{\lambda \|x\|^{5p}}{4|n^5-n^{25p}|} & ; p \neq \frac{1}{5}. \end{cases} \tag{5.54}$$

5.4 Stability Quadratic Functional Equation: Even Case—Fixed Point Method

Theorem 5.15 *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition*

$$\lim_{k \to \infty} \frac{\alpha(\eta_i^k x_1, \eta_i^k x_2, \eta_i^k x_3, \eta_i^k x_4, \eta_i^k x_5)}{\eta_i^{2k}} = 0, \tag{5.55}$$

where $\eta_i = \begin{cases} n, & i = o; \\ \frac{1}{n}, & i = 1 \end{cases}$ *satisfying the functional inequality*

$$\| Df_q(x_1, x_2, x_3, x_4, x_5) \| \leq \alpha(x_1, x_2, x_3, x_4, x_5), \tag{5.56}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function $x \to \gamma(x) = \frac{1}{2}\alpha\left(\frac{x}{n}, 0, 0, 0, 0\right)$ has the property

$$\frac{\gamma(\eta_i x)}{\eta_i} = L\gamma(x), \tag{5.57}$$

for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the functional equation (2.6) and

$$\| f_q(x) - Q(x) \| \leq \frac{L^{1-i}}{1-L}\gamma(x), \tag{5.58}$$

for all $x \in X$.

Proof Let us consider the set $\Omega = \{p | p : U^2 \to V,\ p(0) = 0\}$ and d be a generalized metric on Ω, such that

$$d(p, q) = inf\left\{k \in (0, \infty) :\| p(x) - q(x) \| \leq k\gamma(x), x \in X\right\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tg(x) = \frac{1}{\eta_i^2} g(\eta_i x)$, for all $x \in X$. For $p, q \in \Omega$ and $x \in X$, we have

$$\begin{aligned}
& d(p, q) = k \Rightarrow \| p(x) - q(x) \| \leq k\gamma(x), \\
& \Rightarrow \left|\left|\frac{p(\eta_i x)}{\eta_i^2} - \frac{q(\eta_i x)}{\eta_i^2}\right|\right| \leq \frac{1}{\eta_i^2} k\gamma(\eta_i x) \\
& \Rightarrow \left|\left|Tp(x) - Tq(x)\right|\right| \leq \frac{1}{\eta_i^2} k\gamma(\eta_i x) \\
& \Rightarrow \left|\left|Tp(x) - Tq(x)\right|\right| \leq Lk\gamma(x) \Rightarrow d(Tp(x), Tq(x)) \leq kL.
\end{aligned}$$

That is, $d(Tp, Tq) \leq Ld(p, q)$. Therefore T is a strictly contractive mapping on Ω with Lipschitz constant L. It follows from (5.44) that

$$\| 2n^2 f_q(x) - 2 f_q(nx) \| \leq \alpha(x, 0, 0, 0, 0), \tag{5.59}$$

for all $x \in X$. It is follows from (5.59) that

$$\| n^2 f_q(x) - f_q(nx) \| \leq \frac{\alpha(x, 0, 0, 0, 0)}{2}, \tag{5.60}$$

for all $x \in X$. Using the definition of $\gamma(x)$ in the above equation and for $i = 0$, we get

$$\left|\left|f_q(x) - \frac{f_q(nx)}{n^2}\right|\right| \leq \frac{1}{n^2} L\gamma(x) \Rightarrow \| f_q(x) - Tf_q(x) \| \leq L\gamma(x),$$

for all $x \in X$. Hence, we obtain

$$d(Tf_q(x) - f_q(x)) \leq L = L^{1-i}, \tag{5.61}$$

for all $x \in X$. Replacing x by $\frac{x}{n}$ in (5.60), we have

$$\left\|n^2 f_q\left(\frac{x}{n}\right) - f_q(x)\right\| \leq \frac{1}{2}\alpha\left(\frac{x}{n}, 0, 0, 0, 0\right), \tag{5.62}$$

for all $x \in X$. Using the definition of $\gamma(x)$ in the above equation for $i = 0$, we have

$$\left\|n f_q\left(\frac{x}{n}\right) - f_q(x)\right\| \leq \gamma(x) \Rightarrow \| T f_q(x) - f_q(x) \| \leq \gamma(x),$$

for all $x \in X$. Hence we get

$$d(f_q(x), T f_q(x)) \leq n^2 = L^{1-i}, \tag{5.63}$$

for all $x \in X$. From (5.61) and (5.63), we can conclude

$$d(f_q(x), T f_q(x)) \leq L^{1-i} < \infty, \tag{5.64}$$

for all $x \in X$. Now from the fixed point alternative in both cases, it follows that there exists a fixed point Q of T in Ω such that

$$Q(x) = \lim_{k\to\infty} \frac{f_q(\eta_i^k x)}{\eta_i^{2k}}, \tag{5.65}$$

for all $x \in X$. In order to prove $Q : X \to Y$ satisfies the functional equation (2.6), the proof is similar to that of Theorem (5.2). Since Q is a unique fixed point of T in the set $\Delta = \left\{f_q \in \Omega / d(f_q, Q) < \infty\right\}$, A is a unique mapping such that

$$\begin{aligned} d(f_q, Q) &\leq \frac{1}{1-L} d(f_q, T f_q) \\ d(f_q, Q) &\leq \frac{L^{1-i}}{1-L} \\ \text{i.e., } \| f_q(x) - Q(x) \| &\leq \frac{L^{1-i}}{1-L}\gamma(x), \end{aligned}$$

for all $x \in X$. This completes the proof of the theorem. □

Example 5.16 Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition (5.55) where $\eta_i = \begin{cases} 2, & if\ i = o; \\ \frac{1}{2}, & if\ \ i = 1 \end{cases}$ satisfying the functional inequality (5.56), for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function

$$x \to \gamma(x) = \frac{1}{2}\alpha\left(\frac{x}{2}, 0, 0, 0, 0\right)$$

has the property (5.57), for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the Eqs. (2.6) and (5.58), for all $x \in X$.

Proposition 5.17 *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition (5.55) where $\eta_i = \begin{cases} n^2, & i = 0; \\ \frac{1}{n^2}, & i = 1 \end{cases}$ satisfying the functional inequality (5.56), for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function*

$$x \to \gamma(x) = \frac{1}{2}\alpha\Big(0, \frac{x}{n^2}, 0, 0, 0\Big)$$

has the property (5.57), for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the Eqs. (2.6) and (5.58), for all $x \in X$.

Example 5.18 Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition (5.55) where $\eta_i = \begin{cases} 2^2, & if\ i = 0; \\ \frac{1}{2^2}, & if\ i = 1 \end{cases}$ satisfying the functional inequality (5.56), for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function

$$x \to \gamma(x) = \frac{1}{2}\alpha\Big(0, \frac{x}{2^2}, 0, 0, 0\Big)$$

has the property (5.57), for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the Eqs. (2.6) and (5.58), for all $x \in X$.

Proposition 5.19 *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition (5.55) where $\eta_i = \begin{cases} n^3, & i = 0; \\ \frac{1}{n^3}, & i = 1 \end{cases}$ satisfying the functional inequality (5.56), for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function*

$$x \to \gamma(x) = \frac{1}{2}\alpha\Big(0, 0, \frac{x}{n^3}, 0, 0\Big)$$

has the property (5.57), for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the Eqs. (2.6) and (5.58), for all $x \in X$.

Example 5.20 Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition (5.55) where $\eta_i = \begin{cases} 2^3, & if\ i = 0; \\ \frac{1}{2^3}, & if\ i = 1 \end{cases}$ satisfying the functional inequality (5.56), for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function

$$x \to \gamma(x) = \frac{1}{2}\alpha\Big(0, 0, \frac{x}{2^3}, 0, 0\Big)$$

has the property (5.57), for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the Eqs. (2.6) and (5.58), for all $x \in X$.

Proposition 5.21 *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition (5.55) where $\eta_i = \begin{cases} n^4, & i = 0; \\ \frac{1}{n^4}, & i = 1 \end{cases}$ satisfying the functional inequality (5.56), for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function*

$$x \to \gamma(x) = \frac{1}{2}\alpha\Big(0, 0, 0, \frac{x}{n^4}, 0\Big)$$

has the property (5.57), for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the Eqs. (2.6) and (5.58), for all $x \in X$.

Proposition 5.22 *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the condition (5.55) where $\eta_i = \begin{cases} n^5, & i = 0; \\ \frac{1}{n^5}, & i = 1 \end{cases}$ satisfying the functional inequality (5.56), for all $x_1, x_2, x_3, x_4, x_5 \in X$ and $n \geq 1$ and assume that there exists $L = L(i)$ such that the function*

$$x \to \gamma(x) = \frac{1}{2}\alpha\Big(0, 0, 0, 0, \frac{x}{n^5}\Big)$$

has the property (5.57), for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the Eqs. (2.6) and (5.58), for all $x \in X$.

The following corollaries are the immediate consequence of the Theorem 5.15 and Propositions 5.17–5.22, respectively, concerning the stability of (2.6).

Corollary 5.23 *Let λ and p be non-negative real numbers. If a mapping $f_q : X \to Y$ satisfies the inequality (5.49), for all $x_1, x_2, x_3, x_4, x_5 \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the inequality (5.50), for all $x \in X$.*

Proof Set

$$\alpha(x_1, x_2, x_3, x_4, x_5) \leq \begin{cases} \lambda, \\ \lambda\Big\{ \sum_{i=1}^{5} \| x_i \|^p \Big\}, \\ \lambda\Big\{ \prod_{i=1}^{5} \| x_i \|^p + \sum_{i=1}^{5} \| x_i \|^{5p} \Big\} \end{cases}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Now

$$\frac{\alpha(\eta_i^k x_1, \eta_i^k x_2, \eta_i^k x_3, \eta_i^k x_4, \eta_i^k x_5)}{\eta_i^{2k}} = \begin{cases} \frac{\lambda}{\eta_i^{2k}}, \\ \frac{\lambda}{\eta_i^{2k}}\left\{ \sum_{i=1}^{5} \| \eta_i x_i \|^p \right\}, \\ \frac{\lambda}{\eta_i^{2k}}\left\{ \prod_{i=1}^{5} \| \eta_i x_i \|^p + \sum_{i=1}^{5} \| \eta_i x_i \|^{5p} \right\}, \end{cases}$$
$$= \begin{cases} \to 0 \;\; as \;\; k \to \infty \\ \to 0 \;\; as \;\; k \to \infty \\ \to 0 \;\; as \;\; k \to \infty. \end{cases}$$

That is, (5.59) holds. Since we have

$$\gamma(x) = \frac{1}{2}\alpha\left(\frac{x}{n}, 0, 0, 0, 0\right),$$

$$\gamma(x) = \tfrac{1}{2}\alpha\left(\tfrac{x}{n}, 0, 0, 0, 0\right) = \begin{cases} \frac{\lambda}{2} \\ \frac{\lambda\|x\|^p}{2n^p} \\ \frac{\lambda\|x\|^{5p}}{2n^{5p}}, \end{cases}$$

Also,

$$\frac{1}{\eta_i^2}\gamma(\eta_i x) = \begin{cases} \frac{1}{\eta_i^2}\frac{\lambda}{2} \\ \frac{1}{\eta_i^2}\frac{\lambda\|x\|^p\eta_i^p}{2n^s} \\ \frac{1}{\eta_i^2}\frac{\lambda\|x\|^{5p}\eta_i^{5p}}{2n^{5p}} \end{cases} = \begin{cases} \eta_i^{-2}\gamma(x) \\ \eta_i^{p-2}\gamma(x) \\ \eta_i^{5p-2}, \gamma(x) \end{cases}$$

for all $x \in X$. Hence the inequality (2.6) holds for following cases:
$L = n^{-2} \;\; if \;\; i = 0 \;\; and \;\; L = n^2 \;\; if \;\; i = 1.$
$L = n^{p-2} \;\; for \;\; p < 2 \;\; if \;\; i = 0 \;\; and \;\; L = n^{2-p} \;\; for \;\; p > 2 \;\; if \;\; i = 1.$
$L = n^{5p-2} \;\; for \;\; p < \frac{2}{5} \;\; if \;\; i = 0 \;\; and \;\; L = n^{2-5p} \;\; for \;\; p > \frac{2}{5} \;\; if \;\; i = 1.$
Now from (5.59), we prove the following cases.

Case 1. $L = n^{-2} \;\; if \;\; i = 0$

$$\left|\left| f_q(x) - Q(x) \right|\right| \le \frac{L^{1-i}}{1-L}\gamma(x) = \frac{n^{-2}}{1-n^{-2}}\frac{\lambda}{2} = \frac{\lambda}{2(n^2-1)}.$$

Case 2. $L = n^2 \;\; if \;\; i = 1$

$$\left|\left| f_q(x) - Q(x) \right|\right| \le \frac{L^{1-i}}{1-L}\gamma(x) = \frac{1}{1-n^2}\frac{\lambda}{2} = \frac{\lambda}{2(1-n^2)}.$$

Case 3. $L = n^{p-2} \;\; for \;\; p < 2 \;\; if \;\; i = 1$

$$\left|\left| f_q(x) - Q(x) \right|\right| \le \frac{L^{1-i}}{1-L}\gamma(x) = \frac{n^{p-2}}{1-n^{p-2}}\frac{\lambda \parallel x \parallel^p}{2n^p} = \frac{\lambda \parallel x \parallel^p}{2(n^2-n^p)}.$$

Case 4. $L = n^{2-p} \quad for \quad p > 2 \quad if \quad i = 1$

$$\big|\big|f_q(x) - Q(x)\big|\big| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{1}{1-n^{2-p}} \frac{\lambda \parallel x \parallel^p}{2n^p} = \frac{\lambda \parallel x \parallel^p}{2(n^2 - n^p)}.$$

Case 5. $L = n^{5p-2} \quad for \quad p < \frac{2}{5} \quad if \quad i = 1$

$$\big|\big|f_q(x) - Q(x)\big|\big| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{n^{5p-2}}{1-n^{5p-2}} \frac{\lambda \parallel x \parallel^{5p}}{2n^{5p}} = \frac{\lambda \parallel x \parallel^{5p}}{2(n^2 - n^{5p})}.$$

Case 6. $L = n^{2-5p} \quad for \quad p > \frac{2}{5} \quad if \quad i = 1$

$$\big|\big|f_q(x) - Q(x)\big|\big| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{n^{2-5p}}{1-n^{2-5p}} \frac{\lambda \parallel x \parallel^{5p}}{2n^{5p}} = \frac{\lambda \parallel x \parallel^{5p}}{2(n^{5p} - n^2)}.$$

Hence the proof is complete. □

Remark: Let λ and p be non-negative real numbers and $f_q : X \to Y$ satisfy the inequality (5.49), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the inequality

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{|6|} \\ \frac{\lambda \|x\|^p}{2|2^2 - 2^p|} \\ \frac{\lambda \|x\|^{5p}}{2|2^2 - 2^{5p}|}, \end{cases}$$

for all $x \in X$.

Corollary 5.24 *Let λ and p be non-negative real numbers. If a mapping $f_q : X \to Y$ satisfies the inequality (5.49), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique mapping $Q : X \to Y$ satisfying the inequality (5.3), for all $x \in X$.*

Remark: Let λ and p be non-negative real numbers. If a mapping $f_q : X \to Y$ satisfies the inequality (5.49), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying the inequality

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\lambda}{|30|} \\ \frac{\lambda \|x\|^p}{2|2^4 - 2^2 p|} \\ \frac{\lambda \|x\|^{5p}}{2|2^4 - 2^{10p}|}, \end{cases}$$

for all $x \in X$.

Corollary 5.25 *Let λ and p be non-negative real numbers. If a mapping $f_q : X \to Y$ satisfies the inequality (5.49), for all $x_1, x_2, x_3, x_4, x_5 \in X$, then there exists an unique mapping $Q : X \to Y$ satisfying the inequality (5.52), for all $x \in X$.*

Corollary 5.26 *Let λ and p be non-negative real numbers. If a mapping $f_q : X \to Y$ satisfies the inequality (5.49), for all $x_1, x_2, x_3, x_4, x_5 \in X$, then there exists an unique quadratic mapping $Q : X \to Y$ satisfying the inequality (5.53), for all $x \in X$.*

Corollary 5.27 *Let λ and p be non-negative real numbers. If a mapping $f_q : X \to Y$ satisfies the inequality (5.49), for all $x_1, x_2, x_3, x_4, x_5 \in X$, then there exists an unique quadratic mapping $Q : X \to Y$ satisfying the inequality (5.54), for all $x \in X$.*

5.5 Applications

Consider the quadratic functional equation

$$\sum_{i=1}^{n} g\left(\sum_{j=1}^{i} x_j\right) - \sum_{i=1}^{n}(n-i+1)g(x_i) = \frac{1}{2}\sum_{i=1}^{n-1}(n-i)\sum_{j=1}^{i}\left(g(x_j + x_{i+1}) - g(x_j - x_{i+1})\right).$$

Since $g(x) = x^2$ is the solution of the functional equation, the above equation can be rewritten as follows

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{i} x_j\right)^2 = \frac{1}{2}\sum_{i=1}^{n-1}(n-i)\sum_{j=1}^{i}\left((x_j + x_{i+1})^2 - (x_j - x_{i+1})^2\right) + \sum_{i=1}^{n}(n-i+1)(x_i)^2.$$

Now, let us take the variables as consecutive terms, we arrive that the partial sums of the consecutive terms is equal to the right hand side terms. Mathematically

$$\begin{aligned} &[x_1]^2 + [x_1 + x_2]^2 + \cdots \\ &\quad + [x_1 + x_2 + \ldots + x_n]^2 = \frac{1}{2}\sum_{i=1}^{n-1}(n-i)\sum_{j=1}^{i}\left([x_j + x_{i+1}]^2 - [x_j - x_{i+1}]^2\right) \\ &\qquad\qquad + \left(n[x_1]^2 + (n-1)[x_2]^2 + \ldots + [x_n]^2\right). \end{aligned}$$

Next, we show the following counter example replaced by the well-known counter example of the functional equation (2.6).

Example 5.28 Let a mapping $\phi : E \to F$ defined by

$$\phi(x) = \sum_{l=0}^{\infty} \frac{\xi(2^m x)}{2^{2m}}, \tag{5.66}$$

where

$$\xi(x) = \begin{cases} \psi x^2, & \text{if } -1 < x < 1 \\ \psi, & \text{otherwise,} \end{cases} \tag{5.67}$$

where ψ is a constant, then the mapping $\phi : E \to F$ satisfies the inequality

$$|D_\phi(x_1, x_2, x_3, ..., x_m)| \leq (-m^2 + 7m - 7)\frac{64}{3}\psi\left(\sum_{j=1}^{m}|x_j|^2\right), \tag{5.68}$$

for all $x_1, x_2, x_3, ..., x_l \in E$, but there does not exist a quadratic mapping $Q_2 : E \to F$ with a constant ϵ such that

$$|\phi(x) - Q_2(x)| \leq \epsilon|s|^2, \tag{5.69}$$

for all $s \in E$.

Proof It is easy to notice that ϕ is bounded by $\frac{4}{3}\psi$ on E.

If $\sum_{j=1}^{l}|x_j|^2 \geq \frac{1}{2^2}$ or 0, then the left side of (5.68) is less than $(-m^2 + 7m - 7)\frac{4}{3}\psi$ and thus (5.68) is true.

Assume that $0 < \sum_{j=1}^{l}|x_j|^2 < \frac{1}{2^2}$. Then there exists an integer l such that

$$\frac{1}{2^{2(l+2)}} \leq \sum_{j=1}^{l}|x_j|^2 < \frac{1}{2^{2(l+1)}}. \tag{5.70}$$

So that $2^{2l}|x_1| < \frac{1}{2^2}, 2^{2l}|x_2| < \frac{1}{2^2}, ..., 2^{2l}|x_m| < \frac{1}{2^2}$ and $2^m x_1, 2^m x_2, ..., 2^m x_m \in (-1, 1)$, for all $m = 0, 1, 2, ..., l-1$.

So, for $m = 0, 1, 2, ..., l-1$

$$\sum_{a=1}^{m}\xi\left(2^m\left(-x_a + \sum_{b=1; b\neq a}^{m} x_b\right)\right) - (m-4)\sum_{1\leq a<b\leq m}\phi(2^m(x_a + x_b)) - (-m^2 + 6m - 4)\sum_{a=1}^{m}\phi(2^m x_a) = 0.$$

By the definition of ϕ, we obtain

$$\begin{aligned}|D_\phi(x_1, x_2, x_3, ..., x_m)| &\leq \sum_{j=1}^{\infty}\frac{1}{2^{2j}}|\xi(2^j x_1, 2^j x_2, 2^j x_3, ..., 2^j x_m)| \\ &\leq \sum_{j=1}^{\infty}\frac{1}{2^{2j}}(-m^2 + 7m - 7)\psi \\ &\leq (-m^2 + 7m - 7)\frac{2^{2(1-l)}}{3}\psi.\end{aligned}$$

It follows from (5.70) that

$$|D_\phi(x_1, x_2, x_3, ..., x_m)| \le (-m^2 + 7m - 7)\frac{64}{3}\psi\left(\sum_{j=1}^{m}|x_j|^2\right), \tag{5.71}$$

for all $x_1, x_2, x_3, ..., x_m \in E$.

Thus the function ϕ satisfies the inequality (5.68), for all $x_1, x_2, x_3, ..., x_m \in E$.

We propose that there exists an quadratic mapping $Q_2 : E \to F$ with a constant $\epsilon > 0$ satisfying the inequality (5.69).

Since the function ϕ is bounded and continuous, for all $s \in E$, Q_2 is bounded on every open interval containing the origin and continuous at the origin. Thus we have $Q_2(x) = \gamma s^2$, for all $\gamma \in \mathbb{Q}$, $x \in E$ and

$$|\phi(x)| \le (\epsilon + |\gamma|)|s^2|,$$

for all $s \in E$. However, we can select a non-negative integer l and $l\psi > \epsilon + |\gamma|$. If $x \in (0, 2^{-l})$, then $2^m x \in (0, 1)$, for all $m = 0, 1, 2, ..., l - 1$ and for this x, we obtain

$$\begin{aligned}\phi(x) &= \sum_{m=0}^{\infty}\frac{\xi(2^m x)}{2^{2m}}\\ &\ge \sum_{m=0}^{l-1}\frac{\psi(2^m x)}{2^{2m}}\\ &= l\psi x\\ &> (\epsilon + |\gamma|)|x^2|,\end{aligned}$$

which is contradictory. □

Chapter 6
Quadratic Functional Equations in Banach Algebras

6.1 Banach Algebra—Quadratic Functional Equation

In fact, assume that a function satisfies a functional equation approximately according to some convention. Is it then possible to find near this function a function satisfying the equation accurately?

In this chapter, we prove the stability of the quadratic functional equation (2.6) for even cases in Banach algebra with the help of direct and fixed point methods (see [13, 60, 85, 86]).

6.2 Quadratic Functional Equation: Even Case—Direct Method

Definition 6.1 Let X be a Banach algebra. A mapping $Q : X \to X$ is said to be a quadratic derivation if the quadratic mapping Q satisfies

$$Q(x_1x_2) = Q(x_1)x_2^2 + x_1^2Q(x_2), \tag{6.1}$$

for all $x_1, x_2 \in X$. Also the quadratic derivation for five variables satisfies

$$\begin{aligned} Q(x_1x_2x_3x_4x_5) =& Q(x_1)x_2^2x_3^2x_4^2x_5^2 + x_1^2Q(x_2)x_3^2x_4^2x_5^2 + x_1^2x_2^2Q(x_3)x_4^2x_5^2 \\ &+ x_1^2x_2^2x_3^2Q(x_4)x_5^2 + x_1^2x_2^2x_3^2x_4^2Q(x_5), \end{aligned} \tag{6.2}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$.

Theorem 6.2 *Let* $j = \pm 1$. *Let* $f_q : X \to X$ *be a mapping for which there exist functions* $\alpha, \beta : X^5 \to [0, \infty)$ *such that*

H. Dutta et al., *Stability of Some Advanced Functional Equations in Various Spaces*, Studies in Systems, Decision and Control 479,
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$$\sum_{k=0}^{\infty} \frac{\alpha(n^{kj}x_1, n^{kj}x_2, n^{kj}x_3, n^{kj}x_4, n^{kj}x_5)}{n^{2kj}} \tag{6.3}$$

converges in $\mathbb{R}$ *and*

$$\sum_{k=0}^{\infty} \frac{\beta(n^{kj}x_1, n^{kj}x_2, n^{kj}x_3, n^{kj}x_4, n^{kj}x_5)}{n^{10kj}} \tag{6.4}$$

converges in $\mathbb{R}$ *and satisfying the functional inequalities*

$$\| Df_q(x_1, x_2, x_3, x_4, x_5) \| \leq \alpha(x_1, x_2, x_3, x_4, x_5) \tag{6.5}$$

and

$$\begin{aligned}&\| f_q(x_1x_2x_3x_4x_5) - f_a(x_1)x_2^2x_3^2x_4^2x_5^2 - x_1^2 f_q(x_2)x_3^2x_4^2x_5^2 - x_1^2x_2^2 f_q(x_3)x_4^2x_5^2 \\ &-x_1^2x_2^2x_3^2 f_q(x_4)x_5^2 - x_1^2x_2^2x_3^2x_4^2 f_q(x_5) \| \leq \beta(x_1, x_2, x_3, x_4, x_5),\end{aligned} \tag{6.6}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. *Then there exists a unique quadratic derivation* $Q : X \to X$ *satisfying the functional equation (2.6) and*

$$\| f_q(x) - Q(x) \| \leq \frac{1}{2n^2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(n^{kj}x, 0, 0, 0, 0)}{n^{2kj}}, \tag{6.7}$$

6 for all $x \in X$. *The mapping* $Q(x)$ *is defined by*

$$Q(x) = \lim_{k\to\infty} \frac{f_q(n^{kj}x)}{n^{2kj}}, \tag{6.8}$$

for all $x \in X$.

Proof It follows from Theorem 5.2 that Q is a unique quadratic mapping and satisfies (2.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$. It follows from (6.6) that

$$\begin{aligned}&\| Q(x_1x_2x_3x_4x_5) - Q(x_1)x_2^2x_3^2x_4^2x_5^2 - x_1^2 Q(x_2)x_3^2x_4^2x_5^2 - x_1^2x_2^2 Q(x_3)x_4^2x_5^2 \\ &-x_1^2x_2^2x_3^2 Q(x_4)x_5^2 - x_1^2x_2^2x_3^2x_4^2 Q(x_5), \| \\ &\leq \frac{1}{n^{10k}} \big|\big| f_q(n^k(x_1x_2x_3x_4x_5)) \\ &\quad - f_q(n^kx_1)(n^{2k}x_2n^{2k}x_3n^{2k}x_4n^{2k}x_5) - n^{2k}x_1 f_q(n^kx_2)n^{2k}x_3n^{2k}x_4n^{2k}x_5 \\ &\quad -n^{2k}x_1n^{2k}x_2 f_q(n^kx_3)n^{2k}x_4n^{2k}x_5 - n^{2k}x_1n^{2k}x_2n^{2k}x_3n^{2k} f_q(n^kx_4)n^{2k}x_5 \\ &\quad -n^{2k}x_1n^{2k}x_2n^{2k}x_3n^{2k}x_4 f_q(n^kx_5)\big|\big|, \\ &\leq \frac{1}{n^{10k}} \beta(n^kx_1, n^kx_2, n^kx_3, n^kx_4, n^kx_5) \quad \to 0 \;\; as \quad k \to \infty,\end{aligned}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Hence, the mapping $Q : X \to X$ is a unique quadratic derivation satisfying (6.7).

Example 6.3 Let $j = \pm 1$. Let $f_q : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ such that

$$\sum_{k=0}^{\infty} \frac{\alpha(2^{kj}x_1, 2^{kj}x_2, 2^{kj}x_3, 2^{kj}x_4, 2^{kj}x_5)}{2^{2kj}}$$

converges in $\mathbb{R}$ and

$$\sum_{k=0}^{\infty} \frac{\beta(2^{kj}x_1, 2^{kj}x_2, 2^{kj}x_3, 2^{kj}x_4, 2^{kj}x_5)}{2^{10kj}}$$

converges in $\mathbb{R}$ and satisfying the functional inequalities (6.5) and (6.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6), for all $x \in X$ and

$$\| f_q(x) - Q(x) \| \leq \frac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(2^{kj}x, 0, 0, 0, 0)}{n^{2kj}}.$$

The mapping $Q(x)$ is defined by $Q(x) = \lim_{k\to\infty} \frac{f_q(2^{kj}x)}{2^{2kj}}$, for all $x \in X$.

Proposition 6.4 *Let $j = \pm 1$. Let $f_q : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{2kj}x_1, n^{2kj}x_2, n^{2kj}x_3, n^{2kj}x_4, n^{2kj}x_5)}{n^{4kj}}$$

converges in $\mathbb{R}$ and

$$\sum_{k=0}^{\infty} \frac{\beta(n^{2kj}x_1, n^{2kj}x_2, n^{2kj}x_3, n^{2kj}x_4, n^{2kj}x_5)}{n^{20kj}}$$

converges in $\mathbb{R}$ and satisfying the functional inequalities (6.5) and (6.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6), for all $x \in X$ and

$$\| f_q(x) - Q(x) \| \leq \frac{1}{2n^4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, n^{2kj}x, 0, 0, 0)}{n^{4kj}}.$$

The mapping $Q(x)$ *is defined by* $Q(x) = \lim_{k\to\infty} \frac{f_q(n^{2kj}x)}{n^{4kj}}$, *for all* $x \in X$.

Example 6.5 Let $j = \pm 1$. Let $f_q : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ such that

$$\sum_{k=0}^{\infty} \frac{\alpha(2^{2kj}x_1, 2^{2kj}x_2, 2^{2kj}x_3, 2^{2kj}x_4, 2^{2kj}x_5)}{2^{4kj}}$$

converges in $\mathbb{R}$ and also

$$\sum_{k=0}^{\infty} \frac{\beta(2^{2kj}x_1, 2^{2kj}x_2, 2^{2kj}x_3, 2^{2kj}x_4, 2^{2kj}x_5)}{2^{20kj}}$$

converges in $\mathbb{R}$ and satisfying the functional inequalities (6.5) and (6.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6), for all $x \in X$ and

$$\| f_q(x) - Q(x) \| \le \frac{1}{32} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 2^{2kj}x, 0, 0, 0)}{2^{4kj}}.$$

The mapping $Q(x)$ is defined by $Q(x) = \lim_{k\to\infty} \frac{f_q(2^{2kj}x)}{2^{4kj}}$, for all $x \in X$.

Proposition 6.6 *Let* $j = \pm 1$. *Let* $f_q : X \to X$ *be a mapping for which there exist functions* $\alpha, \beta : X^5 \to [0, \infty)$ *such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{3kj}x_1, n^{3kj}x_2, n^{3kj}x_3, n^{3kj}x_4, n^{3kj}x_5)}{n^{6kj}},$$

converges in $\mathbb{R}$ *and*

$$\sum_{k=0}^{\infty} \frac{\beta(n^{3kj}x_1, n^{3kj}x_2, n^{3kj}x_3, n^{3kj}x_4, n^{3kj}x_5)}{n^{30kj}}$$

converges in $\mathbb{R}$ *and satisfying the functional inequalities (6.5) and (6.6), for all* $x_1, x_2, x_3, x_4, x_5 \in X$. *Then there exists a unique quadratic derivation* $Q : X \to X$ *satisfying the functional equation (2.6), for all* $x \in X$ *and*

$$\| f_q(x) - Q(x) \| \le \frac{1}{2n^6} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 0, n^{3kj}x, 0, 0)}{n^{6kj}}.$$

The mapping $Q(x)$ *is defined by* $Q(x) = \lim_{k\to\infty} \frac{f_q(n^{3kj}x)}{n^{6kj}}$, *for all* $x \in X$.

Proposition 6.7 *Let* $j = \pm 1$. *Let* $f_q : X \to X$ *be a mapping for which there exist functions* $\alpha, \beta : X^5 \to [0, \infty)$ *such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{4kj}x_1, n^{4kj}x_2, n^{4kj}x_3, n^{4kj}x_4, n^{4kj}x_5)}{n^{8kj}},$$

converges in $\mathbb{R}$ *and*

$$\sum_{k=0}^{\infty} \frac{\beta(n^{4kj}x_1, n^{4kj}x_2, n^{4kj}x_3, n^{4kj}x_4, n^{4kj}x_5)}{n^{40kj}},$$

converges in $\mathbb{R}$ *and satisfying the functional inequalities (6.5) and (6.6), for all* $x_1, x_2, x_3, x_4, x_5 \in X$. *Then there exists a unique quadratic derivation* $Q : X \to X$ *satisfying the functional equation (2.6), for all* $x \in X$ *and*

$$\| f_q(x) - Q(x) \| \leq \frac{1}{2n^8} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 0, 0, n^{4kj}x, 0)}{n^{8kj}}.$$

The mapping $Q(x)$ *is defined by* $Q(x) = \lim_{k\to\infty} \frac{f_q(n^{4kj}x)}{n^{8kj}}$, *for all* $x \in X$.

Proposition 6.8 *Let* $j = \pm 1$. *Let* $f_q : X \to X$ *be a mapping for which there exist functions* $\alpha, \beta : X^5 \to [0, \infty)$ *such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{5kj}x_1, n^{5kj}x_2, n^{5kj}x_3, n^{5kj}x_4, n^{5kj}x_5)}{n^{10kj}},$$

converges in $\mathbb{R}$ *and*

$$\sum_{k=0}^{\infty} \frac{\beta(n^{5kj}x_1, n^{5kj}x_2, n^{5kj}x_3, n^{5kj}x_4, n^{5kj}x_5)}{n^{50kj}},$$

converges in $\mathbb{R}$ *and satisfying the functional inequalities (6.5) and (6.6), for all* $x_1, x_2, x_3, x_4, x_5 \in X$. *Then there exists a unique quadratic derivation* $Q : X \to X$ *satisfying the functional equation (2.6) and*

$$\| f_q(x) - Q(x) \| \leq \frac{1}{2n^{10}} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(0, 0, 0, 0, n^{5kj}x)}{n^{10kj}},$$

for all $x \in X$.

The mapping $Q(x)$ *is defined by* $Q(x) = \lim_{k\to\infty} \frac{f_q(n^{5kj}x)}{n^{10kj}}$, *for all* $x \in X$.

The following corollaries are immediate consequences of Theorem 6.2, Propositions 6.4–6.8 respectively, concerning the stability (2.6).

Corollary 6.9 *Let* $f_q : X \to X$ *be a mapping and there exist real numbers* $\Diamond$ *and* r *such that*

$$\| Df_q(x_1, x_2, x_3, x_4, x_5) \| \leq \begin{cases} \Diamond, \\ \Diamond\left\{ \sum_{i=1}^{5} \| x_i \|^r \right\}, & r \neq 2 \\ \Diamond\left\{ \prod_{i=1}^{5} \| x_i \|^r + \sum_{i=1}^{5} \| x_i \|^{5r} \right\}, & r \neq \frac{2}{5}, \end{cases} \tag{6.9}$$

$$\begin{aligned} &\| f_q(x_1x_2x_3x_4x_5) - f_q(x_1)x_2^2x_3^2x_4^2x_5^2 - x_1^2 f_q(x_2)x_3^2x_4^2x_5^2 - x_1^2x_2^2 f_q(x_3)x_4^2x_5^2 \\ &-x_1^2x_2^2x_3^2 f_q(x_4)x_5^2 - x_1^2x_2^2x_3^2x_4^2 f_q(x_5) \| \\ &\leq \begin{cases} \Diamond, \\ \Diamond\left\{ \sum_{i=1}^{5} \| x_i \|^r \right\}, & r \neq 2 \\ \Diamond\left\{ \prod_{i=1}^{5} \| x_i \|^r + \sum_{i=1}^{5} \| x_i \|^{5r} \right\}, & r \neq \frac{2}{5}, \end{cases} \end{aligned} \tag{6.10}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. *Then there exists a unique quadratic derivation* $Q : X \to X$, *such that*

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\Diamond}{2|n^2-1|} \\ \frac{\Diamond \|x\|^r}{2|n^2-n^r|} \\ \frac{\Diamond \|x\|^{5r}}{2|n^2-n^{5r}|}, \end{cases} \tag{6.11}$$

for all $x \in X$.

Remark: Let $f_q : X \to X$ be a mapping and assume that there exist real numbers $\Diamond$ and r such that (6.9) and (6.10), for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to X$, such that

$$\|f_q(x) - Q(x)\| \leq \begin{cases} \frac{\diamond}{|6|} \\ \frac{\diamond \|x\|^r}{2|2^2-2^r|} \\ \frac{\diamond \|x\|^{5r}}{2|2^2-2^{5r}|}, \end{cases}$$

for all $x \in X$.

Corollary 6.10 *Let $f_q : X \to X$ be a mapping and assume that there exist real numbers $\diamondsuit$ and r such that (6.9) and (6.10), for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to X$, such that*

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\diamondsuit}{2|n^4-1|} \\ \frac{\diamondsuit \|x\|^r}{2|n^4-n^{2r}|} \\ \frac{\diamondsuit \|x\|^{5r}}{2|n^4-n^{10r}|}, \end{cases} \tag{6.12}$$

for all $x \in X$.

Remark: Let $f_q : X \to X$ be a mapping and assume that there exist real numbers $\diamondsuit$ and r such that (6.9) and (6.10), for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to X$, such that

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\diamond}{|30|} \\ \frac{\diamond \|x\|^r}{2|2^4-2^{2r}|} \\ \frac{\diamond \|x\|^{5r}}{2|2^4-2^{10r}|}. \end{cases}$$

Corollary 6.11 *Let $f_q : X \to X$ be a mapping and assume that there exist real numbers $\diamondsuit$ and r such that (6.9) and (6.10), for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to X$, such that*

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\diamondsuit}{2|n^6-1|} \\ \frac{\diamondsuit \|x\|^r}{2|n^6-n^{3r}|} \\ \frac{\diamondsuit \|x\|^{5r}}{2|n^6-n^{15r}|}. \end{cases} \tag{6.13}$$

for all $x \in X$.

Corollary 6.12 *Let $f_q : X \to X$ be a mapping and assume that there exist real numbers $\diamondsuit$ and r such that (6.9) and (6.10), for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to X$, such that, for all $x \in X$*

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\diamondsuit}{2|n^8-1|} \\ \frac{\diamondsuit \|x\|^r}{2|n^8-n^{4r}|} \\ \frac{\diamondsuit \|x\|^{5r}}{2|n^8-n^{20r}|}. \end{cases} \tag{6.14}$$

Corollary 6.13 *Let $f_q : X \to X$ be a mapping and assume that there exist real numbers $\diamondsuit$ and r such that (6.9) and (6.10), for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ such that, for all $x \in X$*

$$\| f_q(x) - Q(x) \| \leq \begin{cases} \frac{\diamondsuit}{2|n^{10}-1|} \\ \frac{\diamondsuit \|x\|^r}{2|n^{10}-n^{5r}|} \\ \frac{\diamondsuit \|x\|^{5r}}{2|n^{10}-n^{25r}|}. \end{cases} \tag{6.15}$$

Remark: Let $f_q : X \to X$ be a mapping and assume that there exist real numbers $\diamondsuit$ and r such that (6.9) and (6.10), for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ such that, for all $x \in X$

$$\|f_q(x) - Q(x)\| \leq \begin{cases} \frac{\diamond}{|126|} \\ \frac{\diamond \|x\|^r}{2|2^6-2^{3r}|} \\ \frac{\diamond \|x\|^{5r}}{2|2^6-2^{15r}|}. \end{cases}$$

6.3 Quadratic Functional Equation: Even Case—Fixed Point Method

Theorem 6.14 *Let $j = \pm 1$. Let $f_q : X \to X$ be an even mapping for which there exist function $\alpha, \beta : X^5 \to [0, \infty)$ such that*

$$\sum_{k=0}^{\infty} \frac{\alpha(\eta_i^{kj} x_1, \eta_i^{kj} x_2, \eta_i^{kj} x_3, \eta_i^{kj} x_4, \eta_i^{kj} x_5)}{\eta_i^{2kj}} \tag{6.16}$$

converges in $\mathbb{R}$ and also

$$\sum_{k=0}^{\infty} \frac{\beta(\eta_i^{kj} x_1, \eta_i^{kj} x_2, \eta_i^{kj} x_3, \eta_i^{kj} x_4, \eta_i^{kj} x_5)}{\eta_i^{10kj}} \tag{6.17}$$

converges in $\mathbb{R}$, where $\eta_i = \begin{cases} n & i = 0; \\ \frac{1}{n} & i = 1 \end{cases}$ and satisfying the functional inequalities (6.5) and (6.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Assume that there exists $L = L(i) < 1$ such that the function $x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{n}, 0, 0, 0, 0\right)$ has the property

$$\frac{1}{\eta_i^2}\beta(\eta_i x) = L\beta(x), \tag{6.18}$$

for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6) and

$$\| f_q(x) - Q(x) \| \leq \frac{L^{1-i}}{1-L}\beta(x), \tag{6.19}$$

for all $x \in X$.

Proof It follows from Theorem 5.15 that Q is a quadratic mapping and satisfies (2.6), for all $x \in X$. It follows from (6.16), (6.17) and (6.6) that

$$\begin{aligned}
&\| Q(x_1x_2x_3x_4x_5) - Q(x_1)x_2^2x_3^2x_4^2x_5^2 - x_1^2Q(x_2)x_3^2x_4^2x_5^2 - x_1^2x_2^2Q(x_3)x_4^2x_5^2 \\
&-x_1^2x_2^2x_3^2Q(x_4)x_5^2 - x_1^2x_2^2x_3^2x_4^2Q(x_5) \| \\
&\leq \frac{1}{\eta_i^{2k}}\Big|\Big| f_q(\eta_i^k(x_1x_2x_3x_4x_5)) - f_q(\eta_i^kx_1)(\eta_i^{2k}x_2\eta_i^{2k}x_3\eta_i^{2k}x_4\eta_i^{2k}x_5) \\
&\quad -\eta_i^{2k}x_1f_q(\eta_i^kx_2)\eta_i^{2k}x_3\eta_i^{2k}x_4\eta_i^{2k}x_5 - \eta_i^{2k}x_1\eta_i^{2k}x_2f_q(\eta_i^kx_3)\eta_i^{2k}x_4\eta_i^{2k}x_5 \\
&\quad -\eta_i^{2k}x_1\eta_i^{2k}x_2\eta_i^{2k}x_3f_q(\eta_i^kx_4)\eta_i^{2k}x_5 - \eta_i^{2k}x_1\eta_i^{2k}x_2\eta_i^{2k}x_3\eta_i^{2k}x_4f_q(\eta_i^kx_5)\Big|\Big| \\
&\leq \frac{1}{\eta_i^{2k}}\beta(\eta_i^kx_1, \eta_i^kx_2, \eta_i^kx_3, \eta_i^kx_4, \eta_i^kx_5) \ \to 0 \ \ as \ \ k \to \infty.
\end{aligned} \tag{6.20}$$

Thus the mapping $Q : X \to X$ is a unique quadratic derivation satisfying (2.6).

Example 6.15 Let $j = \pm 1$. Let $f_q : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with the conditions (6.16) and (6.17), where $\eta_i = \begin{cases} 2 \ if \ i = 0; \\ \frac{1}{2} \ if \ i = 1 \end{cases}$ and satisfying the functional inequalities (6.5) and (6.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$, and assume that there exists $L = L(i) < 1$ such that the function $x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{2}, 0, 0, 0, 0\right)$ has the property (6.18), for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ satisfying the functional equations (2.6) and (6.19), for all $x \in X$.

Proposition 6.16 *Let* $j = \pm 1$. *Let* $f_q : X \to X$ *be a mapping for which there exist functions* $\alpha, \beta : X^5 \to [0, \infty)$ *with the conditions (6.16) and (6.17), where* $\eta_i = \begin{cases} n^2 & i = 0; \\ \frac{1}{n^2} & i = 1 \end{cases}$ *and satisfying the functional inequalities (6.5) and (6.6), for all* $x_1, x_2, x_3, x_4, x_5 \in X$*, and assume that there exists* $L = L(i) < 1$ *such that the function* $x \to \beta(x) = \frac{1}{2}\alpha\left(0, \frac{x}{n^2}, 0, 0, 0\right)$ *has property (6.18), for all* $x \in X$. *Then there exists a unique quadratic derivation* $Q : X \to X$ *satisfying the functional equations (2.6) and (6.19), for all* $x \in X$.

Example 6.17 Let $j = \pm 1$. Let $f_q : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with the conditions (6.16) and (6.17), where $\eta_i = \begin{cases} 2^2 \ if \ i = 0; \\ \frac{1}{2^2} \ if \ i = 1 \end{cases}$ and satisfying the functional inequalities (6.5) and (6.6), for all

$x_1, x_2, x_3, x_4, x_5 \in X$, and assume that there exists $L = L(i) < 1$ such that the function $x \to \beta(x) = \frac{1}{2}\alpha\left(0, \frac{x}{2^2}, 0, 0, 0\right)$ has the property (6.18), for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ satisfying the functional equations (2.6) and (6.19), for all $x \in X$.

Proposition 6.18 *Let $j = \pm 1$. Let $f_q : X \to X$ be a mapping for which there exists functions $\alpha, \beta : X^5 \to [0, \infty)$ with the conditions (6.16) and (6.17), where $\eta_i = \begin{cases} n^3 & i = 0; \\ \frac{1}{n^3} & i = 1 \end{cases}$ and satisfying the functional inequalities (6.5) and (6.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$, and assume that there exists $L = L(i) < 1$ such that the function $x \to \beta(x) = \frac{1}{2}\alpha\left(0, 0, \frac{x}{n^3}, 0, 0\right)$ has property (6.18), for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ satisfying the functional equations (2.6) and (6.19), for all $x \in X$.*

Proposition 6.19 *Let $j = \pm 1$. Let $f_q : X \to X$ be a mapping for which there exists functions $\alpha, \beta : X^5 \to [0, \infty)$ with the conditions (6.16) and (6.17), where $\eta_i = \begin{cases} n^4 & i = 0; \\ \frac{1}{n^4} & i = 1 \end{cases}$ and satisfying the functional inequalities (6.5) and (6.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$, and assume that there exists $L = L(i) < 1$ such that the function $x \to \beta(x) = \frac{1}{2}\alpha\left(0, 0, 0, \frac{x}{n^4}, 0\right)$ has property (6.18), for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ satisfying the functional equations (2.6) and (6.19), for all $x \in X$.*

Proposition 6.20 *Let $j = \pm 1$. Let $f_q : X \to X$ be a mapping for which there exists functions $\alpha, \beta : X^5 \to [0, \infty)$ with the conditions (6.16) and (6.17), where $\eta_i = \begin{cases} n^5 & i = 0; \\ \frac{1}{n^5} & i = 1 \end{cases}$ and satisfying the functional inequalities (6.5) and (6.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$, and assume that there exists $L = L(i) < 1$ such that the function $x \to \beta(x) = \frac{1}{2}\alpha\left(0, 0, 0, 0, \frac{x}{n^5}\right)$ has property (6.18), for all $x \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ satisfying the functional equations (2.6) and (6.19), for all $x \in X$.*

This following corollaries are the immediate consequences of Theorem 6.14, Propositions 6.16–6.20 respectively, concerning the stability of (2.6).

Corollary 6.21 *Let $f_q : X \to X$ be a mapping and there exist real numbers $\diamondsuit$ and r such that the inequalities (6.9) and (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ such that the functional inequality (6.11), for all $x \in X$.*

Remark: Let $f_q : X \to X$ be a mapping and there exist real numbers $\diamondsuit$ and r such that the inequalities (6.9) and (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ such that the functional inequality

$$\|f_q(x) - Q(x)\| \le \begin{cases} \frac{\diamond}{|6|} \\ \frac{\diamond\|x\|^r}{2|2^2-2^r|} \\ \frac{\diamond\|x\|^{5r}}{2|2^2-2^{5r}|}, \end{cases}$$

for all $x \in X$.

Corollary 6.22 *Let $f_q : X \to X$ be a mapping and there exist real numbers $\diamondsuit$ and r such that the inequalities (6.9) and (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ such that the functional inequality (6.12), for all $x \in X$.*

Remark: Let $f_q : X \to X$ be a mapping and there exist real numbers $\diamondsuit$ and r such that the inequalities (6.9) and (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ such that the functional inequality

$$\|f_q(x) - A(x)\| \le \begin{cases} \frac{\diamond}{|30|} \\ \frac{\diamond\|x\|^r}{2|2^4-2^{2r}|} \\ \frac{\diamond\|x\|^{5r}}{2|2^4-2^{10r}|}, \end{cases}$$

for all $x \in X$.

Corollary 6.23 *Let $f_q : X \to X$ be a mapping and there exist real numbers $\diamondsuit$ and r such that the inequalities (6.9) and (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ such that the functional inequality (6.13), for all $x \in X$.*

Corollary 6.24 *Let $f_q : X \to X$ be a mapping and there exist real numbers $\diamondsuit$ and r such that the inequalities (6.9) and (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ such that the functional inequality (6.14), for all $x \in X$.*

Remark: Let $f_q : X \to X$ be a mapping and there exist real numbers $\diamondsuit$ and r such that the inequalities (6.9) and (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ such that the functional inequality

$$\|f_q(x) - A(x)\| \le \begin{cases} \frac{\diamond}{|126|} \\ \frac{\diamond\|x\|^r}{2|2^6-2^{3r}|} \\ \frac{\epsilon\|x\|^{5r}}{2|2^6-2^{15r}|}, \end{cases}$$

for all $x \in X$.

Corollary 6.25 *Let $f_q : X \to X$ be a mapping and there exist real numbers $\Diamond$ and r such that the inequalities (6.9) and (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exists a unique quadratic derivation $Q : X \to X$ such that the functional inequality (6.15), for all $x \in X$.*

Chapter 7
n-Dimensional Quadratic Functional Equations in Generalized 2-Normed Spaces

7.1 *n*-Dimensional Quadratic Functional Equation

A Hyers-Ulam stability problem for the above quadratic functional equation was proved by Skof for mapping $f : E_1 \to E_2$, where E_1 is a normed space and E_2 a Banach space (see [90]). Cholewa [15] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an abelian group. The quadratic functional equation and several other functional equations are useful to characteristic inner product space.

$$f\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1}^{n} f\left(-x_j + \sum_{i=1, i\neq j}^{n} x_i\right) = (n-3) \sum_{1\leq i<j\leq n} f(x_i + x_j) + (-n^2 + 5n - 2)\sum_{i=1}^{n} f(x_i) \quad (7.1)$$

where n is a positive integer with $n \geq 3$.

In this chapter, we discuss about the general solution and stability of the n-dimensional quadratic functional equation (7.1) in generalized 2-normed space.

Theorem 7.1 *Let X and Y be real vector spaces. A mapping $f : X \to Y$ satisfies the functional equation (5.1), for all $x, y \in X$ if and only if $f : X \to Y$ satisfies the functional equation (7.1), for all $x_1, x_2, \ldots x_n \in X$.*

Proof Let $f : X \to Y$ satisfy the functional equation (7.1). Replacing $(x_1, x_2, \ldots x_n)$ by $(0, 0, \ldots 0)$ in (7.1), we get $f(0) = 0$. Now replacing $(x_1, x_2, \ldots x_n)$ by $(0, 0, \ldots 0)$ in (7.1), we obtain

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$$f(x_1+x_2)+f(-x_1+x_2)+f(x_1-x_2)+f(x_1+x_2)+f(x_1+x_2)$$
$$=2f(x_1+x_2)+2f(x_2)+2f(x_1)+2f(x_1)+2f(x_2)$$
$$+2f(x_2-2f(x_2)-2f(x_2)+2f(x_2), \tag{7.2}$$

for all $x \in X$. It follows from (7.2) that

$$f(x_1+x_2)+f(x_{-}x_2)+(n-1)f(x_1+x_2)=(n-3)[f(x_1+x_2)$$
$$+(n-2)f(x_1)+(n-2)f(x_2)]+(-n^2+5n-2)$$
$$+[f(x_1)+f(x_2)], \tag{7.3}$$

for all $x_1, x_2 \in X$. Replacing (x_1, x_2) by (x, y) in (7.1) and using evenness of f and rearranging, our result is obtained.

Conversely, assume that $f : X \to Y$ satisfies (5.1) and using evenness of f we get $f(0) = 0$. Letting $y = 0$ in (5.1), we obtain $f(-x) = f(x)$ for all $x \in X$. Therefore f is an even mapping. Replacing y by x and $2x$ respectively in (5.1), we get $f(2x) = 4f(x)$ and $f(3x) = 9f(x)$. In general for any positive integer a, we have $f(ax) = a^2 f(x)$ for all $x \in X$. Replacing (x, y) by (x_1, x_2) in (5.1) and using (5.1) we get,

$$f(x_1+x_2)+f(x_1-x_2)=2f(x_1)+2f(x_2), \tag{7.4}$$

for all $x_1, x_2 \in X$. Replacing (X_1, X_2) by (x_1+x_2, x_3) in (7.4), we obtain

$$f(x_1+x_2+x_3)+f(x_1+x_2-x_3)=2f(x_1+x_2)+2f(x_3), \tag{7.5}$$

for all $x_1, x_2, x_3 \in X$. Replacing x_2 by $-x_2$ in (7.5), we get

$$f(x_1-x_2+x_3)+f(x_1-x_2-x_3)=2f(x_1-x_2)+2f(x_3), \tag{7.6}$$

for all $x_1, x_2, x_3 \in X$. Using evenness in (7.6), we get

$$f(x_1-x_2+x_3)+f(-x_1+x_2+x_3)=2f(x_1-x_2)+2f(x_3), \tag{7.7}$$

for all $x_1, x_2, x_3 \in X$. Adding (7.5), (7.7) and using (7.4), we obtain

$$f(x_1+x_2+x_3)+f(-x_1+x_2+x_3)+f(x_1-x_2+x_3)$$
$$+f(x_1+x_2-x_3)=4f(x_1)+4f(x_2)+4f(x_3), \tag{7.8}$$

for all $x_1, x_2, x_3 \in X$. Replacing (x, y) by (x_1+x_4, x_2+x_3) in (5.1), we get

$$f(x_1+x_2+x_3+x_4)+f(x_1-x_2+x_3+x_4)=2f(x_1+x_4)$$
$$+f(x_2+x_3), \tag{7.9}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(x_1 + x_2 + x_3 + x_4)$ on both sides in (7.9), we get

$$2f(x_1 + x_2 + x_3 + x_4) + f(x_1 - x_2 - x_3 + x_4) = f(x_1 + x_2 + x_3 + x_4) + 2f(x_1 + x_4) + f(x_2 + x_3), \tag{7.10}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(x_1 - x_2 + x_3 - x_4)$ on both sides in (7.10), we get

$$2f(x_1 + x_2 + x_3 + x_4) + f(x_1 - x_2 - x_3 + x_4) + f(x_1 - x_2 + x_3 - x_4) = f(x_1 + x_2 + x_3 + x4) + f(x_1 - x_2 + x_3 - x_4) + 2f(x_1 + x_4) + f(x_2 + x_3), \tag{7.11}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (7.10) in (7.11) and rearranging, we get

$$2f(x_1 + x_2 + x_3 + x_4) + 2f(x_1 - x_2) + 2f(x_3 - x_4) = 2f(x_1 + x_3) + 2f(x_2 + x_4) + 2f(x_1 + x_4) + 2f(x_2 + x_3), \tag{7.12}$$

for all $x_1, x_2, x_3, x_4 \in X$. Dividing by 2 in (7.12) on both sides, we have

$$f(x_1 + x_2 + x_3 + x_4) + f(x_1 - x_2) + f(x_3 - x_4) = f(x_1 + x_3) + f(x_2 + x_4) + f(x_1 + x_4) + f(x_2 + x_3), \tag{7.13}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(x_1 + x_2) + f(x_3 + x_4)$ on both sides, we obtain

$$f(x_1 + x_2 + x_3 + x_4) + f(x_1 + x_2) + f(x_3 + x_4) + f(x_1 - x_2) + f(x_3 - x_4) = f(x_1 + x_2) + f(x_3 + x_4) + f(x_1 + x_3) + f(x_2 + x_4) + f(x_1 + x_4) + f(x_2 + x_3), \tag{7.14}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (5.1) in (7.14) and rearranging, we have

$$f(x_1 + x_2 + x_3 + x_4) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) = f(x_1 + x_2) + f(x_3 + x_4) + f(x_1 + x_3) + f(x_2 + x_4) + f(x_1 + x_4) + f(x_2 + x_3), \tag{7.15}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4)$ on both sides in (7.15), we have

$$f(x_1 + x_2 + x_3 + x_4) + 4f(x_1) + 4f(x_2) + 4f(x_3) + 4f(x_4) = f(x_1 + x_2) + f(x_3 + x_4) + f(x_1 + x_3) + f(x_2 + x_4) + f(x_1 + x_4) + f(x_2 + x_3) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4), \tag{7.16}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (7.8) and rearranging, we obtain

$$\begin{aligned} &f(x_1+x_2+x_3+x_4)+2[f(x_1)+f(x_2)]+2[f(x_3)+f(x_4)]=f(x_1+x_2)\\ &+f(x_3+x_4)+f(x_1+x_3)+f(x_2+x_4)+f(x_1+x_4)+f(x_2+x_3)\\ &+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4), \end{aligned} \tag{7.17}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (5.1) in (7.17), we get

$$\begin{aligned} &f(x_1+x_2+x_3+x_4)+2[f(x_1+x_2)+f(x_1-x_2)]+2[f(x_3+x_4)\\ &+f(x_3-x_4)]=f(x_1+x_2)+f(x_3+x_4)+f(x_1+x_3)+f(x_2+x_4)\\ &+f(x_1+x_4)+f(x_2+x_3)+2f(x_1)+2f(x_2)\\ &+2f(x_3)+2f(x_4), \end{aligned} \tag{7.18}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (7.10) and remodifying, we get

$$\begin{aligned} &f(x_1+x_2+x_3+x_4)+2f(x_1+x_2)+2f(x_1-x_2)+2f(x_3+x_4)+\\ &2f(x_3-x_4)=f(x_1+x_2)+f(x_3+x_4)+f(x_1+x_3)+f(x_2+x_4)\\ &+f(x_1+x_4)+f(x_2+x_3)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4), \end{aligned} \tag{7.19}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (5.1) in (7.19), we obtain

$$\begin{aligned} &f(x_1+x_2+x_3+x_4)+f(-x_1+x_2+x_3+x_4)+f(x_1-x_2+x_3\\ &+x_4)+f(x_1+x_2-x_3+x_4)+f(x_1+x_2+x_3-x_4)=f(x_1+x_2)\\ &+f(x_1+x_3)+f(x_1+x_4)+f(x_2+x_3)+f(x_2+x_4)\\ &+f(x_3+x_4)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4), \end{aligned} \tag{7.20}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Similarly, one can easily verify for five variables that

$$\begin{aligned} &f(x_1+x_2+x_3+x_4+x_5)+f(-x_1+x_2+x_3+x_4+x_5)\\ &+f(x_1-x_2+x_3+x_4+x_5)+f(x_1+x_2-x_3+x_4+x_5)\\ &+f(x_1+x_2+x_3-x_4+x_5)+f(x_1+x_2+x_3+x_4-x_5)=(5-3)\\ &\sum_{1\le i<j\le 5} f(x_i+y_i)+(-5^2+5.5-2)\sum_{i=1}^{5} f(x_i), \end{aligned} \tag{7.21}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Extending this result, for any positive n, we can get (7.1), for all $x_1, x_2, x_3, \ldots, x_n \in X$.

