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# Existence theory and Ulam's stabilities for switched coupled system of implicit impulsive fractional order Langevin equations

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**Abstract**
In this work, a system of nonlinear, switched, coupled, implicit, impulsive Langevin equations with two Hilfer fractional derivatives is introduced. The suitable conditions and results are established to discuss existence, uniqueness, and Ulam-type stability results of the mentioned model, with the help of nonlinear functional analysis techniques and Banach's fixed-point theorem. Furthermore, we examine our results with the help of example.

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**Keywords:** Langevin equation; Hilfer fractional derivative; Ulam–Hyers–Rassias stability

## 1 Introduction

One of the most representative field in mathematical sciences is stability analysis, which has many types, but the most interesting and important type is Ulam–Hyers (UH) stability. UH stability problem was identified by Ulam [35] and Hyers [14] solved the UH stability problem partially for Banach spaces case. Rassias generalized this concept [22] in 1978 and it was named as UH–Rassias (UHR) stability. Numerous papers have been published representing UH and UHR stability concepts, and the reader is referred to see [28, 31, 36, 43–46, 48].

Obloza seems to have been the first to study the UH stability of linear differential equations (DEs) $f'(\wp) + g(\wp)f(\wp) - r(\wp) = 0$ (see [19, 20]). After that, Alsina and Ger [7] investigated that if a differentiable $f : (a, b) \to \mathbb{R}$ satisfies $|f'(\wp) - f(\wp)| \leq \varepsilon$, then there exists $f_0 : (a, b) \to \mathbb{R}$ such that $f_0'(\wp) = f_0(\wp)$ and $|f(\wp) - f_0(\wp)| \leq 3\varepsilon$ for all $\wp \in (a, b)$ and this work was generalized by Takahasi et al. [32]. For further detail, see [26, 28, 31, 45, 46, 48].

Fractional differential equations (FDEs) of integer order are the classical DEs generalized form. The calculus of fractional derivatives is presently a created region and has numerous applications in electromagnetics, physical sciences, electrochemistry, medicine, porous media, economics etc. Progressively, FDEs have very essential role in signal processing, control, defence, optics, viscoelasticity, astronomy, electrical circuits, statistical physics, etc. The main theoretical tools for this area's qualitative analysis and interconnection, as well as the distinction between classical, integral models, and FDEs are provided by some interesting articles [1, 4, 15, 21, 27, 33, 34, 40].

The Langevin FDEs is one of the most useful and essential subjects in electrical engineering, chemistry, and physics. The Langevin equation (LE), formulated by Langevin, has proven to be an effective tool for describing physical phenomena's evolution under vacillating circumstances. For complex media systems, LE with ordinary derivative does not give the right dynamics description. Various LEs generalizations have been suggested for the description of fractal medium dynamical processes. Generalized LE is one of such generalization, which associates the fractal and memory properties with immoderate memory kernel into the LE. Further achievable generalization needs replacing ordinary derivative with a fractional derivative in the LE to form fractional LE, see [3, 12, 17, 18, 23] for more details.

The use of impulses with DEs is very well incorporated in mathematical modeling. In our day to day life, a wide range of phenomena and procedures exists that are defined knowing that at some instances these phenomena undergo immediate system changes. Also these procedures are exhibited for temporary disruptions, a process called impulsive effects in the system. DEs together with impulses have been keenly observed by many authors, e.g., the reader can see the contribution [9, 11, 16, 25, 37, 42, 44, 47].