Remark: Every normed space $(X, \|.\|)$ defines a random normed space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t+\|x\|},$$

and T_M is the minimum t- norm. This space is called the induced random normed space.

7.2 Solution of n-Dimensional Quadratic Functional Equation

Definition 7.2 Let X be a linear space. A function $N(.,.) : X \times X \to [0, \infty)$ is called a generalized 2-normed space if it satisfies the following.

(G1) $N(x, y) = 0$ if and only if x and y are linearly independent vectors.
(G2) $N(x, y) = N(y, x)$, for all $x, y \in X$.
(G3) $N(\lambda x, y) = |\lambda| N(x, y)$ for all $x, y \in X$ and $X = \varphi$, φ is a real or complex field.
(G4) $N(x + y, z) \leq N(x, z) + N(y, z)$, for all $x, y, z \in X$.

The generalized 2- normed space is denoted by $(X, N(.,.))$.

Remark: If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence in X such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.

Definition 7.3 A sequence $\{x_n\}$ in a generalized 2-normed space $(X, N(.,.))$ is called convergent if there exists $x \in X$ such that $\lim\limits_{k\to\infty} N(x_n - x, y) = 0$ and we will denote $\lim\limits_{k\to\infty} N(x_n, y) = N(x, y)$, for all $x, y \in X$.

Definition 7.4 A sequence $\{x_n\}$ in a generalized 2-normed linear space $(X, N(.,.))$ is called a Cauchy sequence if there exist two linearly independent elements y and z in X, such that $\{N(x, y)\}$ and $\{N(x_n, z)\}$ are real Cauchy sequences.

Remark: Let $(X, \|.\|)$ be a normed space. Let $T(a, b) = (a, b, \min(a_2 + b_2, 1))$, for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, v be membership and non - membership degree of an intuitionistic fuzzy set defined by

$$P_{\mu,v}(x, t) = (\mu_x)(t), v_x(t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\right), \forall t \in R^+.$$

Then $(X, P_{\mu,v}, T)$ is an IFN—space.

Definition 7.5 A generalized 2-normed space $(X, N(.,.))$ is called generalized 2-Banach space if every Cauchy sequence is convergent.

Theorem 7.6 *Let $j \in \{-1, 1\}$. Let $\theta : X^n \to [0, \infty)$ be a function such that*

$$\sum_{l=0}^{\infty} \frac{\theta(2^{lj}x_1, 2^{lj}x_2, 2^{lj}x_3, \ldots, 2^{lj}x_n)}{2^{mj}}, \tag{7.22}$$

converges for all $x_1, x_2, x_3, x_4, \ldots, x_n \in X$. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$N\left(f\left(\sum_{i=1}^{x_n} x_i\right) + \sum_{j=1}^{n} f\left(-x_j + \sum_{i=1, i\neq j}^{n} x_i\right) - (n-3) \sum_{i,j=1,\ 1\leq i<j\leq n} f(x_i + x_j) \right.$$
$$\left. -(n^2+5n+2)\sum_{i=1}^{n} f(x_i), t\right) \leq \theta(x_1, x_2, x_3, x_4, \ldots, x_n), \tag{7.23}$$

for all $x_1, x_2, x_3, x_4, \ldots, x_n \in X$ and $t \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \leq \frac{1}{8(n-5)} \sum_{l=\frac{1-j}{2}}^{\infty} \frac{\theta(2^{lj}x, -2^{lj}x, 2^{lj}x, -2^{lj}x, 2^{lj}x, 0, \ldots, 0, t)}{2^{2lj}}, \tag{7.24}$$

for all $x \in X$ and all $t \in X$. The mapping $Q(x)$ is defined by

$$\lim_{l\to\infty} N\left(Q(x) - \frac{f(2^{lj}x)}{2^{2lj}}, t\right) = 0, \tag{7.25}$$

for all $x \in X$ and all $t \in X$.

Proof Assume that $j = -1$. Replacing $(x_1, x_2, x_3, x_4, x_5, x_6, \ldots, x_n)$ by $(x, -x, x, -x, x, -x, 0, .., 0, t)$ in (7.23), we get

$$N\left(\frac{f(2x)}{2^2} - f(x), t\right) \leq \frac{\theta(x, -x, x, -x, x, -x, 0, .., 0, t)}{8(n-5)2^2}, \tag{7.26}$$

for all $x \in X$ and all $t \in X$. Replacing x by $2x$ and dividing by 2^2 in (7.26), we get

$$N\left(\frac{f(2^2x)}{2^4} - \frac{f(2x)}{2^2}, t\right) \leq \frac{\theta(2x, -2x, 2x, -2x, 2x, -2x, 0, \ldots, 0, t)}{8(n-5)2^4}, \tag{7.27}$$

for all $x \in X$ and all $t \in X$. Combining (7.26) and (7.27) and using (G4), we get

$$N\left(\frac{f(2^2x)}{2^4} - f(x), t\right) \leq N\left(\frac{f(2^2x)}{2^4} - \frac{f(2x)}{2^2}, t\right) + N\left(\frac{f(2x)}{2^2} - f(x), t\right)$$
$$\leq \frac{1}{8(n-5)2^2}\left[\theta(x, -x, x, -x, x, -x, 0, \ldots, 0, t) \right.$$
$$\left. + \frac{\theta(2x, -2x, 2x, -2x, 2x, -2x, 0, \ldots, 0, t)}{2^2}\right], \tag{7.28}$$

for all $x \in X$ and all $t \in X$. In general, for any positive integer l, we have

$$
\begin{aligned}
&N\left(\frac{f(2^m x)}{2^{2m}} - f(x), t\right)\\
&\quad \le \frac{1}{8(n-5)2^2}\sum_{k=0}^{l-i}\frac{\theta(2^k x, -2^k x, 2^k x, -2^k x, 2^k x, -2^k x, 0, \ldots, 0, t)}{2^{2k}},\\
&\quad \le \frac{1}{8(n-5)2^2}\sum_{k=0}^{\infty}\frac{\theta(2^k x, -2^k x, 2^k x, -2^k x, 2^k x, -2^k x, 0, \ldots, 0, t)}{2^{2k}},
\end{aligned}
\tag{7.29}
$$

for all $x \in X$ and all $t \in X$. In order to prove the convergence of the sequence $\left\{\frac{f(2^l x)}{2^{2l}}\right\}$, replace x by $2^r x$ and divide by 2^{2r} in (7.29), for any $m, l > 0$. Then we obtain

$$
\begin{aligned}
&N\left(\frac{f(2^{l+r} x)}{2^{2(l+r)}} - \frac{f(2^r x)}{2^{2r}}, t\right)\\
&= \frac{1}{8(n-5)2^{2r}} N\left(\frac{f(2^l.2^r x)}{2^{2l}} - f(2^r x), t\right),\\
&\le \frac{1}{8(n-5)2^2}\sum_{k=0}^{\infty}\frac{\theta(2^{k+r} x, -2^{k+r} x, 2^{k+r} x, -2^{k+r} x, 2^{k+r} x, 0, \ldots, 0, t)}{2^{2(k+r)}}\\
&\to 0 \ \ as \ \ r \to \infty,
\end{aligned}
\tag{7.30}
$$

for all $x \in X$ and all $t \in X$. So

$$
\begin{aligned}
&N\left(\frac{f(2^{l+r} x)}{2^{2(l+r)}} - \frac{f(2^r x)}{2^{2r}}, i\right)\\
&= \frac{1}{8(n-5)2^{2r}} N\left(\frac{f(2^l.2^r x)}{2^{2l}} - f(2^r x), t\right),\\
&\le \frac{1}{8(n-5)2^2}\sum_{k=0}^{\infty}\frac{\theta(2^{k+r} x, -2^{k+r} x, 2^{k+r} x, -2^{k+r} x, 2^{k+r} x, 0, \ldots, 0, t)}{2^{2(k+r)}}\\
&\to 0 \ \ as \ \ r \to \infty,
\end{aligned}
\tag{7.31}
$$

for all $x \in X$ and all $i \in X$. Hence, there exist two linearly independent elements t and i in X such that $\left\{N\left(\frac{f(2^l x)}{2^{2l}}, t\right)\right\}$ and $\left\{N\left(\frac{f(2^l x)}{2^{2l}}, i\right)\right\}$ are real Cauchy sequences. Thus the sequence $\left\{\frac{f(2^l x)}{2^{2l}}\right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $Q : X \to Y$ such that

$$
\lim_{k\to\infty} N\left(Q(x) - \frac{f(2^l x)}{2^2 l}, t\right) = 0,
\tag{7.32}
$$

for all $x \in X$ and all $t \in X$. Now we need to prove that Q satisfies (5.1). Replacing $(x_1, x_2, x_3, x_4, x_5, x_6, \ldots, x_n)$ by $(2^l x, -2^l x, 2^l x, -2^l x, 2^l x, 0, \ldots, 0)$ in (7.23), we get

$$N\Bigg(\frac{1}{2^{2l}}\Bigg[f\Bigg(2^l\Bigg(\sum_{i=1}^{n} x_i\Bigg)\Bigg) + \sum_{j=1}^{n} f\Bigg(2^l\Bigg(-x_j + \sum_{i=1, i\neq j}^{n} x_i\Bigg)\Bigg) - (n-3)$$
$$\sum_{i,j=1, 1\leq i<j\leq n} f(2^l(x_i + x_j)) - (-n^2 + 5n + 2)\sum_{i=1}^{n} f(2^l(x_i))\Bigg], t\Bigg)$$
$$\leq \frac{1}{2^{2l}}\theta(2^l x_1, 2^l x_2, 2^l x_3, 2^l x_4, \ldots, 2^l x_n), \tag{7.33}$$

for all $x_1, x_2, x_3, x_4, x_5, x_6, \ldots, x_n \in X$ and all $t \in X$. Now

$$N\Bigg(Q\Bigg(\sum_{i=1}^{n})\Bigg) + \sum_{j=1}^{n} Q\Bigg(-x_j + \sum_{i=1, i\neq j}^{n} x_i\Bigg) - (n-3)$$
$$\sum_{i,j=1, 1\leq i<j\leq n} Q(x_i + x_j) - (n^2 + 5n + 2)\sum_{i=1}^{n} Q(x_i), t\Bigg)$$
$$= N\Bigg(Q\Bigg(\sum_{i=1}^{n} x_i\Bigg) - \frac{1}{2^{2l}} f\Bigg(2^l\Bigg(\sum_{i=1}^{n} x_i\Bigg)\Bigg), t\Bigg)$$
$$+N\Bigg(\sum_{j=1}^{n} Q\Bigg(-x_j + \sum_{i=1, i\neq j}^{n} x_i\Bigg)\Bigg) - \frac{1}{2^{2l}}\sum_{j=1}^{n} Q\Bigg(2^l\Bigg(-x_j + \sum_{i=1, i\neq j}^{n} x_i\Bigg)\Bigg), t\Bigg)$$
$$+N\Bigg((n-3)\sum_{1\leq i<j\leq n} Q(x_i + x_j - \frac{1}{2^{2l}}(n-3)\sum_{1\leq i<j\leq n} f(2^l(x_i + x_j)\Bigg), t\Bigg)$$
$$+N\Bigg((-n^2 + 5n + 2)\sum_{i=1}^{n} Q(x_i) - \frac{1}{2^{2l}}(-n^2 + 5n + 2)\sum_{i=1}^{n} f(2^l(x_i)), t\Bigg)$$
$$+N\Bigg(\frac{1}{2^{2l}}\Bigg[Q\Bigg(2^l\Bigg(\sum_{i=1}^{n} x_i\Bigg)\Bigg) + \sum_{j=1}^{n} Q\Bigg(2^l\Bigg(-x_j + \sum_{i=1, i\neq j} x_i\Bigg)\Bigg)$$
$$-(n-3)\sum_{1\leq i<j\leq n} f(2^l(x_i + x_j)) - (-n^2 + 5n + 2)\sum_{i=1}^{n} f(2^l x_i)\Bigg], t\Bigg), \tag{7.34}$$

for all $x_1, x_2, x_3, x_4, x_5, x_6, \ldots, x_n \in X$ and all $t \in X$. Hence it follows from (7.32), (7.33) and (7.34) that

$$N\left(Q\left(\sum_{i=1}^{n})\right)+\sum_{j=1}^{n}Q\left(-x_j+\sum_{i=1,i\neq j}^{n}x_i\right)-(n-3)\right.$$
$$\left.\sum_{i,j=1,1\le i<j\le n}Q(x_i+x_j)-(n^2+5n+2)\sum_{i=1}^{n}Q(x_i),t\right)$$
$$=0+0+0+0+\frac{1}{2^{2l}}\theta(2^l x_1,2^l x_2,2^l x_3,\ldots,2^l x_n), \tag{7.35}$$

for all $x_1, x_2, x_3, x_4, x_5, x_6, \ldots, x_n \in X$ and all $t \in X$. Letting $l \to \infty$ in (7.35) and using (7.22), we see that

$$Q\left(\sum_{i=1}^{n})x_i\right)+\sum_{j=1}^{n}Q\left(-x_j+\sum_{i=1,i\neq j}^{n}x_i\right)=(n-3)\sum_{i,j=1,1\le i<j\le n}Q(x_i+x_j)$$
$$+(n^2+5n+2)\sum_{i=1}^{n}Q(x_i),$$

Using (G1), we see that Q satisfies (7.1). In order to prove that $Q(x)$ is unique, let $R(x)$ be another quadratic mapping satisfying (7.1) and (7.24). We get

$$N(Q(x)-R(x),u)=\frac{1}{2^{2l}}N(Q(2^l x)-R(2^l x),u)$$
$$\le\frac{1}{2^{2l}}\{N(Q(2^l x)-f(2^l x),u)+N(f(2^l x)-R(2^l x),u)\}$$
$$\le\sum_{k=0}^{l-1}\frac{2\theta(2^l x_1,2^l x_2,2^l x_3,\ldots,2^l x_n)}{2^{2(k+l)}}\ \to 0\ \ as\ \ l\to\infty,$$

for all $x \in X$ and all $t \in X$. Hence Q is unique. For $j = -1$, we can prove that the similar stability result. This completes the proof of the theorem.

The following Corollary is an immediate consequence of Theorem 7.6 concerning the stability of (7.1).

Corollary 7.7 *Let λ and s be positive real numbers and suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$N\left(f\left(\sum_{i=1}^{x_n}x_i\right)+\sum_{j=1}^{n}f\left(-x_j+\sum_{i=1,i\neq j}^{n}x_i\right)-(n-3)\sum_{i,j=1,\ 1\le i<j\le n}f(x_i+x_j)\right.$$
$$\left.-(n^2+5n+2)\sum_{i=1}^{n}f(x_i),t\right),$$

$$\leq \begin{cases} \lambda, \\ \lambda \sum_{i=1}^{n} \parallel x_i \parallel^s, & s \neq 2 \\ \lambda \left\{ \prod_{i=1}^{n} x_i \parallel^s + \sum_{i=1}^{n} x_i \parallel^{ns} \right\}, & s \neq \frac{2}{n}, \end{cases}$$

for all $x_1, x_2, x_3, x_4, \ldots, x_n \in X$ *and all* $t \in X$*. Then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$N(f(x) - Q(x), t) \leq \begin{cases} \frac{\lambda}{||(n-5)} \\ \frac{\lambda \|x\|^s}{2(n-5)|2^2-2^s|} \\ \frac{\lambda \|x\|^{ns}}{2(n-5)|2^2-2^{ns}|}, \end{cases}$$

for all $x \in X$ *and* $t \in X$*.*

Chapter 8
Additive-Quadratic Functional Equations

8.1 Additive Quadratic Functional Equation

The Hyers-Ulam stability of mappings is an interesting application of this theory to various mathematical problems. The Hyers-Ulam stability has been mainly used to study problems concerning approximate isometries or quasi-isometries, and the stability of Lorentz and conformal mappings, the stability of stationary points, the stability of convex mappings, or homogeneous mappings, etc [3, 15, 41, 43, 93].

In this chapter, we discuss the stability of the quadratic functional equation (2.6) for mixed cases in Banach space with the help of direct and fixed point methods.

8.2 Additive Quadratic Functional Equation: Mixed Case—Direct Method

Theorem 8.1 *Let $j \in \{-1, 1\}$ and $\alpha : X^5 \to [0, \infty)$ be a function satisfying (2.24) and (5.40), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f : X \to Y$ be a mapping satisfying the inequality*

$$\| Df(x_1, x_2, x_3, x_4, x_5) \| \leq \alpha(x_1, x_2, x_3, x_4, x_5). \tag{8.1}$$

Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (2.6) and

$$\| f(x) - A(x) - Q(x) \| \leq \frac{1}{2}\Bigg[\frac{1}{4n} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(n^{kj}x,0,0,0,0)}{n^{kj}} + \frac{\alpha(n^{-kj}x,0,0,0,0)}{n^{kj}} \right) + \frac{1}{2n^2} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(n^{kj}x,0,0,0,0)}{n^{2kj}} + \frac{\alpha(n^{-kj}x,0,0,0,0)}{n^{2kj}} \right) \Bigg], \tag{8.2}$$

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for all $x \in X$. The mappings $A(x)$ and $Q(x)$ are defined in (2.27) and (5.43) respectively, for all $x \in X$.

Proof Let $f_0(x) = \frac{f_a(x)-f_a(-x)}{2}$. Then $f_0(0) = 0$ *and* $f_0(-x) = -f_0(x)$, for all $x \in X$. Then

$$\begin{aligned} \| Df_0(x, y, z) \| &\leq \frac{1}{2}\big(\| Df(x_1, x_2, x_3, x_4, x_5) \| \\ &\quad + \| Df(-x_1, -x_2, -x_3, -x_4, -x_5) \| \big) \\ &\leq \frac{\alpha(x_1, x_2, x_3, x_4, x_5)}{2} + \frac{\alpha(-x_1, -x_2, -x_3, -x_4, -x_5)}{2} \end{aligned} \tag{8.3}$$

for all $x, y, z \in X$. By Theorem 2.2, we have

$$\| f_0(x) - A(x) \| \leq \frac{1}{8n} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(n^{kj}x, 0, 0, 0, 0)}{n^{kj}} + \frac{\alpha(n^{-kj}x, 0, 0, 0, 0)}{n^{kj}} \right). \tag{8.4}$$

Moreover $f_e(-x) = f_e(x)$, for all $x \in X$. Hence,

$$\begin{aligned} \| Df_e(x, y, z) \| &\leq \frac{1}{2}\big(\| Df_e(x_1, x_2, x_3, x_4, x_5) \| \\ &\quad + \| Df_e(-x_1, -x_2, -x_3, -x_4, -x_5) \| \big\} \\ &\leq \frac{\alpha(x_1, x_2, x_3, x_4, x_5)}{2} + \frac{\alpha(-x_1, -x_2, -x_3, -x_4, -x_5)}{2}, \end{aligned} \tag{8.5}$$

for all $x, y, z \in X$. By Theorem 5.2, we have

$$\| f_e(x) - Q(x) \| \leq \frac{1}{4n^2} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(n^{kj}x, 0, 0, 0, 0)}{n^{2kj}} + \frac{\alpha(n^{-kj}x, 0, 0, 0, 0)}{n^{2kj}} \right), \tag{8.6}$$

for all $x \in X$. Define

$$f(x) = f_e(x) + f_0(x), \tag{8.7}$$

for all $x \in X$. It follows from (8.4), (8.6) and (8.7) that

$$\begin{aligned} \| f(x) - A(x) - Q(x) \| &= \| f_e(x) + f_a(x) - A(x) - Q(x) \| \\ &\leq \| f_0(x) - A(x) \| + \| f_e(x) - Q(x) \| \\ &\leq \frac{1}{8n} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(n^{kj}x, 0, 0, 0, 0)}{n^{kj}} + \frac{\alpha(n^{-kj}x, 0, 0, 0, 0)}{n^{kj}} \right) \\ &\quad + \frac{1}{4n^2} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(n^{kj}x, 0, 0, 0, 0)}{n^{2kj}} + \frac{\alpha(n^{-kj}x, 0, 0, 0, 0)}{n^{2kj}} \right), \end{aligned} \tag{8.8}$$

for all $x \in X$. Hence the theorem is proved.

Example 8.2 Let $j \in \{-1, 1\}$ and $\alpha : X^5 \to [0, \infty)$ be a function satisfying (2.24) and (5.40), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f : X \to Y$ be a mapping satisfying the inequality (8.1), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ which satisfies the functional equation (2.6) and

$$\begin{aligned}\| f(x) - A(x) - Q(x) \| \leq \frac{1}{2}\Bigg[\frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(2^{kj}x, 0, 0, 0, 0)}{2^{2kj}} + \frac{\alpha(2^{-kj}x, 0, 0, 0, 0)}{2^{kj}}\right) \\ + \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(2^{2kj}x, 0, 0, 0)}{2^{2kj}} + \frac{\alpha(2^{-kj}x, 0, 0, 0)}{2^{2kj}}\right)\Bigg],\end{aligned}$$

for all $x \in X$. The mappings $A(x)$ and $Q(x)$ are defined by $A(x) = \lim_{k\to\infty} \frac{f_a(2^{kj}x)}{2^{2kj}}$ and $Q(x) = \lim_{k\to\infty} \frac{f_q(2^{kj}x)}{n^{2kj}}$, for all $x \in X$.

Proposition 8.3 *Let $j \in \{-1, 1\}$ and $\alpha : X^5 \to [0, \infty)$ be a function satisfying (2.24) and (5.40), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f : X \to Y$ be a mapping satisfying the inequality (8.1), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (2.6) and*

$$\begin{aligned}\| f(x) - A(x) - Q(x) \| \leq \frac{1}{2}\Bigg[\frac{1}{4n^2}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(0, n^{2kj}x, 0, 0, 0)}{n^{2kj}} + \frac{\alpha(0, n^{-2kj}x, 0, 0, 0)}{n^{2kj}}\right) \\ + \frac{1}{2n^4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(0, n^{2kj}x, 0, 0, 0)}{n^{4kj}} + \frac{\alpha(0, n^{-2kj}x, 0, 0, 0)}{n^{4kj}}\right)\Bigg],\end{aligned}$$

for all $x \in X$. The mappings $A(x)$ and $Q(x)$ are defined by $A(x) = \lim_{k\to\infty} \frac{f_a(n^{2kj}x)}{n^{2kj}}$ and $Q(x) = \lim_{k\to\infty} \frac{f_q(n^{2kj}x)}{n^{4kj}}$, for all $x \in X$.

Example 8.4 Let $j \in \{-1, 1\}$ and $\alpha : X^5 \to [0, \infty)$ be a function satisfying (2.24) and (5.40), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f : X \to Y$ be a mapping satisfying the inequality (8.1), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ which satisfies the functional equation (2.6) and

$$\| f(x) - A(x) - Q(x) \| \le \frac{1}{2}\Bigg[\frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(0, 2^{2kj}x, 0, 0, 0, 0)}{2^{2kj}} + \frac{\alpha(2^{-2kj}x, 0, 0, 0, 0)}{2^{2kj}}\right) + \frac{1}{32}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(2^{2kj}x, 0, 0, 0)}{2^{4kj}} + \frac{\alpha(2^{-kj}x, 0, 0, 0)}{2^{4kj}}\right)\Bigg],$$

for all $x \in X$. The mappings $A(x)$ and $Q(x)$ are defined by $A(x) = \lim_{k\to\infty} \frac{f_a(2^{kj}x)}{2^{2kj}}$ and $Q(x) = \lim_{k\to\infty} \frac{f_q(2^{kj}x)}{n^{4kj}}$, for all $x \in X$.

Proposition 8.5 *Let $j \in \{-1, 1\}$ and $\alpha : X^5 \to [0, \infty)$ be a mapping satisfying (2.24) and (5.40), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f : X \to Y$ be a function satisfying the inequality (8.1), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique mapping $A : X \to Y$ and a unique mapping $Q : X \to Y$ which satisfies the functional equation (2.6) and*

$$\| f(x) - A(x) - Q(x) \| \le \frac{1}{2}\Bigg[\frac{1}{4n^3}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(0, 0, n^{3kj}x, 0, 0)}{n^{3kj}} + \frac{\alpha(0, 0, n^{-3kj}x, 0, 0)}{n^{3kj}}\right) + \frac{1}{2n^6}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(0, 0, n^{3kj}x, 0, 0)}{n^{6kj}} + \frac{\alpha(0, 0, n^{-3kj}x, 0, 0)}{n^{6kj}}\right)\Bigg],$$

for all $x \in X$. The mappings $A(x)$ and $Q(x)$ are defined by $A(x) = \lim_{k\to\infty} \frac{f_a(n^{3kj}x)}{n^{3kj}}$ and $Q(x) = \lim_{k\to\infty} \frac{f_q(n^{3kj}x)}{n^{6kj}}$, for all $x \in X$.

Proposition 8.6 *Let $j \in \{-1, 1\}$ and $\alpha : X^5 \to [0, \infty)$ be a function satisfying (2.24) and (5.40), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f : X \to Y$ be a mapping satisfying the inequality (8.1), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (2.6) and*

$$\| f(x) - A(x) - Q(x) \| \le \frac{1}{2}\Bigg[\frac{1}{4n^4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(0, 0, 0, n^{4kj}x, 0)}{n^{4kj}} + \frac{\alpha(0, 0, 0, n^{-4kj}x, 0)}{n^{4kj}}\right) + \frac{1}{2n^8}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(0, 0, 0, n^{4kj}x, 0)}{n^{8kj}} + \frac{\alpha(0, 0, 0, n^{-4kj}x, 0)}{n^{8kj}}\right)\Bigg],$$

for all $x \in X$. The mappings $A(x)$ and $Q(x)$ are defined by $A(x) = \lim_{k\to\infty} \frac{f_a(n^{4kj}x)}{n^{4kj}}$ and $Q(x) = \lim_{k\to\infty} \frac{f_q(n^{4kj}x)}{n^{8kj}}$, for all $x \in X$.

Proposition 8.7 *Let $j \in \{-1, 1\}$ and $\alpha : X^5 \to [0, \infty)$ be a function satisfying (2.24) and (5.40), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Let $f : X \to Y$ be a mapping satisfying the inequality (8.1), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ which satisfies the functional equation (2.6) and*

$$\| f(x) - A(x) - Q(x) \| \leq \frac{1}{2}\Bigg[\frac{1}{4n^5}\sum_{k=\frac{1-j}{2}}^{\infty}\Big(\frac{\alpha(0,0,0,0,n^{5kj}x)}{n^{5kj}} + \frac{\alpha(0,0,0,0,n^{-5kj}x)}{n^{5kj}}\Big) + \frac{1}{2n^{10}}\sum_{k=\frac{1-j}{2}}^{\infty}\Big(\frac{\alpha(0,0,0,0,n^{5kj}x)}{n^{10kj}} + \frac{\alpha(0,0,0,0,n^{-5kj}x)}{n^{10kj}}\Big)\Bigg],$$

for all $x \in X$. The mapping $A(x)$ and $Q(x)$ is defined by $A(x) = \lim_{k\to\infty} \frac{f_a(n^{5kj}x)}{n^{5kj}}$ and $Q(x) = \lim_{k\to\infty} \frac{f_q(n^{5kj}x)}{n^{10kj}}$, for all $x \in X$.

Using Theorem 8.1 and Propositions 8.3–8.7 respectively, we have the following corollaries, concerning the stability of (2.6).

Corollary 8.8 *Let θ and t be non-negative real numbers. Let $f : X \to Y$ be a mapping satisfying the inequality*

$$\| Df(x_1, x_2, x_3, x_4, x_5) \| \leq \begin{cases} \theta, & \\ \theta\Big\{\sum_{i=1}^{5} \| x_i \|^t\Big\}, & t \neq 1, 2 \\ \theta\Big\{\prod_{i=1}^{5} \| x_i \|^t + \sum_{i=1}^{5} \| x_i \|^{5t}\Big\}, & \frac{1}{5}, \frac{2}{5} \end{cases} \tag{8.9}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - A(x) - Q(x) \| \leq \begin{cases} \frac{\theta}{2}\Big[\frac{1}{2|n-1|} + \frac{1}{|n^2-1|}\Big], \\ \frac{\theta\|x\|^t}{2}\Big[\frac{1}{2|n-n^t|} + \frac{1}{|n^2-n^t|}\Big], \\ \frac{\theta\|x\|^{5t}}{2}\Big[\frac{1}{2|n-n^{5t}|} + \frac{1}{|n^2-n^{5t}|}\Big], \end{cases} \tag{8.10}$$

for all $x \in X$.

Remark: Let θ and t be non-negative real numbers. Let $f : X \to Y$ be a mapping satisfying the inequality (8.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - A(x) - Q(x)\| \leq \begin{cases} \frac{\theta}{2}\left[\frac{1}{|2|} + \frac{1}{|3|}\right] \\ \frac{\theta\|x\|^t}{2}\left[\frac{1}{2|2-2^t|} + \frac{1}{|4-2^t|}\right] \\ \frac{\theta\|x\|^{5t}}{2}\left[\frac{1}{2|2-2^{5t}|} + \frac{1}{|4-2^{5t}|}\right], \end{cases}$$

for all $x \in X$.

Corollary 8.9 *Let θ and t be non-negative real numbers. Let $f : X \to Y$ be a mapping satisfying the inequality (8.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$\| f(x) - A(x) - Q(x) \| \leq \begin{cases} \frac{\theta}{2}\left[\frac{1}{2|n^2-1|} + \frac{1}{|n^4-1|}\right], \\ \frac{\theta\|x\|^t}{2}\left[\frac{1}{2|n^2-n^{2t}|} + \frac{1}{|n^4-n^{2t}|}\right], \\ \frac{\theta\|x\|^{5t}}{2}\left[\frac{1}{2|n^2-n^{10t}|} + \frac{1}{|n^4-n^{10t}|}\right], \end{cases} \tag{8.11}$$

for all $x \in X$.

Remark: Let θ and t be non-negative real numbers. Let $f : X \to Y$ be a mapping satisfying the inequality (8.9) ,for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - A(x) - Q(x)\| \leq \begin{cases} \frac{\theta}{6}\left[\frac{1}{|2|} + \frac{1}{|15|}\right] \\ \frac{\theta\|x\|^t}{2}\left[\frac{1}{2|2^2-2^t|} + \frac{1}{|2^4-2^t|}\right] \\ \frac{\theta\|x\|^{5t}}{2}\left[\frac{1}{2|2^2-2^{5t}|} + \frac{1}{|2^4-2^{5t}|}\right], \end{cases}$$

for all $x \in X$.

Corollary 8.10 *Let θ and t be non-negative real numbers. Let $f : X \to Y$ be a mapping satisfying the inequality (8.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that, for all $x \in X$*

$$\| f(x) - A(x) - Q(x) \| \leq \begin{cases} \frac{\theta}{2}\left[\frac{1}{2|n^3-1|} + \frac{1}{|n^6-1|}\right], \\ \frac{\theta\|x\|^t}{2}\left[\frac{1}{2|n^3-n^{3t}|} + \frac{1}{|n^6-n^{3t}|}\right], \\ \frac{\theta\|x\|^{5t}}{2}\left[\frac{1}{2|n^3-n^{15t}|} + \frac{1}{|n^6-n^{15t}|}\right]. \end{cases} \tag{8.12}$$

Corollary 8.11 *Let θ and t be non-negative real numbers. Let $f : X \to Y$ be a mapping satisfying the inequality (8.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that, for all $x \in X$*

$$\| f(x) - A(x) - Q(x) \| \leq \begin{cases} \frac{\theta}{2}\left[\frac{1}{2|n^4-1|} + \frac{1}{|n^8-1|}\right], \\ \frac{\theta\|x\|^t}{2}\left[\frac{1}{2|n^4-n^{4t}|} + \frac{1}{|n^8-n^{4t}|}\right], \\ \frac{\theta\|x\|^{5t}}{2}\left[\frac{1}{2|n^4-n^{20t}|} + \frac{1}{|n^8-n^{20t}|}\right]. \end{cases} \tag{8.13}$$

Corollary 8.12 *Let θ and t be a non-negative real numbers. Let $f : X \to Y$ be a mapping satisfying the inequality (8.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that, for all $x \in X$*

$$\| f(x) - A(x) - Q(x) \| \leq \begin{cases} \frac{\theta}{2}\left[\frac{1}{2|n^5-1|} + \frac{1}{|n^{10}-1|}\right], \\ \frac{\theta\|x\|^t}{2}\left[\frac{1}{2|n^5-n^{5t}|} + \frac{1}{|n^{10}-n^{5t}|}\right], \\ \frac{\theta\|x\|^{5t}}{2}\left[\frac{1}{2|n^5-n^{25t}|} + \frac{1}{|n^{10}-n^{25t}|}\right]. \end{cases} \tag{8.14}$$

8.3 Additive Quadratic Functional Equation: Mixed Case—Fixed Point Method

Theorem 8.13 *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the conditions (2.41) and (5.55) where $\eta_i = \begin{cases} n, & i = 0; \\ \frac{1}{n}, & i = 1; \end{cases}$ satisfying the functional inequality*

$$\| Df(x_1, x_2, x_3, x_4, x_5) \| \leq \alpha(x_1, x_2, x_3, x_4, x_5), \tag{8.15}$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$. If there exists $L = L(i)$ such that the function $x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{n}, 0, 0, 0, 0\right)$, has the properties (2.43) and (5.57), for all $x \in X$, then there exist a unique additive function $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying the equation (2.6) and

$$\| f(x) - A(x) - Q(x) \| \leq \frac{L^{1-i}}{1-L}[\beta(x) + \beta(-x)], \tag{8.16}$$

for all $x \in X$.

Proof It follows from (8.3) and Theorem 2.14 that

$$\| f_0(x) - A(x) \| \leq \frac{1}{2}\frac{L^{1-i}}{1-L}[\beta(x) + \beta(-x)]. \tag{8.17}$$

Similarly, it follows from (8.5) and Theorem 5.15 that

$$\| f_e(x) - Q(x) \| \leq \frac{1}{2}\frac{L^{1-i}}{1-L}[\beta(x) + \beta(-x)], \tag{8.18}$$

for all $x \in X$. Define

$$f(x) = f_0(x) + f_e(x) \tag{8.19}$$

for all $x \in X$. From (8.17), (8.18) and (8.19), we have

$$\| f(x) - A(x) - Q(x) \| \leq \| f_e(x) + f_o(x) - A(x) - Q(x) \| \tag{8.20}$$

$$\leq \| f_0(x) - A(x) \| + \| f_e(x) - Q(x) \| \tag{8.21}$$

$$\leq \frac{L^{1-i}}{1-L}[\beta(x) + \beta(-x)], \tag{8.22}$$

for all $x \in X$. Hence the theorem is proved.

Example 8.14 Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the conditions (2.41) and (5.55), where

$$\eta_i = \begin{cases} 2, & if\ i = 0; \\ \frac{1}{2}, & if\ i = 1; \end{cases}$$

satisfying (8.15) for all $x_1, x_2, x_3, x_4, x_5 \in X$. If there exists $L = L(i)$ such that the function $x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{2}, 0, 0, 0, 0\right)$, has the properties (2.43) and (5.57), for all $x \in X$, then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying the equations (2.6) and (8.16), for all $x \in X$.

Proposition 8.15 *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\alpha : X^5 \to [0, \infty)$ *with the conditions (2.41) and (5.55) where*

$$\eta_i = \begin{cases} n^2, & i = 0; \\ \frac{1}{n^2}, & i = 1; \end{cases}$$

satisfying (8.15), for all $x_1, x_2, x_3, x_4, x_5 \in X$. *If there exists* $L = L(i)$ *such that the function* $x \to \beta(x) = \frac{1}{2}\alpha\left(0, \frac{x}{n^2}, 0, 0, 0\right)$, *has the properties (2.43) and (5.57), for* *all $x \in X$, then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying the equations (2.6) and (8.16), for all $x \in X$.*

Example 8.16 Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the conditions (2.41) and (5.55), where

$$\eta_i = \begin{cases} 2^2, & if\ i = 0; \\ \frac{1}{2^2}, & if\ i = 1; \end{cases}$$

satisfying (8.15) for all $x_1, x_2, x_3, x_4, x_5 \in X$. If there exists $L = L(i)$ such that the function $x \to \beta(x) = \frac{1}{2}\alpha\left(0, \frac{x}{2^2}, 0, 0, 0\right)$, has the properties (2.43) and (5.57), for all $x \in X$, then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying the equations (2.6) and (8.16), for all $x \in X$.

Proposition 8.17 *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the conditions (2.41) and (5.55) where $\eta_i = \begin{cases} n^3, & i = 0; \\ \frac{1}{n^3}, & i = 1; \end{cases}$ satisfying (8.15) for all $x_1, x_2, x_3, x_4, x_5 \in X$. If there exists $L = L(i)$ such that the function $x \to \beta(x) = \frac{1}{2}\alpha\left(0, 0, \frac{x}{n^3}, 0, 0\right)$, has the properties (2.43) and (5.57), for all $x \in X$, then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying the equations (2.6) and (8.16), for all $x \in X$.*

Proposition 8.18 *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the conditions (2.41) and (5.55) where $\eta_i = \begin{cases} n^4, & i = 0; \\ \frac{1}{n^4}, & i = 1; \end{cases}$ satisfying (8.15) for all $x_1, x_2, x_3, x_4, x_5 \in X$. If there exists $L = L(i)$ such that the function $x \to \beta(x) = \frac{1}{2}\alpha\left(0, 0, 0, \frac{x}{n^4}, 0\right)$, has the properties (2.43) and (5.57), for all $x \in X$, then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying the equations (2.6) and (8.16), for all $x \in X$.*

Proposition 8.19 *Let $f : X \to Y$ be a mapping for which there exists a function $\alpha : X^5 \to [0, \infty)$ with the conditions (2.41) and (5.55) where $\eta_i = \begin{cases} n^5, & i = 0; \\ \frac{1}{n^5}, & i = 1; \end{cases}$ satisfying (8.15) for all $x_1, x_2, x_3, x_4, x_5 \in X$. If there exists $L = L(i)$ such that the function $x \to \beta(x) = \frac{1}{2}\alpha\left(0, 0, 0, 0, \frac{x}{n^5}\right)$, has the properties (2.43) and (5.57), for all $x \in X$, then there exist a unique additive function $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying the equations (2.6) and (8.16), for all $x \in X$.*

Using Theorem 8.13, Propositions 8.15, and 8.17–8.19 respectively, we have the following corollaries, concerning the stability of (2.6).

Corollary 8.20 *Let $f : X \to Y$ be a mapping and assume that there exist real numbers θ and t satisfying (8.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying (8.10), for all $x \in X$.*

Remark: Let $f : X \to Y$ be a mapping and assume that there exist real numbers θ and t satisfying (8.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying

$$\|f(x) - A(x) - Q(x)\| \leq \begin{cases} \frac{\theta}{2}\left[\frac{1}{|2|} + \frac{1}{|3|}\right] \\ \frac{\theta\|x\|^t}{2}\left[\frac{1}{2|2-2^t|} + \frac{1}{|4-2^t|}\right] \\ \frac{\theta\|x\|^{5t}}{2}\left[\frac{1}{2|2-2^{5t}|} + \frac{1}{|4-2^{5t}|}\right], \end{cases}$$

for all $x \in X$.

Corollary 8.21 *Let $f : X \to Y$ be a mapping and assume that there exist real numbers θ and t satisfying (8.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying (8.11), for all $x \in X$.*

Remark: Let $f : X \to Y$ be a mapping and assume that there exist real numbers θ and t satisfying (8.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying

$$\|f(x) - A(x) - Q(x)\| \leq \begin{cases} \frac{\theta}{2}\left[\frac{1}{|6|} + \frac{1}{|15|}\right] \\ \frac{\theta\|x\|^t}{2}\left[\frac{1}{2|2^2-2^{2t}|} + \frac{1}{|2^4-2^{2t}|}\right] \\ \frac{\theta\|x\|^{5t}}{2}\left[\frac{1}{2|2^2-2^{20t}|} + \frac{1}{|2^4-2^{20t}|}\right], \end{cases}$$

for all $x \in X$.

Corollary 8.22 *Let $f : X \to Y$ be a mapping and assume that there exist real numbers θ and t satisfying (8.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying (8.12), for all $x \in X$.*

Corollary 8.23 *Let $f : X \to Y$ be a mapping and assume that there exist real numbers θ and t satisfying (8.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying (8.13), for all $x \in X$.*

Corollary 8.24 *Let $f : X \to Y$ be a mapping and assume that there exist real numbers θ and t satisfying (8.9), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying (8.14), for all $x \in X$.*

Chapter 9
Additive-Quadratic Functional Equations in Banach Algebras

9.1 Banach Algebra—Additive Quadratic Functional Equation

Gavruta [34] obtained a generalisation of Rassias theorem by replacing the bound $\in \left(\| x \|^p + \| y \|^p \right)$ with a general control function $\varepsilon(x, y)$. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem [23, 26, 34, 57, 63, 69, 70, 83].

Let Γ^+ denote the set of all probability distribution functions $F : \mathbb{R} \cup [-\infty, +\infty] \to [0, 1]$ such that F is left-continuous and non decreasing on $\mathbb{R}$ and $F(0) = 0$, $F(+\infty) = 1$.

It is clear that the set $D^+ = \{F \in \Gamma^+ : l^- F(-\infty) = 1\}$, where $l^- f(x) = \lim_{t \to x^-} f(t)$, is a subset of Γ^+. The set Γ^+ is partially ordered by the usual point wise ordering of functions, that is, $F \le G$ if and only if $F(t) \le G(t)$, for all $t \in \mathbb{R}$. For any $a \ge 0$, the element $H_a(t)$ of D^+ is defined by

$$H_a(t) = \begin{cases} 0, & if \;\; t \le a, \\ 1, & if \;\; t > a. \end{cases}$$

We show that the maximal element in Γ^+ is the distribution function $H_0(t)$.

we establish stability of the additive quadratic functional equation (2.6) for mixed case in Banach algebra with the help of direct and fixed point method.

H. Dutta et al., *Stability of Some Advanced Functional Equations in Various Spaces*, Studies in Systems, Decision and Control 479,
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9.2 Additive Quadratic Functional Equation: Mixed Case—Direct Method

Proposition 9.1 *Let $j = \pm 1$. Let $f : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with the conditions (3.3),(3.4) and (6.3), (6.4) such that the functional inequalities,*

$$\| Df(x_1, x_2, x_3, x_4, x_5) \| \leq \alpha(x_1, x_2, x_3, x_4, x_5), \tag{9.1}$$

and (3.6) and satisfying the inequality (6.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and unique quadratic derivation $Q : X \to X$ which satisfies the functional equation (2.6) and

$$\| f(x) - A(x) - Q(x) \| \leq \frac{1}{2}\Bigg[\frac{1}{4n}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(n^{kj}x, 0, 0, 0, 0)}{n^{kj}} + \frac{\alpha(-n^{kj}x, 0, 0, 0, 0)}{n^{kj}}\right) + \frac{1}{2n^2}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\alpha(n^{kj}x, 0, 0, 0, 0)}{n^{2kj}} + \frac{\alpha(-n^{kj}x, 0, 0, 0, 0)}{n^{2kj}}\right)\Bigg],$$

for all $x \in X$. The mappings $A(x)$ and $Q(x)$ are defined by $A(x) = \lim_{k\to\infty} \frac{f_a(n^{kj}x)}{n^{kj}}$ and $Q(x) = \lim_{k\to\infty} \frac{f_q(n^{kj}x)}{n^{2kj}}$, for all $x \in X$.

Proof The proof follows from Theorems 8.1, 3.2, and 6.2. □

Example 9.2 Let $j = \pm 1$. Let $f : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with condition

$$\sum_{k=0}^{\infty} \frac{\alpha(2^{kj}x_1, 2^{kj}x_2, 2^{kj}x_3, 2^{kj}x_4, 2^{kj}x_5)}{2^{kj}}$$

converges in $\mathbb{R}$,

$$\sum_{k=0}^{\infty} \frac{\beta(2^{kj}x_1, 2^{kj}x_2, 2^{kj}x_3, 2^{kj}x_4, 2^{kj}x_5)}{2^{5kj}}$$

converges in $\mathbb{R}$,

$$\sum_{k=0}^{\infty} \frac{\alpha(2^{kj}x_1, 2^{kj}x_2, 2^{kj}x_3, 2^{kj}x_4, 2^{kj}x_5)}{2^{2kj}}$$

converges in $\mathbb{R}$ and also

$$\sum_{k=0}^{\infty} \frac{\beta(2^{kj}x_1, 2^{kj}x_2, 2^{kj}x_3, 2^{kj}x_4, 2^{kj}x_5)}{2^{10kj}},$$

converges in $\mathbb{R}$, for all $x_1, x_2, x_3, x_4, x_5 \in X$ such that the functional inequality (3.5) and satisfying the inequalities (3.6), (6.5) and (6.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6) and

$$\| f(x) - A(x) - Q(x) \| \leq \frac{1}{2}\Bigg[\frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\Big(\frac{\alpha(2^{kj}x, 0, 0, 0, 0)}{2^{kj}} + \frac{\alpha(-2^{kj}x, 0, 0, 0, 0,)}{2^{kj}}\Big) + \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\Big(\frac{\alpha(2^{kj}x, 0, 0, 0, 0)}{2^{2kj}} + \frac{\alpha(-2^{kj}x, 0, 0, 0, 0)}{2^{2kj}}\Big)\Bigg],$$

for all $x \in X$. The mappings $A(x)$ and $Q(x)$ are defined by $A(x) = \lim_{k\to\infty} \frac{f_a(2^{kj}x)}{2^{kj}}$ and $Q(x) = \lim_{k\to\infty} \frac{f_q(2^{kj}x)}{2^{2kj}}$, for all $x \in X$.

Proposition 9.3 *Let $j = \pm 1$. Let $f : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with condition*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{2kj}x_1, n^{2kj}x_2, n^{2kj}x_3, n^{2kj}x_4, n^{2kj}x_5)}{n^{2kj}},$$

converges in $\mathbb{R}$ and also

$$\sum_{k=0}^{\infty} \frac{\beta(n^{2kj}x_1, n^{2kj}x_2, n^{2kj}x_3, n^{2kj}x_4, n^{2kj}x_5)}{n^{10kj}},$$

converges in $\mathbb{R}$ and also

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{2kj}x_1, n^{2kj}x_2, n^{2kj}x_3, n^{2kj}x_4, n^{2kj}x_5)}{n^{4kj}},$$

converges in $\mathbb{R}$ and also

$$\sum_{k=0}^{\infty} \frac{\beta(n^{2kj}x_1, n^{2kj}x_2, n^{2kj}x_3, n^{2kj}x_4, n^{2kj}x_5)}{n^{20kj}},$$

converges in $\mathbb{R}$ for all $x_1, x_2, x_3, x_4, x_5 \in X$ such that the functional inequality (3.5) and satisfying the inequalities (3.6), (6.5) and (6.6) for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6) and

$$\| f(x) - A(x) - Q(x) \| \leq \frac{1}{2}\Big[\frac{1}{4n^2}\sum_{k=\frac{1-j}{2}}^{\infty}\Big(\frac{\alpha(0, n^{2kj}x, 0, 0, 0)}{n^{2kj}} + \frac{\alpha(0, -n^{2kj}x, 0, 0, 0,)}{n^{2kj}}\Big) + \frac{1}{2n^4}\sum_{k=\frac{1-j}{2}}^{\infty}\Big(\frac{\alpha(0, n^{2kj}x, 0, 0, 0)}{n^{4kj}} + \frac{\alpha(0, -n^{2kj}x, 0, 0, 0)}{n^{4kj}}\Big)\Big],$$

for all $x \in X$. The mappings $A(x)$ and $Q(x)$ are defined by $A(x) = \lim_{k\to\infty} \frac{f_a(n^{2kj}x)}{n^{2kj}}$ and $Q(x) = \lim_{k\to\infty} \frac{f_q(n^{2kj}x)}{n^{4kj}}$, for all $x \in X$.

Example 9.4 Let $j = \pm 1$. Let $f : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with condition

$$\sum_{k=0}^{\infty} \frac{\alpha(2^{2kj}x_1, 2^{2kj}x_2, 2^{2kj}x_3, 2^{2kj}x_4, 2^{2kj}x_5)}{2^{2kj}}$$

converges in $\mathbb{R}$,

$$\sum_{k=0}^{\infty} \frac{\beta(2^{2kj}x_1, 2^{2kj}x_2, 2^{2kj}x_3, 2^{2kj}x_4, 2^{2kj}x_5)}{2^{10kj}}$$

converges in $\mathbb{R}$,

$$\sum_{k=0}^{\infty} \frac{\alpha(2^{2kj}x_1, 2^{2kj}x_2, 2^{2kj}x_3, 2^{2kj}x_4, 2^{2kj}x_5)}{2^{4kj}}$$

converges in $\mathbb{R}$ and also

$$\sum_{k=0}^{\infty} \frac{\beta(2^{2kj}x_1, 2^{2kj}x_2, 2^{2kj}x_3, 2^{2kj}x_4, 2^{2kj}x_5)}{2^{20kj}},$$

converges in $\mathbb{R}$, for all $x_1, x_2, x_3, x_4, x_5 \in X$ such that the functional inequality (3.5) and satisfying the inequalities (3.6), (6.5) and (6.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6) and

$$\| f(x) - A(x) - Q(x) \| \leq \frac{1}{2}\Big[\frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\Big(\frac{\alpha(0, 2^{2kj}x, 0, 0, 0)}{2^{2kj}} + \frac{\alpha(0, -2^{2kj}x, 0, 0, 0,)}{2^{2kj}}\Big) + \frac{1}{32}\sum_{k=\frac{1-j}{2}}^{\infty}\Big(\frac{\alpha(0, 2^{2kj}x, 0, 0, 0)}{2^{4kj}} + \frac{\alpha(0, -2^{2kj}x, 0, 0, 0)}{2^{4kj}}\Big)\Big],$$

for all $x \in X$. The mappings $A(x)$ and $Q(x)$ are defined by $A(x) = \lim_{k\to\infty} \frac{f_a(2^{2kj}x)}{2^{2kj}}$ and $Q(x) = \lim_{k\to\infty} \frac{f_q(2^{2kj}x)}{2^{4kj}}$, for all $x \in X$.

Proposition 9.5 *Let $j = \pm 1$. Let $f : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with condition*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{3kj}x_1, n^{3kj}x_2, n^{3kj}x_3, n^{3kj}x_4, n^{3kj}x_5)}{n^{3kj}},$$

converges in $\mathbb{R}$ and also

$$\sum_{k=0}^{\infty} \frac{\beta(n^{3kj}x_1, n^{3kj}x_2, n^{3kj}x_3, n^{3kj}x_4, n^{3kj}x_5)}{n^{15kj}}$$

converges in $\mathbb{R}$ and

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{3kj}x_1, n^{3kj}x_2, n^{3kj}x_3, n^{3kj}x_4, n^{3kj}x_5)}{n^{6kj}},$$

converges in $\mathbb{R}$ and also

$$\sum_{k=0}^{\infty} \frac{\beta(n^{3kj}x_1, n^{3kj}x_2, n^{3kj}x_3, n^{3kj}x_4, n^{3kj}x_5)}{n^{30kj}},$$

converges in $\mathbb{R}$ for all $x_1, x_2, x_3, x_4, x_5 \in X$ such that the functional inequality (3.5) and satisfying the inequalities (3.6), (6.5) and (6.6) for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6), for all $x_1, x_2, x_3, x_4, x_5 \in X$ and

$$\| f(x) - A(x) - Q(x) \| \leq \frac{1}{2}\Bigg[\frac{1}{4n^3} \sum_{k=\frac{1-j}{2}}^{\infty} \Big(\frac{\alpha(0, 0, n^{3kj}x, 0, 0)}{n^{3kj}} + \frac{\alpha(0, 0, -n^{3kj}x, 0, 0,)}{n^{3kj}}\Big)$$
$$+ \frac{1}{2n^6} \sum_{k=\frac{1-j}{2}}^{\infty} \Big(\frac{\alpha(0, 0, n^{3kj}x, 0, 0)}{n^{6kj}} + \frac{\alpha(0, 0, -n^{3kj}x, 0, 0)}{n^{6kj}}\Big)\Bigg],$$

for all $x \in X$. The mappings $A(x)$ and $Q(x)$ are defined by $A(x) = \lim_{k\to\infty} \frac{f_a(n^{3kj}x)}{n^{3kj}}$ and $Q(x) = \lim_{k\to\infty} \frac{f_q(n^{3kj}x)}{n^{6kj}}$, for all $x \in X$.

Proposition 9.6 *Let $j = \pm 1$, $f : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with condition*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{4kj}x_1, n^{4kj}x_2, n^{4kj}x_3, n^{4kj}x_4, n^{4kj}x_5)}{n^{4kj}}$$

converges in $\mathbb{R}$ *and also*

$$\sum_{k=0}^{\infty} \frac{\beta(n^{4kj}x_1, n^{4kj}x_2, n^{4kj}x_3, n^{4kj}x_4, n^{4kj}x_5)}{n^{20kj}}$$

converges in $\mathbb{R}$ *and also*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{4kj}x_1, n^{4kj}x_2, n^{4kj}x_3, n^{4kj}x_4, n^{4kj}x_5)}{n^{8kj}}$$

converges in $\mathbb{R}$ *and also*

$$\sum_{k=0}^{\infty} \frac{\beta(n^{4kj}x_1, n^{4kj}x_2, n^{4kj}x_3, n^{4kj}x_4, n^{4kj}x_5)}{n^{40kj}}$$

converges in $\mathbb{R}$ *for all* $x_1, x_2, x_3, x_4, x_5 \in X$ *such that the functional inequality (3.5) and satisfying the inequalities (3.6), (6.5) and (6.6) for all* $x_1, x_2, x_3, x_4, x_5 \in X$. *Then there exist a unique additive derivation* $A : X \to X$ *and a unique quadratic derivation* $Q : X \to X$ *satisfying the functional equation (2.6), for all* $x_1, x_2, x_3, x_4, x_5 \in X$ *and*

$$\| f(x) - A(x) - Q(x) \| \le \frac{1}{2}\Bigg[\frac{1}{4n^4} \sum_{k=\frac{1-j}{2}}^{\infty} \Big(\frac{\alpha(0,0,0,n^{4kj}x,0)}{n^{4kj}} + \frac{\alpha(0,0,0,-n^{4kj}x,0)}{n^{4kj}}\Big) + \frac{1}{2n^8} \sum_{k=\frac{1-j}{2}}^{\infty} \Big(\frac{\alpha(0,0,0,n^{4kj}x,0)}{n^{8kj}} + \frac{\alpha(0,0,0,-n^{4kj}x,0)}{n^{8kj}}\Big)\Bigg],$$

for all $x \in X$. *The mappings* $A(x)$ *and* $Q(x)$ *are defined by* $A(x) = \lim_{k\to\infty} \frac{f_a(n^{4kj}x)}{n^{4kj}}$ *and* $Q(x) = \lim_{k\to\infty} \frac{f_q(n^{4kj}x)}{n^{8kj}}$, *for all* $x \in X$.

Proposition 9.7 *Let* $j = \pm 1$. *Let* $f : X \to X$ *be a mapping for which there exist functions* $\alpha, \beta : X^5 \to [0, \infty)$ *with condition*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{5kj}x_1, n^{5kj}x_2, n^{5kj}x_3, n^{5kj}x_4, n^{5kj}x_5)}{n^{5kj}}$$

converges in $\mathbb{R}$ *and also*

$$\sum_{k=0}^{\infty} \frac{\beta(n^{5kj}x_1, n^{5kj}x_2, n^{5kj}x_3, n^{5kj}x_4, n^{5kj}x_5)}{n^{25kj}}$$

converges in $\mathbb{R}$ *and also*

$$\sum_{k=0}^{\infty} \frac{\alpha(n^{5kj}x_1, n^{5kj}x_2, n^{5kj}x_3, n^{5kj}x_4, n^{5kj}x_5)}{n^{10kj}}$$

converges in $\mathbb{R}$ *and also*

$$\sum_{k=0}^{\infty} \frac{\beta(n^{5kj}x_1, n^{5kj}x_2, n^{5kj}x_3, n^{5kj}x_4, n^{5kj}x_5)}{n^{50kj}}$$

converges in $\mathbb{R}$ *for all* $x_1, x_2, x_3, x_4, x_5 \in X$ *such that the functional inequality (3.5) and satisfying the inequalities (3.6), (6.5) and (6.6) for all* $x_1, x_2, x_3, x_4, x_5 \in X$. *Then there exist a unique additive derivation* $A : X \to X$ *and a unique quadratic derivation* $Q : X \to X$ *satisfying the functional equation (2.6), for all* $x_1, x_2, x_3, x_4, x_5 \in X$ *and*

$$\| f(x) - A(x) - Q(x) \| \leq \frac{1}{2}\left[\frac{1}{4n^5} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(0,0,0,0,n^{5kj}x,)}{n^{5kj}} + \frac{\alpha(0,0,0,0,-n^{5kj}x)}{n^{5kj}}\right) + \frac{1}{2n^{10}} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(0,0,0,0,n^{5kj}x)}{n^{10kj}} + \frac{\alpha(0,0,0,0,-n^{5kj}x)}{n^{10kj}}\right)\right],$$

for all $x \in X$. *The mappings* $A(x)$ *and* $Q(x)$ *are defined by* $A(x) = \lim_{k\to\infty} \frac{f_a(n^{5kj}x)}{n^{5kj}}$ *and* $Q(x) = \lim_{k\to\infty} \frac{f_q(n^{5kj}x)}{n^{10kj}}$, *for all* $x \in X$.

Using Propositions 9.1–9.7 respectively, we have the following corollaries concerning the stability of (2.6).

Corollary 9.8 *Let* $f : X \to X$ *be a mapping and assume that there exist real numbers* s *and* r *such that,*

$$\| Df(x_1, x_2, x_3, x_4, x_5) \| \leq \begin{cases} s, \\ s\left\{ \sum_{i=1}^{5} \| x_i \|^r \right\}, & r \neq 1, 2 \\ s\left\{ \prod_{i=1}^{5} \| x_i \|^r + \sum_{i=1}^{5} \| x_i \|^{5r} \right\}, & r \neq \frac{1}{5}, \frac{2}{5} \end{cases} \tag{9.2}$$

and (3.9), (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. *Then there exist a unique additive derivation* $A : X \to X$ *and a unique quadratic derivation* $Q : X \to X$ *such that*

$$\| f(x) - A(x) - Q(x) \| \leq \begin{cases} \frac{s}{2}\left[\frac{1}{2|n-1|} + \frac{1}{|n^2-1|}\right] \\ \frac{s\|x\|^r}{2}\left[\frac{1}{2|n-n^r|} + \frac{1}{|n^2-n^r|}\right] \\ \frac{s\|x\|^{5r}}{2}\left[\frac{1}{2|n-n^{5r}|} + \frac{1}{|n^2-n^{5r}|}\right], \end{cases} \tag{9.3}$$

for all $x \in X$.

Remark: Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.9), (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique quadratic derivation $Q : X \to X$ such that

$$\|f_q(x) - Q(x)\| \le \begin{cases} \frac{\diamond}{|6|} \\ \frac{\diamond\|x\|^r}{2|2^2-2^r|} \\ \frac{\diamond\|x\|^{5r}}{2|2^2-2^{5r}|}, \end{cases}$$

for all $x \in X$.

Corollary 9.9 *Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.9), (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ such that*

$$\| f(x) - A(x) - Q(x) \| \le \begin{cases} \frac{s}{2}\left[\frac{1}{2|n^2-1|} + \frac{1}{|n^4-1|}\right] \\ \frac{s\|x\|^r}{2}\left[\frac{1}{2|n^2-n^{2r}|} + \frac{1}{|n^4-n^{2r}|}\right] \\ \frac{s\|x\|^{5r}}{2}\left[\frac{1}{2|n^2-n^{10r}|} + \frac{1}{|n^4-n^{10r}|}\right], \end{cases} \tag{9.4}$$

for all $x \in X$.

Remark: Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.9), (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ such that

$$\|f_q(x) - A(x)\| \le \begin{cases} \frac{\diamond}{|30|} \\ \frac{\diamond\|x\|^r}{2|2^4-2^{2r}|} \\ \frac{\diamond\|x\|^{5r}}{2|2^4-2^{10r}|}, \end{cases}$$

for all $x \in X$.

Corollary 9.10 *Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.9), (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$, such that for all $x \in X$*

$$\| f(x) - A(x) - Q(x) \| \leq \begin{cases} \frac{s}{2}\left[\frac{1}{2|n^3-1|} + \frac{1}{|n^6-1|}\right] \\ \frac{s\|x\|^r}{2}\left[\frac{1}{2|n^3-n^{3r}|} + \frac{1}{|n^6-n^{3r}|}\right] \\ \frac{s\|x\|^{5r}}{2}\left[\frac{1}{2|n^3-n^{15r}|} + \frac{1}{|n^6-n^{15r}|}\right]. \end{cases} \tag{9.5}$$

Remark: Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.9), (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ such that

$$\|f_q(x) - A(x)\| \leq \begin{cases} \frac{\diamond}{|126|} \\ \frac{\diamond\|x\|^r}{2|2^6-2^{3r}|} \\ \frac{\epsilon\|x\|^{5r}}{2|2^6-2^{15r}|}, \end{cases}$$

for all $x \in X$.

Corollary 9.11 *Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.9), (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$, such that*

$$\| f(x) - A(x) - Q(x) \| \leq \begin{cases} \frac{s}{2}\left[\frac{1}{2|n^4-1|} + \frac{1}{|n^8-1|}\right] \\ \frac{s\|x\|^r}{2}\left[\frac{1}{2|n^4-n^{4r}|} + \frac{1}{|n^8-n^{4r}|}\right] \\ \frac{s\|x\|^{5r}}{2}\left[\frac{1}{2|n^4-n^{20r}|} + \frac{1}{|n^8-n^{20r}|}\right], \end{cases} \tag{9.6}$$

for all $x \in X$

Corollary 9.12 *Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.9), (6.10), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$, such that*

$$\| f(x) - A(x) - Q(x) \| \leq \begin{cases} \frac{s}{2}\left[\frac{1}{2|n^5-1|} + \frac{1}{|n^{10}-1|}\right] \\ \frac{s\|x\|^r}{2}\left[\frac{1}{2|n^5-n^{5r}|} + \frac{1}{|n^{10}-n^{5r}|}\right] \\ \frac{s\|x\|^{5r}}{2}\left[\frac{1}{2|n^5-n^{25r}|} + \frac{1}{|n^{10}-n^{25r}|}\right], \end{cases} \tag{9.7}$$

for all $x \in X$.

9.3 Additive Quadratic Functional Equation: Mixed Case—Fixed Point Method

Proposition 9.13 *Let $j = \pm 1$. Let $f : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with conditions (3.15) and (3.16), (6.16) and (6.17), where $\eta_i = \begin{cases} n & i = 0; \\ \frac{1}{n} & i = 1 \end{cases}$ and satisfying the functional inequalities (9.1) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$.*

Assume that there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{4}\alpha\left(\frac{x}{n}, 0, 0, 0, 0\right)$$

and

$$x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{n}, 0, 0, 0, 0\right)$$

has properties (3.19) and (6.18), for all $x \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6), for all $x \in X$ and

$$\| f(x) - A(x) - Q(x) \| \leq \frac{L^{1-i}}{1-L}[\beta(x) + \beta(-x)]. \tag{9.8}$$

Example 9.14 Let $j = \pm 1$. Let $f_q : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with conditions (3.15) and (3.16), (6.16) and (6.17), where $\eta_i = \begin{cases} 2 \; if \; i = 0; \\ \frac{1}{2} \; if \; i = 1; \end{cases}$ and satisfying the functional inequalities (9.1) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$.

assume that there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{4}\alpha\left(\frac{x}{2}, 0, 0, 0, 0\right)$$

and

$$x \to \beta(x) = \frac{1}{2}\alpha\left(\frac{x}{2}, 0, 0, 0, 0\right)$$

have properties (3.19) and (6.18), for all $x \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6) and

$$\| f(x) - A(x) - Q(x) \| \leq \frac{L^{1-i}}{1-L}[\beta(x) + \beta(-x)], \tag{9.9}$$

for all $x \in X$.

Proposition 9.15 *Let $j = \pm 1$. Let $f : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with conditions (3.15) and (3.16), (6.16) and (6.17), where $\eta_i = \begin{cases} n^2 & i = 0; \\ \frac{1}{n^2} & i = 1 \end{cases}$ and satisfying the functional inequalities (9.1) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$.*

assume that there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{4}\alpha\left(0, \frac{x}{n^2}, 0, 0, 0\right)$$

and

$$x \to \beta(x) = \frac{1}{2}\alpha\left(0, \frac{x}{n^2}, 0, 0, 0\right)$$

have properties (3.19) and (6.18), for all $x \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6) and the inequality (9.8), for all $x \in X$.

Example 9.16 Let $j = \pm 1$. Let $f : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with conditions (3.15) and (3.16), (6.16) and (6.17), where $\eta_i = \begin{cases} 2^2 \; if \; i = 0; \\ \frac{1}{2^2} \; if \; i = 1 \end{cases}$ and satisfying the functional inequalities (9.1) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$.

assume that there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{4}\alpha\left(0, \frac{x}{2^2}, 0, 0, 0\right)$$

and

$$x \to \beta(x) = \frac{1}{2}\alpha\left(0, \frac{x}{2^2}, 0, 0, 0\right)$$

have properties (3.19) and (6.18), for all $x \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6) and the inequality (9.8), for all $x \in X$.

Proposition 9.17 *Let $j = \pm 1$, $f : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with conditions (3.15) and (3.16), (6.16) and (6.17), where $\eta_i = \begin{cases} n^3 & i = 0; \\ \frac{1}{n^3} & i = 1 \end{cases}$ and satisfying the functional inequalities (9.1) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$.*

assume that there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{4}\alpha\left(0, 0, \frac{x}{n^3}, 0, 0\right)$$

and

$$x \to \beta(x) = \frac{1}{2}\alpha\left(0, 0, \frac{x}{n^3}, 0, 0\right)$$

have properties (3.19) and (6.18), for all $x \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6) and the inequality (9.8), for all $x \in X$.

Proposition 9.18 *Let $j = \pm 1$, $f : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with conditions (3.15) and (3.16), (6.16) and (6.17), where $\eta_i = \begin{cases} n^4 & i = 0; \\ \frac{1}{n^4} & i = 1 \end{cases}$ and satisfying the functional inequalities (9.1) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$.*

assume that there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{4}\alpha\left(0, 0, 0, \frac{x}{n^4}, 0\right)$$

and

$$x \to \beta(x) = \frac{1}{2}\alpha\left(0, 0, 0, \frac{x}{n^4}, 0\right)$$

have properties (3.19) and (6.18), for all $x \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6) and the inequality (9.8), for all $x \in X$.

Proposition 9.19 *Let $j = \pm 1$, $f : X \to X$ be a mapping for which there exist functions $\alpha, \beta : X^5 \to [0, \infty)$ with conditions (3.15) and (3.16), (6.16) and (6.17), where $\eta_i = \begin{cases} n^5 & i = 0; \\ \frac{1}{n^5} & i = 1 \end{cases}$ and satisfying the functional inequalities (9.1) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$.*

assume that there exists $L = L(i) < 1$ such that the function

$$x \to \beta(x) = \frac{1}{4}\alpha\left(0, 0, 0, 0, \frac{x}{n^5}\right)$$

and

$$x \to \beta(x) = \frac{1}{2}\alpha\left(0, 0, 0, 0, \frac{x}{n^5}\right)$$

have properties (3.19) and (6.18), for all $x \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ satisfying the functional equation (2.6) and the inequality (9.8), for all $x \in X$.

The following corollaries are the immediate consequence of Propositions 9.13–9.19 respectively, concerning the stability of (2.6).

Corollary 9.20 *Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ such that (9.3), for all $x \in X$.*

Remark: Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ such that

$$\|f(x) - A(x) - Q(x)\| \leq \begin{cases} \frac{\diamond}{2}\left[\frac{1}{|2|} + \frac{1}{|3|}\right] \\ \frac{\diamond\|x\|^r}{2}\left[\frac{1}{2|2-2^r|} + \frac{1}{|4-2^r|}\right] \\ \frac{\diamond\|x\|^{5r}}{2}\left[\frac{1}{2|2-2^{5r}|} + \frac{1}{|4-2^{5r}|}\right], \end{cases}$$

for all $x \in X$.

Corollary 9.21 *Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$, such that (9.4), for all $x \in X$.*

Remark: Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ such that

$$\|f(x) - A(x) - Q(x)\| \leq \begin{cases} \frac{\diamond}{2}\left[\frac{1}{|6|} + \frac{1}{|15|}\right] \\ \frac{\diamond\|x\|^r}{2}\left[\frac{1}{2|2^2-2^{2r}|} + \frac{1}{|2^4-2^{2r}|}\right] \\ \frac{\diamond\|x\|^{5r}}{2}\left[\frac{1}{2|2-2^{10r}|} + \frac{1}{|2^4-2^{10r}|}\right], \end{cases}$$

for all $x \in X$.

Corollary 9.22 *Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$, such that (9.5), for all $x \in X$.*

Remark: Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$ such that

$$\|f(x) - A(x) - Q(x)\| \leq \begin{cases} \frac{\diamond}{2}\left[\frac{1}{|14|} + \frac{1}{|63|}\right] \\ \frac{\diamond\|x\|^r}{2}\left[\frac{1}{2|2^3-2^{3r}|} + \frac{1}{|2^6-2^{3r}|}\right] \\ \frac{\diamond\|x\|^{5r}}{2}\left[\frac{1}{2|2^3-2^{15r}|} + \frac{1}{|2^6-2^{15r}|}\right], \end{cases}$$

for all $x \in X$.

Corollary 9.23 *Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$, such that (9.6), for all $x \in X$.*

Corollary 9.24 *Let $f : X \to X$ be a mapping and assume that there exist real numbers s and r such that (9.2) and (3.18), (6.19), for all $x_1, x_2, x_3, x_4, x_5 \in X$. Then there exist a unique additive derivation $A : X \to X$ and a unique quadratic derivation $Q : X \to X$, such that (9.7), for all $x \in X$.*

Chapter 10
3-Dimensional Cubic Functional Equations

10.1 3-Dimensional Cubic Functional Equation

Rassias [82] introduced the new cubic functional equation.

$$C(x + 2y) + 3C(x) = 3C(x + y) + C(x - y) + 6C(y), \tag{10.1}$$

and investigated Ulam stability problem.

Sahoo [88] determined the general solution of the functional equation

$$f(x + 2y) + f(x - 2y) + 6f(x) = 4f(x + y) + 4f(x - y), \tag{10.2}$$

for all $x, y \in \mathbb{R}$ without assuming any regularity conditions on the unknown function f. The Eq. (10.2) characterizes polynomials of degree three. The method used for solving this functional equation is the Frechet functional equation [36].

Also, the solution and stability of the following cubic functional equations

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x), \tag{10.3}$$

$$f(x + y + 2z) + f(x + y - 2z) + f(2x) + f(2y) = 2[f(x + y) + 2f(x + z) + 2f(y + z) + 2f(x - z) + 2f(y - z)], \tag{10.4}$$

$$f\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + f\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + \sum_{j=1}^{n-1} f(2x_j) = 2f\left(\sum_{j=1}^{n-1} 2x_j\right) + 4\sum_{j=1}^{n-1}\big(f(x_j + x_n) + f(x_j - x_n)\big), \tag{10.5}$$

$$f(3x + y) + f(3x - y) = 3f(x + y) + 3f(x - y) + 48f(x), \tag{10.6}$$

$$f(x + y + 2z) + f(x + y - 2z) + f(2x) + f(2y) = 2f(x + y) + 4f(x + z) + 4f(x - z) + 4f(y + z) + 4f(y - z), \tag{10.7}$$

H. Dutta et al., *Stability of Some Advanced Functional Equations in Various Spaces*, Studies in Systems, Decision and Control 479,
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were argued by Jun and Kim [47], Jung and Chang [51], Park and Jung [72], Baak and Moslenian [7]

$$\begin{aligned}&3f(nx_1+n^2x_2+n^3x_3)+f(-nx_1+n^2x_2+n^3x_3)+f(nx_1-n^2x_2+n^3x_3)\\&\quad+f(nx_1+n^2x_2-n^3x_3)=4\big[f(nx_1+n^2x_2)+f(nx_1+n^3x_3)\\&\quad+f(n^2x+n^3x_3)-n^3f(x_1)-n^6f(x_2)-n^9f(x_3)\big]\end{aligned}\tag{10.8}$$

In this chapter, we discuss the solution and stability of the 3-dimensional cubic functional equation for (10.8) (with 3 variables) in fuzzy Banach space with the help of the direct and fixed point methods (see [6, 24, 56, 89]).

10.2 General Solution of 3-Dimensional Cubic Functional Equation

Theorem 10.1 *An odd mapping $f : X \to Y$ satisfies the functional equation (10.3), for all $x, y \in X$, if and only if $f : X \to Y$ satisfies the functional equation (10.8), for all $x_1, x_2, x_3 \in X$.*

Proof Let $f : X \to Y$ satisfies the functional equation (10.3). Replacing (x, y) by $(0, 0)$ in (10.3), we get $f(0) = 0$. Replacing (x, y) by $(x, 0)$, (x, x) and $(x, 2x)$ respectively in (10.3), we obtain

$$f(2x)=2^3f(x),\quad f(3x)=3^3f(x)\ \ and\ \ f(4x)=4^3f(x),\tag{10.9}$$

for all $x \in X$. In general for any positive integer a, we have

$$f(ax)=a^3f(x),\tag{10.10}$$

for all $x \in X$. It is easy to verify from (10.10) that,

$$f(a^2x)=a^6f(x)\ \text{ and }\ f(a^3x)=a^9f(x),\tag{10.11}$$

for all $x \in X$. Replacing (x, y) by $(nx_1 + n^2x_2, n^3x_3)$ in (10.3), we get

$$\begin{aligned}&f(2nx_1+2n^2x_2+n^3x_3)+f(2nx_1+2n^2x_2-n^3x_3)+2f(-nx_1-n^2x_2\\&-n^3x_3)+2f(-nx_1-n^2x_2+n^3x_3)=12f(nx_1+n^2x_2),\end{aligned}\tag{10.12}$$

for all $x_1, x_2, x_3 \in X$. Again replacing (x, y) by $(n^2x_2 + n^3x_3, nx_1)$ in (10.3), we have

$$\begin{aligned}&f(nx_1+2n^2x_2+n_3^x)+f(-nx_1+2n^2x_2+2n^3x_3)+2f(-nx_1-n^2x_2\\&-n^3x_3)+2f(nx_1-n^2x_2-n^3x_3)=12f(n^2x_2+n^3x_3),\end{aligned}\tag{10.13}$$

for all $x_1, x_2, x_3 \in X$. Again replacing (x, y) by $(nx_1 + n^3x_3, n^2x_2)$ in (10.3), we arrive

$$f(2nx_1 + n^2x_2 + 2n^3x_3) + f(2nx_1 - n^2x_2 + 2n^3x_3) + 2f(-nx_1 - n^2x_2 - n^3x_3) + 2f(-nx_1 + n^2x_2 - n^3x_3) = 12f(nx_1 + n^3x_3), \tag{10.14}$$

for all $x_1, x_2, x_3 \in X$. Adding (10.12), (10.13) and (10.14), we obtain

$$\begin{aligned}
&12f(nx_1 + n^2x_2) + 12f(n^2x_2 + n^3x_3) + 12f(nx_1 + n^3x_3)\\
&= f(2nx_1 + 2n^2x_2 + n^3x_3) + f(2nx_1 + 2n^2x_2 - n^3x_3)\\
&+2f(-nx_1 - n^2x_2 - n^3x_3) + 2f(-nx_1 - n^2x_2 + n^3x_3)\\
&+f(nx_1 + 2n^2x_2 + 2n^3x_3) + f(-nx_1 - 2n^2x_2 + 2n^3x_3)\\
&+2f(-nx_1 - n^2x_2 - n^3x_3) + 2f(nx_1 - n^2x_2 - n^3x_3)\\
&+f(2nx_1 + n^2x_2 + 2n^3x_3) + f(2nx_1 - n^2x_2 + 2n^3x_3)\\
&+2f(-nx_1 - n^2x_2 - n^3x_3) + 2f(-nx_1 + n^2x_2 - n^3x_3),
\end{aligned} \tag{10.15}$$

for all $x_1, x_2, x_3 \in X$. Replacing (x, y) by $(nx_1, 2n^2x_2 + n^3x_3)$ in (10.3), we reach

$$f(2nx_1 + 2n^2x_2 + n^3x_3) = f(-2nx_1 + 2n^2x_2 + n^3x_3) + 2f(nx_1 + 2n^2x_2 + n^3x_3) + 2f(nx_1 - 2n^2x_2 - n^3x_3) + 12f(nx_1), \tag{10.16}$$

for all $x_1, x_2, x_3 \in X$. Adding $f(2nx_1 + 2n^2x_2 - n^3x_3)$ on both sides of (10.16) and using (10.3), we get

$$\begin{aligned}
&f(2nx_1 + 2n^2x_2 + n^3x_3) + f(2nx_1 + 2n^2x_2 - n^3x_3) = 2f(nx_1 + 2n^2x_2\\
&+n^3x_3) + 2f(nx_1 - 2n^2x_2 - n^3x_3) + 12f(nx_1) + 2f(-2nx_1 + n^2x_2 + n^3x_3)\\
&+2f(2nx_1 + n^2x_2 - n^3x_3) + 12f(n^2x_2),
\end{aligned} \tag{10.17}$$

for all $x_1, x_2, x_3 \in X$. Interchanging (x, y) by $(n^2x_2, nx_1 + 2n^3x_3)$ in (10.3), we have

$$f(nx_1 + 2n^2x_2 + 2n^3x_3) = f(nx_1 + 2n^2x_2 + 2n^3x_3) + 2f(-nx_1 + n^2x_2 - 2n^3x_3) + 12f(n^2x_2) + f(nx_1 - 2n^2x_2 + 2n^3x_3), \tag{10.18}$$

for all $x_1, x_2, x_3 \in X$. Adding $f(-nx_1 + 2n^2x_2 + 2n^3x_3)$ on both sides of (10.18), we have

$$\begin{aligned}
&f(nx_1 + 2n^2x_2 + 2n^3x_3) + f(-nx_1 + 2n^2x_2 + 2n^3x_3) = 2f(nx_1 + n^2x_2\\
&+2n^3x_3) + 2f(-nx_1 + n^2x_2 - 2n^3x_3) + 12f(n^2x_2) + 2f(nx_1 - 2n^2x_2\\
&+n^3x_3) + 2f(-nx_1 + 2n^2x_2 + n^3x_3) + 12f(n^3x_3),
\end{aligned} \tag{10.19}$$

for all $x_1, x_2, x_3 \in X$. Replacing (x, y) by $(n^3x_3, 2nx_1 + n^2x_2)$ in (10.3), we obtain

$$f(2nx_1+n^2x_2+2n^3x_3)=2f(2nx_1+n^2x_2+n^3x_3)1+2f(-2nx_1-n^2x_2 \\ +n^3x_3)+12f(n^3x_3)+f(2nx_1+n^2x_2-2n^3x_3), \tag{10.20}$$

for all $x_1, x_2, x_3 \in X$. Adding $f(2nx_1-n^2x_2+2n^3x_3)$ on both sides of (10.20), we have

$$\begin{aligned} &f(2nx_1+n^2x_2+2n^3x_3)+f(2nx_1-n^2x_2+2n^3x_3)\\ &=2f(2nx_1+n^2x_2+n^3x_3)+2f(-2nx_1-n^2x_2+n^3x_3)+12f(n^3x_3)\\ &+2f(nx_1+n^2x_2-2n^3x_3)+2f(nx_1-n^2x_2+2n^3x_3)\\ &+12f(nx_1), \end{aligned} \tag{10.21}$$

for all $x_1, x_2, x_3 \in X$. Using (10.17), (10.19) and (10.21) in (10.15), we get

$$\begin{aligned} &12f(nx_1+n^2x_2)+12f(n^2x_2+n^3x_3)+12f(nx_1+n^3x_3)\\ &=2f(nx_1+2n^2x_2+n^3x_3)+2f(nx_1-2n^2x_2-n^3x_3)\\ &+12f(nx_1)+2f(-2nx_1+n^2x_2+n^3x_3)+2f(2nx_1+n^2x_2-n^3x_3)\\ &+12f(n^2x_2)+2f(-nx_1-n^2x_2-n^3x_3)+2f(-nx_1-n^2x_2\\ &+n^3x_3)+2f(nx_1+n^2x_2+2n^3x_3)+2f(-nx_1+n^2x_2-2n^3x_3)\\ &+12f(n^2x_2)+2f(nx_1-2n^2x_2+n^3x_3)+2f(-nx_1+2n^2x_2+n^3x_3)\\ &12f(n^3x_3)+2f(-nx_1-n^2x_2-n^3x_3)+2f(nx_1-n^2x_2-n^3x_3)\\ &+2f(2nx_1+n^2x_2+n^3x_3)+2f(-2nx_1-n^2x_2+n^3x_3)+12f(n^3x_3)\\ &+2f(nx_1+n^2x_2-2n^3x_3)+2f(nx_1-n^2x_2+2n^3x_3)+12f(nx_1)\\ &+2f(-nx_1-n^2x_2-n^3x_3)+2f(-nx_1+n^2x_2-n^3x_3), \end{aligned} \tag{10.22}$$

for all $x_1, x_2, x_3 \in X$. Replacing (x, y) by $(n^3x_3, 2nx_1+n^3x_3)$ in (10.3), we get

$$f(2nx_1+2n^2x_2+n^3x_3)=2f(2nx_1+n^2x_2+n^3x_3)+2f(-2nx_1+n^2x_2 \\ -n^3x_3)+12f(n^2x_2)+f(2nx_1-2n^2x_2+n^3x_3), \tag{10.23}$$

for all $x_1, x_2, x_3 \in X$. Adding $f(2nx_1+2n^2x_2-n^3x_3)$ on both sides of (10.23), we obtain

$$\begin{aligned} &f(2nx_1+2n^2x_2+n^3x_3)+f(2nx_1+2n^2x_2-n^3x_3)\\ &=2f(2nx_1+n^2x_2+n^3x_3)+2f(-2nx_1-n^2x_2-n^3x_3)+12f(n^2x_2)\\ &+2f(nx_1-2n^2x_2+n^3x_3)+2f(nx_1+2n^2x_2-n^3x_3)\\ &+12f(nx_1), \end{aligned} \tag{10.24}$$

for all $x_1, x_2, x_3 \in X$. Replacing (x, y) by $(n^3x_3, nx_1+2n^2x_2)$ in (10.3), we obtain

$$f(nx_1+2n^2x_2+2n^3x_3)=2f(nx_1+2n^2x_2+n^3x_3)+2f(-nx_1-2n^2x_2 +n^3x_3)+12f(n^3x_3)+f(nx_1+2n^2x_2-2n^3x_3), \tag{10.25}$$

for all $x_1, x_2, x_3 \in X$. Adding $f(-nx_1+2n^2x_2+2n^3x_3)$ on both sides of (10.25), we attain

$$\begin{aligned}&f(nx_1+2n^2x_2+2n^3x_3)+f(-nx_1+2n^2x_2+2n^3x_3)\\&=2f(2nx_1+2n^2x_2+n^3x_3)+2f(-nx_1-2n^2x_2+n^3x_3)+12f(n^3x_3)\\&+2f(nx_1+n^2x_2-2n^3x_3)+2f(-nx_1+n^2x_2+2n^3x_3)\\&+12f(n^2x_2),\end{aligned} \tag{10.26}$$

for all $x_1, x_2, x_3 \in X$. Replacing (x, y) by $(nx_1, n^2x_2+2n^3x_3)$ in (10.3), we get

$$f(2nx_1+n^2x_2+2n^3x_3)=2f(nx_1+n^2x_2+2n^3x_3)+2f(nx_1-n^2x_2 -2n^3x_3)+12f(nx_1)+f(-2nx_1+n^2x_2+2n^3x_3), \tag{10.27}$$

for all $x_1, x_2, x_3 \in X$. Adding $f(2nx_1-n^2x_2+2n^3x_3)$ on both sides of (10.27), we obtain

$$\begin{aligned}&f(2nx_1+n^2x_2+2n^3x_3)+f(2nx_1-n^2x_2+2n^3x_3)\\&=2f(2nx_1+n^2x_2+n^3x_3)+2f(nx_1-n^2x_2-2n^3x_3)+12f(nx_1)\\&+2f(-2nx_1+n^2x_2+n^3x_3)+2f(2nx_1-n^2x_2+n^3x_3)\\&+12f(n^3x_3),\end{aligned} \tag{10.28}$$

for all $x_1, x_2, x_3 \in X$. Using (10.24), (10.26) and (10.28) in (10.15), we have

$$\begin{aligned}&12f(nx_1+n^2x_2)+12f(n^2x_2+n^3x_3)+12f(nx_1+n^3x_3)\\&=2f(nx_1-2n^2x_2+n^3x_3)+2f(nx_1+2n^2x_2-n^3x_3)\\&+12f(nx_1)+2f(2nx_1+n^2x_2+n^3x_3)+2f(-2nx_1+n^2x_2-n^3x_3)\\&+12f(n^2x_2)+2f(-nx_1-n^2x_2-n^3x_3)+2f(-nx_1-n^2x_2\\&+n^3x_3)+2f(nx_1+n^2x_2-2n^3x_3)+2f(-nx_1+n^2x_2+2n^3x_3)\\&+12f(n^2x_2)+2f(nx_1+2n^2x_2+n^3x_3)+2f(-nx_1-2n^2x_2+n^3x_3)\\&12f(n^3x_3)+2f(-nx_1-n^2x_2-n^3x_3)+2f(nx_1-n^2x_2-n^3x_3)\\&+2f(-2nx_1+n^2x_2+n^3x_3)+2f(2nx_1-n^2x_2+n^3x_3)+12f(n^3x_3)\\&+2f(nx_1+n^2x_2+2n^3x_3)+2f(nx_1-n^2x_2-2n^3x_3)+12f(nx_1)\\&+2f(-nx_1-n^2x_2-n^3x_3)+2f(-nx_1+n^2x_2-n^3x_3),\end{aligned} \tag{10.29}$$

for all $x_1, x_2, x_3 \in X$. Adding (10.22) and (10.29), we get

$$\begin{aligned}&48f(nx_1+n^2x_2)+48f(n^2x_2+n^3x_3)+48f(nx_1+n^3x_3)\\&-48f(nx_1)-48f(n^2x_2)-48f(n^3x_3)=36f(nx_1+n^2x_2+n^3x_3)\\&+12f(-nx_1+n^2x_2+n^3x_3)+12f(nx_1-n^2x_2+n^3x_3)\\&+12f(nx_1+n^2x_2-n^3x_3),\end{aligned} \tag{10.30}$$

for all $x_1, x_2, x_3 \in X$. Remodifying (10.30), we obtain

$$\begin{aligned}&3f(nx_1+n^2x_2+n^3x_3)+f(-nx_1+n^2x_2+n^3x_3)\\&+f(nx_1-n^2x_2+n^3x_3)+f(nx_1+n^2x_2-n^3x_3)\\&=4f(nx_1+n^2x_2)+4f(nx_1+n^3x_3)+4f(n^2x_2+n^3x_3)\\&-4nf(x_1)-4n^6f(x_2)-4n^9f(x_3),\end{aligned} \tag{10.31}$$

for all $x_1, x_2, x_3 \in X$.

Conversely, assume that $f : X \to Y$ satisfies the functional equation (10.8). Replacing (x_1, x_2, x_3) by $(x, 0, 0)$, $(0, x, 0)$ *and* $(0, 0, x)$ respectively in (10.31), we obtain

$$f(nx) = n^3 f(x), \quad f(n^2x) = n^6 f(x) \;\; and \;\; f(n^3x) = n^9 f(x), \tag{10.32}$$

for all $x \in X$. One can easy to verify from (10.32) that

$$f\left(\frac{x}{n^i}\right) = \left(\frac{1}{n^i}\right)^3 f(x_i), \tag{10.33}$$

for all $x, y \in X$. Replacing (x_1, x_2, x_3) by $\left(\frac{x}{n}, \frac{x}{n^2}, \frac{y}{n^3}\right)$ in (10.8), we get

$$3f(2x+y)+f(2x-y)=24f(x)-6f(y)+8f(x+y), \tag{10.34}$$

for all $x, y \in X$. Interchanging y by $-y$ in (10.34), we have

$$3f(2x-y)+f(2x+y)=24f(x)+6f(y)+8f(x-y). \tag{10.35}$$

Adding the Eqs. (10.34) and (10.35), we obtain

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x).$$

Define a mapping $Df : X \to y$ by

$$\begin{aligned}Df(x_1, x_2, x_3) &= 3f(nx_1 + n^2x_2 + n^3x_3) + f(-nx_1 + n^2x_2 + n^3x_3)\\ &+f(nx_1 - n^2x_2 + n^3x_3) + f(nx_1 + n^2x_2 - n^3x_3)\\ &-4f[f(nx_1 + n^2x_2) + f(nx_1 + n^3x_3) + f(n^2x_2 + n^3x_3)\\ &-n^3 f(nx_1) - n^6 f(x_2) - n^9 f(x_3)],\end{aligned} \tag{10.36}$$

for all $x_1, x_2, x_3 \in X$. □

10.3 3-Dimensional Cubic Functional Equation—Direct Method

Definition 10.2 Let x be a real linear space. A function $F : X \to R \cup [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $p, q \in \mathbb{R}$

(N1) $F(x, c) = 0 \;\; for \;\; c \le 0$
(N2) $x = 0$ if and only if $F(x, c) = 1 \;\; for \;\; all \;\; c > 0$
(N3) $F(cx, q) = F\left(x, \frac{q}{|c|}\right) \;\; if \;\; c \neq 0$
(N4) $F(x + y, p + q) \ge min\{F(x, p), F(y, q)\}$;
(N5) $F(x, .)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{q\to\infty} F(x, q) = 1$;
(N6) for $x \neq 0$, $F(x, .)$ is continuous on $\mathbb{R}$.

The pair (X, F) is called fuzzy normed linear space one may regard $F(x, q)$ as the truth value of the statement the norm of x is less than or equal to the real number q [58].

Definition 10.3 Let (X, F) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} (x_n - x, q) = 1$, for all $t > 0$. In that case x is called the limit of the sequence x_n and we denote it by $F - \lim_{n\to\infty} x_n = x$.

Definition 10.4 A sequence $\{x_n\}$ be in x called Cauchy if for each $\epsilon > 0$ and each $q > 0$ there exists n_0 such that for all $n \ge n_0$ and all $r > 0$, we have $F(x_{n+r} - x_n, q) > 1 - \epsilon$.

Definition 10.5 Every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and fuzzy normed space is called a fuzzy Banach space.

Note: Assume that X, (Z, F') and (Y, F) are linear space, fuzzy normed space and fuzzy Banach space respectively.
We define a mapping $Df_C : X \to Y$ by

$$
\begin{aligned}
Df_c(x_1,x_2,x_3) &= 3f(nx_1+n^2x_2+n^3x_3)+f(-nx_1+n^2x_2+n^3x_3)\\
&+f(nx_1-n^2x_2+n^3x_3)+f(nx_1+n^2x_2-n^3x_3)-4\big[f(nx_1+n^2x_2)\\
&+f(nx_1+n^3x_3)+f(n^2x_+n^3x_3)-n^3f(x_1)-n^6f(x_2)-n^9f(x_3)\big],
\end{aligned}
$$

for all $x_1, x_2, x_3 \in X$.

Theorem 10.6 *Let $\beta \in \{-1, 1\}$. Let $\chi : X^3 \to Z$ be a mapping with $0 < \left(\frac{d}{n^3}\right) < 1$*

$$F'(\chi(n^{\beta k}x_1, n^{\beta k}x_2, n^{\beta k}x_3), r) \geq F'(d^{\beta}\chi(x,0,0), r), \tag{10.37}$$

for all $x \in X$ and all $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} F'(\chi(n^{\beta k}x_1, n^{\beta k}x_2, n^{\beta k}x_3), n^{3\beta k}r) = 1, \tag{10.38}$$

for all $x \in X$ and all $r > 0$. Suppose that a function $Df_C : X \to Y$ satisfies the inequality

$$F(Df_c(x_1,x_2,x_3), r) \geq F'(\chi(x_1,x_2,x_3), r), \tag{10.39}$$

for all $r > 0$ and $x_1, x_2, x_3 \in X$. Then the limit

$$G(x) = F - \lim_{k\to\infty} \frac{f(n^{\beta k}x)}{n^{3\beta k}}, \tag{10.40}$$

exists for all $x \in X$ and the mapping $G : X \to Y$ is a unique cubic mapping such that

$$F(f(x) - G(x), r) \geq F'(\chi(x,0,0), 4r|n^3 - d|), \tag{10.41}$$

for all $x \in X$ and for all $r > 0$.

Proof First assume that $\beta = 1$. Replacing (x_1, x_2, x_3) by $(x, 0, 0)$, in (10.39), we have

$$F((4f(nx) - 4n^3f(x)), r) \geq F'(\chi(x,0,0), r), \tag{10.42}$$

for all $x \in X$ and for all $r > 0$. Replacing x by n^kx in (10.42), we obtain

$$F\left(\frac{f(n^{k+1}x)}{n^3} - f(n^kx), \frac{r}{4n^3}\right) \geq F'(\chi(n^kx,0,0), r), \tag{10.43}$$

for all $x \in X$ and for all $r > 0$. Using (10.37), (N3) in (10.43), we have

$$F\left(\frac{f(n^{k+1}x)}{n^3} - f(n^kx), \frac{r}{4n^3}\right) \geq F'(\chi(n^kx,0,0), \frac{r}{d^k}), \tag{10.44}$$

for all $x \in X$ and for all $r > 0$. It is easy to verify from (10.44), that

$$F\left(\frac{f(n^{k+1}x)}{n^{3(k+1)}} - \frac{f(n^k x)}{n^{3k}}, \frac{r}{4n^3 n^{3k}}\right) \geq F'(\chi(n^k x, 0, 0), \frac{r}{d^k}), \tag{10.45}$$

holds for all $x \in X$ and for all $r > 0$. Replacing r by $d^k r$ in (10.45)

$$F\left(\frac{f(n^{k+1}x)}{n^{3(k+1)}} - \frac{f(n^k x)}{n^{3k}}, \frac{r}{4n^3 n^{3k}}\right) \geq F'(\chi(n^k x, 0, 0), r), \tag{10.46}$$

for all $x \in X$ and for all $r > 0$, it is easy to see that

$$\frac{f(n^{k+1}x)}{n^{3(k+1)}} - f(x) = \sum_{i=0}^{k-1}\left[\frac{f(n^{i+1}x)}{n^{3(i+1)}} - \frac{f(n^i x)}{n^{3i}}\right], \tag{10.47}$$

for all $x \in X$. From the Eqs. (10.46) and (10.47), we get

$$\begin{aligned} F\left(\frac{f(n^k x)}{n^{3k}} - f(x), \sum_{i=0}^{k-1} \frac{d^i r}{4n^3 n^{3i}}\right) &\geq min \cup_{i=1}^{k-1} \left\{\frac{f(n^{i+1}x)}{n^{3(i+1)}} - \frac{f(n^i x)}{n^{3i}}, \frac{d^i r}{4n^3 n^{3i}}\right\} \\ &\geq min \cup_{i=1}^{k-1} F'(\chi(x, 0, 0), r) \\ &\geq F'(\chi(x, 0, 0), r), \end{aligned} \tag{10.48}$$

for all $x \in X$ and for all $r > 0$. Replacing x by $n^m x$ in (10.48) and using (10.37) and (N3), we obtain

$$F\left(\frac{f(n^{k+m}x)}{n^{3(k+m)}} - \frac{f(n^m x)}{n^{3m}}, \sum_{i=m}^{m+k-1} \frac{d^i r}{4n^3 n^{3i}}\right) \geq F\big(\chi(x, 0, 0), \frac{r}{d^m}\big), \tag{10.49}$$

for all $x \in X$ and for all $r > 0$. And all $m, k \geq 0$. Replacing r by $d^m r$ in (10.49), we get

$$F\left(\frac{f(n^{k+m}x)}{n^{3(k+m)}} - \frac{f(n^m x)}{n^{3m}}, \sum_{i=m}^{m+k-1} \frac{d^i r}{4n^3 n^{3i}}\right) \geq F\big(\chi(x, 0, 0), r\big), \tag{10.50}$$

for all $x \in X$ and for all $r > 0$. And all $m, k \geq 0$. Using (N3) in (10.49), we have

$$F\left(\frac{f(n^{k+m}x)}{n^{3(k+m)}} - \frac{f(n^m x)}{n^{3m}}, r\right) \geq F\left(\chi(x, 0, 0), \frac{r}{\sum_{i=m}^{m+k-1} \frac{d^i}{4n^3 n^{3i}}},\right), \tag{10.51}$$

for all $x \in X$ and for all $r > 0$. The Cauchy criterion for convergence and (N5) imply that $\left\{\frac{f(n^k x)}{n^{3k}}\right\}$, which is a Cauchy sequence in (Y, F) is a fuzzy Banach space. This sequence converges to some point $G(x) \in Y$. So one can define the mapping $G : X \to Y$ by

$$G(x) = F - \lim_{k\to\infty} \frac{f(n^{\beta k} x)}{n^{3\beta k}},$$

for all $x \in X$. Letting $m = 0$ in (10.51), we get

$$F\left(\frac{f(n^k x)}{n^{3k}} - f(x), r\right) \geq F'\left(\chi(x, 0, 0), \frac{r}{\sum_{i=0}^{k-1} \frac{d^i r}{4n^3 n^{3i}}}\right), \tag{10.52}$$

for all $x \in X$. Letting $k \to \infty$ in (10.52) and using (N6), we have

$$F(f(x) - G(x), r) \geq F'(\chi(x, 0, 0), 4r(n^3 - d)),$$

for all $x \in X$ and for all $r > 0$. To prove G satisfies (10.8), replacing (x_1, x_2, x_3) by $(n^k x_1, n^k x_2, n^k x_3)$ in (10.39), we get

$$F\left(\frac{1}{n^{3k}} Df_c(n^k x_1, n^k x_2, n^k x_3), r\right) \geq F'(\chi(n^k x_1, n^k x_2, n^k x_3), n^{3k} r), \tag{10.53}$$

for all $r > 0$ and all $x_1, x_2, x_3 \in X$. Now

$$\begin{aligned}
&F\big(3G(nx_1 + n^2x_2 + n^3x_3) + G(-nx_1 + n^2x_2 + n^3x_3) \\
&+G(nx_1 - n^2x_2 + n^3x_3) + G(nx_1 + n^2x_2 - n^3x_3) - 4\big[G(nx_1 + n^2x_2) \\
&+G(nx_1 + n^3x_3) + G(n^2x_+ n^3x_3) - n^3G(x_1) - n^6G(x_2) - n^9G(x_3)\big], r\big) \\
&\geq min\big\{F((3G(nx_1 + n^2x_2 + n^3x_3) + G(-nx_1 + n^2x_2 + n^3x_3) \\
&+G(nx_1 - n^2x_2 + n^3x_3) + G(nx_1 + n^2x_2 - n^3x_3))) \\
&-\frac{1}{n^{3k}}\bigg(3f(n^k(nx_1 + n^2x_2 + n^3x_3)) + f(n^k(-nx_1 + n^2x_2 + n^3x_3)) \\
&+f(n^k(nx_1 - n^2x_2 + n^3x_3)) + f(n^k(nx_1 + n^2x_2 - n^3x_3))\bigg) \\
&F\bigg(4\big[G(nx_1 + n^2x_2) + G(nx_1 + n^3x_3) + G(n^2x_+ n^3x_3) - n^3G(x_1) \\
&-n^6G(x_2) - n^9G(x_3)\big] - \frac{1}{n^{3k}}\bigg(4\big[f(n^k(nx_1 + n^2x_2)) + f(n^k(nx_1 + n^3x_3)) \\
&+f(n^k(n^2x_2 + n^3x_3)) - n^3f(n^k(x_1)) - n^6f(n^k(x_2)) - n^9f(n^k(x_3))\big]\bigg)\bigg)
\end{aligned}$$

$$F\Bigg(\frac{1}{n^{3k}}\Bigg(3f(n^k(nx_1+n^2x_2+n^3x_3))+f(n^k(-nx_1+n^2x_2+n^3x_3))$$
$$+f(n^k(nx_1-n^2x_2+n^3x_3))+f(n^k(nx_1+n^2x_2-n^3x_3))$$
$$-\frac{1}{n^{3k}}\Bigg(4\big[f(n^k(nx_1+n^2x_2))+f(n^k(nx_1+n^3x_3))+f(n^k(n^2x_2+n^3x_3))$$
$$-n^3f(n^k(x_1))-n^6f(n^k(x_2))-n^9f(n^k(x_3))\big]\Bigg)\Bigg),\frac{r}{3}\},\tag{10.54}$$

for all $x_1, x_2, x_3 \in X$ and all $r > 0$. Using (10.43) and (N5) in (10.54), we see that

$$\begin{aligned}
&F\big(3G(nx_1+n^2x_2+n^3x_3)+G(-nx_1+n^2x_2+n^3x_3)\\
&+G(nx_1-n^2x_2+n^3x_3)+G(nx_1+n^2x_2-n^3x_3)-4\big[G(nx_1+n^2x_2)\\
&+G(nx_1+n^3x_3)+G(n^2x_+n^3x_3)-n^3G(x_1)-n^6G(x_2)-n^9G(x_3)\big],r\big)\\
&\qquad\ge min\{1,1,F'(\chi(n^kx_1,n^kx_2,n^kx_3),n^{3k}r)\},\\
&\qquad\ge F'(\chi(n^kx_1,n^kx_2,n^kx_3),n^{3k}r),
\end{aligned}\tag{10.55}$$

for all $x_1, x_2, x_3 \in X$ and all $r > 0$. Letting $k \to \infty$ in (10.55) and using (10.38), we have

$$\begin{aligned}
&F\big(3G(nx_1+n^2x_2+n^3x_3)+G(-nx_1+n^2x_2+n^3x_3)+G(nx_1-n^2x_2\\
&+n^3x_3)+G(nx_1+n^2x_2-n^3x_3)-4\big[G(nx_1+n^2x_2)+G(nx_1+n^3x_3)\\
&+G(n^2x_+n^3x_3)-n^3G(x_1)-n^6G(x_2)-n^9G(x_3)\big],r\big)=1,
\end{aligned}$$

for all $x_1, x_2, x_3 \in X$ and all $r > 0$. Using (N2), we get

$$\begin{aligned}
&F\big(3G(nx_1+n^2x_2+n^3x_3)+G(-nx_1+n^2x_2+n^3x_3)+G(nx_1-n^2x_2\\
&+n^3x_3)+G(nx_1+n^2x_2-n^3x_3)=4\big[G(nx_1+n^2x_2)+G(nx_1+n^3x_3)\\
&+G(n^2x_+n^3x_3)-n^3G(x_1)-n^6G(x_2)-n^9G(x_3)\big]\big),
\end{aligned}$$

for all $x_1, x_2, x_3 \in X$. Hence G satisfies the cubic functional equation (10.8). In order to prove $G(x)$ is unique. We let $G'(x)$ be another cubic functional equation satisfying (10.8) and (10.41). Then

$$\begin{aligned}
F(G(x)-G'(x),r)&=F\left(\frac{G(n^kx)}{n^{3k}}-\frac{G'(n^kx)}{n^{3k}}\right)\\
&\ge\left\{F\left(\frac{G(n^kx)}{n^{3k}}-\frac{f(n^kx)}{n^{3k}},\frac{r}{2}\right),F\left(\frac{f(n^kx)}{n^{3k}}-\frac{G'(n^kx)}{n^{3k}},\frac{r}{2}\right)\right\}\\
&\ge F'\left(\chi(x,0,0),\frac{4n^{3k}r(n^3-d)}{2}\right)
\end{aligned}$$

$$\geq F'\left(\chi(x, 0, 0), \frac{4n^{3k}r(n^3 - d)}{2d^k}\right),$$

for all $x_1, x_2, x_3 \in X$ and all $r > 0$. Since,

$$\lim_{k\to\infty} \frac{4n^{3k}r(n^3 - d)}{2d^k} = \infty,$$

we obtain

$$F'\left(\chi(n^k x, 0, 0), \frac{4n^{3k}r(n^3 - d)}{2d^k}\right) = 1.$$

Thus $F(G(x) - G'(x), r) = 1$, for all $x \in X$ and for $r > 0$. Hence $G(x) = G'(x)$. Therefore $G(x)$ is unique.

For $\beta = -1$, we can prove the result by a similar method. This completes the proof of the theorem. □

Example 10.7 Let $\beta \in \{-1, 1\}$, $\chi : X^3 \to Z$ be a mapping with $0 < \left(\frac{d}{2^3}\right) < 1$

$$F'(\chi(2^{\beta k}x_1, 2^{\beta k}x_2, 2^{\beta k}x_3), r) \geq F'(d^\beta \chi(x, 0, 0), r),$$

for all $x \in X$ and all $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} F'(\chi(2^{\beta k}x_1, 2^{\beta k}x_2, 2^{\beta k}x_3), 2^{3\beta k}r) = 1,$$

for all $x \in X$ and all $r > 0$. Suppose that a function $Df_c : X \to Y$ satisfies the inequality

$$F(Df_c(x_1, x_2, x_3), r) \geq F'(\chi(x_1, x_2, x_3), r),$$

for all $r > 0$ and $x_1, x_2, x_3 \in X$. Then the limit

$$G(x) = F - \lim_{k\to\infty} \frac{f(2^{\beta k}x)}{2^{3\beta k}},$$

exists for all $x \in X$ and the mapping $G : X \to Y$ is a unique cubic mapping such that

$$F(f(x) - G(x), r) \geq F'(\chi(x, 0, 0), 4r|2^3 - d|),$$

for all $x \in X$ and for all $r > 0$.

Proposition 10.8 *Let* $\beta \in \{-1, 1\}$, $\chi : X^3 \to Z$ *be a mapping with* $0 < \left(\frac{d}{n^6}\right) < 1$

$$F'(\chi(n^{2\beta k}x_1, n^{2\beta k}x_2, n^{2\beta k}x_3), r) \geq F'(d^\beta \chi(0, x, 0), r),$$

for all $x \in X$ and all $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} F'(\chi(n^{2\beta k}x_1, n^{2\beta k}x_2, n^{2\beta k}x_3), n^{6\beta k}r) = 1,$$

for all $x \in X$ and all $r > 0$. Suppose that a function $Df_c : X \to Y$ satisfies the inequality

$$F(Df_c(x_1, x_2, x_3), r) \geq F'(\chi(x_1, x_2, x_3), r),$$

for all $r > 0$ and $x_1, x_2, x_3 \in X$. Then the limit

$$G(x) = F - \lim_{k\to\infty} \frac{f(n^{2\beta k}x)}{n^{6\beta k}},$$

exists for all $x \in X$ and the mapping $G : X \to Y$ is a unique cubic mapping such that

$$F(f(x) - G(x), r) \geq F'(\chi(0, x, 0), 4r|n^6 - d|),$$

for all $x \in X$ and for all $r > 0$.

Example 10.9 Let $\beta \in \{-1, 1\}$, $\chi : X^3 \to Z$ be a mapping with $0 < \left(\frac{d}{2^6}\right) < 1$

$$F'(\chi(2^{2\beta k}x_1, 2^{2\beta k}x_2, 2^{2\beta k}x_3), r) \geq F'(d^{\beta}\chi(0, x, 0), r),$$

for all $x \in X$ and all $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} F'(\chi(2^{2\beta k}x_1, 2^{2\beta k}x_2, 2^{2\beta k}x_3), 2^{6\beta k}r) = 1,$$

for all $x \in X$ and all $r > 0$. Suppose that a function $Df_c : X \to Y$ satisfies the inequality

$$F(Df_c(x_1, x_2, x_3), r) \geq F'(\chi(x_1, x_2, x_3), r),$$

for all $r > 0$ and $x_1, x_2, x_3 \in X$. Then the limit

$$G(x) = F - \lim_{k\to\infty} \frac{f(2^{2\beta k}x)}{2^{6\beta k}},$$

exists for all $x \in X$ and the mapping $G : X \to Y$ is a unique cubic mapping such that

$$F(f(x) - G(x), r) \geq F'(\chi(0, x, 0), 4r|2^6 - d|),$$

for all $x \in X$ and for all $r > 0$.

Proposition 10.10 *Let $\beta \in \{-1, 1\}$, $\chi : X^3 \to Z$ be a mapping with* $0 < \left(\frac{d}{n^9}\right) < 1$

$$F'(\chi(n^{3\beta k}x_1, n^{3\beta k}x_2, n^{3\beta k}x_3), r) \geq F'(d^{\beta}\chi(0, 0, x), r),$$

for all $x \in X$ and all $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} F'(\chi(n^{3\beta k}x_1, n^{3\beta k}x_2, n^{3\beta k}x_3), n^{9\beta k}r) = 1,$$

for all $x \in X$ and all $r > 0$. Suppose that a function $Df_c : X \to Y$ satisfies the inequality

$$F(Df_c(x_1, x_2, x_3), r) \geq F'(\chi(x_1, x_2, x_3), r),$$

for all $r > 0$ and $x_1, x_2, x_3 \in X$. Then the limit

$$G(x) = F - \lim_{k\to\infty} \frac{f(n^{3\beta k}x)}{n^{9\beta k}},$$

exists for all $x \in X$ and the mapping $G : X \to Y$ is a unique cubic mapping such that

$$F(f(x) - G(x), r) \geq F'(\chi(0, x, 0), 4r|n^9 - d|),$$

for all $x \in X$ and for all $r > 0$.

The following corollaries are the immediate consequence of Theorem 10.6, Propositions 10.8 and 10.10, concerning the stability of (10.8).

Corollary 10.11 *Suppose that $f : X \to Y$ satisfies the inequality*

$$F(Df_c(x_1, x_2, x_3)) \geq \begin{cases} F'(\epsilon, r), \\ F'\left(\epsilon \sum_{i=1}^{3} \parallel x_i \parallel^s, r\right), \quad s \neq 3 \\ F'\left(\epsilon\left(\sum_{i=1}^{3} \parallel x_i \parallel^{3s} + \prod_{i=1}^{3} \parallel x_i \parallel^s\right), r\right), \quad s \neq 1 \end{cases} \tag{10.56}$$

for all $x_1, x_2, x_3 \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $G : X \to Y$ such that

$$F(f(x) - G(x), r) \geq \begin{cases} F'(\epsilon, 4r|n^3 - 1|) \\ F'(\epsilon \parallel x \parallel^s, 4r|n^3 - n^s|) \\ F'(\epsilon \parallel x \parallel^{3s}, 4r|n^3 - n^{3s}|), \end{cases} \tag{10.57}$$

for all $x \in X$ and for $r > 0$.

Remark: Suppose that $f : X \to Y$ satisfies the inequality (10.56), for all $x_1, x_2, x_3 \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $G : X \to Y$ such that

$$F\left(f(x) - G(x), r\right) \geq \begin{cases} F'\left(\epsilon, 4r|7|\right) \\ F'\left(\epsilon\|x\|^s, 4r|2^3 - 2^s|\right) \\ F'\left(\epsilon\|x\|^{ns}, 4r|2^3 - 2^{3s}|\right). \end{cases}$$

Corollary 10.12 *Suppose that $f : X \to Y$ satisfies the inequality (10.56), for all $x_1, x_2, x_3 \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $G : X \to Y$ such that for all $x \in X,\ r > 0$*

$$F(f(x) - G(x), r) \geq \begin{cases} F'(\epsilon, 4r|n^6 - 1|) \\ F'(\epsilon \parallel x \parallel^s, 4r|n^6 - n^{2s}|) \\ F'(\epsilon \parallel x \parallel^{3s}, 4r|n^6 - n^{6s}|). \end{cases} \tag{10.58}$$

Remark: Suppose that $f : X \to Y$ satisfies the inequality (10.56), for all $x_1, x_2, x_3 \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $G : X \to Y$ such that for all $x \in X,\ r > 0$

$$F\left(f(x) - G(x), r\right) \geq \begin{cases} F'\left(\epsilon, 4r|63|\right) \\ F'\left(\epsilon\|x\|^s, 4r|2^6 - 2^{2s}|\right) \\ F'\left(\epsilon\|x\|^{ns}, 4r|2^6 - 2^{6s}|\right). \end{cases}$$

Corollary 10.13 *Suppose that $f : X \to Y$ satisfies the inequality (10.56), for all $x_1, x_2, x_3 \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $G : X \to Y$ such that for all $x \in X,\ r > 0$*

$$F(f(x) - G(x), r) \geq \begin{cases} F'(\epsilon, 4r|n^9 - 1|) \\ F'(\epsilon \parallel x \parallel^s, 4r|n^9 - n^{3s}|) \\ F'(\epsilon \parallel x \parallel^{3s}, 4r|n^9 - n^{9s}|). \end{cases} \tag{10.59}$$

10.4 3-Dimensional Cubic Functional Equation—Fixed Point Method

Theorem 10.14 *Let $f : X \to Y$ be a mapping for which there exists a function $\chi : X^3 \to Z$ with condition*

$$\lim_{k\to\infty} F'\left(\chi(\psi_i^k x_1, \psi_i^k x_2, \psi_i^k x_3), \psi_i^{3k} r\right) = 1, \tag{10.60}$$

where $\psi_i = \begin{cases} n & if \;\; i = 0 \\ \frac{1}{n} & if \; \imath = 1 \end{cases}$ *and* Ω *is the set such that*

$$\Omega = \{p \backslash p : X \to Y, p(0) = 0\},$$

for all $x_1, x_2, x_3 \in X$, $r > 0$ *and satisfying the inequality*

$$F(Df_c(x_1, x_2, x_3), r) \geq F'(\chi(x_1, x_2, x_3), r), \tag{10.61}$$

for all $x_1, x_2, x_3 \in X$ *and* $r > 0$. *If there exists* $L = L[i]$ *such that the function* $x \to \rho(x)$ *has the property*

$$F'\left(L\frac{1}{\psi_i^3}\rho(\psi_i x), r\right) = F'(\rho(x), r), \tag{10.62}$$

for all $x \in X$, $r > 0$*, then there exists unique cubic mapping* $G : X \to Y$ *satisfying the functional equation (10.8) and*

$$F(f(x) - G(x), r) \geq F'\left(\frac{L^{1-i}}{1-L}\rho(x), r\right), \tag{10.63}$$

for all $x \in X$, $r > 0$.