Nowadays, the existence along with uniqueness and different types of UH stability of implicit nonlinear FDEs with fractional Caputo derivative have considerable attention, see [8, 10, 29, 31, 39, 41]. Wang et al. [38], studied generalized Ulam–Hyers–Rassias stability of FDE:

$$\begin{cases} {}^cD^{\alpha}_{0,\xi}x(\xi) = f(\xi, x(\xi)), & \xi \in (\xi_\ell, s_\ell], \ell = 0,1,2,3,4,\dots,m, 0<\alpha<1,\\ x(\xi) = g_\ell(\xi, x(\xi)), & \xi \in (\varsigma_{i-1}, \xi_\ell], \ell = 1,2,3,4,5,\dots,m. \end{cases}$$

Zada et al. [43], studied existence along with uniqueness of solutions by utilizing Diaz Margolis's fixed-point theorem and established different kinds of UH stability for a class of implicit nonlinear FDE with nonlinear integral boundary conditions and noninstantaneous integral impulses:

$$\begin{cases} {}^cD^{\alpha}_{0,\xi}x(\xi) = f(\xi, x(\xi), {}^cD^{\alpha}_{0,\xi}x(\xi)),\\ \quad \xi \in (\xi_\ell, s_\ell], \ell = 0,1,2,3,4,\dots,m, 0<\alpha<1, \xi\in(0,1],\\ x(\xi) = I^{\alpha}_{s_{i-1},\xi_\ell}(\xi_\ell(\xi, x(\xi))), \quad \xi \in (s_{i-1}, \xi_\ell], \ell = 1,2,3,4,\dots,m,\\ x(0) = \frac{1}{\Gamma(\alpha)}\int_0^T (T-\varrho)^{\alpha-1}\eta(\varrho, x(\varrho))\,d\varrho. \end{cases}$$

Ali et al. [6] presented the existence as well as uniqueness of solutions and various types of Ulam stabilities for a coupled nonlinear systems of implicit FDEs containing Caputo derivative by using Banach contraction principle and Leray–Schauder of cone type,

$$\begin{cases} {}^cD^{\nu}x(\xi) - f(\xi, y(\xi), {}^cD^{\nu}x(\xi)) = 0, & \nu \in (2,3], \xi\in J,\\ {}^cD^{\mu}y(\xi) - f(\xi, x(\xi), {}^cD^{\nu}y(\xi)) = 0, & \mu \in (2,3], \xi\in J,\\ \acute{x}(\xi)|_{\xi=0} = \acute{\acute{x}}(\xi)|_{\xi=0}, \quad x(\xi)|_{\xi=1} = \lambda x(\eta), & \lambda,\eta\in(0,1),\\ \acute{y}(\xi)|_{\xi=0} = \acute{\acute{y}}(\xi)|_{\xi=0}, \quad y(\xi)|_{\xi=1} = \lambda y(\eta), & \lambda,\eta\in(0,1). \end{cases}$$

Omar et al. [30], study the existence as well as uniqueness of solutions of anti–periodic boundary problem of switched coupled system of nonlinear implicit Langevin equations with two fractional derivatives and then, by using Banach's fixed-point theorem, UH type are also discussed.

$$\begin{cases} \begin{cases} \mathcal{D}^{\Omega_1}(\mathcal{D}^{\Psi_1}+\kappa_1)u(w)=f_1(w,v(w),\mathcal{D}^{\Omega_1}u(w)), & 0\le w\le 1, 0<\Psi_1\le 1, 1<\Omega_1\le 2,\\ w(0)+w(1)=0, \quad \mathcal{D}^{\Psi_1}w(0)+\mathcal{D}^{\Psi_1}w(1)=0, & \mathbb{D}^{2\Psi_1}w(0)+\mathbb{D}^{2\Psi_1}w(1)=0,\end{cases}\\ \begin{cases} \mathcal{D}^{\Omega_2}(\mathcal{D}^{\Psi_2}+\kappa_2)u(w)=f_2(w,u(w),\mathcal{D}^{\Omega_2}v(w)), & 0\le w\le 1, 0<\Psi_2\le 1, 1<\Omega_2\le 2,\\ w(0)+w(1)=0, \quad \mathcal{D}^{\Psi_2}w(0)+\mathcal{D}^{\Psi_2}w(1)=0, & \mathbb{D}^{2\Psi_2}w(0)+\mathbb{D}^{2\Psi_2}w(1)=0,\end{cases}\end{cases}$$