Proof Let us consider the set $\Omega = \{p | p : U^2 \to V, \;\; p(0) = 0\}$ and d be a generalized metric on Ω, such that

$$d(p, q) = inf\{k \in (0, \infty)/F(p(x) - q(x), r) \geq F'(\rho(x), kr), x \in X, r > 0\},$$

It is easy to see that (Ω, d) is complete.
Define $T : \Omega \to \Omega$ by $Tp(x) = \frac{1}{\psi_i^3}\rho(\psi_i x), \;\; \forall x \in X$. For $p, q \in \Omega$, we get

$$\begin{aligned} d(p, q) = k \;\; &\Rightarrow F(p(x) - q(x)) \geq F'(\rho(x), kr) \\ &\Rightarrow F\left(\frac{p(\psi_i x)}{\psi_i^3} - \frac{q(\psi_i x)}{\psi_i^3}, r\right) \geq F'(\rho(\psi(x)), k\psi_i^3 r) \\ &\Rightarrow F(Tp(x) - Tq(x), r) \geq F'(\rho(\psi(x)), k\psi_i^3 r) \\ &\Rightarrow F(Tp(x) - Tq(x), r) \geq F'(\rho(x), kLr) \\ &\Rightarrow d(Tp(x) - Tq(x), r) \geq kL \\ &\Rightarrow d(Tp - Tq, r) \geq kd(p, q), \;\; \forall p, q \in \Omega. \end{aligned} \tag{10.64}$$

Therefore T is strictly contractive mapping on Ω with Lipschitz constant L. Replacing (x_1, x_2, x_3) by $(x, 0, 0)$ in (10.61), we get

$$F(4f(nx) - 4n^3 f(x), r) \geq F'(\chi(x, 0, 0), r), \tag{10.65}$$

for all $x \in X, r > 0$. Using (N3) in (10.65), we have

$$F\left(\frac{f(nx)}{n^3} - f(x), r\right) \geq F'\left(\frac{1}{4n^3}\chi(x, 0, 0), r\right), \tag{10.66}$$

for all $x \in X, r > 0$ with the help of (10.62), when $i = 0$. It follows from (10.66) that

$$\begin{aligned} &F\left(\frac{f(nx)}{n^3} - f(x), r\right) \geq F'(L\rho(x), r), \\ &d(Tf(x), r) \geq L = L^1 = L^{1-i}, \end{aligned} \tag{10.67}$$

for all $x \in X, r > 0$. Replacing x by $\frac{x}{n}$ in (10.65), we get

$$F\left(f(x) - n^3 f\left(\frac{x}{n}\right), r\right) \geq F'\left(\frac{1}{4}\chi\left(\frac{x}{n}, 0, 0\right), r\right), \tag{10.68}$$

for all $x \in X, r > 0$ when $i = 1$. It follows from (10.68), that

$$\begin{aligned} &F\left(f(x) - n^3 f\left(\frac{x}{n}\right), r\right) \geq F'(\rho(x), r), \\ &T(f - Tf) \leq 1 = L^0 = L^{1-i}. \end{aligned} \tag{10.69}$$

Then from (10.67) and (10.69), we get

$$T(f, Tf) \leq L^{1-i} < \infty.$$

Now from the fixed point alternative in both cases it follows that there exists a fixed point G of T in Ω, such that

$$G(x) = F - \lim_{k\to\infty} \frac{f(\psi_i^k x)}{\psi_i^{3k}}, \tag{10.70}$$

for all $x \in X, r > 0$. Replacing (x_1, x_2, x_3) by $(\psi_i^k x_1, \psi_i^k x_2, \psi_i^k x_3)$ in (10.61), we get

$$F\left(\frac{1}{\psi_i^{3k}} Df_c(\psi_i^k x_1, \psi_i^k x_2, \psi_i^k x_3), r\right) \geq F'(\chi(\psi_i^k x_1, \psi_i^k x_2, \psi_i^k x_3), \psi_i^{3k} r)$$

for all $r > 0$ and all $x_1, xx_2, x_3 \in X$. By proceeding the some procedure in Theorem 10.6, we can prove the mapping $G : X \to Y$ is cubic and its satisfies the functional equation (10.8) by a fixed point alternative. Since G is unique fixed point of T in the set $\Delta = \{f \in \Omega/d(f, G) < \infty\}$, G is a unique function such that

$$F(f(x) - G(x), r) \geq F'(\rho(x), kr), \tag{10.71}$$

for all $x \in X$, $r > 0$. Again using the fixed point alternative, we get

$$\begin{aligned} & d(f, G) \leq \frac{1}{1-L} d(f, Tf), \\ & \Rightarrow d(f, G) \leq \frac{L^{1-i}}{1-L} \\ & \Rightarrow F(f(x) - G(x), r) \geq F'\left(\rho(x)\frac{L^{1-i}}{1-L}, r\right). \end{aligned} \tag{10.72}$$

This completes the proof of the theorem. □

Example 10.15 Let $f : X \to Y$ be a mapping for which there exists a function $\chi : X^3 \to Z$ with condition (10.60) where $\psi_i = \begin{cases} 2 \ if \ i = 0; \\ \frac{1}{2} \ if \ \imath = 1; \end{cases}$ and Ω is the set such that $\Omega = \{p|p : X \to Y, p(0) = 0\}$, for all $x \in X$, $r > 0$ and satisfying the inequality (10.61) for all $x_1, x_2, x_3 \in X$ and $r > 0$. If there exists $L = L[i]$ such that the function $x \to \rho(x)$ has the property (10.62) for all $x \in X$, $r > 0$, then there exists unique cubic mapping $G : X \to Y$ satisfying the functional equation (10.8) and (10.63), for all $x \in X$, $r > 0$.

Proposition 10.16 *Let $f : X \to Y$ be a mapping for which there exists a function $\chi : X^3 \to Z$ with condition (10.60) where $\psi_i = \begin{cases} n^2 \ if \ i = 0 \\ \frac{1}{n^2} \ if \ \imath = 1 \end{cases}$ and Ω is the set such that $\Omega = \{p|p : X \to Y, p(0) = 0\}$, for all $x \in X$, $r > 0$ and satisfying the inequality (10.61) for all $x_1, x_2, x_3 \in X$ and $r > 0$. If there exists $L = L[i]$ such that the function $x \to \rho(x)$ has the property (10.62) for all $x \in X$, $r > 0$, then there exists unique cubic mapping $G : X \to Y$ satisfying the functional equation (10.8) and (10.63), for all $x \in X$, $r > 0$.*

Example 10.17 Let $f : X \to Y$ be a mapping for which there exists a function $\chi : X^3 \to Z$ with condition (10.60) where $\psi_i = \begin{cases} 2^2 \ if \ i = 0; \\ \frac{1}{2^2} \ if \ \imath = 1; \end{cases}$ and Ω is the set such that $\Omega = \{p|p : X \to Y, p(0) = 0\}$, for all $x \in X$, $r > 0$ and satisfying the inequality (10.61) for all $x_1, x_2, x_3 \in X$ and $r > 0$. If there exists $L = L[i]$ such that the function $x \to \rho(x)$ has the property (10.62) for all $x \in X$, $r > 0$, then there exists unique cubic mapping $G : X \to Y$ satisfying the functional equation (10.8) and (10.63), for all $x \in X$, $r > 0$.

Proposition 10.18 *Let $f : X \to Y$ be a mapping for which there exists a function $\chi : X^3 \to Z$ with condition (10.60) where $\psi_i = \begin{cases} n^3 \ if \ i = 0 \\ \frac{1}{n^3} \ if \ \imath = 1 \end{cases}$ and Ω is the set such that $\Omega = \{p|p : X \to Y, p(0) = 0\}$, for all $x \in X$, $r > 0$ and satisfying the* *inequality (10.61) for all $x_1, x_2, x_3 \in X$ and $r > 0$. If there exists $L = L[i]$ such that the function $x \to \rho(x)$ has the property (10.62) for all $x \in X$, $r > 0$, then there exists unique cubic mapping $G : X \to Y$ satisfying the functional equation (10.8) and (10.63), for all $x \in X$, $r > 0$.*

The following corollaries are in the immediate consequence of Theorem 10.14, Propositions 10.16 and 10.18 respectively, concerning the stability of (10.8).

Corollary 10.19 *Suppose that $f : X \to Y$ satisfies the inequality (10.56), for all $x_1, x_2, x_3 \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $G : X \to Y$ such that (10.57), for all $x \in X$ and $r > 0$.*

Proof Set

$$\chi(x_1, x_2, x_3) \leq \begin{cases} \epsilon \\ \epsilon\left\{\sum_{i=1}^{3} \| x_i \|^s, r\right\} \\ \epsilon\left\{\sum_{i=1}^{3} \| x_i \|^{3s} + \prod_{i=1}^{3} \| x_i \|^s \right\}, \end{cases}$$

for all $x_1, x_2, x_3 \in X$. Then

$$F'(\chi(\psi_i^k x_1, \psi_i^k x_2, \psi_i^k x_3), \psi_i^{3k} r) = \begin{cases} F'(\epsilon, \psi_i^{3k} r) \\ F'\left(\epsilon \sum_{i=1}^{3} \| x_i \|^s, \psi_i^{(3-s)k} r\right), \\ F'\left(\epsilon\left(\sum_{i=1}^{3} \| x_i \|^{3s} + \prod_{i=1}^{3} \| x_i \|^s \right), \right. \\ \qquad \left. \psi_i^{(3-s)k} r\right) \end{cases}$$
$$= \begin{cases} \to 1 \;\; as \;\; k \to \infty \\ \to 1 \;\; as \;\; k \to \infty \\ \to 1 \;\; as \;\; k \to \infty, \end{cases}$$

i.e., (10.60) holds. Since we have

$$\rho(x) = \frac{1}{4}\chi\left(\frac{x}{n}, 0, 0\right),$$

$$F'\left(L\frac{1}{\psi_i^3}\rho(\psi_i x), r\right) = F'(\rho(x), r),$$

for all $x \in X$ and $r > 0$. Hence

$$F'(\rho(x), r) = F'\left(\chi\left(\frac{x}{n}, 0, 0\right), 4r\right) = \begin{cases} F'(\epsilon, 4n^3 r) \\ F'(\epsilon \| x \|^s, 4n^s r) \\ F'(\epsilon \| x \|^{3s}, 4n^{3s} r). \end{cases}$$

Now

$$F'\left(L\frac{1}{\psi_i^3}\rho(\psi_i x), r\right) = \left\{\begin{array}{l} F'\left(\frac{\epsilon}{\psi_i^3}, 4r\right) \\ F'\left(\frac{\epsilon\|x\|^s}{\psi_i^3 n^s}, 4r\right) \\ F'\left(\frac{\epsilon\|x\|^s\psi_i^{3s}}{\psi_i^3 n^{3s}}, 4r\right) \end{array}\right\} = \left\{\begin{array}{l} \psi_i^{-3}\rho(x) \\ \psi_i^{s-3}\rho(x) \\ \psi_i^{3s-3}\rho(x), \end{array}\right\}$$

for all $x \in X$. Now setting
$L = n^{-3}\ if\ i = 0\ and\ L = n^3\ if\ i = 1.$
$L = n^{s-3}\ for\ s > 3\ if\ i = 0\ and\ L = n^{3-s}\ for\ s < 3\ if\ i = 1.$
$L = n^{3s-3}\ for\ s > 1\ if\ i = 0\ and\ L = n^{3-3s}\ for\ s < 1\ if\ i = 1.$

Case 1. $L = n^{-3}\ if\ i = 0$

$$\begin{aligned} F(f(x) - G(x), r) &\leq F'\left(\frac{L^{1-i}}{1-L}\rho(x), r\right) \\ &= F'\left(\frac{n^{-3}}{1-n^{-3}}\frac{\epsilon}{4}, r\right) \\ &= F'(\epsilon, 4(n^3 - 1)r). \end{aligned}$$

Case 2. $L = n^3\ if\ i = 1$

$$\begin{aligned} F(f(x) - G(x), r) &\leq F'\left(\frac{L^{1-i}}{1-L}\rho(x), r\right) \\ &= F'\left(\frac{1}{1-n^3}\frac{\epsilon}{4}, r\right) \\ &= F'(\epsilon, 4(1 - n^3)r). \end{aligned}$$

Case 3. $L = n^{s-3}\ for\ s > 3\ if\ i = 0$

$$\begin{aligned} F(f(x) - G(x), r) &\leq F'\left(\frac{L^{1-i}}{1-L}\rho(x), r\right) \\ &= F'\left(\frac{n^{s-3}}{1-n^{s-3}}\frac{\epsilon \parallel x \parallel^s}{4n^s}, r\right) \\ &= F'(\epsilon \parallel x \parallel^s, 4(n^3 - n^s)r). \end{aligned}$$

Case 4. $L = 2^{3-s}\ for\ s < 3\ if\ i = 1$

$$\begin{aligned} F(f(x)-G(x),r) &\le F'\left(\frac{L^{1-i}}{1-L}\rho(x),r\right) \\ &= F'\left(\frac{1}{1-n^{3-s}}\frac{\epsilon \parallel x \parallel^s}{4n^s},r\right) \\ &= F'(\epsilon \parallel x \parallel^{3s}, 4(n^s-n^3)r). \end{aligned}$$

Case 5. $L = 2^{3s-3}\ \ for\ \ s > 1\ \ if\ \ i = 0$

$$\begin{aligned} F(f(x)-G(x),r) &\le F'\left(\frac{L^{1-i}}{1-L}\rho(x),r\right) \\ &= F'\left(\frac{n^{3s-3}}{1-n^{3s-3}}\frac{\epsilon \parallel x \parallel^{3s}}{4n^{3s}},r\right) \\ &= F'(\epsilon \parallel x \parallel^{3s}, 4(n^3-n^{3s})r). \end{aligned}$$

Case 6. $L = 2^{3-3s}\ \ for\ \ s < 1\ \ if\ \ i = 1$

$$\begin{aligned} F(f(x)-G(x),r) &\le F'\left(\frac{L^{1-i}}{1-L}\rho(x),r\right) \\ &= F'\left(\frac{1}{1-n^{3-3s}}\frac{\epsilon \parallel x \parallel^{3s}}{4n^{3s}},r\right) \\ &= F'(\epsilon \parallel x \parallel^{3s}, 4(n^{3s}-n^3)r). \end{aligned}$$

Hence the proof is completed. □

Remark: Suppose that a mapping $f : X \to y$ satisfies the inequality (10.56), for all $x_1, x_2, x_3 \in X$ and $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $G : X \to Y$, such that

$$F\left(f(x)-G(x),r\right) \ge \begin{cases} F'\left(\epsilon, 4r|7|\right) \\ F'\left(\epsilon\|x\|^s, 4r|2^3-2^s|\right) \\ F'\left(\epsilon\|x\|^{ns}, 4r|2^3-2^{3s}|\right), \end{cases}$$

for all $x \in X$ and $r > 0$.

Corollary 10.20 *Suppose that a mapping $f : X \to y$ satisfies the inequality (10.56), for all $x_1, x_2, x_3 \in X$ and $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $G : X \to Y$ such that (10.58), for all $x \in X$ and $r > 0$.*

Remark: Suppose that a mapping $f : X \to y$ satisfies the inequality (10.56), for all $x_1, x_2, x_3 \in X$ and $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $G : X \to Y$ such that

$$F\left(f(x)-G(x),r\right)\geq\begin{cases}F'\left(\epsilon,4r|63|\right)\\F'\left(\epsilon\|x\|^{s},4r|2^{6}-2^{2s}|\right)\\F'\left(\epsilon\|x\|^{ns},4r|2^{6}-2^{6s}|\right),\end{cases}$$

for all $x \in X$ and $r > 0$.

Corollary 10.21 *Suppose that a mapping $f : X \to y$ satisfies the inequality (10.56), for all $x_1, x_2, x_3 \in X$ and $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $G : X \to Y$ such that (10.59) for all $x \in X$ and $r > 0$.*

10.5 Stability of 3-Dimensional Cubic Functional Equation—Direct Method

Proposition 10.22 *Let $\beta \in \{-1, 1\}$. Let $\chi : X^3 \to [0, \infty)$ be a mapping such that*

$$\sum_{k=0}^{\infty}\frac{\chi(n^{k\beta}x_1, n^{k\beta}x_2, n^{k\beta}x_3)}{n^{3k\beta}}$$

converges in $\mathbb{R}$ for all $x_1, x_2, x_3 \in X$. Let $f : X \to Y$ be a mapping satisfying the inequality $\| Df_c(x_1, x_2, x_3) \| \leq \chi(x_1, x_2, x_3)$, for all $x_1, x_2, x_3 \in X$. Then there exists a unique cubic mapping $C : X \to Y$ which satisfies the functional equation (10.8) and

$$\| f(x) - C(x) \| \leq \frac{1}{4n^3}\sum_{k=\frac{1-\beta}{2}}^{\infty}\frac{\chi(n^{k\beta}x, 0, 0)}{n^{3k\beta}},$$

for all $x \in X$. The mapping $C : X \to Y$ is defined by $C(x) = \lim_{k\to\infty}\frac{f(n^{k\beta}x)}{n^{3k\beta}}$, for all $x \in X$.

Example 10.23 Let $\beta \in \{-1, 1\}$. Let $\chi : X^3 \to [0, \infty)$ be a mapping such that

$$\sum_{k=0}^{\infty}\frac{\chi(2^{k\beta}x_1, 2^{k\beta}x_2, 2^{k\beta}x_3)}{2^{3k\beta}}$$

converges in $\mathbb{R}$ for all $x_1, x_2, x_3 \in X$. Let $f : X \to Y$ be function satisfying the inequality $\| Df_c(x_1, x_2, x_3) \| \leq \chi(x_1, x_2, x_3)$, for all $x_1, x_2, x_3 \in X$. Then there exists a unique cubic mapping $C : X \to Y$ which satisfies the functional equation (10.8) and

$$\| f(x) - C(x) \| \leq \frac{1}{32}\sum_{k=\frac{1-\beta}{2}}^{\infty}\frac{\chi(2^{k\beta}x, 0, 0)}{2^{3k\beta}}$$

for all $x \in X$. The mapping $C : X \to Y$ is defined by $C(x) = \lim_{k\to\infty} \frac{f(2^{k\beta}x)}{2^{3k\beta}}$, for all $x \in X$.

Proposition 10.24 *Let $\beta \in \{-1, 1\}$, $\chi : X^3 \to [0, \infty)$ be a mapping such that*

$$\sum_{k=0}^{\infty} \frac{\chi(n^{2k\beta}x_1, n^{2k\beta}x_2, n^{2k\beta}x_3)}{n^{6k\beta}}$$

converges in $\mathbb{R}$ for all $x_1, x_2, x_3 \in X$. Let $f : X \to Y$ be function satisfying the inequality $\| f(x_1, x_2, x_3) \| \leq \chi(x_1, x_2, x_3)$, for all $x_1, x_2, x_3 \in X$. Then there exists a unique mapping $C : X \to Y$ which satisfies the functional equation (10.8) and

$$\| f(x) - C(x) \| \leq \frac{1}{4n^6} \sum_{k=\frac{1-\beta}{2}}^{\infty} \frac{\chi(0, n^{2k\beta}x, 0)}{n^{6k\beta}}$$

for all $x \in X$. The mapping $C : X \to Y$ is defined by $C(x) = \lim_{k\to\infty} \frac{f(n^{2k\beta}x)}{n^{6k\beta}}$, for all $x \in X$.

Proposition 10.25 *Let $\beta \in \{-1, 1\}$, $\chi : X^3 \to [0, \infty)$ be a mapping such that*

$$\sum_{k=0}^{\infty} \frac{\chi(n^{3k\beta}x_1, n^{3k\beta}x_2, n^{3k\beta}x_3)}{n^{9k\beta}}$$

converges in $\mathbb{R}$ for all $x_1, x_2, x_3 \in X$. Let $f : X \to Y$ be function satisfying the inequality $\| f(x_1, x_2, x_3) \| \leq \chi(x_1, x_2, x_3)$, for all $x_1, x_2, x_3 \in X$. Then there exists a unique mapping $C : X \to Y$ which satisfies the functional equation (10.8) and

$$\| f(x) - C(x) \| \leq \frac{1}{4n^9} \sum_{k=\frac{1-\beta}{2}}^{\infty} \frac{\chi(0, 0, n^{2k\beta}x)}{n^{9k\beta}}$$

for all $x \in X$. The mapping $C : X \to Y$ is defined by $C(x) = \lim_{k\to\infty} \frac{f(n^{3k\beta}x)}{n^{9k\beta}}$, for all $x \in X$.

The following corollaries are the immediate consequence of Propositions 10.22–10.25 respectively, concerning the stability of (10.8).

Corollary 10.26 *Let ϵ and s be non-negative real numbers. If a mapping $f : X \to Y$ satisfies the inequality*

$$Df_c(x_1, x_2, x_3) \leq \begin{cases} \epsilon, \\ \epsilon \sum_{i=1}^{3} \| x_i \|^s, \quad s \neq 3 \\ \epsilon \left(\sum_{i=1}^{3} \| x_i \|^{3s} + \prod_{i=1}^{3} \| x_i \|^{3s} \right), \quad s \neq 1 \end{cases} \tag{10.73}$$

for all $x_1, x_2, x_3 \in X$, then there exists a unique cubic mapping $C : X \to Y$ such that, for all $x \in X$

$$\| f(x) - C(x) \| \leq \begin{cases} \frac{\epsilon}{4|n^3-1|} \\ \frac{\epsilon \|x\|^s}{4|n^3-n^s|} \\ \frac{\epsilon \|x\|^{ns}}{4|n^3-n^{3s}|}. \end{cases} \tag{10.74}$$

Remark: Let ϵ and s be non-negative real numbers. If a mapping $f : X \to Y$ satisfies the inequality (10.73) for all $x_1, x_2, x_3 \in X$, then there exists a unique cubic mapping $C : X \to Y$ such that, for all $x \in X$

$$\| f(x) - C(x) \| \leq \begin{cases} \frac{\epsilon}{4|7|} \\ \frac{\epsilon \|x\|^s}{4|2^3-2^s|} \\ \frac{\epsilon \|x\|^{ns}}{4|2^3-2^{3s}|}. \end{cases}$$

Corollary 10.27 *Let ϵ and s be non-negative real numbers. If a mapping $f : X \to Y$ satisfies the inequality (10.73) for all $x_1, x_2, x_3 \in X$, then there exists a unique cubic mapping $C : X \to Y$ such that, for all $x \in X$*

$$f(x) - C(x) \leq \begin{cases} \frac{\epsilon}{4|n^6-1|} \\ \frac{\epsilon \|x\|^s}{4|n^6-n^{2s}|} \\ \frac{\epsilon \|x\|^{ns}}{4|n^6-n^{6s}|}. \end{cases} \tag{10.75}$$

Corollary 10.28 *Let ϵ and s be non-negative real numbers. If a mapping $f : X \to Y$ satisfies the inequality (10.73) for all $x_1, x_2, x_3 \in X$, then there exists a unique cubic mapping $C : X \to Y$ such that, for all $x \in X$*

$$f(x) - C(x) \leq \begin{cases} \frac{\epsilon}{4|n^9-1|} \\ \frac{\epsilon \|x\|^s}{4|n^9-n^{3s}|} \\ \frac{\epsilon \|x\|^{ns}}{4|n^9-n^{9s}|}. \end{cases} \tag{10.76}$$

10.6 Stability of 3-Dimensional Cubic Functional Equation—Fixed Point Method

Proposition 10.29 *Let $f : X \to Y$ be a mapping for which there exists a function $\chi : X^3 \to [0, \infty)$ with condition*

$$\lim_{k\to\infty} \frac{\chi(\eta_i^k x_1, \eta_i^k x_2, \eta_i^k x_3)}{\eta_i^{3k}} = 0, \tag{10.77}$$

where $\eta_i = \begin{cases} n & if \;\; i = 0 \\ \frac{1}{n} & if \; \imath = 1 \end{cases}$ *and satisfying the inequality*

$$\| Df_c(x_1, x_2, x_3) \| \leq \chi(x_1, x_2, x_3), \tag{10.78}$$

for all $x_1, x_2, x_3 \in X$ *and* $n \geq 4$. *If there exists* $L = L(i)$ *such that the function* $x \to \rho(x) = \frac{1}{4}\chi\left(\frac{x}{n}, 0, 0\right)$ *has the property*

$$\frac{\rho(n^i x)}{\eta_i^3} = L\rho(x), \tag{10.79}$$

for all $x \in X$, *then there exists a unique cubic mapping* $C : X \to Y$ *satisfying the functional equation (10.8) and*

$$\| f(x) - C(x) \| \leq \frac{L^{1-i}}{1-L}\rho(x), \tag{10.80}$$

for all $x \in X$.

Example 10.30 Let $f : X \to Y$ be a mapping for which there exists a function $\chi : X^3 \to [0, \infty)$ with condition (10.77) where $\eta_i = \begin{cases} 2 & if \;\; i = 0; \\ \frac{1}{2} & if \; \imath = 1; \end{cases}$ and satisfying the functional inequality (10.78) for all $x_1, x_2, x_3 \in X$ and $n \geq 4$. Then there exists $L = L(i)$ such that the function $x \to \rho(x) = \frac{1}{4}\chi\left(\frac{x}{2}, 0, 0\right)$ has the property (10.79), for all $x \in X$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equations (10.8) and (10.80), for all $x \in X$.

Proposition 10.31 *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\chi : X^3 \to [0, \infty)$ *with condition (10.77) where* $\eta_i = \begin{cases} n^2 & if \;\; i = 0 \\ \frac{1}{n^2} & if \; \imath = 1 \end{cases}$ *and satisfying the inequality (10.78) for all* $x_1, x_2, x_3 \in X$ *and* $n \geq 4$. *If there exists* $L = L(i)$ *such that the function* $x \to \rho(x) = \frac{1}{4}\chi\left(0, \frac{x}{n^2}, 0\right)$ *has the property (10.79), for all* $x \in X$, *then there exists a unique cubic mapping* $C : X \to Y$ *satisfying the functional equations (10.8) and (10.80), for all* $x \in X$.

Proposition 10.32 *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\chi : X^3 \to [0, \infty)$ *with condition (10.77) where* $\eta_i = \begin{cases} n^3 & if \;\; i = 0 \\ \frac{1}{n^3} & if \; \imath = 1 \end{cases}$ *and satisfying the inequality (10.78), for all* $x_1, x_2, x_3 \in X$ *and* $n \geq 4$. *Assume that there exists* $L = L(i)$ *such that the function* $x \to \rho(x) = \frac{1}{4}\chi\left(0, 0, \frac{x}{n^3}\right)$ *has the property (10.79), for all* $x \in X$. *Then there exists a unique cubic mapping* $C : X \to Y$ *satisfying the functional equations (10.8) and (10.80), for all* $x \in X$.

The following corollaries are the immediate consequence of Propositions 10.29–10.32 respectively, concerning the stability of (10.8).

Corollary 10.33 *Let ϵ and s be non-negative real numbers. If a mapping $f : X \to Y$ satisfies the inequality (10.73) for all $x_1, x_2, x_3 \in X$, then there exists a unique cubic mapping $C : X \to Y$ such that (10.74), for all $x \in X$.*

Remark: Let ϵ and s be non-negative real numbers. If a mapping $f : X \to Y$ satisfies the inequality (10.73) for all $x_1, x_2, x_3 \in X$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\| f(x) - C(x) \| \leq \begin{cases} \frac{\epsilon}{4|7|} \\ \frac{\epsilon \|x\|^s}{4|2^3 - 2^s|} \\ \frac{\epsilon \|x\|^{ns}}{4|2^3 - 2^{3s}|}, \end{cases}$$

for all $x \in X$.

Corollary 10.34 *Let ϵ and s be non-negative real numbers. If a mapping $f : X \to Y$ satisfies the inequality (10.73) for all $x_1, x_2, x_3 \in X$, then there exists a unique cubic mapping $C : X \to Y$ such that (10.75), for all $x \in X$.*

Corollary 10.35 *Let ϵ and s be non-negative real numbers. If a mapping $f : X \to Y$ satisfies the inequality (10.73) for all $x_1, x_2, x_3 \in X$, then there exists a unique cubic mapping $C : X \to Y$ such that (10.76), for all $x \in X$.*

10.7 Examples

Now we will provide an example to illustrate that the functional equation is not stable for $x = 3$, $x = \frac{n}{3}$.

Example 10.36 Let $\upsilon : R \to R$ be a function defined by

$$\Upsilon(x)^+_\beta = \begin{cases} \mu x^3, & \text{if } \|x\|^+_\beta < 1 \\ \mu, & \text{otherwise,} \end{cases}$$

where $\mu > 0$ is a constant and define a function $f : R \to R$ by $f(x) = \sum_{k=0}^{\infty} \frac{\Upsilon(2^k x)^+_\beta}{8^k}$, for all $x \in R$. Then f satisfies the functional inequality

$$\|\aleph f(x)\|^+_\beta \leq \sum_{i=1}^{n} \left(1 + \frac{(i-2)(i-3)(23i-20)}{48}\right) \mu e \left(\sum_{i=1}^{n} (\|x_i\|^+_\beta)^3\right), \quad (10.81)$$

for all $x_1, x_2, x_3, ..., x_n \in R$. Then there exists a cubic function $C : R \to R$ and a constant $\delta > 0$ such that

$$\| f(x) - C(x)\|_\beta^+ \leq \delta(\|x_i\|_\beta^+)^3, \quad \forall\, x \in R. \tag{10.82}$$

Proof Now,

$$\begin{aligned}\| f(x)\|_\beta^+ &\leq \sum_{l=0}^{\infty} \frac{|\Upsilon(2^l x)|}{|8^l|}\\ &= \sum_{l=0}^{\infty} \frac{\mu}{8^l}\\ &= \frac{8}{7}\mu.\end{aligned}$$

Therefore we see that f is bounded. We are going to prove that f satisfies (10.81).

If $x_1 = x_2 = x_3 = \cdots = x_n = 0$, then (10.81) is trivial.

If $\sum_{i=1}^{n}(\|x_i\|_\beta^+)^3 \geq 1$, then the left hand side of (10.81) is less than

$$\sum_{i=1}^{n}\left(1 + \frac{(i-2)(i-3)(23i-20)}{48}\right)\mu.$$

Now suppose that $0 < \sum_{i=1}^{n}(\|x_i\|_\beta^+)^3 < 1$. Then there exists a positive integer p such that

$$\frac{1}{8^{p-1}} \leq \sum_{i=1}^{n}(\|x_i\|_\beta^+)^3 < \frac{1}{8^p}. \tag{10.83}$$

So that $8^{p-1}x_1 < 1, 8^{p-1}x_2 < 1, 8^{p-1}x_3 < 1, ..., 8^{p-1}x_n < 1$ and consequently

$$8^{p-1}\sum_{j=1}^{i}(x_j), 8^{p-1}(2x_j), 8^{p-1}(x_j + x_k + x_l), 8^{p-1}(x_j + x_k) \in (-1, 1),$$

for all $1 \leq j < k < l \leq i,\ i = 1, 2, 3, ..., n$ and

$$\sum_{i=1}^{n}\Upsilon\left(8^q\sum_{j=1}^{i}(x_j)\right) - \sum_{i=1}^{n}\left(\frac{i^2-5i+6}{16}\right)\sum_{j=1}^{i}\Upsilon(8^q 2x_j)$$

$$-\sum_{i=1}^{n}\sum_{1\le j<k<l\le i}\Upsilon(8^q(x_j+x_k+x_l))+\sum_{i=1}^{n}(i-3)\sum_{1\le j<k\le i}\Upsilon(8^q(x_j+x_k))=0 \tag{10.84}$$

for $q=0,1,2,3,...,(p-1)$. From the definition of f and (10.83), we obtain that

$$\begin{aligned}
\|\aleph f(x)\|_\beta^+ &\le \sum_{q=0}^{\infty}\frac{1}{8^q}\Bigg\|\sum_{i=1}^{n}\Upsilon\left(8^q\sum_{j=1}^{i}(x_j)\right)-\sum_{i=1}^{n}\left(\frac{i^2-5i+6}{16}\right)\sum_{j=1}^{i}\Upsilon(8^q 2x_j)\\
&\quad -\sum_{i=1}^{n}\sum_{1\le j<k<l\le i}\Upsilon(8^q(x_j+x_k+x_l))+\sum_{i=1}^{n}(i-3)\sum_{1\le j<k\le i}\Upsilon(8^q(x_j+x_k))\Bigg\|_\beta^+\\
&\le \sum_{q=p}^{\infty}\frac{1}{8^q}\Bigg\|\sum_{i=1}^{n}\Upsilon\left(8^q\sum_{j=1}^{i}(x_j)\right)-\sum_{i=1}^{n}\left(\frac{i^2-5i+6}{16}\right)\sum_{j=1}^{i}\Upsilon(8^q 2x_j)\\
&\quad -\sum_{i=1}^{n}\sum_{1\le j<k<l\le i}\Upsilon(8^q(x_j+x_k+x_l))+\sum_{i=1}^{n}(i-3)\sum_{1\le j<k\le i}\Upsilon(8^q(x_j+x_k))\Bigg\|_\beta^+\\
&\le \mu\sum_{i=1}^{n}\left(1+\frac{(i-2)(i-3)(23i-20)}{48}\right)\sum_{q=p}^{\infty}\frac{1}{8^q}\\
&= \frac{8}{7}\mu\sum_{i=1}^{n}\left(1+\frac{(i-2)(i-3)(23i-20)}{48}\right)\frac{1}{8^q}\\
&\le \frac{8}{7}\mu\sum_{i=1}^{n}\left(1+\frac{(i-2)(i-3)(23i-20)}{48}\right)e\left(\sum_{i=1}^{n}(\|x_i\|_\beta^+)^3\right).
\end{aligned}$$

Thus f satisfies (10.81), for all $x_1, x_2, x_3, ..., x_n \in R$ with $0<\sum_{i=1}^{n}|x_i|^3<\frac{1}{8}$.

We claim that the cubic functional equation (2.6) is not stable for $s=3$.

Suppose on the contrary that there exist a cubic mapping $C: R\to R$ and a constant $\delta>0$ satisfying (10.82). Since f is bounded and continuous for all $x\in R$, C is bounded on any open interval containing the origin and continuous at the origin. Then C must have the form $C(x)=cx^3$ for any x in R. Thus we obtain that

$$\|f(x)\|_\beta^+\le(\delta+|c|)(\|x_i\|_\beta^+)^3. \tag{10.85}$$

But we can choose a positive integer m with $m\mu>\delta+|c|$. If $x\in\left(0,\frac{1}{2^{m+1}}\right)$, the $2^n x\in(0,1)$, for all $n=0,1,2,...,m-1$.

For this x, we get

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} \frac{\Upsilon(2^i x)}{|8^i|} \\ &\geq \sum_{i=0}^{m-1} \frac{\mu(2^i x)^3}{8^i} \\ &= m\mu x^3 \\ &> (\delta + |c|)x^3 \end{aligned}$$

which contradicts (10.85). Therefore the cubic functional equation (2.6) is not stable in sense of Ulam-Hyers if s = 3. □

Example 10.37 Let $\Upsilon : R \to R$ be a function defined by

$$\Upsilon(x)_\beta^+ = \begin{cases} \mu x^3, & \text{if } \|x\|_\beta^+ < 1 \\ \mu, & \text{otherwise,} \end{cases}$$

where $\mu > 0$ is a constant and define a function $f : R \to R$ by $f(x) = \sum_{k=0}^{\infty} \frac{\Upsilon(2^k x)_\beta^+}{8^k}$, for all $x \in R$. Then f satisfies the functional inequality

$$\|\aleph f(x)\|_\beta^+ \leq \sum_{i=1}^{n} \left(1 + \frac{(i-2)(i-3)(23i-20)}{48}\right) \frac{24\mu}{7n} e \left(\sum_{i=1}^{n} (\|x_i\|_\beta^+)^{3n} \oplus \prod_{i=1}^{n} (\|x_i\|_\beta^+)^3\right), \tag{10.86}$$

for all $x_1, x_2, x_3, ..., x_n \in R$. Then there do not exist a cubic function $C : R \to R$ and a constant $\delta > 0$ such that

$$\|f(x) - C(x)\|_\beta^+ \leq \delta(\|x_i\|_\beta^+)^3, \quad \forall\, x \in R. \tag{10.87}$$

Chapter 11
4-Dimensional Cubic Functional Equations

11.1 4-Dimensional Cubic Functional Equation

In the last few decades, the stability problems of several cubic functional equations in various spaces such as menger probabilistic normed spaces, random normed spaces, non-Archimedean fuzzy normed spaces, Banach spaces, and orthogonal spaces have been extensively investigated by a number of mathematicians (see [25, 28, 31, 39, 44, 61, 66, 73]).

Katrasas [53] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a linear space from various points of view [26, 94]. In particular, Cheng and Mordeson [16] gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces. Recently, [65] given to the problem of the fuzzy stability of functional equations. Several various fuzzy stability results concerning Cauchy, Jensen, simple quadratic, and cubic functional equations have been investigated.

The study of approximate additive mappings and approximate linear mapping allowing the Cauchy difference operator

$$CDf(x, y) = f(x + y) + [f(x) + f(y)]$$

to be controlled $\epsilon\big(\parallel x \parallel^p + \parallel y \parallel^p \big)$.

$$\begin{aligned} f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4) &= f(nx_1 + n^2x_2 + n^3x_3) + f(nx_1 + n^2x_2 \\ &+n^4x_4) + f(nx_1 + n^3x_3 + n^4x_4) + f(n^2x_2 + n^3x_+n^4x_4) - f(nx_1 + n^2x_2) \\ &-f(nx_1 + n^3x_3) - f(nx_1 + n^4x_4) - f(n^2x_2 + n^3x_3) - f(n^2x_2 + n^4x_4) \\ &-f(n^3x_3 + n^4x_4) + n^3f(x_1) + n^6f(x_2) + n^9f(x_3) + n^{12}f(x_4). \end{aligned} \tag{11.1}$$

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In this chapter, we inspect the general solution and stability of the 4-dimensional cubic functional equation (11.1) in intuitionistic fuzzy normed space and fuzzy normed space with the help of direct and fixed point methods.

11.2 Solution of 4-Dimensional Cubic Functional Equation

Theorem 11.1 *An odd mapping $f : X \to Y$ satisfies the Eq. (10.3), for all $x, y \in X$ if and only if $f : X \to Y$ satisfies the functional equation (11.1), for all $x_1, x_2, x_3, x_4 \in X$.*

Proof Let $f : X \to Y$ satisfies the functional equation (10.3). Replacing (x, y) by $(0, 0)$ in (10.3), we get $f(0) = 0$. Replacing (x, y) by $(x, 0)$, (x, x) and $(x, 2x)$ respectively in (10.3), we obtain

$$f(2x) = 2^3 f(x) \ \ and \ \ f(3x) = 3^3 f(x), \tag{11.2}$$

for all $x \in X$. In general for any positive integer a, we have

$$f(ax) = a^3 f(x), \tag{11.3}$$

for all $x \in X$. It is easy to verify from (11.3) that,

$$f(a^2x) = a^6 f(x) \ \ and \ \ f(a^3x) = a^9 f(x), \tag{11.4}$$

for all $x \in X$. Replacing (x, y) by $(nx_1 + n^2x_2, n^3x_3 + n^4x_4)$ in (10.3), we get

$$\begin{aligned} &f(2nx_1 + 2n^2x_2 + n^3x_3 + n^4x_4) + f(2nx_1 + 2n^2x_2 - n^3x_3 - n^4x_4) \\ &+2f(-nx_1 - n^2x_2 - n^3x_3 - n^4x_4) + 2f(-nx_1 - n^2x_2 + n^3x_3 + n^4x_4) \\ &= 12f(nx_1 + n^2x_2), \end{aligned} \tag{11.5}$$

for all $x_1, x_2, x_3, x_4 \in X$. Again replacing (x, y) by $(nx_1 + n^3x_3, n^2x_2 + n^4x_4)$ in (10.3), we have

$$\begin{aligned} &f(2nx_1 + n^2x_2 + 2n^3x_3 + n^4x_4) + f(2nx_1 - n^2x_2 + 2n^3x_3 - n^4x_4) \\ &+2f(-nx_1 - n^2x_2 - n^3x_3 - n^4x_4) + 2f(-nx_1 + n^2x_2 - n^3x_3 + n^4x_4) \\ &= 12f(nx_1 + n^3x_3), \end{aligned} \tag{11.6}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(nx_1 + n^4x_4, n^2x_2 + n^3x_3)$ in (10.3), we obtain

$$\begin{aligned}
&f(2nx_1 + n^2x_2 + n^3x_3 + 2n^4x_4) + f(2nx_1 - n^2x_2 - n^3x_3 + 2n^4x_4)\\
&+2f(-nx_1 - n^2x_2 - n^3x_3 - n^4x_4) + 2f(-nx_1 + n^2x_2 + n^3x_3 - n^4x_4)\\
&= 12f(nx_1 + n^4x_4), \qquad (11.7)
\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(n^2x_2 + n^3x_3, nx_1 + n^4x_4)$ in (10.3), we attain

$$\begin{aligned}
&f(nx_1 + 2n^2x_2 + 2n^3x_3 + n^4x_4) + f(-nx_1 + 2n^2x_2 + 2n^3x_3 - n^4x_4)\\
&+2f(-nx_1 - n^2x_2 - n^3x_3 - n^4x_4) + 2f(nx_1 - n^2x_2 - n^3x_3 + n^4x_4)\\
&= 12f(n^2x_2 + n^3x_3), \qquad (11.8)
\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$. Letting (x, y) by $(n^2x_2 + n^4x_4, nx_1 + n^3x_3)$ in (10.3), we obtain

$$\begin{aligned}
&f(nx_1 + 2n^2x_2 + n^3x_3 + 2n^4x_4) + f(-nx_1 + 2n^2x_2 - n^3x_3 + 2n^4x_4)\\
&+2f(-nx_1 - n^2x_2 - n^3x_3 - n^4x_4) + 2f(nx_1 - n^2x_2 + n^3x_3 - n^4x_4)\\
&= 12f(n^2x_2 + n^4x_4), \qquad (11.9)
\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(n^3x_3 + n^4x_4, nx_1 + n^2x_2)$ in (10.3), we get

$$\begin{aligned}
&f(nx_1 + n^2x_2 + 2n^3x_3 + 2n^4x_4) + f(-nx_1 - n^2x_2 + 2n^3x_3 + 2n^4x_4)\\
&+2f(-nx_1 - n^2x_2 - n^3x_3 - n^4x_4) + 2f(nx_1 + n^2x_2 - n^3x_3 - n^4x_4)\\
&= 12f(n^3x_3 + n^4x_4), \qquad (11.10)
\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding (11.5), (11.6), (11.7), (11.8), (11.9) and (11.10), we get

$$\begin{aligned}
&12f(nx_1 + n^2x_2) + 12f(nx_1 + n^3x_3) + 12f(nx_1 + n^4x_4)\\
&+12f(n^2x_2 + n^3x_3) + 12f(n^2x_2 + n^4x_4) + 12f(n^3x_3 + n^4x_4)\\
&= f(2nx_1 + 2n^2x_2 + n^3x_3 + n^4x_4) + f(2nx_1 + 2n^2x_2 - n^3x_3 - n^4x_4)\\
&+2f(-nx_1 - n^2x_2 - n^3x_3 - n^4x_4) + 2f(-nx_1 - n^2x_2 + n^3x_3 + n^4x_4)\\
&+f(2nx_1 + n^2x_2 + 2n^3x_3 + n^4x_4) + f(2nx_1 - n^2x_2 + 2n^3x_3 - n^4x_4)\\
&+2f(-nx_1 - n^2x_2 - n^3x_3 - n^4x_4) + 2f(-nx_1 + n^2x_2 - n^3x_3 + n^4x_4)\\
&+f(2nx_1 + n^2x_2 + n^3x_3 + 2n^4x_4) + f(2nx_1 - n^2x_2 - n^3x_3 + 2n^4x_4)\\
&+2f(-nx_1 - n^2x_2 - n^3x_3 - n^4x_4) + 2f(-nx_1 + n^2x_2 + n^3x_3 - n^4x_4)\\
&+f(nx_1 + 2n^2x_2 + 2n^3x_3 + n^4x_4) + f(-nx_1 + 2n^2x_2 + 2n^3x_3 - n^4x_4)\\
&+2f(-nx_1 - n^2x_2 - n^3x_3 - n^4x_4) + 2f(nx_1 - n^2x_2 - n^3x_3 + n^4x_4)\\
&+f(nx_1 + 2n^2x_2 + n^3x_3 + 2n^4x_4) + f(-nx_1 + 2n^2x_2 - n^3x_3 + 2n^4x_4)\\
&+2f(-nx_1 - n^2x_2 - n^3x_3 - n^4x_4) + 2f(nx_1 - n^2x_2 + n^3x_3 - n^4x_4)
\end{aligned}$$

$$+f(nx_1+n^2x_2+2n^3x_3+2n^4x_4)+f(-nx_1-n^2x_2+2n^3x_3+2n^4x_4)$$
$$+2f(-nx_1-n^2x_2-n^3x_3-n^4x_4)$$
$$+2f(nx_1+n^2x_2-n^3x_3-n^4x_4), \tag{11.11}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(nx_1, 2n^2x_2+n^3x_3+n^4x_4)$ in (10.3), we have

$$f(2nx_1+2n^2x_2+n^3x_3+n^4x_4)=2f(nx_1+2n^2x_2+n^3x_3+n^4x_4)$$
$$+2f(nx_1-2n^2x_2-n^3x_3-n^4x_4)+f(-2nx_1+2n^2x_2+n^3x_3+n^4x_4)$$
$$+12f(nx_1), \tag{11.12}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(2nx_1+2n^2x_2-n^3x_3-n^4x_4)$ on both sides of (11.12), we have

$$f(2nx_1+2n^2x_2+n^3x_3+n^4x_4)+f(2nx_1+2n^2x_2-n^3x_3-n^4x_4)$$
$$=2f(nx_1+2n^2x_2+n^3x_3+n^4x_4)+2f(nx_1-2n^2x_2-n^3x_3-n^4x_4)$$
$$+12f(nx_1)+2f(-2nx_1+n^2x_2+n^3x_3+n^4x_4)$$
$$+2f(2nx_1+n^2x_2-n^3x_3-n^4x_4)+12f(nx_2), \tag{11.13}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(n^3x_3, 2nx_1+n^2x_2+n^4x_4)$ in (10.3), we obtain

$$f(2nx_1+n^2x_2+2n^3x_3+n^4x_4)=2f(2nx_1+n^2x_2+n^3x_3+n^4x_4)$$
$$+2f(-2nx_1-n^2x_2+n^3x_3-n^4x_4)+f(2nx_1+n^2x_2-2n^3x_3+n^4x_4)$$
$$+12f(n^3x_3), \tag{11.14}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(2nx_1-n^2x_2+2n^3x_3+n^4x_4)$ on both sides of (11.14), we attain

$$f(2nx_1+n^2x_2-2n^3x_3+n^4x_4)+f(2nx_1-n^2x_2+2n^3x_3-n^4x_4)$$
$$=2f(2nx_1+n^2x_2+n^3x_3+n^4x_4)+2f(-2nx_1-n^2x_2+n^3x_3-n^4x_4)$$
$$+12f(n^3x_3)+2f(nx_1+n^2x_2-2n^3x_3+n^4x_4)$$
$$+2f(nx_1-n^2x_2+2n^3x_3-n^4x_4)+12f(nx_1), \tag{11.15}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(n^4x_4, 2nx_1+n^2x_2+n^3x_3)$ in (10.3), we get

$$f(2nx_1+n^2x_2+n^3x_3+2n^4x_4)=2f(2nx_1+n^2x_2+n^3x_3+n^4x_4)$$
$$+2f(-2nx_1-n^2x_2-n^3x_3+n^4x_4)+f(2nx_1+n^2x_2+n^3x_3-2n^4x_4)$$
$$+12f(n^4x_4), \tag{11.16}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(2nx_1 - n^2x_2 - n^3x_3 + 2n^4x_4)$ on both sides of (11.16), we have

$$\begin{aligned}
&f(2nx_1 + n^2x_2 + n^3x_3 + 2n^4x_4) + f(2nx_1 - n^2x_2 - n^3x_3 + 2n^4x_4)\\
&= 2f(2nx_1 + n^2x_2 + n^3x_3 + n^4x_4) + 2f(-2nx_1 - n^2x_2 - n^3x_3 + n^4x_4)\\
&+12f(n^4x_4) + 2f(nx_1 + n^2x_2 + n^3x_3 - 2n^4x_4)\\
&+2f(nx_1 - n^2x_2 - n^3x_3 + 2n^4x_4) + 12f(nx_1),
\end{aligned} \tag{11.17}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(n^2x_2, nx_1 + 2n^3x_3 + n^4x_4)$ in (10.3), we obtain

$$\begin{aligned}
&f(nx_1 + 2n^2x_2 + 2n^3x_3 + n^4x_4) = 2f(nx_1 + n^2x_2 + 2n^3x_3 + n^4x_4)\\
&+2f(-nx_1 + n^2x_2 - 2n^3x_3 - n^4x_4) + f(nx_1 - 2n^2x_2 + 2n^3x_3 + n^4x_4)\\
&+12f(n^2x_2),
\end{aligned} \tag{11.18}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(-nx_1 + 2n^2x_2 + 2n^3x_3 - n^4x_4)$ on both sides of (11.18), we get

$$\begin{aligned}
&f(nx_1 + 2n^2x_2 + 2n^3x_3 + n^4x_4) + f(-nx_1 + 2n^2x_2 + 2n^3x_3 - n^4x_4)\\
&= 2f(nx_1 + n^2x_2 + 2n^3x_3 + n^4x_4) + 2f(-nx_1 + n^2x_2 - 2n^3x_3 - n^4x_4)\\
&+12f(n^2x_2) + 2f(nx_1 - 2n^2x_2 + n^3x_3 + n^4x_4)\\
&+2f(-nx_1 + 2n^2x_2 + n^3x_3 - n^4x_4) + 12f(n^3x_3),
\end{aligned} \tag{11.19}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(n^2x_2, nx_1 + n^3x_3 + 2n^4x_4)$ in (10.3), we get

$$\begin{aligned}
&f(nx_1 + 2n^2x_2 + n^3x_3 + 2n^4x_4) = 2f(nx_1 + n^2x_2 + n^3x_3 + 2n^4x_4)\\
&+2f(-nx_1 + n^2x_2 - n^3x_3 - 2n^4x_4) + f(nx_1 - 2n^2x_2 + n^3x_3 + 2n^4x_4)\\
&+12f(n^2x_2),
\end{aligned} \tag{11.20}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(-nx_1 + 2n^2x_2 - n^3x_3 + 2n^4x_4)$ on both sides of (11.20), we obtain

$$\begin{aligned}
&f(nx_1 + 2n^2x_2 + n^3x_3 + 2n^4x_4) + f(-nx_1 + 2n^2x_2 - n^3x_3 + 2n^4x_4)\\
&= 2f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4) + 2f(-nx_1 + n^2x_2 - n^3x_3 - 2n^4x_4)\\
&+12f(n^2x_2) + 2f(nx_1 - 2n^2x_2 + n^3x_3 + n^4x_4)\\
&+2f(-nx_1 + 2n^2x_2 - n^3x_3 + n^4x_4) + 12f(n^4x_4),
\end{aligned} \tag{11.21}$$

for all $x_1, x_2, x_3, x_4 \in X$. Substituting (x, y) by $(n^3x_3, nx_1 + n^2x_2 + 2n^4x_4)$ in (10.3), we have

$$
\begin{aligned}
&f(nx_1+n^2x_2+2n^3x_3+2n^4x_4)=2f(nx_1+n^2x_2+n^3x_3+2n^4x_4)\\
&+2f(-nx_1-n^2x_2+n^3x_3-2n^4x_4)+f(nx_1+n^2x_2-2n^3x_3+2n^4x_4)\\
&+12f(n^3x_3),
\end{aligned} \tag{11.22}
$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(-nx_1-n^2x_2+2n^3x_3+2n^4x_4)$ on both sides of (11.22), we obtain

$$
\begin{aligned}
&f(nx_1+n^2x_2+2n^3x_3+2n^4x_4)+f(-nx_1-n^2x_2+2n^3x_3+2n^4x_4)\\
&=2f(nx_1+n^2x_2+n^3x_3+2n^4x_4)+2f(-nx_1-n^2x_2+n^3x_3-2n^4x_4)\\
&+12f(n^3x_3)+2f(nx_1+n^2x_2-2n^3x_3+n^4x_4)\\
&+2f(-nx_1-n^2x_2+2n^3x_3+n^4x_4)+12f(n^4x_4),
\end{aligned} \tag{11.23}
$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (11.13), (11.15), (11.17), (11.19), (11.21) and (11.23) in (11.11), we get

$$
\begin{aligned}
&12f(nx_1+n^2x_2)+12f(nx_1+n^3x_3)+12f(nx_1+n^4x_4)\\
&+12f(n^2x_2+n^3x_3)+12f(n^2x_2+n^4x_4)+12f(n^3x_3+n^4x_4)\\
&=2f(nx_1+2n^2x_2+n^3x_3+n^4x_4)+2f(nx_1-2n^2x_2-n^3x_3-n^4x_4)\\
&+12f(nx_1)+2f(-2nx_1+n^2x_2+n^3x_3+n^4x_4)\\
&+f(2nx_1+n^2x_2-n^3x_3-n^4x_4)+12f(n^2x_2)\\
&+2f(-nx_1-n^2x_2-n^3x_3-n^4x_4)+2f(-nx_1-n^2x_2+n^3x_3+n^4x_4)\\
&2f(2nx_1+n^2x_2+n^3x_3+n^4x_4)+2f(-2nx_1-n^2x_2+n^3x_3-n^4x_4)\\
&+12f(n^3x_3)+2f(nx_1+n^2x_2-2n^3x_3+n^4x_4)\\
&+2f(nx_1-n^2x_2+2n^3x_3-n^4x_4)+12f(nx_1)\\
&+2f(-nx_1-n^2x_2-n^3x_3-n^4x_4)+2f(-nx_1+n^2x_2-n^3x_3+n^4x_4)\\
&+2f(2nx_1+n^2x_2+n^3x_3+n^4x_4)+2f(-2nx_1-n^2x_2-n^3x_3+n^4x_4)\\
&+12f(n^4x_4)+2f(nx_1+n^2x_2+n^3x_3-2n^4x_4)\\
&+2f(nx_1-n^2x_2-n^3x_3+2n^4x_4)+12f(nx_1)\\
&+2f(-nx_1-n^2x_2-n^3x_3-n^4x_4)+2f(-nx_1+n^2x_2+n^3x_3-n^4x_4)\\
&+2f(nx_1+n^2x_2+2n^3x_3+n^4x_4)+2f(-nx_1+n^2x_2-2n^3x_3-n^4x_4)\\
&+12f(n^2x_2)+2f(nx_1-2n^2x_2+n^3x_3+n^4x_4)\\
&+2f(-nx_1+2n^2x_2+n^3x_3-n^4x_4)+12f(n^3x_3)\\
&+2f(-nx_1-n^2x_2-n^3x_3-n^4x_4)+2f(nx_1-n^2x_2-n^3x_3+n^4x_4)\\
&+2f(nx_1+n^2x_2+n^3x_3+2n^4x_4)+2f(-nx_1+n^2x_2-n^3x_3-2n^4x_4)\\
&+12f(n^2x_2)+2f(nx_1-2n^2x_2+n^3x_3+n^4x_4)\\
&+2f(-nx_1+2n^2x_2-n^3x_3+n^4x_4)+12f(n^4x_4)\\
&+2f(-nx_1-n^2x_2-n^3x_3-n^4x_4)+2f(nx_1-n^2x_2+n^3x_3-n^4x_4)\\
&+2f(nx_1+n^2x_2+n^3x_3+2n^4x_4)+2f(-nx_1-n^2x_2+n^3x_3-2n^4x_4),
\end{aligned}
$$

$$+12f(n^3x_3) + 2f(nx_1 + n^2x_2 - 2n^3x_3 + n^4x_4)$$
$$+2f(-nx_1 - n^2x_2 + 2n^3x_3 + n^4x_4) + 12f(n^4x_4) + 2f(-nx_1 - n^2x_2$$
$$-n^3x_3 - n^4x_4) + 2f(nx_1 + n^2x_2 - n^3x_3 - n^4x_4), \tag{11.24}$$

for all $x_1, x_2, x_3, x_4 \in X$. Interchanging nx_1 by n^2x_2 in (11.13), we have

$$f(2nx_1 + 2n^2x_2 + n^3x_3 + n^4x_4) + f(2nx_1 + 2n^2x_2 - n^3x_3 - n^4x_4)$$
$$= 2f(2nx_1 + n^2x_2 + n^3x_3 + n^4x_4) + 2f(-2nx_1 + n^2x_2 - n^3x_3 - n^4x_4)$$
$$+12f(n^2x_2) + 2f(nx_1 - 2n^2x_2 + n^3x_3 + n^4x_4)$$
$$+2f(nx_1 + 2n^2x_2 - n^3x_3 - n^4x_4) + 12f(nx_1), \tag{11.25}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing nx_1 by n^3x_3 in (11.15), we get

$$f(2nx_1 + n^2x_2 + 2n^3x_3 + n^4x_4) + f(2nx_1 - n^2x_2 + 2n^3x_3 - n^4x_4)$$
$$= 2f(nx_1 + n^2x_2 + 2n^3x_3 + n^4x_4) + 2f(nx_1 - n^2x_2 - 2n^3x_3 - n^4x_4)$$
$$+12f(nx_1) + 2f(-2nx_1 + n^2x_2 + n^3x_3 + n^4x_4)$$
$$+2f(2nx_1 - n^2x_2 + n^3x_3 - n^4x_4) + 12f(n^3x_3), \tag{11.26}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing nx_1 by n^4x_4 in (11.17), we have

$$f(2nx_1 + n^2x_2 + n^3x_3 + 2n^4x_4) + f(2nx_1 - n^2x_2 - n^3x_3 + 2n^4x_4)$$
$$= 2f(nx_1 + n^2x_2 + n^3x_3 + 2n^4x_4) + 2f(nx_1 - n^2x_2 - n^3x_3 - 2n^4x_4)$$
$$+12f(nx_1) + 2f(-2nx_1 + n^2x_2 + n^3x_3 + n^4x_4)$$
$$+2f(nx_1 - n^2x_2 - n^3x_3 + n^4x_4) + 12f(n^4x_4), \tag{11.27}$$

for all $x_1, x_2, x_3, x_4 \in X$. Letting n^2x_2 by n^3x_3 in (11.19), we obtain

$$f(nx_1 + 2n^2x_2 + 2n^3x_3 + n^4x_4) + f(-nx_1 + 2n^2x_2 + 2n^3x_3 - n^4x_4)$$
$$= 2f(nx_1 + 2n^2x_2 + n^3x_3 + n^4x_4) + 2f(-nx_1 - 2n^2x_2 + n^3x_3 - n^4x_4)$$
$$+12f(n^3x_3) + 2f(nx_1 + n^2x_2 - 2n^3x_3 + n^4x_4)$$
$$+2f(-nx_1 + n^2x_2 + 2n^3x_3 - n^4x_4) + 12f(n^2x_2), \tag{11.28}$$

for all $x_1, x_2, x_3, x_4 \in X$. Interchanging n^2x_2 by n^4x_4 in(11.21), we get

$$f(nx_1 + 2n^2x_2 + n^3x_3 + 2n^4x_4) + f(-nx_1 + 2n^2x_2 - n^3x_3 + 2n^4x_4)$$
$$= 2f(nx_1 + 2n^2x_2 + n^3x_3 + n^4x_4) + 2f(-nx_1 - 2n^2x_2 - n^3x_3 + n^4x_4)$$
$$+12f(n^4x_4) + 2f(nx_1 + n^2x_2 + n^3x_3 - 2n^4x_4)$$
$$+2f(-nx_1 + n^2x_2 - n^3x_3 + 2n^4x_4) + 12f(n^2x_2), \tag{11.29}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing n^3x_3 by n^4x_4 in (11.21), we have

$$
\begin{aligned}
&f(nx_1+n^2x_2+2n^3x_3+2n^4x_4)+f(-nx_1-n^2x_2+2n^3x_3+2n^4x_4)\\
&=2f(nx_1+n^2x_2+2n^3x_3+n^4x_4)+2f(-nx_1-n^2x_2-2n^3x_3+n^4x_4)\\
&+12f(n^4x_4)+2f(nx_1+n^2x_2+n^3x_3-2n^4x_4)\\
&+2f(-nx_1-n^2x_2+n^3x_3+2n^4x_4)+12f(n^3x_3),
\end{aligned}
\tag{11.30}
$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (11.25), (11.26), (11.27), (11.28), (11.29) and (11.30) in (11.11), we have

$$
\begin{aligned}
&12f(nx_1+n^2x_2)+12f(nx_1+n^3x_3)+12f(nx_1+n^4x_4)\\
&+12f(n^2x_2+n^3x_3)+12f(n^2x_2+n^4x_4)+12f(n^3x_3+n^4x_4)\\
&=2f(2nx_1+n^2x_2+n^3x_3+n^4x_4)+2f(-2nx_1+n^2x_2-n^3x_3-n^4x_4)\\
&+12f(n^2x_2)+2f(nx_1-2n^2x_2+n^3x_3+n^4x_4)\\
&+2f(nx_1+2n^2x_2-n^3x_3-n^4x_4)+12f(nx_1)\\
&+2f(-nx_1-n^2x_2-n^3x_3-n^4x_4)+2f(-nx_1-n^2x_2+n^3x_3+n^4x_4)\\
&+2f(nx_1+n^2x_2+2n^3x_3+n^4x_4)+2f(nx_1-n^2x_2-2n^3x_3-n^4x_4)\\
&+12f(nx_1)+2f(-2nx_1+n^2x_2+n^3x_3+n^4x_4)\\
&+2f(nx_1-n^2x_2+n^3x_3-n^4x_4)\\
&+12f(n^3x_3)+2f(-nx_1-n^2x_2-n^3x_3-n^4x_4)\\
&+2f(-nx_1+n^2x_2-n^3x_3+n^4x_4)+2f(nx_1+n^2x_2+n^3x_3+2n^4x_4)\\
&+2f(nx_1-n^2x_2-n^3x_3-2n^4x_4)+12f(nx_1)\\
&+2f(-2nx_1+n^2x_2+n^3x_3+n^4x_4)+2f(2nx_1-n^2x_2-n^3x_3+n^4x_4)\\
&+12f(n^4x_4)+2f(-nx_1-n^2x_2-n^3x_3-n^4x_4)\\
&+2f(-nx_1+n^2x_2+n^3x_3-n^4x_4)+2f(nx_1+2n^2x_2+n^3x_3+n^4x_4)\\
&2f(-nx_1-2n^2x_2+n^3x_3-n^4x_4)+12f(n^3x_3)\\
&+2f(nx_1+n^2x_2-2n^3x_3+n^4x_4)+2f(-nx_1+n^2x_2+2n^3x_3-n^4x_4)\\
&+12f(n^2x_2)+2f(-nx_1-n^2x_2-n^3x_3-n^4x_4)\\
&+2f(nx_1-n^2x_2-n^3x_3+n^4x_4)+2f(nx_1+2n^2x_2+n^3x_3+n^4x_4)\\
&+2f(-nx_1-2n^2x_2-n^3x_3+n^4x_4)+12f(n^4x_4)\\
&+2f(nx_1+n^2x_2+n^3x_3-2n^4x_4)+2f(-nx_1+n^2x_2-n^3x_3+2n^4x_4)\\
&+12f(n^2x_2)+2f(-nx_1-n^2x_2-n^3x_3-n^4x_4)\\
&+2f(nx_1-n^2x_2+n^3x_3-n^4x_4)+2f(nx_1+n^2x_2+2n^3x_3+n^4x_4)\\
&+2f(-nx_1-n^2x_2-2n^3x_3+n^4x_4)+12f(n^4x_4)\\
&+2f(nx_1+n^2x_2+n^3x_3-2n^4x_4)+2f(-nx_1-n^2x_2+n^3x_3+2n^4x_4)\\
&+12f(n^3x_3)+2f(-nx_1-n^2x_2-n^3x_3-n^4x_4)\\
&+2f(nx_1+n^2x_2-n^3x_3-n^4x_4),
\end{aligned}
\tag{11.31}
$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding (11.24) and (11.31), we obtain

$$
\begin{aligned}
&24f(nx_1+n^2x_2)+24f(nx_1+n^3x_3)+24f(nx_1+n^4x_4)\\
&+24f(n^2x_2+n^3x_3)+24f(n^2x_2+n^4x_4)+24f(n^3x_3+n^4x_4)\\
&-72f(nx_1)-72f(n^2x_2)-72f(n^3x_3)-72f(n^4x_4)\\
&=72f(nx_1+n^2x_2+n^3x_3+n^4x_4)-36f(nx_1+n^3x_3+n^4x_4)\\
&+12f(-nx_1+n^3x_3+n^4x_4)+12f(nx_1-n^3x_3+n^4x_4)+12f(nx_1+n^3x_3\\
&-n^4x_4-36f(n^2x_2+n^3x_3+n^4x_4)+12f(-n^2x_2+n^3x_3+n^4x_4)\\
&+12f(n^2x_2-n^3x_3+n^4x_4)+12f(n^2x_2+n^3x_3-n^4x_4)\\
&-36f(nx_1+n^2x_2+n^4x_4)+12f(-nx_1+n^2x_2+n^4x_4)\\
&+12f(nx_1-n^2x_2+n^4x_4)+12f(nx_1+n^2x_2-n^4x_4)\\
&-36f(nx_1+n^2x_2+n^3x_3)+12f(-nx_1+n^2x_2+n^3x_3)\\
&+12f(nx_1-n^2x_2+n^3x_3)+12f(nx_1+n^2x_2-n^3x_3),
\end{aligned}
\tag{11.32}
$$

for all $x_1, x_2, x_3, x_4 \in X$. We know that

$$
\begin{aligned}
&f(-nx_1+n^2x_2+n^3x_3)+f(nx_1-n^2x_2+n^3x_3)\\
&+f(nx_1+n^2x_2-n^3x_3)=-3f(nx_1+n^2x_2+n^3x_3)+4f(nx_1+n^2x_2)\\
&+4f(n^2x_2+n^3x_3)+4f(nx_1+n^3x_3)-4f(nx_1)\\
&-4f(n^2x_2)-4f(n^3x_3),
\end{aligned}
\tag{11.33}
$$

for all $x_1, x_2, x_3 \in X$. Replacing n^3x_3 by n^4x_4 in (11.33), we have

$$
\begin{aligned}
&f(-nx_1+n^2x_2+n^4x_4)+f(nx_1-n^2x_2+n^4x_4)\\
&+f(nx_1+n^2x_2-n^4x_4)=-3f(nx_1+n^2x_2+n^4x_4)+4f(nx_1+n^2x_2)\\
&+4f(n^2x_2+n^4x_4)+4f(nx_1+n^4x_4)-4f(nx_1)\\
&-4f(n^2x_2)-4f(n^4x_4),
\end{aligned}
\tag{11.34}
$$

for all $x_1, x_2, x_4 \in X$. Replacing n^2x_2 by n^3x_3 in (11.34), we get

$$
\begin{aligned}
&f(-nx_1+n^3x_3+n^4x_4)+f(nx_1-n^3x_3+n^4x_4)\\
&+f(nx_1+n^3x_3-n^4x_4)=-3f(nx_1+n^3x_3+n^4x_4)+4f(nx_1+n^3x_3)\\
&+4f(n^3x_3+n^4x_4)+4f(nx_1+n^4x_4)-4f(nx_1)\\
&-4f(n^3x_3)-4f(n^4x_4),
\end{aligned}
\tag{11.35}
$$

for all $x_1, x_3, x_4 \in X$. Replacing nx_1 by n^2x_2 in (11.35), we have

$$
\begin{aligned}
&f(-n^2x_2+n^3x_3+n^4x_4)+f(n^2x_2-n^3x_3+n^4x_4)\\
&+f(n^2x_2+n^3x_3-n^4x_4)=-3f(n^2x_2+n^3x_3+n^4x_4)+4f(n^2x_2+n^3x_3)\\
&+4f(n^3x_3+n^4x_4)+4f(n^2x_2+n^4x_4)-4f(n^2x_2)\\
&-4f(n^3x_3)-4f(n^4x_4),
\end{aligned}
\tag{11.36}
$$

for all $x_2, x_3, x_4 \in X$. Adding (11.33), (11.34), (11.35), (11.36), we get

$$\begin{aligned}
& f(-nx_1 + n^2x_2 + n^3x_3) + f(nx_1 - n^2x_2 + n^3x_3) \\
& + f(nx_1 + n^2x_2 - n^3x_3) + f(-nx_1 + n^2x_2 + n^4x_4) \\
& + f(nx_1 - n^2x_2 + n^4x_4) + f(nx_1 + n^2x_2 - n^4x_4) \\
& + f(-nx_1 + n^3x_3 + n^4x_4) + f(nx_1 - n^3x_3 + n^4x_4) \\
& + f(nx_1 + n^3x_3 - n^4x_4) + f(-n^2x_2 + n^3x_3 + n^4x_4) \\
& + f(n^2x_2 - n^3x_3 + n^4x_4) + f(n^2x_2 + n^3x_3 - n^4x_4) \\
& = -3f(nx_1 + n^2x_2 + n^3x_3) - 3f(nx_1 + n^2x_2 + n^4x_4) \\
& -3f(nx_1 + n^3x_3 + n^4x_4) - 3f(n^2x_2 + n^3x_3 + n^4x_4) \\
& +8f(nx_1 + n^2x_2) + 8f(nx_1 + n^3x_3) + 8f(nx_1 + n^4x_4) \\
& +8f(n^2x_2 + n^3x_3) + 8f(n^2x_2 + n^4x_4) + 8f(n^3x_3 + n^4x_4) \\
& -12f(nx_1) - 12f(n^2x_2) - 12f(n^3x_3) - 12f(n^4x_4), \qquad (11.37)
\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (11.37) in (11.32), we have

$$\begin{aligned}
& f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4) = f(nx_1 + n^2x_2 + n^3x_3) \\
& + f(nx_1 + n^2x_2 + n^4x_4) + f(nx_1 + n^3x_3 + n^4x_4) + f(n^2x_2 + n^3x_3 \\
& + n^4x_4) - f(nx_1 + n^2x_2) - f(nx_1 + n^3x_3) - f(nx_1 + n^4x_4) \\
& - f(n^2x_2 + n^3x_3) - f(n^2x_2 + n^4x_4) - f(n^3x_3 + n^4x_4) \\
& + n^3 f(x_1) + n^6 f(x_2) + n^9 f(x_3) + n^{12} f(x_4), \qquad (11.38)
\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$.

Conversely, assume that $f : X \to Y$ satisfies the functional equation (11.1). Replacing (x_1, x_2, x_3, x_4) by $(x, 0, 0, 0)$, $(0, x, 0, 0)$, $(0, 0, x, 0)$ and $(0, 0, 0, x)$ respectively in (11.38), we obtain

$$\begin{aligned}
& f(nx) = n^3 f(x), \quad f(n^2x) = n^6 f(x), \quad f(n^3x) = n^9 f(x) \ \textit{and} \\
& f(n^4x) = n^{12} f(x), \qquad (11.39)
\end{aligned}$$

for all $x \in X$. One can easy to verify from (11.39) that

$$f\left(\frac{x}{n^i}\right) = \left(\frac{1}{n^i}\right)^3 f(x), \qquad (11.40)$$

for all $x \in X$. Replacing (x_1, x_2, x_3, x_4) by $\left(\frac{x}{n}, \frac{x}{n^2}, \frac{y}{n^3}, \frac{-y}{n^4}\right)$ in (11.38), we get

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x),$$

for all $x, y \in X$. □

11.3 4-Dimensional Cubic Functional Equation—Direct Method

In this section, we present some basic definitions of intuitionistic fuzzy normed space (briefly, IFN-space) and ascertain the stability of the 4-Dimensional cubic functional equation in intuitionistic fuzzy normed space with the help of the direct method.

Definition 11.2 An intuitionistic fuzzy set $A_{u,v}$ in a universal set W is an object $A_{u,v} = \{(u(w), v(w))|w \in W\}$, for all $w \in W$, $u_A(w) \in [0, 1]$ and $v_A(w) \in [0, 1]$ are called the membership degree and the non-membership degree respectively, of u in $A_{u,v}$ and they satisfies $u_a(w) + v_a(w) \leq 1$.

Remark: We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (0, 1)$ classically, a triangular norm $^* = T$ and $[0, 1]$ is defined as an increasing, commutative and associative mapping $T : [0, 1]^2 \to [0, 1]$ satisfying $T(1, x) = 1 * x = x$, for all $x \in [0, 1]$.

A triangular conorm $S = \Diamond$ is defined as an increasing, commutative, associative mapping $S : [0, 1]^2 \to [0, 1]$ satisfying $S(0, x) = 0 \Diamond x = x$, for all $x \in [0, 1]$.

Using the lattice $(L^*, \leq_{L^*})$, these definitions can be straight forwardly extended.

Definition 11.3 A triangular norm (t-norm) on L^* is a mapping $T : (L^*)^2 \to L^*$ satisfying the following conditions:

(i) $(\forall x \in L^*)(T(x, 1_{L^*}) = x)$ (boundary condition);
(ii) $(\forall (x, y) \in (L^*)^2)(T(x, y) = T(y, x))$ (commutative);
(iii) $(\forall (x, y, z) \in (L^*)^3)(T(x, T(y, z)) = T(T(x, y), z))$ (associativity);
(iv) $(\forall (x, x', y, y') \in (L^*)^4)(x \leq_{L^*} x' \quad and \quad y \leq_{L^*} y' \Rightarrow T(x, y) \leq_{L^*} T(x', y'))$ (monotonicity).

Remark: If $(L^*, \leq_{L^*}, T)$ is an abelian topological monoid with unit 1_{L^*}, then L^* is said to be a continuous t-norm.

Definition 11.4 A continuous t-norms T on L^* is said to be continuous t-representable if there exist a continuous t-norm $*$ and a continuous t-conorm $\Diamond$ on $[0, 1]$ such that, for all $x = (x_1, x_2),\ y = (y_1, y_2) \in L^*$,

$$T(x, y) = (x_1 * y_1, x_2 \Diamond y_2).$$

Definition 11.5 A negator on $L*$ is any decreasing mapping $N : L^* \to L^*$ satisfying $N : (0_{L^*}) = 1_{L^*}$ and $N : (1_{L^*}) = 0_{L^*}$. If $N(N(x)) = x$, for all $x \in L^*$, then N is called an involute negator. A negator on $[0, 1]$ is a decreasing mapping $N : [0, 1] \to [0, 1]$ satisfying $P_{\mu,v}(0) = 1$ and $P_{\mu,v}(1) = 0$. N_s denotes the standard negator on $[0, 1]$ defined by $N_s(x) = 1 - x,\ \forall x \in [0, 1]$.

Definition 11.6 Let μ and v be membership and non-membership degree of an intuitionistic fuzzy set from $X \times (0, +\infty)$ to $[0, 1]$ such that $\mu_x(t) + v_x(t) \leq 1$, for all $x \in X$ and all $t > 0$. The triple $(X, P_{\mu,v}, T)$ is said to be an intuitionistic fuzzy normed space (briefly IFN-space) if X is a vector space, T is continuous t-representable and $P_{\mu,v}$ is a mapping $X \times (0, +\infty) \to L^*$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,

(IFN1) $P_{\mu,v}(x, 0) = 0_{L^*}$;
(IFN2) $P_{\mu,v}(x, t) = 1_{L^*}$ if and only if $x = 0$;
(IFN3) $P_{\mu,v}(\alpha x, t) = P_{\mu,v}\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$;
(IFN4) $P_{\mu,v}(x + y, t + s) \geq_{L^*} T(P_{\mu,v}(x, t), P_{\mu,v}(y, s))$.

In this case, $P_{\mu,v}$ is called an intuitionistic fuzzy norm. Here, $P_{\mu,v}(x, t) = (\mu_x(t), v_x(t))$.

Definition 11.7 A sequence $\{x_n\}$ in an IFN-space $(X, P_{\mu,t}, T)$ is called a Cauchy sequence if, for every $\epsilon > 0$ and $t > 0$, there exists $n_0 \in N$ such that

$$P_{\mu,v}(x_n - x_m, t) > L^*(N_s(\epsilon), \epsilon), \quad \forall n, m \in n_0.$$

Here N_S is the standard negator.

Definition 11.8 A sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ (denoted by $x_n \xrightarrow{P_{\mu,v}} x$) if $P_{\mu,v}(x_n - x, t) \to 1_{L^*}$ as $n \to \infty$ for every $t > 0$.

Definition 11.9 An IFN-space $(X, P_{\mu,t}, T)$ is said to be complete if every Cauchy sequence in X is convergent to a point $x \in X$.

Note: Now use the following notation for a given mapping $f : X \to Y$

$$\begin{aligned} C(x_1, x_2, x_3, x_4) &= f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4) - f(nx_1 + n^2x_2 + n^3x_3) \\ &\quad - f(nx_1 + n^2x_2 + n^4x_4) - f(nx_1 + n^3x_3 + n^4x_4) - f(n^2x_2 + n^3x_+ n^4x_4) \\ &\quad + f(nx_1 + n^2x_2) + f(nx_1 + n^3x_3) + f(n^2x_2 + n^3x_3) + f(n^3x_3 + n^4x_4) \\ &\quad + f(n^2x_2 + n^4x_4) - n^3 f(x_1) - n^6 f(x_2) - n^9 f(x_3) - n^{12} f(x_4) \end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$.

Theorem 11.10 *Let $\beta \in \{-1, 1\}$. Let X be a linear space, $(Z, P'_{\mu,v}, T)$ be an IFN-space, $\rho : X^4 \to Z$ with $0 < \left(\frac{d}{n^3}\right)^{\beta} < 1$,*

$$P'_{\mu,v}(\rho(n^{\beta}x_1, 0, 0, 0), r) \geq_{L^*} P'_{\mu,v}(d^{\beta}\rho(x, 0, 0, 0), r), \tag{11.41}$$

for all $x \in X$ $r > 0$, and $d > 0$ and

$$\lim_{k\to\infty} P'_{\mu,v}\left(\rho(n^{\beta k}x_1, n^{\beta k}x_2, n^{\beta k}x_3, n^{\beta k}x_4), n^{3\beta k}r\right) = 1_{L^*}, \tag{11.42}$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$. Let $(Y, P'_{\mu,v}, T)$ be an IFN-space. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$P_{\mu,v}(C(x_1,x_2,x_3,x_4),r) \geq_{L^*} P'_{\mu,v}(\rho(x_1,x_2,x_3,x_4),r), \tag{11.43}$$

for all $r > 0$ *and* $x_1, x_2, x_3, x_4 \in X$. *Then the limit*

$$P_{\mu,v}\left(H(x) - \frac{f(n^{\beta k}x)}{n^{3\beta k}}\right) \to 1_{L^*} \text{ as } k \to \infty, \quad r > 0, \tag{11.44}$$

exists for all $x \in X$ *and the mapping* $H : X \to Y$ *is a unique cubic mapping satisfying (11.1) and*

$$P_{\mu,v}(f(x) - H(x), r) \geq_{L^*} P'_{\mu,v}\left(\rho(x,0,0,0), r|n^3 - d, |\right), \tag{11.45}$$

for all $x \in X$ *and all* $r > 0$.

Proof First assume that $\beta = 1$. Replacing (x_1, x_2, x_3, x_4) by $(x, 0, 0, 0)$, in (11.43), we have

$$P_{\mu,v}((f(nx) - n^3 f(x)), r) \geq_{L^*} P'_{\mu,v}(\rho(x,0,0,0), r), \tag{11.46}$$

for all $x \in X$ and $r > 0$. Again replacing x by $n^k x$ in (11.46) and using (IFN3), we get

$$P_{\mu,v}\left(\frac{f(n^{k+1}x)}{n^3} - f(n^k x), \frac{r}{n^3}\right) \geq_{L^*} P'_{\mu,v}(\rho(n^k x, 0,0,0), r), \tag{11.47}$$

for all $x \in X$ and $r > 0$. Using (11.41), (IFN3) in (11.47), we have

$$P_{\mu,v}\left(\frac{f(n^{k+1}x)}{n^3} - f(n^k x), \frac{r}{n^3}\right) \geq_{L^*} P'_{\mu,v}\left(\rho(x,0,0,0), \frac{r}{d^k}\right), \tag{11.48}$$

for all $x \in X$ and $r > 0$. It is easy to verify from (11.48), that

$$P_{\mu,v}\left(\frac{f(n^{k+1}x)}{n^{3(k+1)}} - \frac{f(n^k x)}{n^{3k}}, \frac{r}{n^3 n^{3k}}\right) \geq_{L^*} P'_{\mu,v}\left(\rho(x,0,0,0), \frac{r}{d^k}\right), \tag{11.49}$$

holds for all $x \in X$ and $r > 0$. Replacing r by $d^k r$ in (11.49), we get

$$P_{\mu,v}\left(\frac{f(n^{k+1}x)}{n^{3(k+1)}} - \frac{f(n^k x)}{n^{3k}}, \frac{d^k r}{n^3 n^{3k}}\right) \geq_{L^*} P'_{\mu,v}(\rho(x,0,0,0), r), \tag{11.50}$$

for all $x \in X$ and $r > 0$. It is easy to see that

$$\frac{f(n^{k+1}x)}{n^{3(k+1)}} - f(x) = \sum_{i=0}^{k-1}\left[\frac{f(n^{i+1}x)}{n^{3(i+1)}} - \frac{f(n^i x)}{n^{3i}}\right], \tag{11.51}$$

for all $x \in X$. From the Eqs. (11.50) and (11.51), we get

$$\begin{aligned} P_{\mu,v}\left(\frac{f(n^k x)}{n^{3k}} - f(x), \sum_{i=0}^{k-1} \frac{d^i r}{n^3 n^{3i}}\right) &\geq_{L^*} T_{i=0}^{n-1}\left\{P_{\mu,v}\left(\frac{f(n^{i+1}x)}{n^{3(i+1)}} - \frac{f(n^i x)}{n^{3i}}, \frac{d^i r}{n^3 n^{3i}}\right)\right\} \\ &\geq_{L^*} T_{i=0}^{n-1}\{P'_{\mu,v}(\rho(x,0,0,0),r)\} \\ &\geq_{L^*} P'_{\mu,v}(\rho(x,0,0,0),r)\}, \end{aligned} \tag{11.52}$$

for all $x \in X$ and $r > 0$. Replacing x by $n^m x$ in (11.52) and using (11.41) and (IFN3), we obtain

$$P_{\mu,v}\left(\frac{f(n^{k+m}x)}{n^{3(k+m)}} - \frac{f(n^m x)}{n^{3m}}, \sum_{i=m}^{m+k-1} \frac{d^i r}{n^3 n^{3i}}\right) \geq_{L^*} P'_{\mu,v}\big(\rho(x,0,0,0), \frac{r}{d^m}\big), \tag{11.53}$$

for all $x \in X$, $r > 0$ and $m, k \geq 0$. Replacing r by $d^m r$ in (11.53), we get

$$P_{\mu,v}\left(\frac{f(n^{k+m}x)}{n^{3(k+m)}} - \frac{f(n^m x)}{n^{3m}}, \sum_{i=m}^{m+k-1} \frac{d^i r}{n^3 n^{3i}}\right) \geq P'_{\mu,v}\big(\rho(x,0,0,0), r\big), \tag{11.54}$$

for all $x \in X$, $r > 0$ and $m, k \geq 0$. Using (IFN3) in (11.53), we have

$$P_{\mu,v}\left(\frac{f(n^{k+m}x)}{n^{3(k+m)}} - \frac{f(n^m x)}{n^{3m}}, r\right) \geq_{L^*} P'_{\mu,v}\left(\rho(x,0,0,0), \frac{r}{\sum_{i=m}^{m+k-1} \frac{d^i}{n^3 n^{3i}}}\right), \tag{11.55}$$

for all $x \in X$, $r > 0$ and $m, k \geq 0$. Since $0 < d < n^3$ and $\sum_{i=0}^{k} \left(\frac{d}{n^3}\right)^i < \infty$, $\left\{\frac{f(n^k x)}{n^{3k}}\right\}$ is a Cauchy sequence in $(Y, P_{\mu,v}, T)$, which is a complete IFN-space. This sequence converges to some point $H(x) \in Y$ so one can define the mapping $H : X \to Y$ by

$$P_{\mu,v}\left(H(x) - \lim_{k\to\infty} \frac{f(n^{\beta k}x)}{n^{3\beta k}}\right) \to 1_{L^*} \;\; as \;\; k \to \infty, \;\; r > 0,$$

for all $x \in X$. Letting $m = 0$ in (11.55), we obtain

$$P_{\mu,v}\left(\frac{f(n^k x)}{n^{3k}} - f(x), r\right) \geq_{L^*} P'_{\mu,v}\left(\rho(x,0,0,0), \frac{r}{\sum_{i=0}^{k-1} \frac{d^i r}{n^3 n^{3i}}}\right), \tag{11.56}$$

for all $x \in X$. Letting $k \to \infty$ in (11.56) and using (N6), we get

$$P_{\mu,v}(f(x) - H(x), r) \geq_{L^*} P'_{\mu,v}(\rho(x,0,0,0), r(n^3 - d)),$$

for all $x \in X$ and $r > 0$. To prove H satisfies (11.1), replacing (x_1, x_2, x_3, x_4) by $(n^k x_1, n^k x_2, n^k x_3, n^k x_4)$ in (11.43), we obtain

$$P_{\mu,v}\left(\frac{1}{n^{3k}}C(n^kx_1, n^kx_2, n^kx_3, n^kx_4), r\right) \geq P'_{\mu,v}(\rho(n^kx_1, n^kx_2, n^kx_3, n^kx_4), n^{3k}r), \quad (11.57)$$

for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Now

$$\begin{aligned}
&P_{\mu,v}\big(H(nx_1+n^2x_2+n^3x_3+n^4x_4) - H(nx_1+n^2x_2+n^3x_3)\\
&-H(nx_1+n^2x_2+n^4x_4) - H(nx_1+n^3x_3+n^4x_4) - H(n^2x_2+n^3x_3\\
&+n^4x_4) + H(nx_1+n^2x_2) + H(nx_1+n^3x_3) + H(nx_1+n^4x_4)\\
&+H(n^2x_2+n^3x_3) + H(n^2x_2+n^4x_4) + H(n^3x_3+n^4x_4)\\
&-n^3H(x_1) - n^6H(x_2) - n^9H(x_3) - n^{12}H(x_4), r\big)\\
&\geq_{L^*} T\left\{P_{\mu,v}\left(H(nx_1+n^2x_2+n^3x_3+n^4x_4) - \frac{1}{n^{3k}}f(n^k(nx_1+n^2x_2\right.\right.\\
&\left.+n^3x_3+n^4x_4)), \frac{r}{5}\right), P_{\mu,v}\left(H(nx_1+n^2x_2+n^3x_3) + H(nx_1+n^2x_2+n^4x_4)\right.\\
&+H(nx_1+n^3x_3+n^4x_4) + H(n^2x_2+n^3x_3+n^4x_4) - \frac{1}{n^{3k}}(f(n^k(nx_1+n^2x_2\\
&+n^3x_3)) + f(n^k(nx_1+n^2x_2+n^4x_4)) + f(n^k(nx_1+n^3x_3+n^4x_4))\\
&\left.+f(n^k(n^2x_2+n^3x_3+n^4x_4)), \frac{r}{5}\right), P_{\mu,v}\left(H(nx_1+n^2x_2) + H(nx_1+n^3x_3)\right.\\
&+H(nx_1+n^4x_4) + H(n^2x_2+n^3x_3) + H(n^2x_2+n^4x_4) + H(n^3x_3+n^4x_4)\\
&-\frac{1}{n^{3k}}(f(n^k(nx_1+n^2x_2)) + f(n^k(nx_1+n^3x_3)) + f(n^k(nx_1+n^4x_4))\\
&\left.+f(n^k(n^2x_2+n^3x_3)) + +f(n^k(n^2x_2+n^4x_4)) + f(n^k(n^3x_3+n^4x_4))), \frac{r}{5}\right),\\
&P_{\mu,v}\left(n^3H(x_1) + n^6H(x_2) + n^9H(x_3) + n^{12}H(x_4) - \frac{1}{n^{3k}}(n^3f(n^kx_1)\right.\\
&\left.+n^6f(n^kx_2) + n^9f(n^kx_3) + n^{12}f(n^kx_4)), \frac{r}{5}\right), P_{\mu,v}\left(\frac{1}{n^{3k}}f(n^k(nx_1+n^2x_2\right.\\
&+n^3x_3+n^4x_4)) - \frac{1}{n^{3k}}(f(n^k(nx_1+n^2x_2+n^3x_3)) + f(n^k(nx_1+n^2x_2\\
&\left.+n^4x_4)) + f(n^k(nx_1+n^3x_3+n^4x_4)) + f(n^k(n^2x_2+n^3x_3+n^4x_4))\right)\\
&+\frac{1}{n^{3k}}(f(n^k(nx_1+n^2x_2)) + f(n^k(nx_1+n^3x_3)) + f(n^k(nx_1+n^4x_4))\\
&+f(n^k(n^2x_2+n^3x_3)) + f(n^k(n^2x_2+n^4x_4)) + f(n^k(n^3x_3+n^4x_4)))\\
&\left.-\frac{1}{n^{3k}}(n^3f(n^kx_1) + n^6f(n^kx_2) + n^9f(n^kx_3) + n^{12}f(n^kx_4)), \frac{r}{5}\right\},
\end{aligned} \quad (11.58)$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$. Using (8.1) and (IFN5) in (11.58), we see that

$$\begin{aligned}
&P_{\mu,v}\big(H(nx_1 + n^2x_2 + n^3x_3 + n^4x_4) - H(nx_1 + n^2x_2 + n^3x_3) \\
&-H(nx_1 + n^2x_2 + n^4x_4) - H(nx_1 + n^3x_3 - n^4x_4) - H(n^2x_2 + n^3x_3 \\
&+n^4x_4) + H(nx_1 + n^2x_2) + H(nx_1 + n^3x_3) + H(nx_1 + n^4x_4) \\
&+H(n^2x_2 + n^3x_3) + H(n^2x_2 + n^4x_4) + H(n^3x_3 + n^4x_4) \\
&-n^3H(x_1) - n^6H(x_2) - n^9H(x_3) - n^{12}H(x_4), r\big) \\
&\qquad \geq_{L^*} T\big\{1_{L^*}, 1_{L^*}, 1_{L^*}, 1_{L^*}, P'_{\mu,v}(\rho(n^kx_1, n^kx_2, n^kx_3, n^kx_4), n^{3k}r)\big\} \\
&\qquad \geq_{L^*} P'_{\mu,v}(\rho(n^kx_1, n^kx_2, n^kx_3, n^kx_4), n^{3k}r),
\end{aligned} \tag{11.59}$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$. Letting $k \to \infty$ in (11.59) and using (11.42), we get

$$\begin{aligned}
&P_{\mu,v}\big(H(nx_1 + n^2x_2 + n^3x_3 + n^4x_4) - H(nx_1 + n^2x_2 + n^3x_3) \\
&-H(nx_1 + n^2x_2 + n^4x_4) - H(nx_1 + n^3x_3 + n^4x_4) - H(n^2x_2 + n^3x_3 \\
&+n^4x_4) + H(nx_1 + n^2x_2) + H(nx_1 + n^3x_3) + H(nx_1 + n^4x_4) \\
&+H(n^2x_2 + n^3x_3) + H(n^2x_2 + n^4x_4) + H(n^3x_3 + n^4x_4) \\
&-n^3H(x_1) - n^6H(x_2) - n^9H(x_3) - n^{12}H(x_4), r\big) = 1,
\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$. Using (IFN2) in the above inequality, we get

$$\begin{aligned}
&H(nx_1 + n^2x_2 + n^3x_3 + n^4x_4) = H(nx_1 + n^2x_2 + n^3x_3) \\
&+H(nx_1 + n^2x_2 + n^4x_4) + H(nx_1 + n^3x_3 + n^4x_4) \\
&+H(n^2x_2 + n^3x_3 + n^4x_4) - H(nx_1 + n^2x_2) - H(nx_1 + n^3x_3) \\
&-H(nx_1 + n^4x_4) - H(n^2x_2 + n^3x_3) - H(n^3x_3 + n^4x_4) \\
&-H(n^2x_2 + n^4x_4) + n^3H(x_1) + n^6H(x_2) + n^9H(x_3) + n^{12}H(x_4),
\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$. Hence H satisfies the cubic functional equation (11.1). In order to prove $H(x)$ is unique. Let $H'(x)$ be another cubic mapping satisfying (11.1) and (11.54). Then

$$P_{\mu,v}(H(x) - H'(x), r) = P_{\mu,v}\left(\frac{H(n^k x)}{n^{3k}} - \frac{H'(n^k x)}{n^{3k}}\right)$$
$$\geq_{L^*} T\left\{P_{\mu,v}\left(\frac{H(n^k x)}{n^{3k}} - \frac{f(n^k x)}{n^{3k}}, \frac{r}{2}\right), P_{\mu,v}\left(\frac{f(n^k x)}{n^{3k}} - \frac{H'(n^k x)}{n^{3k}}, \frac{r}{2}\right)\right\}$$
$$\geq_{L^*} P'_{\mu,v}\left(\rho(n^k x, 0, 0, 0), \frac{n^{3k} r(n^3 - d)}{2}\right)$$
$$\geq_{L^*} P'_{\mu,v}\left(\rho(x, 0, 0, 0), \frac{n^{3k} r(n^3 - d)}{2d^k}\right),$$

for all $x_1, x_2, x_3 \in X$ and $r > 0$. Since

$$\lim_{k\to\infty} \frac{n^{3k} r(n^3 - d)}{2d^k} = \infty.$$

It follows that

$$P'_{\mu,v}\left(\rho(x, 0, 0, 0), \frac{n^{3k} r(n^3 - d)}{2d^k}\right) = 1_{L^*}.$$

Thus $P_{\mu,v}(H(x) - H'(x), r) = 1_{L^*}$, for all $x \in X$ and for $r > 0$. Hence $H(x) = H'(x)$. Therefore $H(x)$ is unique. For $\beta = -1$, we can prove that the result by a similar method. This completes the proof of the theorem. □

Remark: Let $\beta \in \{-1, 1\}$. Let X be a linear space, $(Z, P'_{\mu,v}, T)$ be an IFN-space, $\rho : X^4 \to Z$ with $0 < \left(\frac{d}{2^3}\right) < 1$ satisfying

$$P_{\mu,v}(\rho(2^{\beta k} x_2, 0, 0, 0), r) \geq_{L^*} P'_{\mu,v}(d^\beta \rho(x, 0, 0, 0), r),$$

for all $x \in X$ and all $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} P'_{\mu,v}(\rho(2^{\beta k} x_1, 2^{\beta k} x_2, 2^{\beta k} x_3 2^{\beta k} x_4), 2^{3\beta k} r) = 1_{L^*},$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$. Let $(Y, P'_{\mu,v}, T)$ be an IFN-space. Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.43) for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit

$$P_{\mu,v}\left(H(x) - \frac{f(2^{\beta k} x)}{2^{3\beta k}}\right) \to 1_{L^*},$$

as $k \to \infty$, $r > 0$ exists for all $x \in X$ and the mapping $H : X \to Y$ is a unique cubic mapping satisfying (11.1) and

$$P_{\mu,v}(f(x) - H(x), r) \geq_{L^*} P'_{\mu,v}(\rho(x, 0, 0, 0), r|2^3 - d|),$$

for all $x \in X$ and $r > 0$.

Proposition 11.11 *Let $\beta \in \{-1, 1\}$. Let X be linear space, $(Z, P'_{\mu,v}, T)$ be an IFN-space, $\rho : X^4 \to Z$ with $0 < \left(\frac{d}{n^6}\right) < 1$ satisfying*

$$P_{\mu,v}(\rho(0, n^{2\beta k}x_2, 0, 0), r) \geq_{L^*} P'_{\mu,v}(d^\beta \rho(0, x, 0, 0), r),$$

for all $x \in X$, $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} P'_{\mu,v}(\rho(n^{2\beta k}x_1, n^{2\beta k}x_2, n^{2\beta k}x_3 n^{2\beta k}x_4), n^{6\beta k}r) = 1_{L^*},$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. Let $(Y, P'_{\mu,v}, T)$ be an IFN-space. Suppose a mapping $f : X \to Y$ satisfies the inequality (11.43), for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit

$$P_{\mu,v}\left(H(x) - \frac{f(n^{2\beta k}x)}{n^{6\beta k}}\right) \to 1_{L^*},$$

as $k \to \infty$, $r > 0$ exists for all $x \in X$ and the mapping $H : X \to Y$ is a unique cubic mapping satisfying (11.1) and

$$P_{\mu,v}(f(x) - H(x), r) \geq_{L^*} P'_{\mu,v}(\rho(0, x, 0, 0), r|n^6 - d|),$$

for all $x \in X$ and $r > 0$.

Proposition 11.12 *Let $\beta \in \{-1, 1\}$. Let X be linear space, $(Z, P'_{\mu,v}, T)$ be an IFN-space, $\rho : X^4 \to Z$ with $0 < \left(\frac{d}{n^9}\right) < 1$ satisfying*

$$P_{\mu,v}(\rho(0, 0, n^{3\beta k}x_3, 0), r) \geq_{L^*} P'_{\mu,v}(d^\beta \rho(0, 0, x, 0), r),$$

for all $x \in X$ and $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} P'_{\mu,v}(\rho(n^{3\beta k}x_1, n^{3\beta k}x_2, n^{3\beta k}x_3 n^{3\beta k}x_4), n^{9\beta k}r) = 1_{L^*},$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. Let $(Y, P'_{\mu,v}, T)$ be an IFN-space. Suppose a mapping $f : X \to Y$ satisfies the inequality (11.43), for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit

$$P_{\mu,v}\left(H(x) - \frac{f(n^{3\beta k}x)}{n^{9\beta k}}\right) \to 1_{L^*},$$

as $k \to \infty$, $r > 0$ exists for all $x \in X$ and the mapping $H : X \to Y$ is a unique cubic mapping satisfying (11.1) and

$$P_{\mu,v}(f(x) - H(x), r) \geq_{L^*} P'_{\mu,v}(\rho(0, 0, x, 0), r|n^9 - d|),$$

for all $x \in X$ and $r > 0$.

Proposition 11.13 *Let $\beta \in \{-1, 1\}$. Let X be linear space,$(Z, P'_{\mu,v}, T)$ be an IFN-space, $\rho : X^4 \to Z$ with $0 < \left(\frac{d}{n^{12}}\right) < 1$ satisfying*

$$P_{\mu,v}(\rho(0, 0, 0, n^{4\beta k} x_4), r) \geq_{L^*} P'_{\mu,v}(d^{\beta}\rho(0, 0, 0, x), r),$$

for all $x \in X$, $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} P'_{\mu,v}(\rho(n^{4\beta k}x_1, n^{4\beta k}x_2, n^{4\beta k}x_3 n^{4\beta k}x_4), n^{12\beta k}r) = 1_{L^*},$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$. Let $(Y, P'_{\mu,v}, T)$ be an IFN-space. Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.43), for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit

$$P_{\mu,v}\left(H(x) - \frac{f(n^{4\beta k}x)}{n^{12\beta k}}\right) \to 1_{L^*},$$

as $k \to \infty$, $r > 0$ exists for all $x \in X$ and the mapping $H : X \to Y$ is a unique cubic mapping satisfying (11.1) and

$$P_{\mu,v}(f(x) - H(x), r) \geq_{L^*} P'_{\mu,v}(\rho(0, 0, 0, x), r|n^{12} - d|),$$

for all $x \in X$ and $r > 0$.

The following corollaries are the immediate consequence of Theorems 11.10–11.13 respectively, concerning the stability of (11.1).

Corollary 11.14 *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$P_{\mu,v}(C(x_1, x_2, x_3, x_4)) \geq_{L^*} \begin{cases} P'_{\mu,v}(\tau, r), \\ P'_{\mu,v}\left(\tau \sum_{i=1}^{4} \| x_i \|^s, r\right), & s \neq 3 \\ P'_{\mu,v}\left(\tau\left(\sum_{i=1}^{4} \| x_i \|^{4s} + \prod_{i=1}^{4} \| x_i \|^s\right), r\right), & s \neq \frac{3}{4} \end{cases} \tag{11.60}$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $H : X \to Y$ such that

$$P_{\mu,v}(f(x) - H(x), r) \geq \begin{cases} P'_{\mu,v}(\tau, r|n^3 - 1|) \\ P'_{\mu,v}(\tau \| x \|^s, r|n^3 - n^s|) \\ P'_{\mu,v}(\tau \| x \|^{4s}, r|n^3 - n^{4s}|), \end{cases} \tag{11.61}$$

for all $x \in X$ and $r > 0$.

Example 11.15 Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.60) for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $H : X \to Y$ such that

$$P_{\mu,v}(f(x) - H(x), r) \geq_{L^*} \begin{cases} P_{\mu,v}(\tau, r|7|) \\ P_{\mu,v}(\tau\|x\|^s, r|2^3 - 2^s|) \\ P_{\mu,v}(\tau\|x\|^{4s}, r|2^3 - 2^{4s}|), \end{cases}$$

for all $x \in X$ and $r > 0$.

Corollary 11.16 *Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.60) for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $H : X \to Y$ such that*

$$P_{\mu,v}(f(x) - H(x), r) \geq \begin{cases} P'_{\mu,v}(\tau, r|n^6 - 1|) \\ P'_{\mu,v}(\tau \parallel x \parallel^s, r|n^6 - n^{2s}|) \\ P'_{\mu,v}(\tau \parallel x \parallel^{4s}, r|n^6 - n^{8s}|), \end{cases} \tag{11.62}$$

for all $x \in X$ and for $r > 0$.

Example 11.17 Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.60) for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $H : X \to Y$ such that

$$P_{\mu,v}(f(x) - H(x), r) \geq_{L^*} \begin{cases} P_{\mu,v}(\tau, r|63|) \\ P_{\mu,v}(\tau\|x\|^s, r|2^6 - 2^{2s}|) \\ P_{\mu,v}(\tau\|x\|^{4s}, r|2^6 - 2^{8s}|), \end{cases}$$

for all $x \in X$ and $r > 0$.

Corollary 11.18 *Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.60) for $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $H : X \to Y$ such that*

$$P_{\mu,v}(f(x) - H(x), r) \geq \begin{cases} P'_{\mu,v}(\tau, r|n^9 - 1|) \\ P'_{\mu,v}(\tau \parallel x \parallel^s, r|n^9 - n^{3s}|) \\ P'_{\mu,v}(\tau \parallel x \parallel^{4s}, r|n^9 - n^{12s}|), \end{cases} \tag{11.63}$$

for all $x \in X$ and for $r > 0$.

Corollary 11.19 *Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.60) for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $H : X \to Y$ such that*

$$P_{\mu,v}(f(x) - H(x), r) \geq \begin{cases} P'_{\mu,v}(\tau, r|n^{12} - 1|) \\ P'_{\mu,v}(\tau \parallel x \parallel^s, r|n^{12} - n^{4s}|) \\ P'_{\mu,v}(\tau \parallel x \parallel^{4s}, r|n^{12} - n^{16s}|), \end{cases} \tag{11.64}$$

for all $x \in X$ and for $r > 0$.

11.4 4-Dimensional Cubic Functional Equation—Fixed Point Method

Theorem 11.20 *Let $f : X \to Y$ be a mapping for which there exists a function $\rho : X^4 \to Z$ with condition*

$$\lim_{k\to\infty} P'_{\mu,v}\big(\rho(\beta_i^k x_1, \beta_i^k x_2, \beta_i^k x_3, \beta_i^k x_4), \beta_i^{3k} r\big) = 1_{L^*}, \tag{11.65}$$

where $\beta_i = \begin{cases} n & if \;\; i = 0 \\ \frac{1}{n} & if \;\; \imath = 1 \end{cases}$ and Λ is the set such that

$$\Lambda = \{a\backslash a : X \to Y, a(0) = 0\},$$

for all $x_1, x_2, x_3, x_4 \in X$, $r > 0$ and satisfying the inequality

$$P_{\mu,v}(C(x_1, x_2, x_3, x_4), r) \geq_{L^*} P'_{\mu,v}(\rho(x_1, x_2, x_3, x_4), r), \tag{11.66}$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. If there exists $L = L[i]$ such that the mapping $x \to \epsilon(x)$ has the property

$$P'_{\mu,v}\left(L\frac{1}{\beta_i^3}\epsilon(\beta_i x), r\right) = P'_{\mu,v}(\epsilon(x), r), \tag{11.67}$$

for all $x \in X$, $r > 0$, then there exists a unique cubic mapping $H : X \to Y$ satisfying the functional equation (11.1) and

$$P_{\mu,v}(f(x) - H(x), r) \geq P'_{\mu,v}\left(\epsilon(x), \frac{L^{1-i}}{1-L}r\right), \tag{11.68}$$

for all $x \in X$, $r > 0$.

Proof Let us consider the set $\Lambda = \{p|p : U^2 \to V, \;\; p(0) = 0\}$ and d be a generalized metric on Λ, such that

$$d(a, b) = inf\{k \in (0, \infty)/P_{\mu,v}(a(x) - b(x), r) \geq_{L^*} P'_{\mu,v}(\epsilon(x), kr), x \in X, r > 0\}.$$

It is easy to see that (Λ, d) is complete. Define $T : \Lambda \to \Lambda$ by

$$Ta(x) = \frac{1}{\beta_i^3} a(\beta_i x), \quad \forall x \in X.$$

For $a, b \in \Lambda$, we attain

$$\begin{aligned}
& d(a, b) = k \\
& \Rightarrow P_{\mu,v}(a(x) - b(x)) \geq_{L^*} P'_{\mu,v}(\epsilon(x), kr) \\
& \Rightarrow P_{\mu,v}\left(\frac{a(\beta_i x)}{\beta_i^3} - \frac{q(\beta_i x)}{\epsilon_i^3}, r\right) \geq_{L^*} P'_{\mu,v}(\epsilon(\beta(x)), k\beta_i^3 r) \\
& \Rightarrow P_{\mu,v}(Ta(x) - Tb(x), r) \geq_{L^*} P'_{\mu,v}(\epsilon(\beta(x)), k\beta_i^3 r) \\
& \Rightarrow P_{\mu,v}(Ta(x) - Tb(x), r) \geq_{L^*} P'_{\mu,v}(\epsilon(x), kLr) \\
& \Rightarrow d(Ta(x) - Tb(x), r) \geq_{L^*} kL \\
& \Rightarrow d(Ta - Tb, r) \geq_{L^*} Ld(a, b) \;\; \forall a, b \in \Lambda.
\end{aligned} \tag{11.69}$$

Therefore T is strictly contractive mapping on Λ with Lipschitz constant L. Replacing (x_1, x_2, x_3, x_4) by $(x, 0, 0, 0)$ in (11.66), we get

$$P_{\mu,v}(f(nx) - n^3 f(x), r) \geq_{L^*} P'_{\mu,v}(\rho(x, 0, 0, 0), r), \tag{11.70}$$

for all $x \in X, r > 0$. Using (IFN3) in (11.70), we have

$$P_{\mu,v}\left(\frac{f(nx)}{n^3} - f(x), r\right) \geq_{L^*} P'_{\mu,v}\big(\rho(x, 0, 0, 0), n^3 r\big), \tag{11.71}$$

for all $x \in X, r > 0$ with the help of (11.67), when $i = 0$. It follows from (11.71) that

$$\begin{aligned}
& \Rightarrow P_{\mu,v}\left(\frac{f(nx)}{n^3} - f(x), r\right) \geq_{L^*} P'_{\mu,v}(\epsilon(x), Lr), \\
& \Rightarrow d(Tf(x), r) \geq_{L^*} L = L^1 = L^{1-i},
\end{aligned} \tag{11.72}$$

for all $x \in X, r > 0$. Replacing x by $\frac{x}{n}$ in (11.70), we get

$$P\left(f(x) - n^3 f\big(\frac{x}{n}\big), r\right) \geq_{L^*} P'_{\mu,v}\left(\rho\big(\frac{x}{n}, 0, 0, 0\big), r\right), \tag{11.73}$$

for all $x \in X, r > 0$ when $i = 1$. It follows from (11.73), that

$$\begin{aligned}
& P_{\mu,v}\left(f(x) - n^3 f\big(\frac{x}{n}\big), r\right) \geq_{L^*} P'_{\mu,v}(\epsilon(x), r) \\
& T(f - Tf) \leq 1 = L^0 = L^{1-i}.
\end{aligned} \tag{11.74}$$

Then from (11.72) and (11.74), we obtain

$$T(f, Tf) \leq L^{1-i} < \infty.$$

Now from the fixed point alternative in both cases it follows that there exists a fixed point H of T in Λ such that

$$\lim_{k\to\infty} P_{\mu,v}\left(\lim_{k\to\infty} \frac{f(\beta_i^k x)}{\beta_i^{3k}} - H(x), r\right) \to 1_{L^*}, \tag{11.75}$$

for all $x \in X$, $r > 0$. Replacing (x_1, xx_2, x_3, x_4) by $(\beta_i^k x_1, \beta_i^k x_2, \beta_i^k x_3, \beta_i^k x_4)$ in (11.66), we achieve

$$P_{\mu,v}\left(\frac{1}{\beta_i^{3k}} C(\beta_i^k x_1, \beta_i^k x_2, \beta_i^k x_3, \beta_i^k x_4), r\right) \geq P'_{\mu,v}(\rho(\beta_i^k x_1, \beta_i^k x_2, \beta_i^k x_3, \beta_i^k x_4), \beta_i^{3k} r),$$

for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. By proceeding the same procedure in Theorem 11.49, we can prove the mapping $H : X \to Y$ is cubic and its satisfies the functional equation (11.1) by a fixed point alternative. Since H is unique fixed point of T in the set $\Delta = \{f \in \Lambda / d(f, H) < \infty\}$, H is a unique mapping such that

$$P_{\mu,v}(f(x) - H(x), r) \geq P'_{\mu,v}(\epsilon(x), kr), \tag{11.76}$$

for all $x \in X, r > 0$. Again using the fixed point alternative, we receive

$$\begin{aligned}
& d(f, H) \leq \frac{1}{1-L} d(f, Tf) \\
& \Rightarrow d(f, H) \leq \frac{L^{1-i}}{1-L} \\
& \Rightarrow P_{\mu,v}(f(x) - H(x), r) \geq P'_{\mu,v}\left(\epsilon(x) \frac{L^{1-i}}{1-L}, r\right),
\end{aligned} \tag{11.77}$$

This completes the proof of the theorem. □

Example 11.21 Let $f : X \to Y$ be a mapping for which there exists a function $\rho : X^4 \to Z$ with condition (11.65), for all $x_1, x_2, x_3, x_4 \in X$, $r > 0$, where $\beta_i = \begin{cases} 2 \;\; if \;\; i = 0; \\ \frac{1}{2} \;\; if \; i = 1; \end{cases}$ and Λ is the set such that

$$\Lambda = \{a \backslash a : X \to Y, a(0) = 0\},$$

and satisfying the inequality (11.66), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. If there exists $L = L[i]$ such that the function $x \to \epsilon(x)$ has the property (11.67), for all $x \in X, r > 0$, then there exists a unique cubic mapping $H : X \to Y$ satisfying the functional equations (11.1) and (11.68), for all $x \in X, r > 0$.

Proposition 11.22 *Let $f : X \to Y$ be a mapping for which there exists a function $\rho : X^4 \to Z$ with condition (11.65), for all $x_1, x_2, x_3, x_4 \in X$, $r > 0$, where $\beta_i = \begin{cases} n^2 \ if \ \ i = 0 \\ \frac{1}{n^2} \ if \ \imath = 1 \end{cases}$ and Λ is the set such that*

$$\Lambda = \{a|a : X \to Y, a(0) = 0\},$$

and satisfying the inequality (11.66), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. If there exists $L = L[i]$ such that the function $x \to \epsilon(x)$ has the property (11.67), for all $x \in X$, $r > 0$, then there exists a unique cubic mapping $H : X \to Y$ satisfying the functional equations (11.1) and (11.68), for all $x \in X$, $r > 0$.

Example 11.23 Let $f : X \to Y$ be a mapping for which there exists a mapping $\rho : X^4 \to Z$ with condition (11.65), for all $x_1, x_2, x_3, x_4 \in X$, $r > 0$, where $\beta_i = \begin{cases} 2^2 \ if \ \ i = 0 \\ \frac{1}{2^2} \ if \ \ i = 1 \end{cases}$ and Λ is the set such that

$$\Lambda = \{a \backslash a : X \to Y, a(0) = 0\},$$

and satisfying the inequality (11.66), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. If there exists $L = L[i]$ such that the function $x \to \epsilon(x)$ has the property (11.67), for all $x \in X$, $r > 0$, then there exists a unique cubic mapping $H : X \to Y$ satisfying the functional equations (11.1) and (11.68), for all $x \in X$, $r > 0$.

Proposition 11.24 *Let $f : X \to Y$ be a mapping for which there exists a function $\rho : X^4 \to Z$ with condition (11.65), for all $x_1, x_2, x_3, x_4 \in X$, $r > 0$, where $\beta_i = \begin{cases} n^3 \ if \ \ i = 0 \\ \frac{1}{n^3} \ if \ \imath = 1 \end{cases}$ and Λ is the set such that*

$$\Lambda = \{a \ a : X \to Y, a(0) = 0\},$$

and satisfying the inequality (11.66), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. If there exists $L = L[i]$ such that the function $x \to \epsilon(x)$ has the property (11.67), for all $x \in X$, $r > 0$, then there exists a unique cubic mapping $H : X \to Y$ satisfying the functional equations (11.1) and (11.68), for all $x \in X$, $r > 0$.

Proposition 11.25 *Let $f : X \to Y$ be a mapping for which there exists a function $\rho : X^4 \to Z$ with condition (11.65), for all $x_1, x_2, x_3, x_4 \in X$, $r > 0$, where $\beta_i = \begin{cases} n^4 \ if \ \ i = 0 \\ \frac{1}{n^4} \ if \ \imath = 1 \end{cases}$ and Λ is the set such that*

$$\Lambda = \{a \ a : X \to Y, a(0) = 0\},$$

and satisfying the inequality (11.66), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. If there exists $L = L[i]$ such that the function $x \to \epsilon(x)$ has the property (11.67), for all $x \in X$, $r > 0$, then there exists a unique cubic mapping $H : X \to Y$ satisfying the functional equations (11.1) and (11.68), for all $x \in X$, $r > 0$.

The following corollaries are the immediate consequences of Theorem 11.20, Propositions 11.22–11.25, concerning the stability of (11.1).