where $\mathcal{D}^{\Psi_1}$, $\mathcal{D}^{\Psi_2}$, $\mathcal{D}^{\Omega_1}$, $\mathcal{D}^{\Omega_2}$ represent the Caputo fractional derivative of order $\Psi_1$, $\Psi_2$, $\Omega_1$, $\Omega_2$ respectively $f_1, f_2 : [0,1]\times\Re\times\Re\to\Re$ are continuous functions and $\kappa_1$, $\kappa_2$ are real numbers. Furthermore, $\mathbb{D}^{2\Psi_1}$, $\mathbb{D}^{2\Psi_2}$ are the sequential fractional derivatives:

$$\begin{cases} \mathbb{D}^{\Psi_1}u=\mathcal{D}^{\Psi_1}u,\\ \mathbb{D}^{k\Psi_1}u=\mathbb{D}_1^{\Psi}\mathbb{D}^{(k-1)\Psi_1}u, & k=2,3,\dots,\\ \mathbb{D}^{\Psi_2}u=\mathcal{D}^{\Psi_2}u,\\ \mathbb{D}^{k\Psi_1}u=\mathbb{D}_1^{\Psi}\mathbb{D}^{(k-1)\Psi_2}u, & k=2,3,\dots.\end{cases}$$

This paper is about the study of a system of switched, coupled, implicit, nonlinear, impulsive LEs with four Hilfer fractional derivatives (HFDs) of the form:

$$\begin{cases} \begin{cases} D^{\alpha_1,\gamma_1}(D^{\alpha_2,\gamma_1}+\lambda)x(\xi)=f_1(\xi,y(\xi),D^{\alpha_1,\gamma_1}x(\xi)),\\ \quad \xi\in J=[0,T], 0<\alpha_1,\alpha_2<1, 0\le\gamma_1\le 1,\\ \Delta x(\xi_\ell)=I_\ell(x(\xi_\ell)), \quad \ell=1,2,3,4,5,\dots,m,\\ I^{1-\gamma_1}x(0)=x_0, \quad \eta_1=(\alpha_1+\alpha_2)(1-\gamma_1)+\gamma_1,\end{cases}\\ \begin{cases} D^{\beta_1,\gamma_2}(D^{\beta_2,\gamma_2}+\lambda)y(\xi)=f_2(\xi,x(\xi),D^{\beta_1,\gamma_2}y(\xi)),\\ \quad \xi\in J=[0,T], 0<\beta_1,\beta_2<1, 0\le\gamma_2\le 1,\\ \Delta y(\xi_\ell)=I_\ell(y(\xi_\ell)), \quad \ell=1,2,3,4,5,\dots,m,\\ I^{1-\gamma_2}y(0)=y_0, \quad \eta_2=(\beta_1+\beta_2)(1-\gamma_2)+\gamma_2,\end{cases}\end{cases} \tag{1.1}$$

where $D^{\alpha_1,\gamma_1}$, $D^{\alpha_2,\gamma_1}$, $D^{\beta_1,\gamma_2}$ and $D^{\beta_2,\gamma_2}$ represent four HFDs [15], of order $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ respectively, $\lambda\in\mathbb{R}-0$, $\gamma_1$, $\gamma_2$ determine to the type of initial conditions used in the problem. Furthermore, $f_1, f_2 : J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous and $I_\ell:\mathbb{R}\to\mathbb{R}$ for all $\ell=1,2,3,4,5,\dots,m$, represent impulsive nonlinear mapping, $\Delta x(\xi_\ell)=x(\xi_\ell^+)-x(\xi_\ell^-)$ and $\Delta y(\xi_\ell)=y(\xi_\ell^+)-y(\xi_\ell^-)$ where $x(\xi_\ell^+)$, $x(\xi_\ell^-)$, $y(\xi_\ell^+)$ and $y(\xi_\ell^-)$ represent the right and the left limits, respectively, at $\xi=\xi_\ell$ for $\ell=1,2,3,4,5,\dots,m$.