Corollary 11.26 *Suppose that a mapping $C : X \to Y$ satisfies the inequality (11.60), for all $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $H : X \to Y$ such that (11.61), for all $x \in X$ and $r > 0$.*

Proof Set

$$P_{\mu,v}(C(x_1, x_2, x_3, x_4)) \geq_{L^*} \begin{cases} P'_{\mu,v}(\tau, r), \\ P'_{\mu,v}\left(\tau \sum_{i=1}^{4} \parallel x_i \parallel^s, r\right), \quad s \neq 3 \\ P'_{\mu,v}\left(\tau \left(\sum_{i=1}^{4} \parallel x_i \parallel^{4s} + \prod_{i=1}^{4} \parallel x_i \parallel^s \right), r\right), \\ s \neq \frac{3}{4} \end{cases}$$

for all $x_1, x_2, x_3, x_4 \in X$. Then

$$\begin{aligned} P'_{\mu,v}(\rho(\beta_i^k x_1, \beta_i^k x_2, \beta_i^k x_3, \beta_i^k x_4), \beta_i^{3k} r) &= \begin{cases} P'_{\mu,v}(\tau, \beta_i^{3k} r), \\ P'_{\mu,v}\left(\tau \sum_{i=1}^{4} \parallel x_i \parallel^s, \beta_i^{(3-s)k} r)\right), \\ P'_{\mu,v}\left(\tau \left(\sum_{i=1}^{4} \parallel x_i \parallel^s + \prod_{i=1}^{4} \parallel x_i \parallel^{4s} \right), \right. \\ \left. \beta_i^{(3-4s)k} r)\right) \end{cases} \\ &= \begin{cases} \to 1_{L^*} \ as \ k \to \infty \\ \to 1_{L^*} \ as \ k \to \infty \\ \to 1_{L^*} \ as \ k \to \infty \end{cases} \end{aligned} \tag{11.78}$$

i.e., (11.65) holds. Since we have

$$\epsilon(x) = \rho\left(\frac{x}{n}, 0, 0, 0\right),$$

$$P'_{\mu,v}\left(L \frac{1}{\beta_i^3} \epsilon(\beta_i x), r\right) \geq_{L^*} P'_{\mu,v}(\epsilon(x), r),$$

for all $x \in X$ and $r > 0$. Hence

$$P_{\mu,v}(\epsilon(x), r) = P'_{\mu,v}\left(\rho\left(\frac{x}{n}, 0, 0, 0\right), r\right),$$

Now

$$P'_{\mu,v}\left(\frac{1}{\beta_i^3}\rho(\psi_i x), r\right) = \left\{\begin{array}{l} P'_{\mu,v}\left(\frac{\tau}{\beta_i^3}, r\right) \\ P'_{\mu,v}\left(\frac{\tau\|x\|^s\beta_i^s}{\beta_i^3 n^s}, r\right) \\ P'_{\mu,v}\left(\frac{\tau\|x\|^s\beta_i^{4s}}{\beta_i^3 n^{4s}}, r\right) \end{array}\right\} = \left\{\begin{array}{l} \rho(x), \beta_i^3 \\ \rho(x), \beta_i^{3-s}\rho(x) \\ \rho(x), \beta_i^{3-4s}, \end{array}\right\}$$

for all $x \in X$. Now setting
$L = n^{-3}\ \ if\ \ i = 0\ \ and\ \ L = n^3\ \ if\ \ i = 1.$
$L = n^{s-3}\ \ for\ \ s < 3\ \ if\ \ i = 1\ \ and\ \ L = n^{3-s}\ \ for\ \ s > 3\ \ if\ \ i = 0.$
$L = n^{4s-3}\ \ for\ \ s < \frac{3}{4}\ \ if\ \ i = 1\ \ and\ \ L = n^{3-4s}\ \ for\ \ s > \frac{3}{4}\ \ if\ \ i = 0.$

Case 1. $L = n^{-3}\ \ if\ \ i = 0$

$$\begin{aligned} P_{\mu,v}(f(x) - H(x), r) \geq_{L^*} P'_{\mu,v}\left(\epsilon(x), \frac{L^{1-i}}{1-L}r\right) \\ = P'_{\mu,v}\left(\tau, \frac{n^{-3}}{1-n^{-3}}r\right) \\ = P'_{\mu,v}\left(\tau, \frac{r}{(n^3-1)}\right). \end{aligned}$$

Case 2. $L = n^3\ \ if\ \ i = 1$

$$\begin{aligned} P_{\mu,v}(f(x) - H(x), r) \geq_{L^*} P'_{\mu,v}\left(\epsilon(x), \frac{L^{1-i}}{1-L}r\right) \\ = P'_{\mu,v}\left(\tau, \frac{1}{1-n^{-3}}r\right) \\ = P'_{\mu,v}\left(\tau, \frac{r}{(1-n^3)}\right). \end{aligned}$$

Case 3. $L = n^{3-s}\ \ for\ \ s > 3\ \ if\ \ i = 0$

$$\begin{aligned} P_{\mu,v}(f(x) - H(x), r) \geq_{L^*} P'_{\mu,v}\left(\epsilon(x), \frac{L^{1-i}}{1-L}r\right) \\ = P'_{\mu,v}\left(\frac{\tau \parallel x \parallel^s}{n^s}, \frac{n^{3-s}}{1-n^{3-s}}r\right) \\ = P'_{\mu,v}\left(\tau \parallel x \parallel^s, \frac{n^{s+3}r}{n^s - n^3}\right). \end{aligned}$$

Case 4. $L = 2^{s-3}\ \ for\ \ s < 3\ \ if\ \ i = 1$

$$P_{\mu,v}(f(x)-H(x),r) \geq_{L^*} P'_{\mu,v}\left(\epsilon(x), \frac{L^{1-i}}{1-L}r\right)$$
$$= P'\mu, v\left(\frac{\tau \parallel x \parallel^s}{n^s}, \frac{1}{1-n^{3-s}r}\right)$$
$$= P'_{\mu,v}\left(\tau \parallel x \parallel^s, \frac{n^{s+3}r}{n^3-n^s}\right).$$

Case 5. $L = n^{3-4s} \quad for \quad s > \frac{3}{4} \quad if \quad i = 0$

$$P_{\mu,v}(f(x)-H(x),r) \geq_{L^*} P'_{\mu,v}\left(\epsilon(x), \frac{L^{1-i}}{1-L}r\right)$$
$$= P'_{\mu,v}\left(\frac{\tau \parallel x \parallel^{4s}}{n^{4s}}, \frac{n^{3-4s}}{1-n^{3-4s}}r\right)$$
$$= P'_{\mu,v}\left(\tau \parallel x \parallel^{4s}, \frac{n^{4s+3}r}{n^{4s}-n^3}\right).$$

Case 6. $L = 2^{3-3s} \quad for \quad s < \frac{n^{3s-3}}{1-n^{3s-3}} \quad if \quad i = 1$

$$P_{\mu,v}(f(x)-H(x),r) \geq_{L^*} P'_{\mu,v}\left(\epsilon(x), \frac{L^{1-i}}{1-L}r\right)$$
$$= P'_{\mu,v}\left(\frac{\tau \parallel x \parallel^{4s}}{n^{4s}}, \frac{1}{1-n^{4s-3}}r\right)$$
$$= P'_{\mu,v}\left(\tau \parallel x \parallel^{4s}, \frac{n^{4s+3}r}{n^3-n^{4s}}\right).$$

Hence the proof is complete. □

Remark: Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.60), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $H : X \to Y$ such that

$$P_{\mu,v}(f(x)-H(x),r) \geq_{L*} \begin{cases} P_{\mu,v}(\tau, r|7|) \\ P_{\mu,v}(\tau\|x\|^s, r|2^3-2^s|) \\ P_{\mu,v}(\tau\|x\|^{4s}, r|2^3-2^{4s}|), \end{cases}$$

for all $x \in X$ and $r > 0$.

Corollary 11.27 *Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.60), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $H : X \to Y$ such that (11.62), for all $x \in X$ and $r > 0$.*

Example 11.28 Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.60), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $H : X \to Y$ such that

$$P_{\mu,v}(f(x) - H(x), r) \geq_{L*} \begin{cases} P_{\mu,v}(\tau, r|63|) \\ P_{\mu,v}(\tau\|x\|^s, r|2^6 - 2^{2s}|) \\ P_{\mu,v}(\tau\|x\|^{4s}, r|2^6 - 2^{8s}|) \end{cases}$$

for all $x \in X$ and for $r > 0$.

Corollary 11.29 *Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.60), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $H : X \to Y$ such that (11.63), for all $x \in X$ and $r > 0$.*

Corollary 11.30 *Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.60), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $H : X \to Y$ such that (11.64), for all $x \in X$ and $r > 0$.*

11.5 Stability of 4-Dimensional Cubic Functional Equation—Direct Method

Proposition 11.31 *Let $\beta \in \{-1, 1\}$. Let $\chi : X^4 \to [0, \infty)$ be a function with $0 < \left(\frac{d}{n^3}\right) < 1$,*

$$F'(\chi(n^{\beta k}x_1, n^{\beta k}x_2,\ n^{\beta k}x_3, n^{\beta k}x_4), n^{3\beta k}r) \geq F'(d^\beta \chi(x, 0, 0, 0), r),$$

for all $x \in X$ and all $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} F'(\chi(n^{\beta k}x_1, n^{\beta k}x_2, n^{\beta k}x_3, n^{\beta k}x_4), n^{3\beta k}r) = 1,$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$F(C(x_1, x_2, x_3, x_4), r) \geq F'(\chi(x_1, x_2, x_3, x_4), r),$$

for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit $H(x) = F - \frac{f(n^{\beta k}x)}{n^{3\beta k}}$ exists for all $x \in X$ and the mapping $H : X \to Y$ is a unique cubic mapping such that $F(f(x) - H(x), r) \geq F'(\chi(x, 0, 0, 0), r|n^3 - d|)$, for all $x \in X$ and $r > 0$.

Remark: Let $\beta \in \{-1, 1\}$. Let $\chi : X^4 \to [0, \infty)$ be a function with $0 < \left(\frac{d}{2^3}\right) < 1$,

$$F'(\chi(2^{\beta k}x_1, 2^{\beta k}x_2,\ 2^{\beta k}x_3, 2^{\beta k}x_4), 2^{3\beta k}r) \geq F'(d^{\beta}\chi(x, 0, 0, 0), r)$$

for all $x \in X, r > 0, d > 0$ and

$$\lim_{k\to\infty} F'(\chi(2^{\beta k}x_1, 2^{\beta k}x_2, 2^{\beta k}x_3, 2^{\beta k}x_4), 2^{3\beta k}r) = 1$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$F(C(x_1, x_2, x_3, x_4), r) \geq F'(\chi(x_1, x_2, x_3, x_4), r)$$

for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit $H(x) = F - \frac{f(2^{\beta k}x)}{2^{3\beta k}}$ exists for all $x \in X$ and the mapping $H : X \to Y$ is a unique cubic mapping such that $F(f(x) - H(x), r) \geq F'(\chi(x, 0, 0, 0), r|2^3 - d|)$, for all $x \in X$ and $r > 0$.

Proposition 11.32 *Let $\beta \in \{-1, 1\}$. Let $\chi : X^4 \to [0, \infty)$ be a function with $0 < \left(\frac{d}{n^6}\right) < 1$,*

$$F'(\chi(n^{2\beta k}x_1, n^{2\beta k}x_2,\ n^{2\beta k}x_3, n^{2\beta k}x_4), n^{6\beta k}r) \geq F'(d^{\beta}\chi(0, x, 0, 0), r),$$

for all $x \in X, r > 0, d > 0$ and

$$\lim_{k\to\infty} F'(\chi(n^{2\beta k}x_1, n^{2\beta k}x_2, n^{2\beta k}x_3, n^{2\beta k}x_4), n^{6\beta k}r) = 1$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$F(C(x_1, x_2, x_3, x_4), r) \geq F'(\chi(x_1, x_2, x_3, x_4), r),$$

for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit $H(x) = F - \frac{f(n^{2\beta k}x)}{n^{6\beta k}}$ exists for all $x \in X$ and the mapping $H : X \to Y$ is a unique cubic mapping such that $F(f(x) - H(x), r) \geq F'(\chi(0, x, 0, 0), r|n^6 - d|)$, for all $x \in X$ and $r > 0$.

Remark: Let $\beta \in \{-1, 1\}$. Let $\chi : X^4 \to [0, \infty)$ be a function with $0 < \left(\frac{d}{2^6}\right) < 1$,

$$F'(\chi(2^{2\beta k}x_1, 2^{2\beta k}x_2,\ 2^{2\beta k}x_3, 2^{2\beta k}x_4), 2^{6\beta k}r) \geq F'(d^{\beta}\chi(0, x, 0, 0), r),$$

for all $x \in X, r > 0, d > 0$ and

$$\lim_{k\to\infty} F'(\chi(2^{2\beta k}x_1, 2^{2\beta k}x_2, 2^{2\beta k}x_3, 2^{2\beta k}x_4), 2^{6\beta k}r) = 1,$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$F(C(x_1, x_2, x_3, x_4), r) \geq F'(\chi(x_1, x_2, x_3, x_4), r),$$

for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit $H(x) = F - \frac{f(2^{2\beta k}x)}{2^{6\beta k}}$ exists for all $x \in X$ and the mapping $H : X \to Y$ is a unique cubic mapping such that $F(f(x) - H(x), r) \geq F'(\chi(0, x, 0, 0), r|2^6 - d|)$, for all $x \in X$ and $r > 0$.

Proposition 11.33 *Let $\beta \in \{-1, 1\}$. Let $\chi : X^4 \to [0, \infty)$ be a function with $0 < \left(\frac{d}{n^9}\right) < 1$,*

$$F'(\chi(n^{3\beta k}x_1, n^{3\beta k}x_2,\ n^{3\beta k}x_3, n^{3\beta k}x_4), n^{9\beta k}r) \geq F'(d^\beta \chi(0, 0, x, 0), r),$$

for all $x \in X$, $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} F'(\chi(n^{3\beta k}x_1, n^{3\beta k}x_2, n^{3\beta k}x_3, n^{3\beta k}x_4), n^{9\beta k}r) = 1,$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$F(C(x_1, x_2, x_3, x_4), r) \geq F'(\chi(x_1, x_2, x_3, x_4), r)$$

for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit $H(x) = F - \frac{f(n^{3\beta k}x)}{n^{9\beta k}}$ exists for all $x \in X$ and the mapping $H : X \to Y$ is a unique cubic mapping such that $F(f(x) - H(x), r) \geq F'(\chi(0, 0, x, 0), r|n^9 - d|)$, for all $x \in X$ and $r > 0$.

Proposition 11.34 *Let $\beta \in \{-1, 1\}$. Let $\chi : X^4 \to [0, \infty)$ be a function with $0 < \left(\frac{d}{n^{12}}\right) < 1$,*

$$F'(\chi(n^{4\beta k}x_1, n^{4\beta k}x_2,\ n^{4\beta k}x_3, n^{4\beta k}x_4), n^{12\beta k}r) \geq F'(d^\beta \chi(0, 0, 0, x), r),$$

for all $x \in X$ and all $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} F'(\chi(n^{4\beta k}x_1, n^{4\beta k}x_2, n^{4\beta k}x_3, n^{4\beta k}x_4), n^{12\beta k}r) = 1,$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$F(C(x_1, x_2, x_3, x_4), r) \geq F'(\chi(x_1, x_2, x_3, x_4), r)$$

for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit $H(x) = F - \frac{f(n^{4\beta k}x)}{n^{12\beta k}}$ exists for all $x \in X$ and the mapping $H : X \to Y$ is a unique cubic mapping such that $F(f(x) - H(x), r) \geq F'(\chi(0, 0, 0, x), r|n^{12} - d|)$, for all $x \in X$ and $r > 0$.

The following corollaries are the immediate consequence of Propositions 11.31–11.34 respectively, concerning the stability of (11.1).

Corollary 11.35 *Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$F(C(x_1, x_2, x_3, x_4), r) \geq \begin{cases} F'(\epsilon, r) \\ F'\left(\epsilon \sum_{i=1}^{4} \parallel x_i \parallel^s, r\right), \ s \neq 3 \\ F'\left(\epsilon \left(\sum_{i=1}^{4} \parallel x_i \parallel^{4s} + \prod_{i=1}^{4} \parallel x_i \parallel^s \right), r\right), \ s \neq \frac{3}{4} \end{cases} \tag{11.79}$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where ϵ, s are constants. Then there exists a unique cubic mapping $H : X \to Y$ such that

$$F(f(x) - H(x) \geq \begin{cases} F'(\epsilon, r|n^3 - 1|) \\ F'(\epsilon \parallel x \parallel^s, r|n^3 - n^s|) \\ F'(\epsilon) \parallel x \parallel^{4s}, r|n^3 - n^{4s}|), \end{cases} \tag{11.80}$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$.

Example 11.36 Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.79) all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where ϵ, s are constants. Then there exists a unique cubic mapping $H : X \to Y$ such that, $\forall\, x \in X,\ r > 0$

$$F\left(f(x) - G(x), r\right) \geq \begin{cases} F'\left(\epsilon, r|7|\right) \\ F'\left(\epsilon \|x\|^s, r|2^3 - 2^s|\right) \\ F'\left(\epsilon \|x\|^{4s}, 4r|2^3 - 2^{4s}|\right). \end{cases}$$

Corollary 11.37 *Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.79), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where ϵ, s are constants. Then there exists a unique cubic mapping $H : X \to Y$ such that*

$$F(f(x) - H(x) \geq \begin{cases} F'(\epsilon, r|n^6 - 1|) \\ F'(\epsilon \parallel x \parallel^s, r|n^6 - n^{2s}|) \\ F'(\epsilon) \parallel x \parallel^{4s}, r|n^6 - n^{8s}|), \end{cases} \tag{11.81}$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$.

Example 11.38 Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.79), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where ϵ, s are constants. Then there exists a unique cubic mapping $H : X \to Y$, such that, $\forall\, x \in X,\ r > 0$

$$F\left(f(x) - G(x), r\right) \geq \begin{cases} F'\left(\epsilon, r|63|\right) \\ F'\left(\epsilon \|x\|^s, r|2^6 - 2^{2s}|\right) \\ F'\left(\epsilon \|x\|^{4s}, r|2^6 - 2^{8s}|\right). \end{cases}$$

Corollary 11.39 *Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.79) for all $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$, where ϵ, s are constants. Then there exists a unique cubic mapping $H : X \to Y$, such that*

$$F(f(x) - H(x) \geq \begin{cases} F'(\epsilon, r|n^9 - 1|) \\ F'(\epsilon \parallel x \parallel^s, r|n^9 - n^{3s}|) \\ F'(\epsilon) \parallel x \parallel^{4s}, r|n^9 - n^{12s}|), \end{cases} \tag{11.82}$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$.

Corollary 11.40 *Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.79), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where ϵ, s are constants. Then there exists a unique cubic mapping $H : X \to Y$ such that*

$$F(f(x) - H(x) \geq \begin{cases} F'(\epsilon, r|n^{12} - 1|) \\ F'(\epsilon \parallel x \parallel^s, r|n^{12} - n^{4s}|) \\ F'(\epsilon) \parallel x \parallel^{4s}, r|n^{12} - n^{16s}|), \end{cases} \tag{11.83}$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $r > 0$.

11.6 Stability of the 4-Dimensional Cubic Functional Equation—Fixed Point Method

Proposition 11.41 *Let $f : X \to Y$ be a mapping for which there exists a function $\chi : X^4 \to Z$ with condition*

$$\lim_{k\to\infty} F'(\chi(\psi_i^k x_1, \psi_i^k x_2, \psi_i^k x_3, \psi_i^k x_4), \psi_i^{3k} r) = 1, \tag{11.84}$$

where $\psi_i = \begin{cases} n & if \;\; i = 0 \\ \frac{1}{n} & if \;\; \imath = 1 \end{cases}$ and Ω is the set such that $\Omega = \{p | p : X \to Y, p(0) = 0\}$, for all $x_1, x_2, x_3, x_4 \in X$ satisfying the inequality

$$F(C(x_1, x_2, x_3, x_4), r) \geq F'(\chi(x_1, x_2, x_3, x_4), r), \tag{11.85}$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. If there exists $L = L[i]$ such that the function $x \to \rho(x)$ has the property

$$F'\left(L\frac{1}{\psi_i^3}\rho(\psi_i x), r\right) = F'(\rho(x), r), \tag{11.86}$$

for all $x \in X$, then there exists a unique cubic mapping $H : X \to Y$ satisfying the functional equation (11.1) and

$$\| F(f(x) - H(x), r) \geq F'\left(\frac{L^{1-i}}{1-L}\rho(x), r\right), \tag{11.87}$$

for all $x \in X$.

Proposition 11.42 *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\chi : X^4 \to Z$ *with condition (11.84) where* $\psi_i = \begin{cases} 2 & if \quad i = 0; \\ \frac{1}{2} & if \quad i = 1; \end{cases}$ *and* Ω *is the set such that* $\Omega = \{p \backslash p : X \to Y, p(0) = 0\}$, *for all* $x_1, x_2, x_3, x_4 \in X$ *and satisfying the inequality (11.85), for all* $x_1, x_2, x_3, x_4 \in X$ *and* $r > 0$. *If there exists* $L = L[i]$ *such that the function* $x \to \rho(x)$ *has the property (11.86), for all* $x \in X$, *then there exists a unique cubic mapping* $H : X \to Y$ *satisfying the functional equations (11.1) and (11.87), for all* $x \in X$.

Proposition 11.43 *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\chi : X^4 \to Z$ *with condition (11.84) where* $\psi_i = \begin{cases} n^2 & if \quad i = 0 \\ \frac{1}{n^2} & if \; \imath = 1 \end{cases}$ *and* Ω *is the set such that* $\Omega = \{p|p : X \to Y, p(0) = 0\}$, *for all* $x_1, x_2, x_3, x_4 \in X$ *and satisfying the inequality (11.85), for all* $x_1, x_2, x_3, x_4 \in X$ *and* $r > 0$. *If there exists* $L = L[i]$ *such that the function* $x \to \rho(x)$ *has the property (11.86), for all* $x \in X$, *then there exists a unique cubic mapping* $H : X \to Y$ *satisfying the functional equations (11.1) and (11.87), for all* $x \in X$.

Example 11.44 Let $f : X \to Y$ be a mapping for which there exists a function $\chi : X^4 \to Z$ with condition (11.84) where $\psi_i = \begin{cases} 2^2 & if \quad i = 0; \\ \frac{1}{2^2} & if \quad i = 1; \end{cases}$ and Ω is the set such that $\Omega = \{p \backslash p : X \to Y, p(0) = 0\}$, for all $x_1, x_2, x_3, x_4 \in X$ and satisfying the inequality (11.85), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. If there exists $L = L[i]$ such that the function $x \to \rho(x)$ has the property (11.86), for all $x \in X$, then there exists a unique cubic mapping $H : X \to Y$ satisfying the functional equations (11.1) and (11.87), for all $x \in X$.

Proposition 11.45 *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\chi : X^4 \to Z$ *with condition (11.84) where* $\psi_i = \begin{cases} n^3 & if \quad i = 0 \\ \frac{1}{n^3} & if \; \imath = 1 \end{cases}$ *and* Ω *is the set such that* $\Omega = \{p|p : X \to Y, p(0) = 0\}$, *for all* $x_1, x_2, x_3, x_4 \in X$ *and satisfying the inequality (11.85), for all* $x_1, x_2, x_3, x_4 \in X$ *and* $r > 0$. *If there exists* $L = L[i]$ *such that the function* $x \to \rho(x)$ *has the property (11.86), for all* $x \in X$, *then there exists a unique cubic mapping* $H : X \to Y$ *satisfying the functional equations (11.1) and (11.87), for all* $x \in X$.

Proposition 11.46 *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\chi : X^4 \to Z$ *with condition (11.84) where* $\psi_i = \begin{cases} n^4 & if \quad i = 0 \\ \frac{1}{n^4} & if \; \imath = 1 \end{cases}$ *and* Ω *is the set such that* $\Omega = \{p|p : X \to Y, p(0) = 0\}$, *for all* $x_1, x_2, x_3, x_4 \in X$ *and satisfying the inequality (11.85), for all* $x_1, x_2, x_3, x_4 \in X$ *and* $r > 0$. *If there exists* $L = L[i]$ *such that the function* $x \to \rho(x)$ *has the property (11.86), for all* $x \in X$, *then there* *exists a unique cubic mapping* $H : X \to Y$ *satisfying the functional equations (11.1) and (11.87), for all* $x \in X$.

The following corollaries are the immediate consequence of Propositions 11.41–11.46 respectively, concerning the stability of (11.1).

Corollary 11.47 *Suppose that a mapping* $f : X \to Y$ *satisfies the inequality (11.79), for all* $x_1, x_2, x_3, x_4 \in X$ *and* $r > 0$, *where* τ, s *are the constants with* $\tau > 0$. *Then there exists a unique cubic mapping* $H : X \to Y$ *such that (11.80), for all* $x \in X$ *and* $r > 0$.

Remark: Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.79), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are the constants with $\tau > 0$. Then there exists a unique cubic mapping $H : X \to Y$, such that

$$F\left(f(x) - H(x), r\right) \geq \begin{cases} F'\left(\epsilon, r|7|\right) \\ F'\left(\epsilon\|x\|^s, r|2^3 - 2^s|\right) \\ F'\left(\epsilon\|x\|^{4s}, r|2^3 - 2^{4s}|\right), \end{cases}$$

for all $x \in X$ and $r > 0$.

Corollary 11.48 *Suppose that a mapping* $f : X \to Y$ *satisfies the inequality (11.79), for all* $x_1, x_2, x_3, x_4 \in X$ *and* $r > 0$, *where* τ, s *are the constants with* $\tau > 0$. *Then there exists a unique cubic mapping* $H : X \to Y$ *such that (11.81), for all* $x \in X$ *and* $r > 0$.

Remark: Suppose that a mapping $f : X \to Y$ satisfies the inequality (11.79), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are the constants with $\tau > 0$. Then there exists a unique cubic mapping $H : X \to Y$, such that, for all $x \in X$, $r > 0$

$$F\left(f(x) - H(x), r\right) \geq \begin{cases} F'\left(\epsilon, r|63|\right) \\ F'\left(\epsilon\|x\|^s, r|2^6 - 2^{2s}|\right) \\ F'\left(\epsilon\|x\|^{4s}, r|2^6 - 2^{8s}|\right). \end{cases}$$

Corollary 11.49 *Suppose that a mapping* $f : X \to Y$ *satisfies the inequality (11.79), for all* $x_1, x_2, x_3, x_4 \in X$ *and* $r > 0$, *where* τ, s *are the constants with* $\tau > 0$. *Then there exists a unique cubic mapping* $H : X \to Y$, *such that (11.82), for all* $x \in X$ *and* $r > 0$.

Corollary 11.50 *Suppose that a function* $f : X \to Y$ *satisfies the inequality (11.79), for all* $x_1, x_2, x_3, x_4 \in X$ *and* $r > 0$, *where* τ, s *are the constants with* $\tau > 0$. *Then there exists a unique cubic function* $H : X \to Y$, *such that (11.83), for all* $x \in X$ *and* $r > 0$.

Chapter 12
n-Type Cubic Functional Equations

12.1 General Solution the Functional Equation

Suppose that E and F are real normed spaces and that F is a complete normed space, $f : E \to F$ is a mapping such that for each fixed $x \in E$ the function $t \to f(tx)$ is continuous on $\mathbb{R}$. If there exist a $\epsilon > 0$ and $p \in [0, 1)$ such that

$$\| f(x+y) - f(x) - f(y) \| \leq \epsilon \left(||x||^p + ||y||^p\right),$$

for all $x, y \in E$ then there exists a unique linear mapping $T : E \to F$, such that

$$\| \, ||f(x) - T(x)|| \leq \frac{\epsilon ||x||^p}{(1 - 2^{p-1})},$$

for all $x \in E$. Now setting

$$\begin{aligned}
& f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4) + f(nx_1 - n^2x_2 + n^3x_3 + n^4x_4) \\
& + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4) + f(nx_1 + n^2x_2 + n^3x_3 - n^4x_4) \\
& = -2\,[\, f(nx_1 + n^2x_2 + n^3x_3) + f(nx_1 + n^2x_2 + n^4x_4) + f(nx_1 + n^3x_3 \\
& + n^4x_4) + f(n^2x_2 + n^3x_3 + n^4x_4)\,] \\
& + 6\left[f(nx_1 + n^2x_2) + f(nx_1 + n^3x_3) + f(nx_1 + n^4x_4) + f(n^2x_2 + n^3x_3)\right. \\
& \left. + f(n^2x_2 + n^4x_4) + f(n^3x_3 + n^4x_4)\,\right] \\
& - 10\left[n^3 f(x_1) + n^6 f(x_2) + n^9 f(x_3) + n^{12} f(x_4)\right].
\end{aligned} \tag{12.1}$$

In this chapter, we discuss the solution and stability of the n-type Cubic functional equation (12.1) in random normed space (briefly, RN-space) and intuitionistic fuzzy normed space with the help of direct and fixed point method.

H. Dutta et al., *Stability of Some Advanced Functional Equations in Various Spaces*, Studies in Systems, Decision and Control 479,
https://doi.org/10.1007/978-3-031-33704-8_12

12.2 Solution of Cubic Functional Equation

Theorem 12.1 *An odd mapping $f : X \to Y$ satisfy the functional equation (10.3), for all $x, y \in X$, if and only if $f : X \to Y$ satisfies the fundamental equation (12.1), for all $x_1, x_2, x_3, x_4 \in X$.*

Proof Let $f : X \to Y$ satisfies the functional equation (10.3). Replacing (x, y) by $(0, 0)$ in (10.3) we get $f(0) = 0$. Replacing (x, y) by (x, x) in (12.1), we have

$$f(2x) = 2^3 f(x) \text{ and } f(3x) = 3^3 f(x), \tag{12.2}$$

for all $x \in X$. In general for any positive integer a, we have

$$f(ax) = a^3 f(x), \tag{12.3}$$

for all $x \in X$. It is easy to verify from (12.3) that

$$f(a^2x) = a^6 f(x) \text{ and } f(a^3x) = a^9 f(x), \tag{12.4}$$

for all $x \in X$. Replacing (x, y) by $(nx_1 + n^2x_2,\ n^3x_3 - n^4x_4)$ in (10.3), we get

$$\begin{aligned}&f(2nx_1 + 2n^2x_2 + n^3x_3 - n^4x_4) + f(2nx_1 + 2n^2x_2 - n^3x_3\\&+n^4x_4) + 2f(-nx_1 - n^2x_2 - n^3x_3 + n^4x_4) + 2f(-nx_1 - n^2x_2 +\\&n^3x_3 - n^4x_4) = 12f(nx_1 + n^2x_2),\end{aligned} \tag{12.5}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(nx_1 + n^3x_3,\ n^2x_2 - n^4x_4)$ in (10.3), we obtain

$$\begin{aligned}&f(2nx_1 + n^2x_2 + 2n^3x_3 - n^4x_4) + f(2nx_1 - n^2x_2 + 2n^3x_3\\&+n^4x_4) + 2f(-nx_1 - n^2x_2 - n^3x_3 + n^4x_4) + 2f(-nx_1 + n^2x_2\\&-n^3x_3 - n^4x_4) = 12f(nx_1 + n^3x_3),\end{aligned} \tag{12.6}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(nx_1 + n^4x_4,\ n^2x_2 - n^3x_3)$ in (10.3), we have

$$\begin{aligned}&f(2nx_1 + n^2x_2 - n^3x_3 + 2n^4x_4) + f(2nx_1 - n^2x_2 + n^3x_3\\&+2n^4x_4) + 2f(-nx_1 - n^2x_2 + n^3x_3 - n^4x_4) + 2f(-nx_1 + n^2x_2\\&-n^3x_3 - n^4x_4) = 12f(nx_1 + n^4x_4),\end{aligned} \tag{12.7}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(n^2x_2 + n^3x_3,\ nx_1 - n^4x_4)$ in (10.3), we obtain

$$f(nx_1 + 2n^2x_2 + n^3x_3 - n^4x_4) + f(-nx_1 + n^2x_2 + 2n^3x_3 + n^4x_4) + 2f(-nx_1 - n^2x_2 - n^3x_3 + n^4x_4) + 2f(nx_1 - n^2x_2 - n^3x_3 - n^4x_4) = 12f(n^2x_2 + n^3x_3), \tag{12.8}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(n^2x_2 + n^4x_4,\ nx_1 - n^3x_3)$ in (10.3), we get

$$f(nx_1 + 2n^2x_2 - n^3x_3 + 2n^4x_4) + f(-nx_1 + 2n^2x_2 + n^3x_3 + 2n^4x_4) + 2f(-nx_1 - n^2x_2 + n^3x_3 - n^4x_4) + 2f(nx_1 - n^2x_2 - n^3x_3 - n^4x_4) = 12f(n^2x_2 + n^4x_4), \tag{12.9}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(n^3x_3 + n^4x_4,\ nx_1 - n^2x_2)$ in (10.3), we get

$$f(nx_1 - n^2x_2 + 2n^3x_3 + 2n^4x_4) + f(-nx_1 + n^2x_2 + 2n^3x_3 + 2n^4x_4) + 2f(-nx_1 + n^2x_2 - n^3x_3 - n^4x_4) + 2f(nx_1 - n^2x_2 - n^3x_3 - n^4x_4) = 12f(n^3x_3 + n^4x_4), \tag{12.10}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding (12.5), (12.6), (12.7), (12.8), (12.9) and (12.10), we get

$$\begin{aligned}&12f(nx_1 + n^2x_2) + 12f(nx_1 + n^3x_3) + 12f(nx_1 + n^4x_4) + 12f(n^2x_2 + n^3x_3) + 12f(n^2x_2 + n^4x_4) + 12f(n^3x_3 + n^4x_4) = f(2nx_1 + 2n^2x_2 + n^3x_3 - n^4x_4) + f(2nx_1 + 2n^2x_2 - n^3x_3 + n^4x_4) + 2f(-nx_1 - n^2x_2 - n^3x_3 + n^4x_4) + 2f(-nx_1 - n^2x_2 + n^3x_3 - n^4x_4) + f(2nx_1 + n^2x_2 + 2n^3x_3 - n^4x_4) + f(2nx_1 - n^2x_2 + 2n^3x_3 + n^4x_4) + 2f(-nx_1 - n^2x_2 - n^3x_3 + n^4x_4) + 2f(-nx_1 + n^2x_2 - n^3x_3 - n^4x_4) + f(2nx_1 + n^2x_2 - n^3x_3 + 2n^4x_4) + f(2nx_1 - n^2x_2 + n^3x_3 + 2n^4x_4) + 2f(-nx_1 - n^2x_2 + n^3x_3 - n^4x_4) + 2f(-nx_1 + n^2x_2 - n^3x_3 - n^4x_4) + f(nx_1 + 2n^2x_2 + 2n^3x_3 - n^4x_4) + f(-nx_1 + 2n^2x_2 + 2n^3x_3 + n^4x_4) + 2f(-nx_1 - n^2x_2 - n^3x_3 + n^4x_4) + 2f(nx_1 - n^2x_2 - n^3x_3 - n^4x_4) + f(nx_1 + 2n^2x_2 - n^3x_3 + 2n^4x_4) + f(-nx_1 + 2n^2x_2 + n^3x_3 + 2n^4x_4) + 2f(-nx_1 - n^2x_2 + n^3x_3 - n^4x_4) + 2f(nx_1 - n^2x_2 - n^3x_3 - n^4x_4) + f(nx_1 - n^2x_2 + 2n^3x_3 + 2n^4x_4) + f(-nx_1 + n^2x_2 + 2n^3x_3 + 2n^4x_4) + 2f(-nx_1 + n^2x_2 - n^3x_3 - n^4x_4) + 2f(nx_1 - n^2x_2 - n^3x_3 - n^4x_4),\end{aligned} \tag{12.11}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(nx_1, 2n^2x_2 + n^3x_3 - n^4x_4)$ in (10.3), we obtain

$$f(2nx_1 + 2n^2x_2 + n^3x_3 - n^4x_4) = 2f(nx_1 + 2n^2x_2 + n^3x_3 - n^4x_4) + 2f(nx_1 - 2n^2x_2 - n^3x_3 + n^4x_4) + f(-2nx_1 + 2n^2x_2 + n^3x_3 - n^4x_4) + 12f(nx_1), \tag{12.12}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(2nx_1 + 2n^2x_2 - n^3x_3 + n^4x_4)$ on both sides of (12.12) we get

$$f(2nx_1 + 2n^2x_2 + n^3x_3 - n^4x_4) + f(2nx_1 + 2n^2x_2 - n^3x_3 + n^4x_4) = 2f(nx_1 + 2n^2x_2 + n^3x_3 - n^4x_4) + 2f(nx_1 - 2n^2x_2 - n^3x_3 + n^4x_4) + 2f(-2nx_1 + n^2x_2 + n^3x_3 - n^4x_4) + 2f(2nx_1 + n^2x_2 - n^3x_3 + n^4x_4) + 12f(nx_1) + 12f(n^2x_2), \tag{12.13}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(n^3x_3, 2nx_1 + n^2x_2)$ in (10.3), we obtain

$$f(2nx_1 + n^2x_2 + 2n^3x_3 - n^4x_4) = 2f(2nx_1 + n^2x_2 + n^3x_3 - n^4x_4) + 2f(-2nx_1 - n^2x_2 + n^3x_3 + n^4x_4) + f(2nx_1 + n^2x_2 - 2n^3x_3 - n^4x_4) + 12f(n^3x_3), \tag{12.14}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(2nx_1 - n^2x_2 + 2n^3x_3 + n^4x_4)$ on both sides of (12.14), we obtain

$$f(2nx_1 + n^2x_2 + 2n^3x_3 - n^4x_4) + f(2nx_1 - n^2x_2 + 2n^3x_3 = +n^4x_4) = 2f(2nx_1 + n^2x_2 + n^3x_3 - n^4x_4) + 2f(-2nx_1 - n^2x_2 + n^3x_3 + n^4x_4) + 2f(nx_1 + n^2x_2 - 2n^3x_3 - n^4x_4) + 2f(nx_1 - n^2x_2 + 2n^3x_3 + n^4x_4) + 12f(nx_1) + 12f(n^3x_3), \tag{12.15}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(n^4x_4, 2nx_1 + n^2x_2 - n^3x_3)$ in (10.3), we obtain

$$f(2nx_1 + n^2x_2 - n^3x_3 + 2n^4x_4) = 2f(2nx_1 + n^2x_2 - n^3x_3 + n^4x_4) + 2f(-2nx_1 - n^2x_2 + n^3x_3 + n^4x_4) + f(2nx_1 + n^2x_2 - n^3x_3 - 2n^4x_4) + 12f(n^4x_4), \tag{12.16}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(2nx_1 - n^2x_2 + n^3x_3 + 2n^4x_4)$ on both sides of (12.16) we get

$$f(2nx_1 + n^2x_2 - n^3x_3 + 2n^4x_4) + f(2nx_1 - n^2x_2 + n^3x_3 + 2n^4x_4) = 2f(2nx_1 + n^2x_2 - n^3x_3 + n^4x_4) + 2f(-2nx_1 - n^2x_2 + n^3x_3 + n^4x_4) + 2f(nx_1 + n^2x_2 - 2n^3x_3 - n^4x_4) + 2f(nx_1 - n^2x_2 + n^3x_3 + 2n^4x_4) + 12f(nx_1) + 12f(n^4x_4), \tag{12.17}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(n^2x_2, nx_1 + 2n^3x_3 - n^4x_4)$ in (10.3), we have

$$f(nx_1 + 2n^2x_2 + 2n^3x_3 - n^4x_4) = 2f(nx_1 + n^2x_2 + 2n^3x_3 - n^4x_4) + 2f(-nx_1 + n^2x_2 - 2n^3x_3 + n^4x_4) + f(nx_1 - 2n^2x_2 + 2n^3x_3 - n^4x_4) + 12f(n^2x_2), \tag{12.18}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(-nx_1 + 2n^2x_2 + 2n^3x_3 + n^4x_4)$ on both sides of (12.18) we get

$$f(nx_1 + 2n^2x_2 + 2n^3x_3 - n^4x_4) + f(-nx_1 + n^2x_2 + 2n^3x_3 + n^4x_4) = 2f(nx_1 + n^2x_2 + 2n^3x_3 - n^4x_4) + 2f(-nx_1 + n^2x_2 - 2n^3x_3 + n^4x_4) + 2f(nx_1 - 2n^2x_2 + n^3x_3 - n^4x_4) + 2f(-nx_1 + 2n^2x_2 + n^3x_3 + 2n^4x_4) + 12f(n^2x_2) + 12f(n^3x_3), \tag{12.19}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(n^2x_2, nx_1 - n^3x_3 + 2n^4x_4)$ in (10.3), we have

$$f(nx_1 + 2n^2x_2 - n^3x_3 + 2n^4x_4) = 2f(nx_1 + n^2x_2 - n^3x_3 + 2n^4x_4) + 2f(-nx_1 + n^2x_2 + n^3x_3 - 2n^4x_4) + f(nx_1 - 2n^2x_2 - n^3x_3 + 2n^4x_4) + 12f(n^2x_2), \tag{12.20}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(-nx_1 + 2n^2x_2 + n^3x_3 + 2n^4x_4)$ on both sides of (12.20) we obtain

$$f(nx_1 + 2n^2x_2 - n^3x_3 + 2n^4x_4) + f(-nx_1 + 2n^2x_2 + n^3x_3 + 2n^4x_4) = 2f(nx_1 + n^2x_2 - n^3x_3 + 2n^4x_4) + 2f(-nx_1 + n^2x_2 + n^3x_3 - 2n^4x_4) + 2f(nx_1 - 2n^2x_2 - n^3x_3 + n^4x_4) + 2f(-nx_1 + 2n^2x_2 + n^3x_3 + n^4x_4) + 12f(n^2x_2) + 12f(n^4x_4), \tag{12.21}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing (x, y) by $(n^3x_3, nx_1 - n^2x_2 + 2n^4x_4)$ in (10.3), we get

$$f(nx_1 - n^2x_2 + 2n^3x_3 + 2n^4x_4) = 2f(nx_1 - n^2x_2 + n^3x_3 + 2n^4x_4) + 2f(-nx_1 + n^2x_2 + n^3x_3 - 2n^4x_4) + f(nx_1 - n^2x_2 - 2n^3x_3 + 2n^4x_4) + 12f(n^3x_3), \tag{12.22}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding $f(-nx_1 + n^2x_2 + 2n^3x_3 + 2n^4x_4)$ on both sides of (12.22) we obtain

$$f(nx_1 - n^2x_2 + 2n^3x_3 + 2n^4x_4) + f(-nx_1 - n^2x_2 + 2n^3x_3 + 2n^4x_4) = 2f(nx_1 - n^2x_2 + n^3x_3 + 2n^4x_4) + 2f(-nx_1 + n^2x_2 + n^3x_3 - 2n^4x_4) + 2f(nx_1 - n^2x_2 - 2n^3x_3 + n^4x_4) + 2f(-nx_1 + n^2x_2 + 2n^3x_3 + n^4x_4) + 12f(n^2x_2) + 12f(n^3x_3), \tag{12.23}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (12.13), (12.15), (12.17), (12.19), (12.21), (12.23) in (12.11), we get

$$\begin{aligned}
&12f(nx_1 + n^2x_2) + 12f(nx_1 + n^3x_3) + 12f(nx_1 + n^4x_4) + 12f(n^2x_2 + n^3x_3) + 12f(n^2x_2 + n^4x_4) + 12f(n^3x_3 + n^4x_4)\\
&= 2f(nx_1 + 2n^2x_2 + n^3x_3 - n^4x_4) + 2f(nx_1 - 2n^2x_2 - n^3x_3 + n^4x_4) + 12f(nx_1) + 2f(-2nx_1 + n^2x_2 + n^3x_3 - n^4x_4) +\\
&f(2nx_1 + n^2x_2 - n^3x_3 + n^4x_4) + 12f(n^2x_2) + 2f(-nx_1 - n^2x_2 - n^3x_3 + n^4x_4) + 2f(-nx_1 - n^2x_2 + n^3x_3 - n^4x_4)\\
&+2f(2nx_1 + n^2x_2 + n^3x_3 - n^4x_4) + 2f(-2nx_1 - n^2x_2 + n^3x_3 + n^4x_4) + 12f(n^3x_3) + 2f(nx_1 + n^2x_2 - 2n^3x_3 - n^4x_4)\\
&+2f(nx_1 - n^2x_2 + 2n^3x_3 + n^4x_4) + 12f(nx_1) + 2f(-nx_1 - n^2x_2 - n^3x_3 + n^4x_4) + 2f(-nx_1 + n^2x_2 - n^3x_3 - n^4x_4)\\
&+2f(2nx_1 + n^2x_2 - n^3x_3 + n^4x_4) + 2f(-2nx_1 - n^2x_2 + n^3x_3 + n^4x_4) + 12f(n^4x_4) + 2f(nx_1 + n^2x_2 - n^3x_3 - 2n^4x_4)\\
&+2f(nx_1 - n^2x_2 + n^3x_3 + 2n^4x_4) + 12f(nx_1) + 2f(-nx_1 - n^2x_2 + n^3x_3 - n^4x_4) + 2f(-nx_1 + n^2x_2 - n^3x_3 - n^4x_4)\\
&+2f(nx_1 + n^2x_2 + 2n^3x_3 - n^4x_4) + 2f(-nx_1 + n^2x_2 - 2n^3x_3 + n^4x_4) + 12f(n^2x_2) + 2f(nx_1 - 2n^2x_2 + n^3x_3 - n^4x_4)\\
&+2f(-nx_1 + 2n^2x_2 + n^3x_3 + n^4x_4) + 12f(n^3x_3) + 2f(-nx_1 - n^2x_2 - n^3x_3 + n^4x_4) + 2f(nx_1 - n^2x_2 - n^3x_3 - n^4x_4) +\\
&2f(nx_1 + n^2x_2 - n^3x_3 + 2n^4x_4) + 2f(-nx_1 + n^2x_2 + n^3x_3 - 2n^4x_4) + 12f(n^2x_2) + 2f(nx_1 - 2n^2x_2 - n^3x_3 + n^4x_4) +\\
&2f(-nx_1 + 2n^2x_2 + n^3x_3 + n^4x_4) + 12f(n^4x_4) + 2f(-nx_1 - n^2x_2 + n^3x_3 - n^4x_4) + 2f(nx_1 - n^2x_2 - n^3x_3 - n^4x_4)\\
&+2f(nx_1 - n^2x_2 + n^3x_3 + 2n^4x_4) + 2f(-nx_1 + n^2x_2 + n^3x_3 - 2n^4x_4) + 12f(n^3x_3) + 2f(nx_1 - n^2x_2 - 2n^3x_3 + n^4x_4)\\
&+ 2f(-nx_1 + n^2x_2 + 2n^3x_3 + n^4x_4) + 12f(n^4x_4) + 2f(-nx_1 + n^2x_2 - n^3x_3 - n^4x_4) + 2f(nx_1 - n^2x_2 - n^3x_3 - n^4x_4),
\end{aligned} \tag{12.24}$$

for all $x_1, x_2, x_3, x_4 \in X$. Replacing nx_1 by n^2x_2 in (12.13), we have

$$f(2nx_1 + 2n^2x_2 + n^3x_3 - n^4x_4) + f(2nx_1 + 2n^2x_2 - n^3x_3 + n^4 x_4) = 2f(2nx_1 + n^2x_2 + n^3x_3 - n^4x_4) + 2f(-2nx_1 + n^2x_2 - n^3x_3 + n^4x_4) + 12f(n^2x_2) + 2f(nx_1 - 2n^2x_2 + n^3x_3 - n^4x_4) + 2f(nx_1 + 2n^2x_2 - n^3x_3 + n^4x_4) + 12f(nx_1), \tag{12.25}$$

for all $x_1, x_2, x_3, x_4 \in X$. Setting nx_1 by n^3x_3 in (12.15), we obtain

$$f(2nx_1 + n^2x_2 + 2n^3x_3 - n^4x_4) + f(2nx_1 - n^2x_2 + 2n^3x_3 + n^4 x_4) = 2f(nx_1 + n^2x_2 + 2n^3x_3 - n^4x_4) + 2f(nx_1 - n^2x_2 - 2n^3x_3 + n^4x_4) + 12f(nx_1) + 2f(-2nx_1 + n^2x_2 + n^3x_3 - n^4x_4) + 2f(2n x_1 - n^2x_2 + n^3x_3 + n^4x_4) + 12f(n^3x_3), \tag{12.26}$$

for all $x_1, x_2, x_3, x_4 \in X$. Switching nx_1 by n^4x_4 in (12.17), we arrive

$$f(2nx_1 + n^2x_2 - n^3x_3 + 2n^4x_4) + f(2nx_1 - n^2x_2 + n^3x_3 + 2n^4 x_4) = 2f(nx_1 + n^2x_2 - n^3x_3 + 2n^4x_4) + 2f(nx_1 - n^2x_2 + n^3x_3 - 2n^4x_4) + 12f(nx_1) + 2f(-2nx_1 + n^2x_2 - n^3x_3 + n^4x_4) + 2f(2n x_1 - n^2x_2 + n^3x_3 + n^4x_4) + 12f(n^4x_4), \tag{12.27}$$

for all $x_1, x_2, x_3, x_4 \in X$. Substituting n^2x_2 by n^3x_3 in (12.19), we obtain

$$f(nx_1 + 2n^2x_2 + 2n^3x_3 - n^4x_4) + f(-nx_1 + 2n^2x_2 + 2n^3x_3 + n^4 x_4) = 2f(nx_1 + 2n^2x_2 + n^3x_3 - n^4x_4) + 2f(-nx_1 - 2n^2x_2 + n^3x_3 + n^4x_4) + 12f(n^3x_3) + 2f(nx_1 + n^2x_2 - 2n^3x_3 - n^4x_4) + 2f(-n x_1 + n^2x_2 + 2n^3x_3 + n^4x_4) + 12f(n^2x_2), \ \forall\, x_1, x_2, x_3, x_4 \in X. \tag{12.28}$$

for all $x_1, x_2, x_3, x_4 \in X$. Letting n^2x_2 by n^4x_4 in (12.21), we get

$$f(nx_1 + 2n^2x_2 - n^3x_3 + 2n^4x_4) + f(-nx_1 + 2n^2x_2 + n^3x_3 + 2n^4 x_4) = 2f(nx_1 + 2n^2x_2 - n^3x_3 + n^4x_4) + 2f(-nx_1 - 2n^2x_2 + n^3x_3 + n^4x_4) + 12f(n^4x_4) + 2f(nx_1 + n^2x_2 - n^3x_3 - 2n^4x_4) + 2f(-n x_1 + n^2x_2 + n^3x_3 + 2n^4x_4) + 12f(n^2x_2), \tag{12.29}$$

for all $x_1, x_2, x_3, x_4 \in X$. Interchanging n^3x_3 by n^4x_4 in (12.23), we get

$$f(nx_1 - n^2x_2 + 2n^3x_3 + 2n^4x_4) + f(-nx_1 + n^2x_2 + 2n^3x_3 + 2n^4 x_4) = 2f(nx_1 - n^2x_2 + 2n^3x_3 + n^4x_4) + 2f(-nx_1 + n^2x_2 - 2n^3x_3 + n^4x_4) + 12f(n^4x_4) + 2f(nx_1 - n^2x_2 + n^3x_3 - 2n^4x_4) + 2f(-n x_1 + n^2x_2 + n^3x_3 + 2n^4x_4) + 12f(n^3x_3), \tag{12.30}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (12.25), (12.26), (12.27), (12.28), (12.29), (12.30) in (12.11), we get

$$\begin{aligned}
&12f(nx_1+n^2x_2)+12f(nx_1+n^3x_3)+12f(nx_1+n^4x_4)+\\
&12f(n^2x_2+n^3x_3)+12f(n^2x_2+n^4x_4)+12f(n^3x_3+n^4x_4)\\
=\;&2f(2nx_1+n^2x_2+n^3x_3-n^4x_4)+2f(-2nx_1+n^2x_2-n^3x_3\\
&+n^4x_4)+12f(n^2x_2)+2f(nx_1-n^2x_2+n^3x_3-n^4x_4)+2f(n\\
&x_1+2n^2x_2-n^3x_3+n^4x_4)+12f(nx_1)+2f(-nx_1-n^2x_2-\\
&n^3x_3+n^4x_4)+2f(-nx_1-n^2x_2+n^3x_3-n^4x_4)+2f(nx_1+\\
&n^2x_2+2n^3x_3-n^4x_4)+2f(nx_1-n^2x_2-2n^3x_3+n^4x_4)+\\
&12f(nx_1)+2f(-2nx_1+n^2x_2+n^3x_3-n^4x_4)+2f(nx_1-n^2\\
&x_2+n^3x_3+n^4x_4)+12f(n^3x_3)+2f(-nx_1-n^2x_2-n^3x_3+\\
&n^4x_4)+2f(-nx_1+n^2x_2-n^3x_3-n^4x_4)+2f(nx_1+n^2x_2\\
&-n^3x_3+2n^4x_4)+2f(nx_1-n^2x_2+n^3x_3-2n^4x_4)+12f(n\\
&x_1)+2f(-2nx_1+n^2x_2-n^3x_3+n^4x_4)+2f(2nx_1-n^2x_2\\
&+n^3x_3+n^4x_4)+12f(n^4x_4)+2f(-nx_1-n^2x_2+n^3x_3-\\
&n^4x_4)+2f(-nx_1+n^2x_2-n^3x_3-n^4x_4)+2f(nx_1+2n^2x_2\\
&+n^3x_3-n^4x_4)+2f(-nx_1-2n^2x_2+n^3x_3+n^4x_4)+12f\\
&(n^3x_3)+2f(nx_1+n^2x_2-2n^3x_3-n^4x_4)+2f(-nx_1+\\
&n^2x_2+2n^3x_3+n^4x_4)+12f(n^2x_2)+2f(-nx_1-n^2x_2+n^3\\
&x_3-n^4x_4)+2f(nx_1-n^2x_2-n^3x_3-n^4x_4)+2f(nx_1+2\\
&n^2x_2-n^3x_3+n^4x_4)+2f(-nx_1-2n^2x_2+n^3x_3+n^4x_4)+\\
&12f(n^4x_4)+2f(nx_1+n^2x_2-n^3x_3-2n^4x_4)+2f(-nx_1+n^2\\
&x_2+n^3x_3+2n^4x_4)+12f(n^2x_2)+2f(-nx_1-n^2x_2+n^3x_3-\\
&n^4x_4)+2f(nx_1-n^2x_2-n^3x_3-n^4x_4)+2f(nx_1-n^2x_2+\\
&2n^3x_3+n^4x_4)+2f(-nx_1+n^2x_2-2n^3x_3+n^4x_4)+12f(n^4\\
&x_4)+2f(nx_1-n^2x_2+n^3x_3-2n^4x_4)+2f(-nx_1+n^2x_2+n^3\\
&x_3+2n^4x_4)+12f(n^3x_3)+2f(-nx_1+n^2x_2-n^3x_3-n^4x_4)\\
&+2f(nx_1-n^2x_2-n^3x_3-n^4x_4),
\end{aligned} \tag{12.31}$$

for all $x_1, x_2, x_3, x_4 \in X$. Adding (12.24) and (12.31), we have

$$\begin{aligned}
&24f(nx_1+n^2x_2)+24f(nx_1+n^3x_3)+24f(nx_1+n^4x_4)+24\\
&f(n^2x_2+n^3x_3)+24f(n^2x_2+n^4x_4)+24f(n^3x_3+n^4x_4)-72\\
&f(nx_1)-72f(n^2x_2)-72(n^3x_3)-72f(n^4x_4)=36f(-nx_1+\\
&n^2x_2+n^3x_3+n^4x_4)+36f(nx_1-n^2x_2+n^3x_3+n^4x_4)+36\\
&f(nx_1+n^2x_2-n^3x_3+n^4x_4)+36f(nx_1+n^2x_2+n^3x_3-n^4
\end{aligned}$$

$$x_4) - 24f(-nx_1 + n^3x_3 + n^4x_4) - 24f(nx_1 - n^3x_3 + n^4x_4) - 24f(nx_1 + n^3x_3 - n^4x_4) - 24f(-n^2x_2 + n^3x_3 + n^4x_4) - 24 f(n^2x_2 - n^3x_3 + n^4x_4) - 24f(n^2x_2 + n^3x_3 - n^4x_4) - 24 f(-nx_1 + n^2x_2 + n^4x_4) - 24f(nx_1 - n^2x_2 + n^4x_4) - 24f(n x_1 + n^2x_2 - n^4x_4) - 24f(-nx_1 + n^2x_2 + n^3x_3) - 24f(nx_1 -n^2x_2 + n^3x_3) - 24f(nx_1 + n^2x_2 - n^3x_3), \tag{12.32}$$

for all $x_1, x_2, x_3, x_4 \in X$. We know that

$$f(-nx_1 + n^2x_2 + n^3x_3) + f(nx_1 - n^2x_2 + n^3x_3) + f(nx_1 + n^2 x_2 - n^3x_3) + f(-nx_1 + n^2x_2 + n^4x_4) + f(nx_1 - n^2x_2 + n^4x_4) + f(nx_1 + n^2x_2 - n^4x_4) + f(-nx_1 + n^3x_3 + n^4x_4) + f(nx_1 - n^3x_3 +n^4x_4) + f(nx_1 + n^3x_3 - n^4x_4) + f(-n^2x_2 + n^3x_3 + n^4x_4) + f (n^2x_2 - n^3x_3 + n^4x_4) + f(n^2x_2 + n^3x_3 - n^4x_4) = -3f(nx_1 + n^2 x_2 + n^3x_3) - 3f(nx_1 + n^2x_2 + n^4x_4) - 3f(nx_1 + n^3x_3 + n^4x_4) - 3f(n^2x_2 + n^3x_3 + n^4x_4) + 8f(nx_1 + n^2x_2) + 8f(nx_1 + n^3x_3) +8f(nx_1 + n^4x_4) + 8f(n^2x_2 + n^3x_3) + 8f(n^2x_2 + n^4x_4) + 8f(n^3x_3 +n^4x_4) - 12f(nx_1) - 12f(n^2x_2) - 12f(n^3x_3) - 12f(n^4x_4), \tag{12.33}$$

for all $x_1, x_2, x_3, x_4 \in X$. Using (12.33) in (12.32), we get

$$f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4) + f(nx_1 - n^2x_2 + n^3x_3 + n^4x_4) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4) + f(nx_1 + n^2x_2 + n^3 x_3 - n^4x_4) = -2\left[f(nx_1 + n^2x_2 + n^3x_3) + f(nx_1 + n^2x_2 + n^4 x_4) + f(nx_1 + n^3x_3 + n^4x_4) + f(n^2x_2 + n^3x_3 + n^4x_4)\right] + 6 \left[f(nx_1 + n^2x_2) + f(nx_1 + n^3x_3) + f(nx_1 + n^4x_4) + f(n^2x_2 +n^3x_3) + f(n^2x_2 + n^4x_4) + f(n^3x_3 + n^4x_4)\right] - 10\left[n^3 f(x_1) + n^6 f(x_2) + n^9 f(x_3) + n^{12} f(x_4)\right], \tag{12.34}$$

for all $x_1, x_2, x_3, x_4 \in X$.

Conversely, assume that $f : X \to Y$ satisfies the functional equations (12.1) and replacing (x_1, x_2, x_3, x_4) by $(x, 0, 0, 0)$, $(0, x, 0, 0)$, $(0, 0, x, 0)$ and $(0, 0, 0, x)$ respectively in (12.34), we obtain

$$f(nx) = n^3 f(x), \qquad f(n^2x) = n^6 f(x),$$
$$f(n^3x) = n^9 f(x) \quad \text{and} \quad f(n^4x) = n^{12} f(x), \tag{12.35}$$

for all $x \in X$. One can easy to verify that from (12.35) that

$$f\left(\frac{x}{n^i}\right) = \left(\frac{1}{n^i}\right)^3 f(x) \tag{12.36}$$

for all $x \in X$. Replacing (x_1, x_2, x_3, x_4) by $\left(\frac{x}{n}, \frac{x}{n^2}, \frac{x}{n^3}, 0\right)$ in (12.34), we have

$$3f(2x+y) + f(2x-y) = 8f(x+y) + 24f(x) - 6f(y), \tag{12.37}$$

for all $x, y \in X$. Replacing y by $-y$ in (12.37), we have

$$3f(2x-y) + f(2x+y) = 8f(x-y) + 24f(x) + 6f(y), \tag{12.38}$$

for all $x, y \in X$. Adding (12.37) and (12.38) we obtain

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{12.39}$$

for all $x, y \in X$. □

12.3 Cubic Functional Equation-Direct Method

In this section, we present some basic Definitions of random normed space and we examine the stability of the cubic functional equation (12.1) in RN-Space with the help of direct method.

Definition 12.2 A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangular norm, if T satisfies the following conditions:

(a) T is commutative and associative.
(b) T is continuous.
(c) $T(a, 1) = a$, for all $a \in [0, 1]$.
(d) $T(a, b) \leq T(c, d)$ when $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0, 1]$.

Remark: Examples of continuous t-norms are $T_p(a, b) = ab$, $T_m(a, b) = min(a, b)$ and $T_L(a, b) = max(a + b - 1, 0)$ (The Lukasiewicz t-norm). Recall [9] that if T is a t-norm and n_x is a given sequence of numbers in [0, 1], then $T^n_{i=1}x_{n+i}$ is defined recurrently by $T_{i=1}x_i = x_i$ and $T^n_{i=1}x_i = T(T^n_{i=1}x_i, x_n)$ for $n \geq 2$, $T^\infty_{i=1}x_i$ is defined as $T^\infty_{i=1}x_{n+i}$. It is known that, for the Lukasiewicz t-norm, the following implication holds:

$$\lim_{n\to\infty}(T_L)^\infty_{i=1} = 1 \iff \sum_{n=1}^{\infty}(1 - x_n) < \infty.$$

Definition 12.3 A random normed space (briefly, RN-space) is a triple (X, μ, T), where X is a vector space. T is a continuous t-norm and μ is a mapping from X into D^+ satisfying the following conditions:

$(RN1)$ $\mu_x(t) = \epsilon_0(t)$, for all $t \geq 0$ if and only $x = 0$.
$(RN2)$ $\mu_{ax}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$, for all $x \in X$, and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$.
$(RN3)$ $\mu_{x+y}\ (x + y) \leq T(\mu_x(t), \mu_y(s))$, for all $x, y \in X$ and $t, s \geq 0$.

Definition 12.4 Let (X, μ, T) be an RN-space.

(1) A sequence x_n in X is said to be convergent to a point $x \in X$ if, for any $\epsilon \geq 0$ and $\lambda \geq 0$, there exists a positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ for all $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\epsilon \geq 0$ $\lambda \geq 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ for all $n \geq m \geq N$.

(3) An RN-space (X, μ, T) is said to be complete, if every Cauchy sequence in X is convergent to a point in X.

Note All over in this part we use the following notation for a given mapping $f : X \to Y$ as $Df(x_1, x_2, x_3, x_4)$ is equal to

$$\begin{aligned}
&f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4) + f(-nx_1 - n^2x_2 + n^3x_3 + n^4x_4)\\
&+f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4) + f(nx_1 + n^2x_2 + n^3x_3 - n^4x_4)\\
&+2\,[\ f(nx_1 + n^2x_2 + n^3x_3) + f(nx_1 + n^2x_2 + n^4x_4) + f(nx_1 + n^3x_3\\
&+n^4x_4) + f(n^2x_2 + n^3x_3 + n^4x_4)\ ]\\
&-6\left[f(nx_1 + n^2x_2) + f(nx_1 + n^3x_3) + f(nx_1 + n^4x_4) + f(n^2x_2 + n^3x_3)\right.\\
&\left.+f(n^2x_2 + n^4x_4) + f(n^3x_3 + n^4x_4)\ \right]\\
&+10\left[n^3 f(x_1) + n^6 f(x_2) + n^9 f(x_3) + n^{12} f(x_4)\right],
\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in X$.

All through this section, let X be a linear space and (Y, μ, T) be a complete RN-space.

Theorem 12.5 *Let $j = \pm 1$. Let $f : X \to Y$ be a mapping for there exists a function $\eta : X^4 \to D^+$ with the condition*

$$\lim_{k\to\infty} T_{i=0}^{\infty} \eta_{n^{(k+i)}x_1, n^{(k+i)}x_2, n^{(k+i)}x_3, n^{(k+i)}x_4}\left(n^{3(k+i+1)j}t\right) = 1 \tag{12.40}$$

$$= \lim_{k\to\infty} \eta_{n^{kj}x_1, n^{kj}x_2, n^{kj}x_3, n^{kj}x_4}\left(n^{3kj}t\right), \tag{12.41}$$

such that the functional inequality with $f(0) = 0$ such that

$$\mu_{Df}(x_1, x_2, x_3, x_4)^{(t)\geq\eta}(x_1, x_2, x_3, x_4)^{(t)}, \tag{12.42}$$

for all $x_1, x_2, x_3, x_4 \in X$ and all $t \geq 0$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equation (12.1) and

$$\mu_{C(x)-f(x)^{(t)}} \geq T_{i=0}^{\infty}\left(\eta_{n^{(i+1)j}x,0,0,0}\left(n^{3(i+1)j}t\right)\right), \tag{12.43}$$

for all $x \in X$ and all $t > 0$. The mapping $C(x)$ is defined by

$$\mu_{C(x)}(t) = \lim_{k\to\infty} \mu \frac{f\left(n^{kj}x\right)^{(t)}}{n^{3kj}}, \tag{12.44}$$

for all $x \in X$ and all $t > 0$.

Proof Assume that $j = 1$. Replacing (x_1, x_2, x_3, x_4) by $(x, 0, 0, 0)$ in (10.2), we get

$$\mu_{10f(nx)-10n^3f(x)}(t) \geq \eta_{x,0,0,0}(t), \tag{12.45}$$

for all $x \in X$ and all $t > 0$. It follows from (12.44) and $(RN2)$, that

$$\mu_{\frac{f(nx)}{n^3}-f(x)}(t) \geq \eta_{x,0,0,0}(n^3 10t), \tag{12.46}$$

for all $x \in X$ and all $t > 0$. Replacing x by $n^k x$ in (12.46) we have

$$\mu_{\frac{f(n^{k+1}x)}{n^{3(k+1)}}-\frac{f(n^kx)}{n^{3k}}}(t) \geq \eta_{n^kx,0,0,0}(n^{3k}n^3t) = \eta_{x,0,0,0}\left(\frac{n^{3k}10n^3}{\alpha^k}t\right),$$

for all $x \in X$ and all $t > 0$. It follows from

$$\frac{f(n^lx)}{n^{3l}} - f(x) = \sum_{k=0}^{l-1} \frac{f(n^{k+1}x)}{n^{3(k+1)}} - \frac{f(n^kx)}{n^{3k}},$$

and (12.47), that

$$\mu_{\frac{f(n^lx)}{n^{3l}}-f(x)}\left(t\sum_{k=0}^{l-1}\frac{\alpha^k}{n^{3k}10n^3}\right) \geq T_{k=0}^{l-1}(\eta_{x,0,0,0}(t)) = \eta_{x,0,0,0}(t), \tag{12.47}$$

$$\mu_{\frac{f(n^lx)}{n^{3l}}-f(x)}(t) \geq \eta_{x,0,0,0}\left(\frac{t}{\sum_{k=0}^{l-1}\frac{\alpha^k}{n^{3k}10n^3}}\right), \tag{12.48}$$

for all $x \in X$ and all $t > 0$. Replacing x by $n^m x$ in (12.48), we obtain

$$\mu_{\frac{f(n^{l+m}x)}{n^{3(l+m)}}-\frac{f(n^mx)}{n^{3m}}}(t) \geq \eta_{x,0,0,0}\left(\frac{t}{\sum_{k=0}^{l-1}\frac{\alpha^k}{n^{3k}10n^3}}\right), \tag{12.49}$$

Now,

$$\eta_{x,0,0,0}\left(\frac{t}{\sum_{k=m}^{l+m}\frac{\alpha^k}{n^{3k}10n^3}}\right) \to 1,$$

as $m, n \to \infty$ and so $\frac{f(n^l x)}{n^{3l}}$ is a Cauchy sequence in (Y, μ, T). Since (Y, μ, T) is a complete RN-space, this sequence converges to some point $C(x) \in Y$. Fixing $x \in X$ and puting $m = 0$ in (12.49), we have

$$\mu_{\frac{f(n^l x)}{n^{3l}}-f(x)}(t) \geq \eta_{x,0,0,0}\left(\frac{t}{\sum_{k=0}^{l-1}\frac{\alpha^k}{n^{3k}10n^3}}\right), \tag{12.50}$$

and so, for every $\delta > 0$, we get

$$\begin{aligned}\mu_{C(x)-f(x)}(t+\delta) &\geq T\left(\mu_{C(x)-\frac{f(n^l x)}{n^{3l}}-f(x)}(\delta), \mu_{\frac{f(n^l x)}{n^{3l}}-f(x)}(t)\right),\\ &\geq T\left(\mu_{C(x)-\frac{f(n^l x)}{n^{3l}}}(\delta), \eta_{x,0,0,0}\left(\frac{t}{\sum_{k=0}^{l-1}\frac{\alpha^k}{n^{3k}10n^3}}\right)\right).\end{aligned} \tag{12.51}$$

Taking the limit as $n \to \infty$ and using (12.51) we get

$$\mu_{C(x)-f(x)}(t+\delta) \geq \eta_{x,0,0,0}\left(10(n^3-\alpha)t\right). \tag{12.52}$$

Since δ was arbitrary, by taking $\delta \to 0$ in (12.52), we have

$$\mu_{C(x)-f(x)}(t) \geq \eta_{x,0,0,0}\left(10(n^3-\alpha)t\right). \tag{12.53}$$

Replacing (x_1, x_2, x_3, x_4) by $(n^l x_1, n^l x_2, n^l x_3, n^l x_4)$ in (12.42), we obtain

$$\mu_{Df(n^l x_1, n^l x_2, n^l x_3, n^l x_4)}(t) \geq \eta_{(n^l x_1, n^l x_2, n^l x_3, n^l x_4)}(n^{3l}t), \tag{12.54}$$

for all $x_1, x_2, x_3, x_4 \in X$ and for all $t > 0$. Since

$$\lim_{k\to\infty} T_{i=0}^{\infty}\left(\eta_{n^{(k+i)}x_1, n^{(k+i)}x_2, n^{(k+i)}x_3, n^{(k+i)}x_4} n^{3(k+i+1)j}t\right) = 1,$$

we conclude that C fulfills (12.1).

To prove the uniqueness of the cubic mapping C, assume that there exists a cubic mapping D from X to Y, which satisfies (12.53). Fix $x \in X$. Clearly, $C(n^l)x = n^{3l}C(x)$ and $D(n^l)x = n^{3l}D(x)$, for all $x \in X$. It follows from (12.53) that

$$\mu_{C(x)-D(x)}(t) = \lim_{n\to\infty}\mu_{\frac{C(n^l x)}{n^{3l}}-\frac{D(n^l x)}{n^{3l}}}(t),$$

$$
\begin{aligned}
\mu_{\frac{C(n^l x)}{n^{3l}} - \frac{D(n^l x)}{n^{3l}}}(t) &\geq \min\left\{ \mu_{\frac{C(n^l x)}{n^{3l}} - \frac{f(n^l x)}{n^{3l}}}\left(\tfrac{t}{2}\right), \mu_{\frac{C(n^l x)}{n^{3l}} - \frac{f(n^l x)}{n^{3l}}}\left(\tfrac{t}{2}\right) \right\} \\
&\geq \eta^l_{x,0,0,0}(n^{3l} 10(n^3 - \alpha)t), \\
&\geq \eta_{x,0,0,0}\left(\frac{n^{3l} 10(n^3 - \alpha)t}{\alpha^l}\right). \qquad (12.55)
\end{aligned}
$$

Since $\lim_{n\to\infty}\left(\frac{n^{3l}10(n^3-\alpha)t}{\alpha^l}\right) = \infty$, we get $\lim_{n\to\infty} \eta_{x,0,0,0}\left(\frac{n^{3l}10(n^3-\alpha)t}{\alpha^l}\right) = 1$.

Therefore it follows that $\mu_{C(x)-D(x)^{(t)}} = 1$, for all $t > 0$ and so $C(x) = D(x)$. This completes the proof. □

Remark: Let $j = \pm 1$. Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^4 \to D^+$ with the condition

$$
\begin{aligned}
\lim_{k\to\infty} T^{\infty}_{i=0}\left(\eta_{2^{(k+i)}x_1, 2^{(k+i)}x_2, 2^{(k+i)}x_3, 2^{(k+i)}x_4}\left(2^{3(k+i+1)j} t\right)\right) &= 1 \\
&= \lim_{k\to\infty} \eta_{2^{kj}x_1, 2^{kj}x_2, 2^{kj}x_3, 2^{kj}x_4}\left(2^{3kj} t\right),
\end{aligned}
$$

satisfying $f(0) = 0$ such that (12.42), for all $x_1, x_2, x_3, x_4 \in X$ and all $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying (12.1) and

$$
\mu_{C(x)-f(x)}(t) \geq T^{\infty}_{i=0}\left(\eta_{2^{(i+1)j}x,0,0,0}\left(2^{3(i+1)j} t\right)\right),
$$

for all $x \in X$ and all $t > 0$. The mapping $C(x)$ is defined by $\mu_{C(x)}(t) = \lim_{k\to\infty} \mu_{\frac{f(2^{kj}x)}{2^{3kj}}}(t)$ for all $x \in X$ and all $t > 0$.

Proposition 12.6 *Let $j = \pm 1$. Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^4 \to D^+$ with the condition*

$$
\begin{aligned}
\lim_{k\to\infty} T^{\infty}_{i=0}\left(\eta_{n^{2(k+i)}x_1, n^{2(k+i)}x_2, n^{2(k+i)}x_3, n^{2(k+i)}x_4}\left(n^{6(k+i+1)j} t\right)\right) &= 1, \\
&= \lim_{k\to\infty} \eta_{n^{2kj}x_1, n^{2kj}x_2, n^{2kj}x_3, n^{2kj}x_4}\left(n^{6kj} t\right),
\end{aligned}
$$

satisfying $f(0) = 0$ such that (12.42), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying (12.1) and

$$
\mu_{C(x)-f(x)}(t) \geq T^{\infty}_{i=0}\left(\eta_{0,n^{2(i+1)j}x,0,0}\left(n^{6(i+1)j} t\right)\right),
$$

for all $x \in X$ *and all* $t > 0$. *The mapping* $C(x)$ *is defined by* $\mu_{C(x)}(t) = \lim_{k\to\infty} \mu_{\frac{f(n^{2kj}x)}{n^{6kj}}}(t)$ *for all* $x \in X$ *and* $t > 0$.

Remark: Let $j = \pm 1$. Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^4 \to D^+$ with the condition

$$\lim_{k\to\infty} T_{i=0}^{\infty}\left(\eta_{2^{2(k+i)}x_1, 2^{2(k+i)}x_2, 2^{2(k+i)}x_3, 2^{2(k+i)}x_4}\left(2^{6(k+i+1)j}t\right)\right) = 1,$$
$$= \lim_{k\to\infty} \eta_{2^{2kj}x_1, 2^{2kj}x_2, 2^{2kj}x_3, 2^{2kj}x_4}\left(2^{6kj}t\right),$$

satisfying $f(0) = 0$ such that (12.42), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$.

Then there exists a unique cubic mapping $C : X \to Y$ satisfying (12.1) and

$$\mu_{C(x)-f(x)}(t) \geq T_{i=0}^{\infty}\left(\eta_{0, 2^{2(i+1)j}x, 0, 0}\left(2^{6(i+1)j}t\right)\right),$$

for all $x \in X$ and $t > 0$. The mapping $C(x)$ is defined by $\mu_{C(x)}(t) = \lim_{k\to\infty} \mu_{\frac{f(2^{2kj}x)}{2^{6kj}}}(t)$ for all $x \in X$ and $t > 0$.

Proposition 12.7 *Let* $j = \pm 1$. *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\eta : X^4 \to D^+$ *with the condition*

$$\lim_{k\to\infty} T_{i=0}^{\infty}\left(\eta_{n^{3(k+i)}x_1, n^{3(k+i)}x_2, n^{3(k+i)}x_3, n^{3(k+i)}x_4}\left(n^{9(k+i+1)j}t\right)\right) = 1,$$
$$= \lim_{k\to\infty} \eta_{n^{3kj}x_1, n^{3kj}x_2, n^{3kj}x_3, n^{3kj}x_4}\left(n^{9kj}t\right),$$

satisfying $f(0) = 0$ *such that (12.42), for all* $x_1, x_2, x_3, x_4 \in X$ *and* $t > 0$.

Then there exists a unique cubic mapping $C : X \to Y$ *satisfying* (12.1) *and*

$$\mu_{C(x)-f(x)}(t) \geq T_{i=0}^{\infty}\left(\eta_{0, 0, n^{3(i+1)j}x, 0}\left(n^{9(i+1)j}t\right)\right),$$

for all $x \in X$ *and* $t > 0$. *The mapping* $C(x)$ *is defined by*

$$\mu_{C(x)}(t) = \lim_{k\to\infty} \mu_{\frac{f(n^{3kj}x)}{n^{9kj}}}(t)$$

for all $x \in X$ *and* $t > 0$.

Proposition 12.8 *Let* $j = \pm 1$. *Let* $f : X \to Y$ *be a mapping for which there exists a function* $\eta : X^4 \to D^+$ *with the condition*

$$\lim_{k\to\infty} T_{i=0}^{\infty}\left(\eta_{n^{4(k+i)}x_1,n^{4(k+i)}x_2,n^{4(k+i)}x_3,n^{4(k+i)}x_4}\left(n^{12(k+i+1)j}t\right)\right) = 1,$$
$$= \lim_{k\to\infty} \eta_{n^{4kj}x_1,n^{4kj}x_2,n^{4kj}x_3,n^{4kj}x_4}\left(n^{12kj}t\right),$$

satisfying $f(0) = 0$ such that (12.42), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying (12.1) and

$$\mu_{C(x)-f(x)}(t) \geq T_{i=0}^{\infty}\left(\eta_{0,0,0,n^{4(i+1)j}x}\left(n^{12(i+1)j}t\right)\right),$$

for all $x \in X$ and $t > 0$. The mapping $C(x)$ is defined by

$$\mu_{C(x)}(t) = \lim_{k\to\infty} \mu_{\frac{f\left(n^{4kj}x\right)}{n^{12kj}}}(t)$$

for all $x \in X$ and $t > 0$.

The following corollaries are the immediate consequence of Theorem (12.5) and Propositions (12.6)–(12.8) respectively, concerning the stability of (12.1).

Corollary 12.9 *Let ϵ and s be non-negative real numbers. Let $f : X \to Y$ satisfies the inequality*

$$\mu_{Df_{(x_1,x_2,x_3,x_4)}}(t) \geq \begin{cases} \eta_{\epsilon}(t) \\ \eta_{\epsilon\sum_{i=1}^{4}||x_i||^s}(t), \quad s \neq 3 \\ \eta_{\epsilon(\prod_{i=1}^{4}||x_i||^s + \sum_{i=1}^{4}||x_i||^{4s})}(t), \quad s \neq \frac{3}{4} \end{cases} \tag{12.56}$$

for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \begin{cases} \eta_{\frac{\epsilon}{10|n^3-1|}}(t) \\ \eta_{\frac{\epsilon||x||^s}{10|n^3-n^s|}}(t) \\ \eta_{\frac{\epsilon||x||^{4s}}{10|n^3-n^{4s}|}}(t), \end{cases} \tag{12.57}$$

for all $x \in X$ and all $t > 0$.

Corollary 12.10 *Let ϵ and s be non-negative real numbers. Let $f : X \to Y$ satisfies the inequality (12.56), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$, such that*

$$\mu_{f(x)-C(x)}(t) \geq \begin{cases} \eta_{\frac{\epsilon}{10|n^6-1|}}(t) \\ \eta_{\frac{\epsilon||x||^s}{10|n^6-n^{2s}|}}(t) \\ \eta_{\frac{\epsilon||x||^{4s}}{10|n^6-n^{8s}|}}(t) \end{cases} \tag{12.58}$$

for all $x \in X$ and $t > 0$.

Example 12.11 Let ϵ and s be non-negative real numbers. Let $f : X \to Y$ satisfies the inequality (12.56), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-c(x)}(t) \geq \begin{cases} \eta_{\frac{\epsilon}{10|7|}}(t) \\ \eta_{\frac{\epsilon\|x\|^s}{10|2^3-2^s|}}(t) \\ \eta_{\frac{\epsilon\|x\|^{4s}}{10|2^3-2^{4s}|}}(t) \end{cases}$$

for all $x \in X$ and $t > 0$.

Corollary 12.12 *Let ϵ and s be non-negative real numbers. Let $f : X \to Y$ satisfies the inequality (12.56), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\mu_{f(x)-C(x)}(t) \geq \begin{cases} \eta_{\frac{\epsilon}{10|n^9-1|}}(t) \\ \eta_{\frac{\epsilon||x||^s}{10|n^9-n^{3s}|}}(t) \\ \eta_{\frac{\epsilon||x||^{4s}}{10|n^9-n^{12s}|}}(t) \end{cases} \tag{12.59}$$

for all $x \in X$ and $t > 0$.

Example 12.13 Let ϵ and s be non-negative real numbers. Let $f : X \to Y$ satisfies the inequality (12.56), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-c(x)}(t) \geq \begin{cases} \eta_{\frac{\epsilon}{10|63|}}(t) \\ \eta_{\frac{\epsilon\|x\|^s}{10|2^6-2^{2s}|}}(t) \\ \eta_{\frac{\epsilon\|x\|^{4s}}{10|2^6-2^{8s}|}}(t) \end{cases}$$

for all $x \in X$ and $t > 0$.

Corollary 12.14 *Let ϵ and s be non-negative real numbers. Let $f : X \to Y$ satisfies the inequality (12.56), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\mu_{f(x)-C(x)}(t) \geq \begin{cases} \eta_{\frac{\epsilon}{10|n^{12}-1|}}(t) \\ \eta_{\frac{\epsilon||x||^s}{10|n^{12}-n^{4s}|}}(t) \\ \eta_{\frac{\epsilon||x||^{4s}}{10|n^{12}-n^{16s}|}}(t) \end{cases} \tag{12.60}$$

for all $x \in X$ and $t > 0$.

12.4 Cubic Functional Equation—Fixed Point Method

Theorem 12.15 *Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^4 \to D^+$ with the condition*

$$\lim_{k\to\infty} \eta_{\delta_i^k x_1, \delta_i^k x_2, \delta_i^k x_3, \delta_i^k x_4}\left(n_i^{3k} t\right) = 1, \tag{12.61}$$

for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$ and where

$$\delta_i = \begin{cases} n, & i = 0; \\ \frac{1}{n}, & i = 1; \end{cases}$$

satisfying the functional inequality

$$\mu_{Df(x_1,x_2,x_3,x_4)}(t) \geq \eta_{(x_1,x_2,x_3,x_4)}(t), \tag{12.62}$$

for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$. If there exists $L = L(i)$ such that the function

$$x \to \beta(x, t) = \eta_{\frac{x}{n},0,0,0}(10t),$$

has the property,

$$\beta(x, t) \leq L \frac{1}{\delta_i^3} \beta(\delta_i x, t), \tag{12.63}$$

for all $x \in X$ and $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equation (12.1) and for all $x \in X$ and $t > 0$,

$$\mu_{C(x)-f(x)}\left(\frac{L^{1-i}}{1-L} t\right) \geq \beta(x, t). \tag{12.64}$$

Proof Let us consider the set $\Omega = \{p | p : U^2 \to V,\ p(0) = 0\}$ and d be a generalized metric on Ω, such that

$$d(g, h) = \inf\{k \in (0, \infty)/\mu_{g(x)-h(x)}(kt) \geq \beta(x, t), x \in X, t > 0\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{\delta_i^3} g(\delta_i x)$$

for all $x \in X$. Now for $g, h \in \Omega$, we have $d(g, h) \leq K$.

$$\implies \mu_{g(x)-h(x)}(kt) \geq \beta(x,t)$$

$$\implies \mu_{(Tg(x)-Th(x))}\left(\frac{Kt}{\delta_i^3}\right) \geq \beta(x,t)$$

$$\implies d(Tg(x), Th(x)) \leq KL$$

$$\implies d(Tg, Th) \leq Ld(g,h), \tag{12.65}$$

for all $g, h \in \Omega$. Therefore T is strictly contractive mapping on Ω with Lipschitz constant L. It follows from (12.45) that

$$\mu_{10f(nx)-n^3 10f(x)}(t) \geq \eta_{x,0,0,0}(t), \tag{12.66}$$

for all $x \in X$. It follows from (12.66) for the case $i = 0$, that

$$\mu_{\frac{f(nx)}{n^3}-f(x)}(t) \geq \eta_{x,0,0,0}(10n^3 t), \tag{12.67}$$

for all $x \in X$. Using (12.63) for the case $i = 0$, we obtain

$$\mu_{\frac{f(nx)}{n^3}-f(x)}(t) \geq L\beta(x,t),$$

for all $x \in X$. Hence, we obtain

$$d\left(\mu_{Tf(x)-f(x)}\right) \geq L = L^{l-i} < \infty, \tag{12.68}$$

for all $x \in X$. Replacing x by $\frac{x}{n}$ in (12.67), we get

$$\mu_{\frac{f(x)}{n^3}-f(\frac{x}{n})}(t) \geq \eta_{\frac{x}{n},0,0,0}(10n^3 t), \tag{12.69}$$

for all $x \in X$. Using (12.63) for the case $i = 1$, we obtain

$$\mu_{n^3 f(\frac{x}{n})-f(x)}(t) \geq \beta(x,t) \implies \mu_{Tf(x)-f(x)}(t) \geq \beta(x,t),$$

for all $x \in X$. Hence we get

$$d\left(\mu_{Tf(x)-f(x)}\right) \geq L = L^{l-i} < \infty, \tag{12.70}$$

for all $x \in X$. From (12.68) and (12.70), we can conclude

$$d\left(\mu_{Tf(x)-f(x)}\right) \geq L = L^{l-i} < \infty, \tag{12.71}$$

for all $x \in X$. In order to prove $C : X \to Y$ satisfies the functional equation (12.1), the remaining proof is similar to the proof of Theorem 12.43. Since C is a unique fixed point of T in the set $\Delta = \{f \in \Omega | d(f, C) < \infty\}$, C is a unique mapping such that

$$\mu_{C(x)-f(x)}\left(\frac{L^{1-i}}{1-L}t\right) \geq \beta(x,t),$$

for all $x \in X$ and $t > 0$. This completes the proof of the theorem. □

Example 12.16 Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^4 \to D^+$ with the condition (12.61), for all $x_1, x_2, x_3, x_4 \in X$ and all $t > 0$ and where

$$\delta_i = \begin{cases} 2 \; if \;\; i = 0; \\ \frac{1}{2} \; if \;\; i = 1; \end{cases}$$

satisfying the functional inequality (12.62) for all $x_1, x_2, x_3, x_4 \in X$ and all $t > 0$. If there exists $L = L(i)$ such that the function

$$x \to \beta(x,t) = \eta_{\frac{x}{2},0,0,0}(10t),$$

has the property, (12.63) for all $x \in X$ and $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equations (12.1) and (12.64) for all $x \in X$ and $t > 0$.

Proposition 12.17 *Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^4 \to D^+$ with the condition (12.61), for all $x_1, x_2, x_3, x_4 \in X$ and all $t > 0$ and where*

$$\delta_i = \begin{cases} n^2, \; i = 0; \\ \frac{1}{n^2} \;\;\; i = 1; \end{cases}$$

satisfying the functional inequality (12.62) for all $x_1, x_2, x_3, x_4 \in X$ and all $t > 0$. If there exists $L = L(i)$ such that the function

$$x \to \beta(x,t) = \eta_{0,\frac{x}{n^2},0,0}(10t),$$

has the property, (12.63) for all $x \in X$ and $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equations (12.1) *and (12.64) for all $x \in X$ and $t > 0$.*

Remark: Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^4 \to D^+$ with the condition (12.61), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$ and where

$$\delta_i = \begin{cases} 2^2 \; if \; i = 0; \\ \frac{1}{2^2} \; if \; i = 1; \end{cases}$$

satisfying the functional inequality (12.62) for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$. If there exists $L = L(i)$ such that the function

$$x \to \beta(x, t) = \eta_{0, \frac{x}{2^2}, 0, 0}(10t),$$

has the property, (12.63) for all $x \in X$ and $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equations (12.1) and (12.64) for all $x \in X$ and $t > 0$.

Proposition 12.18 *Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^4 \to D^+$ with the condition (12.61), for all $x_1, x_2, x_3, x_4 \in X$ and all $t > 0$ and where*

$$\delta_i = \begin{cases} n^3, \; i = 0; \\ \frac{1}{n^3} \;\; i = 1; \end{cases}$$

satisfying the functional inequality (12.62) for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$. If there exists $L = L(i)$, such that the function

$$x \to \beta(x, t) = \eta_{0,0, \frac{x}{n^3}, 0}(10t),$$

has the property, (12.63) for all $x \in X$ and $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equations (12.1) and (12.64) for all $x \in X$ and $t > 0$.

Proposition 12.19 *Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^4 \to D^+$ with the condition (12.61), for all $x_1, x_2, x_3, x_4 \in X$ and all $t > 0$ and where*

$$\delta_i = \begin{cases} n^4, \; i = 0; \\ \frac{1}{n^4} \;\; i = 1; \end{cases}$$

satisfying the functional inequality (12.62) for all $x_1, x_2, x_3, x_4 \in X$ and all $t > 0$. If there exists $L = L(i)$, such that the function

$$x \to \beta(x, t) = \eta_{0,0,0, \frac{x}{n^4}}(10t),$$

has the property, (12.63) for all $x \in X$ and $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equations (12.1) and (12.64) for all $x \in X$ and $t > 0$.

From Theorem (12.15) and Propositions (12.17)–(12.19) respectively, we obtain the following corollaries concerning the stability for the functional equation (12.1).

Corollary 12.20 *If a mapping $f : X \to Y$ satisfies the inequality (12.56), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$, where ϵ, s are constants with $\epsilon > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that (12.57), for all $x \in X$ and $t > 0$.*

Proof Set

$$\mu_{Df(x_1,x_2,x_3,x_4)}(t) \geq \begin{cases} \eta_\epsilon(t) \\ \eta_\epsilon \sum_{i=1}^4 ||x_i||^s(t), \\ \eta_\epsilon(\prod_{i=1}^4 ||x_i||^s + \sum_{i=1}^4 ||x_i||^{4s})(t), \end{cases}$$

for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$. Then

$$\eta_{(\delta_i^k x_1, \delta_i^k x_2, \delta_i^k x_3, \delta_i^k x_4,)} \left(n_i^{3k} t\right) = \begin{cases} \eta_\epsilon(t) \\ \eta_\epsilon \sum_{i=1}^4 ||x_i||^s(t), \\ \eta_\epsilon \left(\prod_{i=1}^4 ||x_i||^s + \sum_{i=1}^4 ||x_i||^{4s}\right)(t) \end{cases}$$

$$= \begin{cases} \to 1 \;\; as \;\; k \to \infty \\ \to 1 \;\; as \;\; k \to \infty \\ \to 1 \;\; as \;\; k \to \infty. \end{cases}$$

But we have $\beta(x, t) = \eta_{\frac{n}{2},0,0,0}(10t)$ has the property $L\frac{1}{\delta_i^3}\beta(\delta_i x, t)$, for all $x \in X$ and $t > 0$. Now

$$\beta(x, t) = \begin{cases} \eta_{\frac{\epsilon}{10}}(t) \\ \eta_{\frac{\epsilon||x||^s}{n^s 10}}(t) \\ \eta_{\frac{2\epsilon||x||^{4s}}{n^{4s} 10}}(t) \end{cases}$$

$$L\frac{1}{\delta_i^3}\beta(\delta_i x, t) = \begin{cases} \eta_{\delta^{-3}\beta(x)}(t) \\ \eta_{\delta^{s-3}\beta(x)}(t) \\ \eta_{\delta^{4s-3}\beta(x)}(t), \end{cases}$$

Now setting
$L = n^-3$ if $i = 0$ and $L = n^3$ if $i = 1$.
$L = n^{s-3}$ for $s < 3$ if $i = 0$ and $L = n^{3-s}$ for $s > 3$ if $i = 1$.
$L = 2^{4s-3}$ for $s < \frac{3}{4}$ if $i = 0$ and $L = n^{3-4s}$ for $s > \frac{3}{4}$ if $i = 1$.

Case: 1 $\quad L = n^{-3}$ if $i = 0$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\epsilon}{10\left(n^3-1\right)}\right)}(t).$$

Case: 2 $\quad L = n^3$ if $i = 1$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\epsilon}{10\left(1-n^3\right)}\right)}(t).$$

Case: 3 $L = n^{s-3}$ for $s < 3$ if $i = 0$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\epsilon||x||^s}{10\left(n^3-n^s\right)}\right)}(t).$$

Case: 4 $L = n^{3-s}$ for $s > 3$ if $i = 1$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\epsilon||x||^s}{10\left(n^s-n^3\right)}\right)}(t).$$

Case: 5 $L = 2^{4s-3}$ for $s < \frac{3}{4}$ if $i = 0$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\epsilon||x||^{4s}}{10\left(n^3-n^{4s}\right)}\right)}(t).$$

Case: 6 $L = 2^{3-4s}$ for $s > \frac{3}{4}$ if $i = 1$

$$\mu_{f(x)-C(x)}(t) \geq L\frac{1}{\delta_i^3}\beta(\delta_i x, t)(t) \geq \eta_{\left(\frac{\epsilon||x||^{4s}}{10\left(n^{4s}-n^3\right)}\right)}(t).$$

Hence the proof is complete. □

Remark: Suppose that a mapping $f : X \to Y$ satisfies the inequality (12.56), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-c(x)}(t) \geq \begin{cases} \eta_{\frac{\epsilon}{10|7|}}(t) \\ \eta_{\frac{\epsilon\|x\|^s}{10|2^3-2^s|}}(t) \\ \eta_{\frac{\epsilon\|x\|^{4s}}{10|2^3-2^{4s}|}}(t) \end{cases}$$

for all $x \in X$ and $t > 0$.

Corollary 12.21 *If a mapping $f : X \to Y$ satisfies the inequality (12.56), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$, where ϵ, s are constants with $\epsilon > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that (12.58), for all $x \in X$ and $t > 0$.*

Remark: Suppose that a mapping $f : X \to Y$ satisfies the inequality (12.56), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-c(x)}(t) \geq \begin{cases} \eta_{\frac{\epsilon}{10|63|}}(t) \\ \eta_{\frac{\epsilon\|x\|^s}{10|2^6-2^{2s}|}}(t) \\ \eta_{\frac{\epsilon\|x\|^{4s}}{10|2^6-2^{8s}|}}(t), \end{cases}$$

for all $x \in X$ and $t > 0$.

Corollary 12.22 *If a mapping $f : X \to Y$ satisfies the inequality (12.56), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$, where ϵ, s are constants with $\epsilon > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that (12.59), for all $x \in X$ and $t > 0$.*

Corollary 12.23 *If a mapping $f : X \to Y$ satisfies the inequality (12.56), for all $x_1, x_2, x_3, x_4 \in X$ and $t > 0$, where ϵ, s are constants with $\epsilon > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that (12.60), for all $x \in X$ and $t > 0$.*

12.5 Stability of the Cubic Functional Equation—Direct Method

Proposition 12.24 *Let $j \in \{-1, 1\}$. Let x be a linear space, $(Z, P'_{\mu,v}, T)$ be an IFN-space, $\eta : X^4 \to Z$ with $0 < \left(\frac{d}{n^3}\right)^j < 1$ satisfying*

$$P'_{\mu,v}(\eta_{(n^j x_1,0,0,0),r}) \geq^*_L P'_{\mu,v}(d^j \eta_{(x,0,0,0),r}),$$

for all $x \in X$ and all $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} P'_{\mu,v}(\eta_{(n^{jk}x_1,n^{jk}x_2,n^{jk}x_3,n^{jk}x_4,)}n^{3jk}r) = 1^*_L$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. Let $(Y, P'_{\mu,v})$ be an IFN-space. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$P_{\mu,v}(Df(x_1, x_2, x_3, x_4), r) \geq^*_L P'_{\mu,v}(\eta_{(x_1,x_2,x_3,x_4)),r},$$

for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit

$$P_{\mu,v}\left(C(x) - \frac{f(n^{jk}x)}{n^{3jk}}\right) \to 1_L*,$$

as $k \to \infty$, $r > 0$ exists for all $x \in X$ and the mapping $C : X \to Y$ is a unique cubic mapping satisfying (12.1) *and*

$$P_{\mu,v}(f(x) - C(x), r) \geq_L *P'_{\mu,v}(\eta_{(x,0,0,0)}, 10r|n^3 - d|),$$

for all $x \in X$ and $r > 0$.

Remark: Let $j \in \{-1, 1\}$. Let x be a linear space, $(Z, P'_{\mu,v}, T)$ be an IFN-space, $\eta : X^4 \to Z$ with $0 < \left(\frac{d}{2^3}\right)^j < 1$ satisfying

$$P'_{\mu,v}(\eta_{(2^j x_1,0,0,0),r}) \geq^*_L P'_{\mu,v}(d^j \eta_{(x,0,0,0),r}),$$

for all $x \in X$, $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} P'_{\mu,v}(\eta_{(2^{jk}x_1,2^{jk}x_2,2^{jk}x_3,2^{jk}x_4,),2^{3jk}r}) = 1^*_L,$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. Let $(Y, P'_{\mu,v})$ be an IFN-space. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$P_{\mu,v}(Df(x_1, x_2, x_3, x_4), r) \geq^*_L P'_{\mu,v}(\eta_{(x_1,x_2,x_3,x_4),r}),$$

for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit

$$P_{\mu,v}\left(C(x) - \frac{f(2^{jk}x)}{2^{3jk}}\right) \to 1_L*,$$

as $k \to \infty$, $r > 0$ exists for all $x \in X$ and the mapping $C : X \to Y$ is a unique cubic mapping satisfying (12.1) and

$$P_{\mu,v}(f(x) - C(x), r) \geq_L *P'_{\mu,v}(\eta_{(x,0,0,0)}, 10r|2^3 - d|),$$

for all $x \in X$ and $r > 0$.

Proposition 12.25 *Let $j \in \{-1, 1\}$. Let x be a linear space, $(Z, P'_{\mu,v}, T)$ be an IFN-space, $\eta : X^4 \to Z$ with $0 < \left(\frac{d}{n^6}\right)^j < 1$ satisfying*

$$P'_{\mu,v}(\eta_{(0,n^{2j}x_2,0,0)}, r) \geq^*_L P'_{\mu,v}(d^j \eta_{(0,x,0,0)}, r),$$

for all $x \in X$, $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} P'_{\mu,v}(\eta_{(n^{2jk}x_1,n^{2jk}x_2,n^{2jk}x_3,n^{2jk}x_4,)}, n^{6jk}r) = 1^*_L,$$

for all $x_1, x_2, x_3, x_4 \in X$ *and* $r > 0$. *Let* $(Y, P'_{\mu,v})$ *be an IFN-space. Suppose that a mapping* $f : X \to Y$ *satisfies the inequality*

$$P_{\mu,v}(Df(x_1, x_2, x_3, x_4), r) \geq^*_L P'_{\mu,v}(\eta_{(x_1,x_2,x_3,x_4)}, r),$$

for all $r > 0$ *and* $x_1, x_2, x_3, x_4 \in X$. *Then the limit*

$$P_{\mu,v}\left(C(x) - \frac{f(n^{2jk}x)}{n^{6jk}}\right) \to 1_L*,$$

as $k \to \infty$, $r > 0$ *exists for all* $x \in X$ *and the mapping* $C : X \to Y$ *is a unique cubic mapping satisfying* (12.1) *and*

$$P_{\mu,v}(f(x) - C(x), r) \geq_L *P'_{\mu,v}(\eta_{(0,x,0,0)}, 10r|n^6 - d|),$$

for all $x \in X$ *and* $r > 0$.

Remark: Let $j \in \{-1, 1\}$. Let x be a linear space, $(Z, P'_{\mu,v}, T)$ be an IFN-space, $\eta : X^4 \to Z$ with $0 < \left(\frac{d}{2^6}\right)^j < 1$ satisfying

$$P'_{\mu,v}(\eta_{(0,2^{2j}x_2,0,0),r}) \geq^*_L P'_{\mu,v}(d^j\eta_{(0,x,0,0),r}),$$

for all $x \in X$, $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} P'_{\mu,v}(\eta_{(2^{2jk}x_1,2^{2jk}x_2,2^{2jk}x_3,2^{2jk}x_4,),n^{6jk}r}) = 1^*_L,$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. Let $(Y, P'_{\mu,v})$ be an IFN-space. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$P_{\mu,v}(Df(x_1, x_2, x_3, x_4), r) \geq^*_L P'_{\mu,v}(\eta_{(x_1,x_2,x_3,x_4),r}),$$

for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit

$$P_{\mu,v}\left(C(x) - \frac{f(2^{2jk}x)}{2^{6jk}}\right) \to 1_L*,$$

as $k \to \infty$, $r > 0$ exists for all $x \in X$ and the mapping $C : X \to Y$ is a unique cubic mapping satisfying (12.1) and

$$P_{\mu,v}(f(x) - C(x), r) \geq_L *P'_{\mu,v}(\eta_{(0,x,0,0)}, 10r|2^6 - d|),$$

for all $x \in X$ and $r > 0$.

Proposition 12.26 *Let $j \in \{-1, 1\}$. Let x be a linear space, $(Z, P'_{\mu,v}, T)$ be an IFN-space, $\eta : X^4 \to Z$ with $0 < \left(\frac{d}{n^9}\right)^j < 1$ satisfying*

$$P'_{\mu,v}(\eta_{(0,0,n^{3j}x_3,0)}, r) \geq^*_L P'_{\mu,v}(d^j \eta_{(0,0,x,0)}, r),$$

for all $x \in X$, $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} P'_{\mu,v}(\eta_{(n^{3jk}x_1,n^{3jk}x_2,n^{3jk}x_3,n^{3jk}x_4)}, n^{9jk}r) = 1^*_L$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. Let $(Y, P'_{\mu,v})$ be an IFN-space. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$P_{\mu,v}(Df(x_1, x_2, x_3, x_4), r) \geq^*_L P'_{\mu,v}(\eta_{(x_1,x_2,x_3,x_4)}, r)$$

for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit

$$P_{\mu,v}\left(C(x) - \frac{f(n^{3jk}x)}{n^{9jk}}\right) \to 1_{L^*}$$

as $k \to \infty$, $r > 0$ exists for all $x \in X$ and the mapping $C : X \to Y$ is a unique cubic mapping satisfying (12.1) and

$$P_{\mu,v}(f(x) - C(x), r) \geq_L *P'_{\mu,v}(\eta_{(0,0,x,0)}, 10r|n^9 - d|),$$

for all $x \in X$ and $r > 0$.

Proposition 12.27 *Let $j \in \{-1, 1\}$. Let x be a linear space, $(Z, P'_{\mu,v}, T)$ be an IFN-space, $\eta : X^4 \to Z$ with $0 < \left(\frac{d}{n^{12}}\right)^j < 1$ satisfying*

$$P'_{\mu,v}(\eta_{(0,0,0,n^{4j}x_4)}, r) \geq^*_L P'_{\mu,v}(d^j \eta_{(0,0,0,x)}, r),$$

for all $x \in X$ and $r > 0$, $d > 0$ and

$$\lim_{k\to\infty} P'_{\mu,v}(\eta_{(n^{4jk}x_1,n^{4jk}x_2,n^{4jk}x_3,n^{4jk}x_{4,})}, n^{12jk}r) = 1^*_L$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$. Let $(Y, P'_{\mu,v})$ be an IFN-space. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$P_{\mu,v}(Df(x_1, x_2, x_3, x_4), r) \geq^*_L P'_{\mu,v}(\eta_{(x_1,x_2,x_3,x_4)}, r)$$

for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. Then the limit

$$P_{\mu,v}\left(C(x) - \frac{f(n^{4jk}x)}{n^{12jk}}\right) \to 1_L*$$

as $k \to \infty$, $r > 0$ exists for all $x \in X$ and the mapping $C : X \to Y$ is a unique cubic mapping satisfying (12.1) *and*

$$P_{\mu,v}(f(x) - C(x), r) \geq_L *P'_{\mu,v}(\eta_{(0,0,0,x)}, 10r|n^{12} - d|),$$

for all $x \in X$ and $r > 0$.

From Propositions 12.24–12.27, we obtain the following corollaries concerning the stability for the functional equation (12.1).

Corollary 12.28 *Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$P_{\mu,v}(Df(x_1, x_2, x_3, x_4), r) \geq_L^* \begin{cases} P'_{\mu,v}(\tau, r) \\ P'_{\mu,v}\left(\tau \sum_{i=1}^4 ||x_i||^s r\right), \quad s \neq 3 \\ P'_{\mu,v}\left(\tau\left(\sum_{i=1}^4 ||x_i||^{4s} + \prod_{i=1}^4 ||x_i||^s\right), r\right), \ s \neq \frac{3}{4} \end{cases} \tag{12.72}$$

for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$ where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $C : X \to Y$, such that

$$P_{\mu,v}(f(x) - C(x), r) \geq_L * \begin{cases} P'_{\mu,v}(\tau, 10r|n^3 - 1|) \\ P'_{\mu,v}\left(\tau||x||^s, 10r|n^3 - n^s|\right) \\ P'_{\mu,v}\left(\tau||x||^{4s}, 10r|n^3 - n^{4s}|\right), \end{cases} \tag{12.73}$$

for all $x \in X$ and for $r > 0$.

Example 12.29 Suppose that a mapping $f : X \to Y$ satisfies the inequality (10.73), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$ where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$P_{\mu,v}(f(x) - C(x), r) \geq_{L*} \begin{cases} P_{\mu,v}(\tau, 10r|7|) \\ P_{\mu,v}(\tau\|x\|^s, 10r|2^3 - 2^s|) \\ P_{\mu,v}(\tau\|x\|^{4s}, 10r|2^3 - 2^{4s}|), \end{cases}$$

for all $x \in X$ and for $r > 0$.

Corollary 12.30 *Suppose that a mapping $f : X \to Y$ satisfies the inequality (12.72), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$ where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $C : X \to Y$, such that*

$$P_{\mu,v}(f(x) - C(x), r) \geq_L * \begin{cases} P'_{\mu,v}(\tau, 10r|n^6 - 1|) \\ P'_{\mu,v}\left(\tau||x||^s, 10r|n^6 - n^{2s}|\right) \\ P'_{\mu,v}\left(\tau||x||^{4s}, 10r|n^6 - n^{8s}|\right), \end{cases} \tag{12.74}$$

for all $x \in X$ and for $r > 0$.

Remark: Suppose that a mapping $f : X \to Y$ satisfies the inequality (10.73), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$ where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $C : X \to Y$, such that

$$P_{\mu,v}(f(x) - H(x), r) \geq_{L*} \begin{cases} P_{\mu,v}(\tau, 10r|63|) \\ P_{\mu,v}(\tau\|x\|^s, 10r|2^6 - 2^{2s}|) \\ P_{\mu,v}(\tau\|x\|^{4s}, 10r|2^6 - 2^{8s}|), \end{cases}$$

for all $x \in X$ and for $r > 0$.

Corollary 12.31 *Suppose that a mapping $f : X \to Y$ satisfies the inequality (12.72), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$ where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $C : X \to Y$, such that*

$$P_{\mu,v}(f(x) - C(x), r) \geq_L * \begin{cases} P'_{\mu,v}(\tau, 10r|n^9 - 1|) \\ P'_{\mu,v}\big(\tau||x||^s, 10r|n^9 - n^{3s}|\big) \\ P'_{\mu,v}\big(\tau||x||^{4s}, 10r|n^9 - n^{12s}|\big), \end{cases} \tag{12.75}$$

for all $x \in X$ and for $r > 0$.

Corollary 12.32 *Suppose that a mapping $ff : X \to Y$ satisfies the inequality (12.72), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$ where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $C : X \to Y$, such that*

$$P_{\mu,v}(f(x) - C(x), r) \geq_L * \begin{cases} P'_{\mu,v}(\tau, 10r|n^{12} - 1|) \\ P'_{\mu,v}\big(\tau||x||^s, 10r|n^{12} - n^{4s}|\big) \\ P'_{\mu,v}\big(\tau||x||^{4s}, 10r|n^{12} - n^{16s}|\big), \end{cases} \tag{12.76}$$

for all $x \in X$ and for $r > 0$.

12.6 Stability of the Cubic Functional Equation—Fixed Point Method

Proposition 12.33 *Let $f : X \to Y$ be a mapping for which there exists a mapping $\eta : X^4 \to Z$ with the condition*

$$\lim_{k\to\infty} P'_{\mu,v}(\eta_{(\beta_i^k x_1, \beta_i^k x_2, \beta_i^k x_3, \beta_i^k x_4)}, \beta^{3k} r) = 1_L^*, \tag{12.77}$$

where $\beta_i = \begin{cases} n, \; if \; i = 0 \\ \frac{1}{n} \; \; if \; i = 1 \end{cases}$ and $\wedge$ is the set such that $\wedge = \{a \setminus a : X \to Y, p(0) = 0\}$, for all $x_1, x_2, x_3, x_4 \in X, r > 0$ and satisfying the inequality

$$P_{\mu,v}(Df(x_1, x_2, x_3, x_4), r) \geq_L^* P'_{\mu,v}(\eta_{(x_1,x_2,x_3,x_4)}, r), \tag{12.78}$$

for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. If there exists $L = L[i]$ such that the function $x \to \epsilon(x)$ has the property

$$P'_{\mu,v}\left(L\frac{1}{\beta_i^3}(\beta_i x), r\right) = P'_{\mu,v}(\epsilon(x), r), \tag{12.79}$$

for all $x \in X$ and $r > 0$, then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equation (12.1) and

$$P_{\mu,v}(f(x) - C(x), r) \geq_L *P'_{\mu,v}\left(\epsilon(x), \frac{L^{1-i}}{1-L}r\right), \tag{12.80}$$

for all $x \in X$ and $r > 0$.

Example 12.34 Let $f : X \to Y$ be a mapping, for which there exists a function $\eta : X^4 \to Z$ with the condition (12.77) where $\beta_i = \begin{cases} 2, & if\ i = 0; \\ \frac{1}{2} & if\ i = 1; \end{cases}$ and $\wedge$ is the set such that $\wedge = \{a \setminus a : X \to Y, p(0) = 0\}$ and satisfying the inequality (12.78) for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. If there exists $L = L[i]$ such that the function $x \to \epsilon(x)$ has the property (12.79) for all $x \in X$ and $r > 0$, then there exists a unique cubic function $C : X \to Y$ satisfying the functional equations (12.1) and (12.80), for all $x \in X$ and $r > 0$.

Proposition 12.35 *Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^4 \to Z$ with the condition (12.77) where $\beta_i = \begin{cases} n^2, & if\ i = 0 \\ \frac{1}{n^2} & if\ i = 1 \end{cases}$ and $\wedge$ is the set such that $\wedge = \{a \setminus a : X \to Y, p(0) = 0\}$ and satisfying the inequality (12.78) for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. If there exists $L = L[i]$ such that the function $x \to \epsilon(x)$ has the property (12.79) for all $x \in X$ and $r > 0$, then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equations (12.1) and (12.80), for all $x \in X$ and $r > 0$.*

Example 12.36 Let $f : X \to Y$ be a mapping, there exists a function $\eta : X^4 \to Z$ with the condition (12.77) where $\beta_i = \begin{cases} 2^2, & if\ i = 0; \\ \frac{1}{2^2} & if\ i = 1; \end{cases}$ and $\wedge$ is the set such that $\wedge = \{a \setminus a : X \to Y, p(0) = 0\}$ and satisfying the inequality (12.78) for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. If there exists $L = L[i]$ such that the function $x \to \epsilon(x)$ has the property (12.79) for all $x \in X$ and $r > 0$, then there exists a unique cubic function $C : X \to Y$ satisfying the functional equations (12.1) and (12.80), for all $x \in X$ and $r > 0$.

Proposition 12.37 *Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^4 \to Z$ with the condition (12.77) where $\beta_i = \begin{cases} n^3, & if\ i = 0 \\ \frac{1}{n^3} & if\ i = 1 \end{cases}$ and $\wedge$ is the set such that $\wedge = \{a \setminus a : X \to Y, p(0) = 0\}$ and satisfying the inequality (12.78) for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. If there exists $L = L[i]$ such that the function*

$x \to \epsilon(x)$ has the property (12.79) for all $x \in X$ and $r > 0$, then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equations (12.1) and (12.80), for all $x \in X$ and $r > 0$.

Proposition 12.38 *Let $f : X \to Y$ be a mapping for which there exists a function $\eta : X^4 \to Z$ with the condition (12.77) where $\beta_i = \begin{cases} n^4, & if\ i = 0 \\ \frac{1}{n^4} & if\ i = 1 \end{cases}$ and $\wedge$ is the set such that $\wedge = \{a \setminus a : X \to Y, p(0) = 0\}$ and satisfying the inequality (12.78) for all $r > 0$ and $x_1, x_2, x_3, x_4 \in X$. If there exists $L = L[i]$ such that the function $x \to \epsilon(x)$ has the property (12.79) for all $x \in X$ and $r > 0$, then there exists a unique cubic mapping $C : X \to Y$ satisfying the functional equations (12.1) and (12.80) for all $x \in X$ and $r > 0$.*

The following corollaries are the immediate consequence of Propositions 12.33–12.38 respectively, concerning the stability of (12.1).

Corollary 12.39 *Suppose that a mapping $f : X \to Y$ satisfies the inequality (12.72), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $C : X \to Y$, such that the inequality (12.73), for all $x \in X$ and $r > 0$.*

Example 12.40 Suppose that a mapping $Df : X \to Y$ satisfies the inequality (10.73), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $C : X \to Y$, such that the inequality

$$P_{\mu,v}(f(x) - C(x), r) \geq_{L*} \begin{cases} P_{\mu,v}(\tau, 10r|7|) \\ P_{\mu,v}(\tau\|x\|^s, 10r|2^3 - 2^s|) \\ P_{\mu,v}(\tau\|x\|^{4s}, 10r|2^3 - 2^{4s}|), \end{cases}$$

for all $x \in X$ and for $r > 0$.

Corollary 12.41 *A mapping $f : X \to Y$ satisfies the inequality (12.72), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $C : X \to Y$, such that the inequality (12.74), for all $x \in X$ and $r > 0$.*

Remark: Suppose that a mapping $f : X \to Y$ satisfies the inequality (10.73), for all $x_1, x_2, x_3, x_4 \in X$ and $r > 0$, where τ, s are constants with $\tau > 0$. Then there exists a unique cubic mapping $C : X \to Y$, such that the inequality

$$P_{\mu,v}(f(x) - C(x), r) \geq_{L*} \begin{cases} P_{\mu,v}(\tau, 10r|63|) \\ P_{\mu,v}(\tau\|x\|^s, 10r|2^6 - 2^{2s}|) \\ P_{\mu,v}(\tau\|x\|^{4s}, 10r|2^6 - 2^{8s}|), \end{cases}$$

for all $x \in X$ and for $r > 0$.

Corollary 12.42 *Suppose that a mapping* $f : X \to Y$ *satisfies the inequality (12.72), for all* $x_1, x_2, x_3, x_4 \in X$ *and* $r > 0$*, where* τ, s *are constants with* $\tau > 0$*. Then there exists a unique cubic mapping* $C : X \to Y$*, such that the inequality (12.75), for all* $x \in X$ *and* $r > 0$*.*

Corollary 12.43 *Suppose that a function* $f : X \to Y$ *satisfies the inequality (12.72), for all* $x_1, x_2, x_3, x_4 \in X$ *and* $r > 0$*, where* τ, s *are constants with* $\tau > 0$*. Then there exists a unique cubic mapping* $C : X \to Y$ *such that the inequality (12.76), for all* $x \in X$ *and* $r > 0$*.*

Chapter 13
Cubic Homomorphisms and Cubic Derivations

13.1 Cubic Homomorphisms and Cubic Derivation

The topic of homomorphisms and derivations and their stability theory in the field of functional equations and inequalities have been taken up by several mathematicians.

If X and Y are normed spaces with the norms $\| \,.\, \|_X$ and $\| \,.\, \|_Y$, respectively, and $f : X \to Y$ is a mapping such that

$$\| f(x) + f(y) + f(z) \|_Y \le \left\| \frac{1}{q} f(qx + qy + qz) \right\|_Y,$$

for all $x, y, z \in X$ and for a fixed nonzero rational number q, then f is Cauchy additive.

Park [71] applied Gavruta's result to Banach modules over a C^*-algebra. Many authors have studied the structure of C^*-algebras for different types of functional equations in various settings one can refer [35, 50, 69, 70]. It seems that approximate derivations was first investigated by Jun and Park [47]. The stability of cubic derivations was first time introduced and investigated by Gordji et al. [23, 42, 75].

$$\begin{aligned} &2f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4) + f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4) \\ &+f(nx_1 - n^2x_2 + n^3x_3 + n^4x_4) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4) \\ &+f(nx_1 + n^2x_2 + n^3x_3 - n^4x_4) = 4[f(nx_1 + n^2x_2) + f(nx_1 + n^3x_3) \\ &+f(n^2x_2 + n^3x_3) + f(n^2x_2 + n^4x_4) + f(n^3x_3 + n^4x_4)] - 8[n^3 f(x_1) \\ &+n^6 f(x_2) + n^9 f(x_3) + n^{12} f(x_4)]. \end{aligned} \tag{13.1}$$

In this chapter, we discuss about the general solution and stability of the cubic homomorphisms and derivations of the functional equation (13.1) in quasi-Banach algebra with the help of direct and fixed point methods (see [27, 29, 37, 38]).

H. Dutta et al., *Stability of Some Advanced Functional Equations in Various Spaces*, Studies in Systems, Decision and Control 479,
https://doi.org/10.1007/978-3-031-33704-8_13

13.2 General Solution of Cubic Functional Equation

Theorem 13.1 *An odd mapping $f : X \to Y$ satisfies the functional equation (10.1), for all $x, y \in X$ if and only if $f : X \to Y$ satisfies the functional equation (13.1), for all $x_1, x_2, x_3, x_4 \in x$.*

Proof It follows from (11.1) that

$$\begin{aligned}
&f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4) = f(nx_1 + n^2x_2 + n^3x_3) + f(nx_1 + n^2x_2 \\
&+n^4x_4) + f(nx_1 + n^3x_3 + n^4x_4) + f(n^2x_2 + n^3x_3 + n^4x_4) - f(nx_1 + n^2x_2) \\
&-f(nx_1 + n^3x_3) - f(nx_1 + n^4x_4) - f(n^2x_2 + n^3x_3) - f(n^2x_2 + n^4x_4) \\
&-f(n^3x_3 + n^4x_4) + n^3 f(x_1) + n^6 f(x_2) + n^9 f(x_3) + n^{12} f(x_4),
\end{aligned} \tag{13.2}$$

for all $x_1, x_2, x_3, x_4 \in x$. From (12.1), we obtain

$$\begin{aligned}
&f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4) + f(nx_1 - n^2x_2 + n^3x_3 + n^4x_4) \\
&+f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4) + f(nx_1 + n^2x_2 + n^3x_3 - n^4x_4) \\
&= -2[f(nx_1 + n^2x_2 + n^3x_3) + f(nx_1 + n^2x_2 + n^4x_4) \\
&+f(nx_1 + n^3x_3 + n^4x_4) + f(n^2x_2 + n^3x_3 + n^4x_4)] \\
&+6[f(nx_1 + n^2x_2) + f(nx_1 + n^3x_3) + f(nx_1 + n^4x_4) \\
&+f(n^2x_2 + n^3x_3) + f(n^2x_2 + n^4x_4) + f(n^3x_3 + n^4x_4)] \\
&-10[n^3 f(x_1) + n^6 f(x_2) + n^9 f(x_3) + n^{12} f(x_4)],
\end{aligned} \tag{13.3}$$

for all $x_1, x_2, x_3, x_4 \in x$. Using (13.2) and (13.3), we get

$$\begin{aligned}
&2f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4) + f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4) \\
&+f(nx_1 - n^2x_2 + n^3x_3 + n^4x_4) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4) \\
&+f(nx_1 + n^2x_2 + n^3x_3 - n^4x_4) = 4[f(nx_1 + n^2x_2) + f(nx_1 + n^3x_3) \\
&+f(n^2x_2 + n^3x_3) + f(n^2x_2 + n^4x_4) + f(n^3x_3 + n^4x_4)] - 8[n^3 \\
&f(x_1) + n^6 f(x_2) + n^9 f(x_3) + n^{12} f(x_4)],
\end{aligned} \tag{13.4}$$

for all $x_1, x_2, x_3, x_4 \in X$. □

13.3 Cubic Homomorphisms in Quasi-Banach Algebras—Direct Method

Definition 13.2 Let X be a real linear space. A quasi-norm is real-valued function on X satisfying the following:

(1) $\| x \| > 0$, for all $x \in X$ and $\| x \| = 0$ if and only if $x = 0$.
(2) $\| \lambda x \| = | \lambda | \| x \|$, for all $\lambda \in p$ and for all $x \in X$.
(3) There is a constant $k \geq 1$ such that $\| x + y \| \geq K(\| x \| + \| y \|)$, for all $x, y \in X$.

The smallest possible K is called the modulus of concavity of $\| . \|$. A quasi-Banach space is a complete quasi-normed space.

Definition 13.3 A quasi-norm $\| . \|$ is called a p-norm $(0 < p < 1)$ if $\| x + y \|^p \leq \| x \|^p + \| y \|^p$, for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space.

Given a p-norm, the formula $d(x, y) = \| x - y \|^p$ gives us a translation invariant metric on X. Each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly with p-norms.

Definition 13.4 Let $(A, \| . \|)$ be a quasi-normed space. The quasi-normed space $(A, \| . \|)$ is called a quasi-normed algebra if A is an algebra and there is a constant $C > 0$ such that $\| xy \| \leq C \| x \| \| y \|$, for all $x, y \in X$.

Definition 13.5 A quasi-Banach algebra is a complete quasi-normed algebra. If the quasi norm $\| . \|$ is a p-norm, then the quasi-Banach algebra is called a p-Banach algebra.

Definition 13.6 Let A, B be two algebras. A mapping $f : A \to B$ is called a cubic homomorphism if f is a cubic mapping satisfying $f(xy) = f(x)f(y)$, for all $x, y \in A$. For instance, let A be commutative. Then the mapping $f : A \to A$ defined by $f(x) = x^3$ $(x \in A)$, is a cubic homomorphism.

Definition 13.7 A cubic mapping $f : A \to A$ is called a cubic derivation if f is a cubic mapping satisfying $f(xy) = x^3 f(y) + f(x)y^3$, for all $x, y \in A$.

Remark: We assume that is a quasi-normed algebra with quasi-norm $\| . \|_A$ and that B is a p-Banach algebra with p-norm $\| . \|_B$. For convenience, we define a mapping $F : A \to B$ by

$$\begin{aligned} F(x_1, x_2, x_3, x_4) &= 2f(nx_1 + n^2x_2 + n^3x_3 + n^4x_4) + f(-nx_1 + n^2x_2 + n^3x_3 + n^4x_4) \\ &+ f(nx_1 - n^2x_2 + n^3x_3 + n^4x_4) + f(nx_1 + n^2x_2 - n^3x_3 + n^4x_4) \\ &+ f(nx_1 + n^2x_2 + n^3x_3 - n^4x_4) - 4[f(nx_1 + n^2x_2) + f(nx_1 + n^3x_3) \\ &+ f(nx_1 + n^4x_4) + f(n^2x_2 + n^3x_3) + f(n^2x_2 + n^4x_4) + f(n^3x_3 + n^4x_4)] \\ &+ 8[n^3 f(x_1) + n^6 f(x_2) + n^9 f(x_3) + n^{12} f(x_4)], \end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in A$.

Definition 13.8 A mapping $H : A \to B$ is called a cubic homomorphism in quasi-Banach algebras if

$$H(xy) = H(x)H(y), \tag{13.5}$$

for all $x, y \in A$.

Theorem 13.9 *Let $j \in \{-1, 1\}$, $\varphi : A^4 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\varphi(n^{kj}x_1, n^{kj}x_2, n^{kj}x_3, n^{kj}x_4)}{n^{3kj}} \tag{13.6}$$

converges to $\mathbb{R}$ and $f : A :\to B$ be a mapping satisfying inequality

$$\| f(x_1, x_2, x_3, x_4) \|_B \leq \varphi(x_1, x_2, x_3, x_4), \tag{13.7}$$

for all $x_1, x_2, x_3, x_4 \in A$ and

$$\| f(xy) - f(x)f(y) \|_B \leq \varphi(x, y, 0, 0), \tag{13.8}$$

for all $x, y \in A$. Then there exists a unique cubic homomorphism $H : A \to B$ which satisfies (13.1) and

$$\| f(x) - H(x) \|_B \leq \frac{1}{8n^3} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\varphi(n^{kj}x, 0, 0, 0)}{n^{3kj}}, \tag{13.9}$$

for all $x \in A$. The mapping $H(x)$ is defined by

$$H(x) = \lim_{k\to\infty} \frac{f(n^{kj}x)}{n^{3kj}}, \tag{13.10}$$

for all $x \in A$.

Proof Assume $j = 1$. Replacing (x_1, x_2, x_3, x_4) by $(x, 0, 0, 0)$ and dividing by $8n^3$ in (13.1), we obtain

$$\left\| \frac{f(nx)}{n^3} - f(x) \right\|_B \leq \frac{1}{8n^3}\varphi(x, 0, 0, 0), \tag{13.11}$$

for all $x \in A$. Replacing x by nx in (13.11), we get

$$\left\| \frac{f(n^2x)}{n^6} - \frac{f(x)}{n^3} \right\|_B \leq \frac{1}{8n^6}\varphi(nx, 0, 0, 0), \tag{13.12}$$

for all $x \in A$. Combining (13.11) and (13.12), we get

$$\left\|\frac{f(n^2x)}{n^6} - f(x)\right\|_B \leq \frac{1}{8n^3}\left[\varphi(x, 0, 0, 0) + \frac{\varphi(nx, 0, 0, 0)}{n^3} + \right], \quad (13.13)$$

for all $x \in A$. Using induction on positive integer k, we get

$$\left\|\frac{f(n^i x)}{n^3 i} - f(x)\right\|_B \leq \frac{1}{8n^3}\sum_{k=0}^{i-1}\frac{\varphi(n^k x, 0, 0, 0)}{n^{3k}},$$

$$\left\|\frac{f(n^i x)}{n^3 i} - f(x)\right\|_B \leq \frac{1}{8n^3}\sum_{k=0}^{\infty}\frac{\varphi(n^k x, 0, 0, 0)}{n^{3k}}, \quad (13.14)$$

for all $x \in A$. In order to prove the convergence of the sequence $\left\{\frac{f(n^i x)}{n^3 i}\right\}$, replacing x by $n^m x$ and dividing n^{3m} in (13.14) , for any $m, k > 0$, we get

$$\left\|\frac{f(n^{i+m}x)}{n^{3(i+m)}} - \frac{f(n^m x)}{n^{3m}}\right\|_B \leq \frac{1}{8n^3}\sum_{k=0}^{i-1}\frac{\varphi(n^{k+m}x, 0, 0, 0)}{n^{3(k+m)}}, \to 0 \ \ as \ \ m \to \infty$$

for all $x \in A$. Since the right hand side of the inequality (13.15) tends to 0 as $m \to \infty$, the sequence $\left\{\frac{f(n^i x)}{n^3 i}\right\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $H : A \to B$ such that

$$H(x) = \lim_{i\to\infty}\frac{f(n^i x)}{n^{3i}},$$

for all $x \in A$. Letting $k \to \infty$ in (13.14), we see that (13.9) holds for all $x \in A$. Now, we need to prove H satisfies (13.1), replacing (x_1, x_2, x_3, x_4) by $(n^m x, n^m x, n^m x, n^m x)$ and dividing by n^{3m} in (13.7), we get

$$\frac{1}{n^{3m}} \parallel f(n^m x, n^m x, n^m x, n^m x) \parallel_B \leq \frac{1}{n^{3m}}\varphi(n^m x, n^m x, n^m x, n^m x),$$

for all $x_1, x_2, x_3, x_4 \in A$. Letting $k \to \infty$ in the above inequalities, we have

$$\parallel H(x_1, x_2, x_3, x_4) \parallel_B = 0.$$

Hence H satisfies (13.1), for all $x_1, x_2, x_3, x_4 \in A$. This shows that H is cubic. Also

$$\begin{aligned}\parallel H(xy) - H(x)H(y) \parallel_B &= \lim_{k\to\infty}\frac{1}{n^{9k}} \parallel f(n^{3k}xy) - f(n^{3k}x)f(n^{3k}y) \parallel_B,\\ &\leq \lim_{k\to\infty}\frac{1}{n^{9k}}\varphi(n^k x, 0, 0, 0) = 0, \quad (13.15)\end{aligned}$$

for all $x, y \in A$. Therefore, H is a cubic homomorphism. In order to prove H is unique, let $H'(x)$ be another cubic homomorphism satisfying (13.9) and (13.1). Then

$$\begin{aligned} \| H(x) - H'(x) \|_B &= \frac{1}{n^{3k}} \| H(n^k x) - H'(n^k x) \|_B \\ &\leq \frac{1}{n^{3k}} \left\{ \| H(n^k x) - f(n^k x) \|_B + \| f(n^k x) - H'(n^k x) \|_B \right\} \\ &\leq \frac{2}{8n^3} \sum_{i=0}^{\infty} \frac{1}{n^{3(i+k)}} \varphi(n^{i+k}x, n^{i+k}x, n^{i+k}x, n^{i+k}x) \\ &\to 0 \ \ as \ \ k \to \infty, \end{aligned} \tag{13.16}$$

for all $x \in A$. Hence H is unique. Thus the mapping $H : A \to B$ is a unique cubic homomorphism satisfying (13.9).

For $j = -1$, we can prove the similar stability result. This completes the proof of the theorem. □

Remark: Let $j \in \{-1, 1\}$. Let $\varphi : A^4 \to [0, \infty)$ be a mapping such that

$$\sum_{k=0}^{\infty} \frac{\varphi(2^{kj}x_1, 2^{kj}x_2, 2^{kj}x_3, 2^{kj}x_4)}{n^{3kj}}$$

converges to $\mathbb{R}$ and $f : A :\to B$ be a mapping satisfying (13.7), for all $x_1, x_2, x_3, x_4 \in A$ and (13.8), for all $x, y \in A$. Then there exists a unique cubic homomorphism $H : A \to B$ which satisfies (13.1) and

$$\| f(x) - H(x) \|_B \leq \frac{1}{64} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\varphi(2^{kj}x, 0, 0, 0)}{2^{3kj}},$$

for all $x \in A$. The mapping $H(x)$ is defined by

$$H(x) = \lim_{k\to\infty} \frac{f(2^{kj}x)}{2^{3kj}},$$

for all $x \in A$.

Proposition 13.10 *Let $j \in \{-1, 1\}$. Let $\varphi : A^4 \to [0, \infty)$ be a mapping such that*

$$\sum_{k=0}^{\infty} \frac{\varphi(n^{2kj}x_1, n^{2kj}x_2, n^{2kj}x_3, n^{2kj}x_4)}{n^{6kj}},$$

converges to $\mathbb{R}$ and $f : A \to B$ be a mapping satisfying (13.7), for all $x_1, x_2, x_3, x_4 \in A$ and (13.8), for all $x, y \in A$. Then there exists a unique cubic homomorphism $H : A \to B$ which satisfies (13.1) and

$$\| f(x) - H(x) \|_B \leq \frac{1}{8n^6} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\varphi(0, n^{2kj}x, 0, 0)}{n^{6kj}},$$

for all $x \in A$. The mapping $H(x)$ is defined by

$$H(x) = \lim_{k\to\infty} \frac{f(n^{2kj}x)}{n^{6kj}},$$

for all $x \in A$.

Remark: Let $j \in \{-1, 1\}$. Let $\varphi : A^4 \to [0, \infty)$ be a function such that

$$\sum_{k=0}^{\infty} \frac{\varphi(n^{2kj}x_1, n^{2kj}x_2, n^{2kj}x_3, n^{2kj}x_4)}{n^{6kj}}$$

converges to $\mathbb{R}$ and $f : A \to B$ be a mapping satisfying (13.7), for all $x_1, x_2, x_3, x_4 \in A$ and (13.8), for all $x, y \in A$. Then there exists a unique cubic homomorphism $H : A \to B$ which satisfies (13.1) and

$$\| f(x) - H(x) \|_B \leq \frac{1}{512} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\varphi(0, 2^{2kj}x, 0, 0)}{n^{6kj}},$$

for all $x \in A$. The mapping $H(x)$ is defined by

$$H(x) = \lim_{k\to\infty} \frac{f(2^{2kj}x)}{2^{6kj}},$$

for all $x \in A$.

Proposition 13.11 *Let $j \in \{-1, 1\}$. Let $\varphi : A^4 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\varphi(n^{3kj}x_1, n^{3kj}x_2, n^{3kj}x_3, n^{3kj}x_4)}{n^{9kj}},$$

converges to $\mathbb{R}$ and $f : A \to B$ be a mapping satisfying (13.7), for all $x_1, x_2, x_3, x_4 \in A$ and (13.8), for all $x, y \in A$. Then there exists a unique cubic homomorphism $H : A \to B$ which satisfies (13.1) and

$$\| f(x) - H(x) \|_B \leq \frac{1}{8n^9} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\varphi(0, 0, n^{3kj}x, 0)}{n^{9kj}},$$

for all $x \in A$. The mapping $H(x)$ is defined by

$$H(x) = \lim_{k\to\infty} \frac{f(n^{3kj}x)}{n^{9kj}},$$

for all $x \in A$.

Proposition 13.12 *Let* $j \in \{-1, 1\}$. *Let* $\varphi : A^4 \to [0, \infty)$ *be a function such that*

$$\sum_{k=0}^{\infty} \frac{\varphi(n^{4kj}x_1, n^{4kj}x_2, n^{4kj}x_3, n^{4kj}x_4)}{n^{12kj}},$$

converges to $\mathbb{R}$ *and* $f : A \to B$ *be a mapping satisfying (13.7), for all* $x_1, x_2, x_3, x_4 \in A$ *and (13.8), for all* $x, y \in A$. *Then there exists a unique cubic homomorphism* $H : A \to B$ *which satisfies (13.1) and*

$$\| f(x) - H(x) \|_B \le \frac{1}{8n^{12}} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\varphi(0, 0, 0, n^{4kj}x)}{n^{12kj}},$$

for all $x \in A$. *The mapping* $H(x)$ *is defined by*

$$H(x) = \lim_{k\to\infty} \frac{f(n^{4kj}x)}{n^{12kj}},$$

for all $x \in A$.

The following corollaries are the immediate consequence of Theorems 13.9–13.12 respectively, concerning the stability of (13.1).

Corollary 13.13 *Let* λ *and* s *be non-negative real numbers. If a mapping* $f : A \to B$ *satisfies the stability*

$$\| F(x_1, x_2, x_3, x_4) \|_B \le \begin{cases} \lambda \\ \lambda\left\{ \sum_{i=1}^{4} \| x_i \|_A^s \right\} \\ \lambda\left\{ \prod_{i=1}^{4} \| x_i \|_A^s + \sum_{i=1}^{4} \| x_i \|_A^{4s} \right\}, \end{cases} \tag{13.17}$$

for all $x_1, x_2, x_3, x_4 \in A$ *and*

$$\| f(xy) - f(x)f(y) \|_B \le \begin{cases} \lambda \\ \lambda\left\{ \sum_{i=1}^{4} \| x_i \|_A^s \right\}, \quad s \neq 3 \\ \lambda\left\{ \prod_{i=1}^{4} \| x_i \|_A^s + \sum_{i=1}^{4} \| x_i \|_A^{4s} \right\} \quad s \neq \frac{3}{4}, \end{cases} \tag{13.18}$$

for all $x, y \in A$, then there exists a unique cubic homomorphism $H : A \to B$ such that

$$\| f(x) - H(x) \| \leq \begin{cases} \frac{\lambda}{8|n^3-1|} \\ \frac{\lambda \|x\|_A^s}{8|n^3-n^s|} \\ \frac{\lambda \|x\|_A^{4s}}{8|n^3-n^{4s}|}, \end{cases} \tag{13.19}$$

for all $x \in A$.

Example 13.14 Let λ and s be non-negative real numbers and a mapping $f : A \to B$ satisfy (13.17), for all $x_1, x_2, x_3, x_4 \in A$ and (13.18), for all $x, y \in A$. Then there exists a unique cubic homomorphism $H : A \to B$ such that

$$\|f(x) - C(x)\|_A \leq \begin{cases} \frac{\lambda}{8|7|} \\ \frac{\lambda \|x\|_A^s}{8|2^3-2^s|} \\ \frac{\lambda \|x\|_A^{4s}}{8|2^3-2^{4s}|}, \end{cases}$$

for all $x \in A$.

Corollary 13.15 *Let λ and s be non-negative real numbers. If a mapping $F : A \to B$ satisfies (13.17), for all $x_1, x_2, x_3, x_4 \in A$ and (13.18), for all $x, y \in A$, then there exists a unique cubic homomorphism $H : A \to B$ such that*

$$\| f(x) - H(x) \| \leq \begin{cases} \frac{\lambda}{8|n^6-1|} \\ \frac{\lambda \|x\|_A^s}{8|n^6-n^{2s}|} \\ \frac{\lambda \|x\|_A^{4s}}{8|n^6-n^{8s}|}, \end{cases} \tag{13.20}$$

for all $x \in A$.

Example 13.16 Let λ and s be non-negative real numbers and a mapping $f : A \to B$ satisfies (13.17), for all $x_1, x_2, x_3, x_4 \in A$ and (13.18), for all $x, y \in A$. Then there exists a unique cubic homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \begin{cases} \frac{\lambda}{8|2^6-1|} \\ \frac{\lambda \|x\|_A^s}{8|2^6-2^{2s}|} \\ \frac{\lambda \|x\|_A^{4s}}{8|2^6-2^{8s}|}, \end{cases}$$

for all $x \in A$.

Corollary 13.17 *Let λ and s be non-negative real numbers. If a mapping $F : A \to B$ satisfies (13.17), for all $x_1, x_2, x_3, x_4 \in A$ and (13.18), for all $x, y \in A$, then there exists a unique cubic homomorphism $H : A \to B$ such that*

$$\| f(x) - H(x) \| \leq \begin{cases} \frac{\lambda}{8|n^9-1|} \\ \frac{\lambda\|x\|_A^s}{8|n^9-n^{3s}|} \\ \frac{\lambda\|x\|_A^{4s}}{8|n^9-n^{12s}|}, \end{cases} \tag{13.21}$$

for all $x \in A$.

Corollary 13.18 *Let* λ *and* s *be non-negative real numbers. If a mapping* $F : A \to B$ *satisfies (13.17), for all* $x_1, x_2, x_3, x_4 \in A$ *and (13.18), for all* $x, y \in A$*, then there exists a unique cubic homomorphism* $H : A \to B$ *such that*

$$\| f(x) - H(x) \| \leq \begin{cases} \frac{\lambda}{8|n^{12}-1|} \\ \frac{\lambda\|x\|_A^s}{8|n^{12}-n^{4s}|} \\ \frac{\lambda\|x\|_A^{4s}}{8|n^{12}-n^{16s}|}, \end{cases} \tag{13.22}$$

for all $x \in A$.

13.4 Cubic Homomorphisms in Quasi-Banach Algebras—Fixed Point Method

Theorem 13.19 *Let* $F : A \to B$ *be a mapping for which there exists a function* $\varphi : A^4 \to [0, \infty)$ *with condition*

$$\lim_{k\to\infty} \frac{\varphi(\sigma_i^k x_1, \sigma_i^k x_2, \sigma_i^k x_3, \sigma_i^k x_4)}{\sigma_i^{3k}} = 0, \tag{13.23}$$

where $\sigma_i = \begin{cases} n & if \;\; i = 0 \\ \frac{1}{n} & if \;\; i = 1 \end{cases}$ *such that the functional inequality (13.7), for all* $x_1, x_2,$ $x_3, x_4 \in A$ *and (13.8), for all* $x, y \in A$*. If there exists* $L = L(i) < 1$ *such that the function*

$$x \to \delta(x) = \frac{1}{8}\left(\frac{x}{n}, 0, 0, 0\right),$$

has the property

$$\delta(x) = L\frac{1}{\sigma_i^3}\delta(\sigma_i x), \tag{13.24}$$

then there exists a unique cubic homomorphism $H : A \to B$ *satisfying the functional equation (13.1) and*

$$\| f(x) - H(x) \|_B \leq \frac{L^{1-i}}{1-L}\delta(x), \tag{13.25}$$

for all $x \in A$.

Proof Consider the set $\tau = \{p | p : A \to B, p(0) = 0\}$ and introduce the generalized metric on τ.

$$d(p, q) = inf\{k \in (0, \infty); \| p(x) - q(x) \|_B \leq k\delta(x), x \in A\}.$$

It is easy to see that (τ, d) is complete. Define $T : \tau \to \tau$ by $Tp(x) = \frac{1}{\sigma_i^3} p(\sigma_i x)$, for all $x \in A$.

Now for $p, q \in \tau$, we have

$$\begin{aligned}
&d(p, q) \leq k \Rightarrow \| p(x) - q(x) \|_B \leq k\delta(x), x \in A, \\
&\Rightarrow \left\| \frac{1}{\sigma_i^3} p(\sigma_i x) - \frac{1}{\sigma_i^3} q(\sigma_i x) \right\| \leq \frac{1}{\sigma_i^3} k\delta(\sigma_i x), x \in A, \\
&\Rightarrow \left\| \frac{1}{\sigma_i^3} p(\sigma_i x) - \frac{1}{\sigma_i^3} q(\sigma_i x) \right\| \leq Lk\delta(x), x \in A, \\
&\Rightarrow \| Tp(x) - Tq(x) \|_B \leq Lk\delta(x), x \in A \\
&\Rightarrow d(p, q) \leq Lk.
\end{aligned} \tag{13.26}$$

This implies $d(Tp, Tq) \leq Ld(p, q)$, for all $p, q \in \tau$. That is, T is strictly contractive mapping on τ with Lipschitz contant L. Replacing (x_1, x_2, x_3, x_4) by $(x, 0, 0, 0)$ and dividing by 8 in (13.7), we get

$$\| f(nx) - n^3 f(x) \|_B \leq \frac{1}{8}\varphi(x, 0, 0, 0), \tag{13.27}$$

for all $x \in A$. Hence from the above inequality, we have

$$\| \frac{f(nx)}{n^3} - f(x) \|_B \leq \frac{1}{8n^3}\varphi(x, 0, 0, 0), \tag{13.28}$$

for all $x \in A$. Using (13.24) for the case $i = 0$, we get

$$\| \frac{f(nx)}{n^3} - f(x) \|_B \leq \frac{1}{n^3}\delta(x),$$

for all $x \in A$. ie., $d(f, Tf) \leq L \leq L^1 < \infty$. Again replacing x by $\frac{x}{n}$ in (13.27), we get

$$\| f(x) - n^3 f(\frac{x}{n}) \|_B \leq \frac{1}{8}\varphi(x, 0, 0, 0), \tag{13.29}$$

for all $x \in A$. Using (13.24) for the case $i = 1$, we get

$$\| f(x) - n^3 f(\frac{x}{n}) \|_B \leq \delta(x),$$

for all $x \in A$. i.e., $d(f, Tf) \leq L \leq L^0 < \infty$. In both cases, we obtain

$$d(f, Tf) \leq L^{1-i}.$$

Therefore (A1) holds. By (A2), it follows that there exists a fixed point H of T in τ such that

$$H(x) = \lim_{k\to\infty} \frac{1}{\sigma_i^{3k}} f(\sigma_i^k x), \tag{13.30}$$

for all $x \in A$. To prove $H : A \to B$ is cubic, replacing (x_1, x_2, x_3, x_4) by $(\sigma_i^k x_1, \sigma_i^k x_2, \sigma_i^k x_3, \sigma_i^k x_4)$ in (3.2) and dividing by σ_i^{3k}, it follows from (13.23) that

$$\begin{aligned} \| H(x_1, x_2, x_3, x_4) \|_B &\leq \lim_{k\to\infty} \frac{\| F(\sigma_i^k x_1, \sigma_i^k x_2, \sigma_i^k x_3, \sigma_i^k x_4) \|_B}{\sigma_i^{3k}}, \\ &\leq \lim_{k\to\infty} \frac{\varphi(\sigma_i^k x_1, \sigma_i^k x_2, \sigma_i^k x_3, \sigma_i^k x_4)}{\sigma_i^{3k}}, \end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in A$. That is, H satisfies the functional equation (13.1). Also

$$\begin{aligned} \| H(xy) - H(x)H(y) \|_B &\leq \lim_{k\to\infty} \frac{1}{\sigma_i^{9k}} \| f(\sigma_i^{3k} xy) - f(\sigma_i^{3k} x) f(\sigma_i^{3k} y) \|_B \\ &\leq \lim_{k\to\infty} \frac{\varphi(\sigma_i^k, 0, 0, 0)}{\sigma_i^{9k}} = 0, \end{aligned}$$

for all $x \in A$. Therefore, H is a cubic homomorphism. By (A3), H is the unique fixed point of T in the set $\Delta = \{H \in \tau : d(f, H) < \infty\}$; H is the unique mapping, such that

$$\| f(x) - H(x) \|_B \leq k\delta(x),$$

for all $x \in A$ and $k > 0$. Finally by (A4), we obtain

$$d(f, H) \leq \frac{1}{1-L} d(f, Tf).$$

This implies

$$d(f, H) \leq \frac{L(1-i)}{1-L},$$

which yields

$$\| f(x) - H(x) \|_B \leq \frac{L(1-i)}{1-L}\delta(x).$$

This completes the proof of the theorem. □

Example 13.20 Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^4 \to [0, \infty)$ with condition (13.23) where $\sigma_i = \begin{cases} 2 \; if \; i = 0; \\ \frac{1}{2} \; if \; i = 1; \end{cases}$ such that the functional inequality (13.7), for all $x_1, x_2, x_3, x_4 \in A$ and (13.8), for all $x, y \in A$. If there exists $L = L(i) < 1$, such that the function

$$x \to \delta(x) = \frac{1}{8}\left(\frac{x}{2}, 0, 0, 0\right),$$

has the property (13.24), then there exists a cubic homomorphism $H : A \to B$ satisfying the functional equations (13.1) and (13.25), for all $x \in A$.

Proposition 13.21 *Let $f : A \to B$ be a mapping for which there exists a mapping $\varphi : A^4 \to [0, \infty)$ with condition (13.23) where $\sigma_i = \begin{cases} n^2 \; if \; i = 0 \\ \frac{1}{n^2} \; if \; i = 1 \end{cases}$ such that the functional inequality with (13.7), for all $x_1, x_2, x_3, x_4 \in A$ and (13.8), for all $x, y \in A$. If there exists $L = L(i) < 1$, such that the function*

$$x \to \delta(x) = \frac{1}{8}\left(0, \frac{x}{n^2}, 0, 0\right),$$

has the property (13.24), then there exists a cubic homomorphism $H : A \to B$ satisfying the functional equations (13.1) and (13.25), for all $x \in A$.

Example 13.22 Let $f : A \to B$ be a mapping for which there exists a mapping $\varphi : A^4 \to [0, \infty)$ with condition (13.23) where $\sigma_i = \begin{cases} 2^2 \; if \; i = 0; \\ \frac{1}{2^{2^2}} \; if \; i = 1; \end{cases}$ such that the functional inequality (13.7), for all $x_1, x_2, x_3, x_4 \in A$ and (13.8), for all $x, y \in A$. If there exists $L = L(i) < 1$, such that the function

$$x \to \delta(x) = \frac{1}{8}\left(0, \frac{x}{2^2}, 0, 0\right),$$

has the property (13.24), then there exists a cubic homomorphism $H : A \to B$ satisfying the functional equations (13.1) and (13.25), for all $x \in A$.

Proposition 13.23 *Let $f : A \to B$ be a mapping for which there exists a mapping $\varphi : A^4 \to [0, \infty)$ with condition (13.23) where $\sigma_i = \begin{cases} n^3 \; if \; i = 0 \\ \frac{1}{n^3} \; if \; i = 1 \end{cases}$ such that the functional inequality (13.7), for all $x_1, x_2, x_3, x_4 \in A$ and (13.8), for all $x, y \in A$. If there exists $L = L(i) < 1$, such that the function*

$$x \to \delta(x) = \frac{1}{8}\left(0, 0, \frac{x}{n^3}, 0\right),$$

has the property (13.24), then there exists a cubic homomorphism $H : A \to B$ satisfying the functional equations (13.1) and (13.25), for all $x \in A$.

Proposition 13.24 *Let $f : A \to B$ be a mapping for which there exists a mapping $\varphi : A^4 \to [0, \infty)$ with condition (13.23) where $\sigma_i = \begin{cases} n^4 & if \;\; i = 0 \\ \frac{1}{n^4} & if \;\; i = 1 \end{cases}$ such that the functional inequality (13.7), for all $x_1, x_2, x_3, x_4 \in A$ and (13.8), for all $x, y \in A$. If there exists $L = L(i) < 1$ such that the function*

$$x \to \delta(x) = \frac{1}{8}\left(0, 0, 0, \frac{x}{n^4}\right)$$

has the property (13.24), then there exists a cubic homomorphism $H : A \to B$ satisfying the functional equations (13.1) and (13.25), for all $x \in A$.

The corollaries are the immediate consequence of Theorems 13.20- 13.23 respectively, concerning the stability of (13.1).

Corollary 13.25 *Let $F : A \to B$ be a mapping and there exist real numbers λ and s such that (13.1), for all $x_1, x_2, x_3, x_4 \in A$ and (13.18), for all $x, y \in A$. Then there exists a unique cubic homomorphism $H : A \to B$ such that (13.19), for all $x \in A$.*

Proof Set

$$\varphi(x_1, x_2, x_3, x_4) = \begin{cases} \lambda \\ \lambda\left\{ \sum_{i=1}^4 \| x_i \|_A^s \right\} \\ \lambda\left\{ \prod_{i=1}^4 \| x_i \|_A^s + \sum_{i=1}^4 \| x_i \|_A^{4s} \right\}, \end{cases}$$

for all $x_1, x_2, x_3, x_4 \in A$. Now

$$\begin{aligned} \frac{\varphi(\sigma_i^k x_1, \sigma_i^k x_2, \sigma_i^k x_3, \sigma_i^k x_4)}{\sigma_i^{3k}} &= \begin{cases} \frac{\lambda}{\sigma_i^{3k}} \\ \frac{\lambda}{\sigma_i^{3k}}\left\{ \sum_{i=1}^4 \| \sigma_i^k x_i \|_A^s \right\} \\ \frac{\lambda}{\sigma_i^{3k}}\left\{ \prod_{i=1}^4 \| \sigma_i^k x_i \|_A^s + \sum_{i=1}^4 \| \sigma_i^k x_i \|_A^{4s} \right\} \end{cases} \\ &= \begin{cases} \lambda \sigma_i^{-3k} \\ \lambda \sigma_i^{(s-3)k}\left\{ \sum_{i=1}^4 \| x_i \|_A^s \right\} \\ \lambda \sigma_i^{(4s-3)k}\left\{ \prod_{i=1}^4 \| x_i \|_A^s + \sum_{i=1}^4 \| x_i \|_A^{4s} \right\} \end{cases} \end{aligned}$$

$$= \begin{cases} \to 0 \ as \ k \to \infty \\ \to 0 \ as \ k \to \infty \\ \to 0 \ as \ k \to \infty. \end{cases} \tag{13.31}$$

Thus, (13.23) holds. But we have

$$\delta(x) = \frac{1}{8}\varphi\left(\frac{x}{n}, 0, 0, 0\right),$$

has the property

$$\delta(x) = L\frac{1}{\sigma_i^3}\delta(\sigma_i x),$$

for all $x \in A$. Hence

$$\delta(x) = \frac{1}{8}\varphi\left(\frac{x}{n}, 0, 0, 0\right) = \begin{cases} \frac{\lambda}{8} \\ \frac{\lambda \|x\|_A^s}{8n^3} \\ \frac{\lambda \|x\|_A^{4s}}{8n^3}. \end{cases}$$

Now,

$$\frac{1}{\sigma_i^3}\delta(\sigma_i x) = \begin{Bmatrix} \frac{\lambda}{8\sigma_i^3} \\ \frac{\lambda \|\sigma_i x\|_A^s}{8\sigma_i^3 n^3} \\ \frac{\lambda \|\sigma_i x\|_A^{4s}}{8\sigma_i^3 n^3} \end{Bmatrix} = \begin{Bmatrix} \sigma_i^{-3}\delta(x) \\ \sigma_i^{s-3}\delta(x) \\ \sigma_i^{4s-3}\delta(x) \end{Bmatrix} \tag{13.32}$$

for all $x \in A$. From (13.25), we prove the following six cases:

Case: 1 If $i = 0$ then $L = n^{-3}$

$$\| f(x) - H(x) \|_B \leq \frac{n^{-3}}{1 - n^{-3}}\delta(x) = \frac{\lambda}{8(n^3 - 1)}.$$

Case: 2 If $i = 1$ then $L = n^3$

$$\| f(x) - H(x) \|_B \leq \frac{1}{1 - n^3}\delta(x) = \frac{\lambda}{8(1 - n^3)}$$

Case: 3 If $L = n^{s-3}$ for $s < 3$ if $i = 0$

$$\| f(x) - H(x) \|_B \leq \frac{n^{s-3}}{1 - n^{s-3}}\delta(x) = \frac{\lambda \| x \|_A^s}{8(n^3 - n^s)}$$

Case: 4 If $L = n^{3-s}$ for $s > 3$ if $i = 1$

$$\| f(x) - H(x) \|_B \leq \frac{n^{3-s}}{1 - n^{3-s}} \delta(x) = \frac{\lambda \| x \|_A^s}{8(n^s - n^3)}$$

Case: 5 $L = n^{4s-3}$ for $s < \frac{3}{4}$ if $i = 0$

$$\| f(x) - H(x) \|_B \leq \frac{n^{4s-3}}{1 - n^{4s-3}} \delta(x) = \frac{\lambda \| x \|_A^{4s}}{8(n^3 - n^{4s})}$$

Case: 6 $L = n^{3-4s}$ for $s > \frac{3}{4}$ if $i = 1$

$$\| f(x) - H(x) \|_B \leq \frac{1}{1 - n^{3-4s}} \delta(x) = \frac{\lambda \| x \|_A^{4s}}{8(n^{4s} - n^3)}$$

Hence the proof is completed. □

Remark: Let $f : A \to B$ be a mapping and there exist real numbers λ and s such that (13.17), for all $x_1, x_2, x_3, x_4 \in A$ and (13.18), for all $x, y \in A$. Then there exists a unique cubic homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \begin{cases} \frac{\lambda}{8|7|} \\ \frac{\lambda\|x\|_A^s}{8|2^3-2^s|} \\ \frac{\lambda\|x\|_A^{4s}}{8|2^3-2^{4s}|}, \end{cases}$$

for all $x \in A$.

Corollary 13.26 *Let $f : A \to B$ be a mapping and there exist real numbers λ and s such that (13.17), for all $x_1, x_2, x_3, x_4 \in A$ and (13.18), for all $x, y \in A$. Then there exists a unique cubic homomorphism $H : A \to B$ such that (13.20), for all $x \in A$.*

Remark: Let $f : A \to B$ be a mapping and there exist real numbers λ and s such that (13.17), for all $x_1, x_2, x_3, x_4 \in A$ and (13.18), for all $x, y \in A$. Then there exists a unique cubic homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \begin{cases} \frac{\lambda}{8|2^6-1|} \\ \frac{\lambda\|x\|_A^s}{8|2^6-2^{2s}|} \\ \frac{\lambda\|x\|_A^{4s}}{8|2^6-2^{8s}|}, \end{cases}$$

for all $x \in A$.

Corollary 13.27 *Let $f : A \to B$ be a mapping and there exist real numbers λ and s such that (13.17), for all $x_1, x_2, x_3, x_4 \in A$ and (13.18), for all $x, y \in A$. Then there exists a unique cubic homomorphism $H : A \to B$, such that (13.21), for all $x \in A$.*

Corollary 13.28 *Let $f : A \to B$ be a mapping and there exist real numbers λ and s such that (13.17), for all $x_1, x_2, x_3, x_4 \in A$ and (13.18), for all $x, y \in A$. Then there exists a unique cubic homomorphism $H : A \to B$, such that (13.22), for all $x \in A$.*

13.5 Stability of Cubic Derivation Quasi-Banach Algebras—Direct Method

Definition 13.29 A mapping $D : A \to A$ is called a cubic derivation in quasi-Banach algebras if

$$D(xy) = D(x)y^3 + x^3 D(y) \tag{13.33}$$

for all $x, y \in A$.

Theorem 13.30 *Let $j \in \{-1, 1\}$, $\varphi : A^4 \to [0, \infty)$ be a mapping such that*

$$\sum_{k=0}^{\infty} \frac{\varphi(n^{kj}x_1, n^{kj}x_2, n^{kj}x_3, n^{kj}x_4)}{n^{3kj}}, \tag{13.34}$$

converges to $\mathbb{R}$ and $f : A \to A$ be a mapping satisfying inequality

$$\| f(x_1, x_2, x_3, x_4) \|_A \leq \varphi(x_1, x_2, x_3, x_4), \tag{13.35}$$

for all $x_1, x_2, x_3, x_4 \in A$ and

$$\| f(xy) - x^3 f(y) - f(x)y^3 \|_A \leq \varphi(x, y, 0, 0), \tag{13.36}$$

for all $x, y \in A$. Then there exists a unique cubic derivation $C : A \to A$ which satisfies (13.1) and

$$\| f(x) - C(x) \|_A \leq \frac{1}{8n^3} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\varphi(n^{kj}x, 0, 0, 0)}{n^{3kj}}, \tag{13.37}$$

for all $x \in A$. The mapping $C(x)$ is defined by

$$C(x) = \lim_{k\to\infty} \frac{f(n^{kj}x)}{n^{3kj}}, \tag{13.38}$$

for all $x \in A$.

Proof Assume $j = 1$. By the same reasoning as in the proof of Theorem 13.9, there exists a unique cubic mapping $C : A \to A$ satisfying (13.37). The mapping $C : A \to A$ is given by $C(x) = \lim_{k\to\infty} \frac{f(n^{kj}x)}{n^{3kj}}$. It follows from (13.35) that

$$\begin{aligned}
&\| f(xy) - x^3 f(y) - f(x)y^3 \|_A \\
&\qquad = \lim_{k\to\infty} \frac{1}{n^{9kj}} \| f(n^{3k}xy) - (n^k x)^3 f(n^k y) - f(n^k y)(n^k x)^3 \|_A \\
&\qquad \le \lim_{k\to\infty} \frac{1}{n^{9kj}} \varphi(n^k x, 0, 0, 0) = 0, \qquad (13.39)
\end{aligned}$$

for all $x \in A$. Therefore $C : A \to A$ is a cubic derivation satisfying (13.37). □

Remark: Let $j \in \{-1, 1\}$. Let $\varphi : A^4 \to [0, \infty)$ be a mapping such that

$$\sum_{k=0}^{\infty} \frac{\varphi(2^{kj}x_1, 2^{kj}x_2, 2^{kj}x_3, 2^{kj}x_4)}{2^{3kj}},$$

converges to $\mathbb{R}$ and $f : A \to A$ be a mapping satisfying (13.35), for all $x_1, x_2, x_3, x_4 \in A$ and (13.36), for all $x, y \in A$. Then there exists a unique cubic derivation $C : A \to A$ which satisfies (13.1) and

$$\| f(x) - C(x) \|_A \le \frac{1}{64} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\varphi(2^{kj}x, 0, 0, 0)}{2^{3kj}},$$

for all $x \in A$. The mapping $C(x)$ is defined by $C(x) = \lim_{k\to\infty} \frac{f(2^{kj}x)}{2^{3kj}}$, for all $x \in A$.

Proposition 13.31 *Let $j \in \{-1, 1\}$. Let $\varphi : A^4 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\varphi(n^{2kj}x_1, n^{2kj}x_2, n^{2kj}x_3, n^{2kj}x_4)}{n^{6kj}},$$

converges to $\mathbb{R}$ and $f : A \to A$ be a mapping satisfying (13.35), for all $x_1, x_2, x_3, x_4 \in A$ and (13.36), for all $x, y \in A$. Then there exists a unique cubic derivation $C : A \to A$ which satisfies (13.1) and

$$\| f(x) - C(x) \|_A \le \frac{1}{8n^6} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\varphi(0, n^{2kj}x, 0, 0)}{n^{6kj}},$$

for all $x \in A$. The mapping $C(x)$ is defined by $C(x) = \lim_{k\to\infty} \frac{f(n^{2kj}x)}{n^{3kj}}$, for all $x \in A$.

Remark: Let $j \in \{-1, 1\}$. Let $\varphi : A^4 \to [0, \infty)$ be a mapping such that

$$\sum_{k=0}^{\infty} \frac{\varphi(2^{2kj}x_1, 2^{2kj}x_2, 2^{2kj}x_3, 2^{2kj}x_4)}{2^{6kj}},$$

converges to $\mathbb{R}$ and $f : A \to A$ be a mapping satisfying (13.35), for all $x_1, x_2, x_3, x_4 \in A$ and (13.36), for all $x, y \in A$. Then there exists a unique cubic derivation $C : A \to$

A which satisfies (13.1) and

$$\| f(x) - C(x) \|_A \leq \frac{1}{512} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\varphi(0, 2^{2kj}x, 0, 0)}{2^{6kj}},$$

for all $x \in A$. The mapping $C(x)$ is defined by $C(x) = \lim_{k\to\infty} \frac{f(2^{2kj}x)}{2^{6kj}}$, for all $x \in A$.

Proposition 13.32 *Let $j \in \{-1, 1\}$. Let $\varphi : A^4 \to [0, \infty)$ be a mapping such that*

$$\sum_{k=0}^{\infty} \frac{\varphi(n^{3kj}x_1, n^{3kj}x_2, n^{3kj}x_3, n^{3kj}x_4)}{n^{9kj}},$$

converges to $\mathbb{R}$ and $f : A \to A$ be a mapping satisfying (13.35), for all $x_1, x_2, x_3, x_4 \in A$ and (13.36), for all $x, y \in A$. Then there exists a unique cubic derivation $C : A \to A$ which satisfies (13.1) and

$$\| f(x) - C(x) \|_A \leq \frac{1}{8n^9} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\varphi(0, 0, n^{3kj}x, 0)}{n^{9kj}},$$

for all $x \in A$. The mapping $C(x)$ is defined by $C(x) = \lim_{k\to\infty} \frac{f(n^{3kj}x)}{n^{9kj}}$, for all $x \in A$.

Proposition 13.33 *Let $j \in \{-1, 1\}$. Let $\varphi : A^4 \to [0, \infty)$ be a mapping such that*

$$\sum_{k=0}^{\infty} \frac{\varphi(n^{4kj}x_1, n^{4kj}x_2, n^{4kj}x_3, n^{4kj}x_4)}{n^{12kj}},$$

converges to $\mathbb{R}$ and $f : A \to A$ be a mapping satisfying (13.35), for all $x_1, x_2, x_3, x_4 \in A$ and (13.36), for all $x, y \in A$. Then there exists a unique cubic derivation $C : A \to A$ which satisfies (13.1) and

$$\| f(x) - C(x) \|_A \leq \frac{1}{8n^9} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\varphi(0, 0, 0, n^{4kj}x)}{n^{12kj}},$$

for all $x \in A$. The mapping $C(x)$ is defined by $C(x) = \lim_{k\to\infty} \frac{f(n^{4kj}x)}{n^{12kj}}$, for all $x \in A$.

The following corollaries are the immediate consequence of Theorem 13.30, Propositions 13.31–13.33 respectively, concerning the stability of (13.1).

Corollary 13.34 *Let λ and s be non-negative real numbers. If a mapping $f : A \to A$ satisfies inequality*

$$\| f(x_1, x_2, x_3, x_4) \|_A \leq \begin{cases} \lambda \\ \lambda\left\{ \sum_{i=1}^{4} \| x_i \|_A^s \right\} \\ \lambda\left\{ \prod_{i=1}^{4} \| x_i \|_A^s + \sum_{i=1}^{4} \| x_i \|_A^{4s} \right\}, \end{cases} \tag{13.40}$$

for all $x_1, x_2, x_3, x_4 \in A$ and

$$\| f(xy) - x^3 f(y) - f(x)y^3 \|_A \leq \begin{cases} \lambda \\ \lambda\left\{ \sum_{i=1}^{4} \| x_i \|_A^s \right\}, \quad s \neq 3 \\ \lambda\left\{ \prod_{i=1}^{4} \| x_i \|_A^s + \sum_{i=1}^{4} \| x_i \|_A^{4s} \right\}, s \neq \frac{3}{4}, \end{cases} \tag{13.41}$$

for all $x, y \in A$, then there exists a unique cubic derivation $C : A \to A$ such that

$$\| f(x) - C(x) \| \leq \begin{cases} \frac{\lambda}{8|n^3-1|} \\ \frac{\lambda\|x\|_A^s}{8|n^3-n^s|} \\ \frac{\lambda\|x\|_A^{4s}}{8|n^3-n^{4s}|}, \end{cases} \tag{13.42}$$

for all $x \in A$.

Example 13.35 Let λ and s be non-negative real numbers. If a mapping $f : A \to A$ satisfies inequality (13.40), for all $x_1, x_2, x_3, x_4 \in A$ and (13.41), for all $x, y \in A$, then there exists a unique cubic derivation $C : A \to A$ such that

$$\|f(x) - H(x)\|_A \leq \begin{cases} \frac{\lambda}{8|7|} \\ \frac{\lambda\|x\|_A^s}{8|2^3-2^s|} \\ \frac{\lambda\|x\|_A^{4s}}{8|2^3-2^{4s}|}, \end{cases}$$

for all $x \in A$.

Corollary 13.36 *Let λ and s be non-negative real numbers. If a mapping $f : A \to A$ satisfies inequality (13.40), for all $x_1, x_2, x_3, x_4 \in A$ and (13.41), for all $x, y \in A$, then there exists a unique cubic derivation $C : A \to A$ such that*

$$\| f(x) - C(x) \| \leq \begin{cases} \frac{\lambda}{8|n^6-1|} \\ \frac{\lambda\|x\|_A^s}{8|n^6-n^{2s}|} \\ \frac{\lambda\|x\|_A^{4s}}{8|n^6-n^{8s}|}, \end{cases} \tag{13.43}$$

for all $x \in A$.

Example 13.37 Let λ and s be non-negative real numbers. If a mapping $f : A \to A$ satisfies inequality (13.40), for all $x_1, x_2, x_3, x_4 \in A$ and (13.41), for all $x, y \in A$, then there exists a unique cubic derivation $C : A \to A$ such that

$$\|f(x) - H(x)\|_A \leq \begin{cases} \frac{\lambda}{8|63|} \\ \frac{\lambda \|x\|_A^s}{8|2^6 - 2^{2s}|} \\ \frac{\lambda \|x\|_A^{4s}}{8|2^6 - 2^{8s}|}, \end{cases}$$

for all $x \in A$.

Corollary 13.38 *Let λ and s be non-negative real numbers. If a mapping $f : A \to A$ satisfies inequality (13.40), for all $x_1, x_2, x_3, x_4 \in A$ and (13.41), for all $x, y \in A$, then there exists a unique cubic derivation $C : A \to A$ such that*

$$\| f(x) - C(x) \| \leq \begin{cases} \frac{\lambda}{8|n^9 - 1|} \\ \frac{\lambda \|x\|_A^s}{8|n^9 - n^{3s}|} \\ \frac{\lambda \|x\|_A^{4s}}{8|n^9 - n^{12s}|}, \end{cases} \tag{13.44}$$

for all $x \in A$.

Corollary 13.39 *Let λ and s be non-negative real numbers. If a mapping $F : A \to A$ satisfies inequality (13.40), for all $x_1, x_2, x_3, x_4 \in A$ and (13.41), for all $x, y \in A$, then there exists a unique cubic derivation $C : A \to A$ such that*

$$\| f(x) - C(x) \| \leq \begin{cases} \frac{\lambda}{8|n^{12} - 1|} \\ \frac{\lambda \|x\|_A^s}{8|n^{12} - n^{4s}|} \\ \frac{\lambda \|x\|_A^{4s}}{8|n^{12} - n^{16s}|}, \end{cases} \tag{13.45}$$

for all $x \in A$.

13.6 Stability of Cubic Derivation Quasi-Banach Algebras—Fixed Point Method

Theorem 13.40 *Let $f : A \to A$ be a mapping and assume that there exists a function $\varphi : A^4 \to [0, \infty)$ with condition*

$$\lim_{k \to \infty} \frac{\varphi(\sigma_i^k x_1, \sigma_i^k x_2, \sigma_i^k x_3, \sigma_i^k x_4)}{\sigma_i^{3k}} = 0, \tag{13.46}$$

where $\sigma_i = \begin{cases} n & if \quad i = 0 \\ \frac{1}{n} & if \quad i = 1 \end{cases}$ *such that the functional inequality*

$$\| f(x_1, x_2, x_3, x_4) \|_A \leq \varphi(x_1, x_2, x_3, x_4) \tag{13.47}$$

for all $x_1, x_2, x_3, x_4 \in A$ *and* $\| f(xy) - x^3 f(y) - f(x)y^3 \|_A \leq \varphi(x_1, x_2, x_3, x_4)$, *for all* $x, y \in A$. *If there exists* $L = L(i) < 1$ *such that the function*

$$x \to \delta(x) = \frac{1}{8}\left(\frac{x}{n}, 0, 0, 0\right)$$

has the property

$$\delta(x) = L \frac{1}{\sigma_i^3} \delta(\sigma_i x), \tag{13.48}$$

then there exists a unique cubic derivation $C : A \to A$ *satisfying the functional equation (13.1) and*

$$\| f(x) - H(x) \|_A \leq \frac{L^{1-i}}{1 - L} \delta(x), \tag{13.49}$$

for all $x \in A$.

Proof By the same reasoning as in the proof of Theorem 13.19, there exists a unique cubic mapping $C : A \to A$ satisfying (13.49). The mapping $C : A \to A$ is given by $C(x) = \lim_{k\to\infty} \frac{f(\sigma_i^k)x}{\sigma_i^{3k}}$, for all $x \in A$. It follows that

$$\begin{aligned} &\| C(xy) - x^3 C(y) - C(x)y^3 \|_A \\ &\quad = \lim_{k\to\infty} \frac{1}{\sigma_i^{9k}} \| f(\sigma_i^{3k} xy) - (\sigma_i^k x)^3 f(\sigma_i^k y) - f(\sigma_i^k x)(\sigma_i^k y)^3, \\ &\quad \leq \lim_{\substack{\lim \\ k\to\infty}} \frac{\varphi(n^k x, 0, 0, 0)}{\sigma_i^{9k}} = 0, \end{aligned}$$

for all $x, y \in A$. Therefore $C : A \to A$ is a cubic derivation satisfying (13.49). The rest of the proof is similar to that of Theorem 13.20. □

Example 13.41 Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^4 \to [0, \infty)$ with condition (13.46) where $\sigma_i = \begin{cases} 2 & if \quad i = 0; \\ \frac{1}{2} & if \quad i = 1; \end{cases}$ such that the functional inequality (13.47), for all $x_1, x_2, x_3, x_4 \in A$ and $\| f(xy) - x^3 f(y) - f(x)y^3 \|_A \leq \varphi(x, 0, 0, 0)$, for all $x, y \in A$. If there exists $L = L(i) < 1$ such that the function $x \to \delta(x) = \frac{1}{8}\left(\frac{x}{2}, 0, 0, 0\right)$ has the property (13.48), then there exists a unique cubic derivation $C : A \to A$ satisfying the functional equations (13.1) and (13.49), for all $x \in A$.

Proposition 13.42 *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^4 \to [0, \infty)$ with condition (13.46) where $\sigma_i = \begin{cases} n^2 & if \;\; i = 0 \\ \frac{1}{n^2} & if \;\; i = 1 \end{cases}$ such that the functional inequality (13.47), for all $x_1, x_2, x_3, x_4 \in A$ and $\| f(xy) - x^3 f(y) - f(x)y^3 \|_A \leq \varphi(0, x, 0, 0)$, for all $x, y \in A$. If there exists $L = L(i) < 1$ such that the function $x \to \delta(x) = \frac{1}{8}\left(0, \frac{x}{n^2}, 0, 0\right)$ has the property (13.48), then there exists a unique cubic derivation $C : A \to A$ satisfying the functional equations (13.1) and (13.49), for all $x \in A$.*

Example 13.43 Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^4 \to [0, \infty)$ with condition (13.46) where $\sigma_i = \begin{cases} 2^2 & if \;\; i = 0; \\ \frac{1}{2^2} & if \;\; i = 1; \end{cases}$ such that the functional inequality (13.47), for all $x_1, x_2, x_3, x_4 \in A$ and $\| f(xy) - x^3 f(y) - f(x)y^3 \|_A \leq \varphi(0, x, 0, 0)$, for all $x, y \in A$. If there exists $L = L(i) < 1$ such that the function $x \to \delta(x) = \frac{1}{8}\left(0, \frac{x}{2^2}, 0, 0\right)$ has the property (13.48), then there exists a unique cubic derivation $C : A \to A$ satisfying the functional equations (13.1) and (13.49), for all $x \in A$.

Proposition 13.44 *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^4 \to [0, \infty)$ with condition (13.46) where $\sigma_i = \begin{cases} n^3 & if \;\; i = 0 \\ \frac{1}{n^3} & if \;\; i = 1 \end{cases}$ such that the functional inequality (13.47), for all $x_1, x_2, x_3, x_4 \in A$ and $\| f(xy) - x^3 f(y) - f(x)y^3 \|_A \leq \varphi(0, 0, x, 0)$, for all $x, y \in A$. If there exists $L = L(i) < 1$ such that the function $x \to \delta(x) = \frac{1}{8}\left(0, 0, \frac{x}{n^3}, 0\right)$ has the property (13.48), then there exists a unique cubic derivation $C : A \to A$ satisfying the functional equations (13.1) and (13.49), for all $x \in A$.*

Proposition 13.45 *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^4 \to [0, \infty)$ with condition (13.46) where $\sigma_i = \begin{cases} n^4 & if \;\; i = 0 \\ \frac{1}{n^4} & if \;\; i = 1 \end{cases}$ such that the functional inequality (13.47), for all $x_1, x_2, x_3, x_4 \in A$ and $\| f(xy) - x^3 f(y) - f(x)y^3 \|_A \leq \varphi(0, 0, 0, x)$, for all $x, y \in A$. If there exists $L = L(i) < 1$ such that the function $x \to \delta(x) = \frac{1}{8}\left(0, 0, 0, \frac{x}{n^4}\right)$ has the property (13.48), then there exists a unique cubic derivation $C : A \to A$ satisfying the functional equations (13.1) and (13.49), for all $x \in A$.*

The corollaries are the immediate consequence of Theorem 13.40, Propositions 13.42–13.45 respectively, concerning the stability of (13.1).

Corollary 13.46 *Let λ and s be non-negative real numbers. If a mapping $f : A \to A$ satisfies the inequality (13.40), for all $x_1, x_2, x_3, x_4 \in A$ and (13.41), for all $x, y \in A$, then there exists a unique cubic derivation $C : A \to A$ such that (13.42), for all $x \in A$.*

Remark: Let λ and s be non-negative real numbers. If a mapping $f : A \to A$ satisfies the inequality (13.40), for all $x_1, x_2, x_3, x_4 \in A$ and (13.41), for all $x, y \in A$, then there exists a unique cubic derivation $C : A \to A$ such that

$$\|f(x) - C(x)\|_A \leq \begin{cases} \frac{\lambda}{8|7|} \\ \frac{\lambda\|x\|_A^s}{8|2^3-2^s|} \\ \frac{\lambda\|x\|_A^{4s}}{8|2^3-2^{4s}|}, \end{cases}$$

for all $x \in A$.

Corollary 13.47 *Let λ and s be non-negative real numbers. If a mapping $f : A \to A$ satisfies the inequality (13.40), for all $x_1, x_2, x_3, x_4 \in A$ and (13.41), for all $x, y \in A$, then there exists a unique cubic derivation $C : A \to A$ such that (13.43), for all $x \in A$.*

Remark: Let λ and s be non-negative real numbers. If a function $f : A \to A$ satisfies the inequality (13.40), for all $x_1, x_2, x_3, x_4 \in A$ and (13.41), for all $x, y \in A$, then there exists a unique cubic derivation $C : A \to A$ such that $\forall x \in A$

$$\|f(x) - C(x)\|_A \leq \begin{cases} \frac{\lambda}{8|63|} \\ \frac{\lambda\|x\|_A^s}{8|2^6-2^{2s}|} \\ \frac{\lambda\|x\|_A^{4s}}{8|2^6-2^{8s}|}. \end{cases}$$

Corollary 13.48 *Let λ and s be non-negative real numbers. If a mapping $f : A \to A$ satisfies the inequality (13.40), for all $x_1, x_2, x_3, x_4 \in A$ and (13.41), for all $x, y \in A$, then there exists a unique cubic derivation $C : A \to A$ such that (13.44), for all $x \in A$.*

Corollary 13.49 *Let λ and s be non-negative real numbers. If a mapping $f : A \to A$ satisfies the inequality (13.40), for all $x_1, x_2, x_3, x_4 \in A$ and (13.41), for all $x, y \in A$, then there exists a unique cubic derivation $C : A \to A$ such that (13.45), for all $x \in A$.*

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