## 2 Preliminaries

The definitions of Riemman–Liouville (RL) fractional derivatives, Caputo fractional derivatives, Hilfer fractional derivatives (HFDs), fractional integrals, some useful lemmas, remarks and results are recalled from the preliminaries sections of [15, 21, 24].

## 3 Existence and uniqueness

Here, we investigate the existence and uniqueness of solutions for the proposed LE using two HFDs.

**Theorem 3.1** *Let $f_1 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, x(\cdot), D^{\alpha_1,\gamma_1}x(\cdot)) \in C_{1-\eta_1}[0,T]$ for all $x \in C_{1-\eta_1}[0,T]$. Then the function $x \in C_{1-\eta_1}[0,T]$ is equivalent to*

$$x(\xi) = \begin{cases} \frac{x_0}{\Gamma(\gamma_1)}\xi^{\gamma_1-1} + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^\xi (\xi-\varrho)^{\alpha_1+\alpha_2-1} f_1(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho))\,d\varrho \\ \quad - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^\xi (\xi-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho, \quad \xi \in J_0, \\ \frac{x_0}{\Gamma(\gamma_1)}\xi_1^{\gamma_1-1} + \int_{\xi_1}^\xi \frac{(\xi-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho))\,d\varrho \\ \quad + \int_0^{\xi_1} \frac{(\xi_1-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho))\,d\varrho \\ \quad - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\xi_1} (\xi_1-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho \\ \quad - \frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_1}^{\xi} (\xi-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho + I_1(x(\xi_1)), \quad \xi \in J_1, \\ \frac{x_0}{\Gamma(\gamma_1)}\xi_m^{\gamma_1-1} + \sum_{i=1}^m \int_{\xi_{i-1}}^{\xi_\ell} \frac{(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho))\,d\varrho \\ \quad - \sum_{\ell=1}^m \frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho \\ \quad + \sum_{\ell=1}^m I_\ell(x(\xi_\ell)), \quad \xi \in J_\ell, \ell = 1,2,3,4,5,\dots,m \end{cases} \tag{3.1}$$

*satisfies*

$$\begin{cases} D^{\alpha_1,\gamma_1}(D^{\alpha_2,\gamma_1} + \lambda)x(\xi) = f(\xi, y(\xi), D^{\alpha_1,\gamma_1}x(\xi)), \\ \quad \xi \in J = [0,T], 0 < \alpha_1, \alpha_2 < 1, 0 \le \gamma_1 \le 1, \\ \Delta x(\xi_\ell) = I_\ell(x(\xi_\ell)), \quad \ell = 1,2,3,4,5,\dots,m, \\ I^{1-\gamma_1}x(0) = x_0, \qquad \eta_1 = (\alpha_1+\alpha_2)(1-\gamma_1)+\gamma_1. \end{cases} \tag{3.2}$$

*Proof* Let $x$ satisfy (3.2). Then for any $\xi \in J_0$, there exists $c \in \mathbb{R}$ such that

$$\begin{aligned} x(\xi) &= c + \int_0^\xi \frac{(\xi-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1\big(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho)\big)\,d\varrho \\ &\quad - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^\xi (\xi-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho. \end{aligned} \tag{3.3}$$

By $I^{1-\gamma_1}x(0) = x_0$, Eq. (3.3) implies

$$\begin{aligned} x(\xi) &= \frac{x_0}{\Gamma(\gamma_1)}\xi^{\gamma_1-1} + \int_0^\xi \frac{(\xi-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1\big(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho)\big)\,d\varrho \\ &\quad - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^\xi (\xi-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho, \quad \xi \in J_0. \end{aligned}$$

Similarly, for $\xi \in J_1$, there exists $d_1 \in \mathbb{R}$ such that

$$\begin{aligned} x(\xi) &= d_1 + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\xi_1}^\xi (\xi-\varrho)^{\alpha_1+\alpha_2-1} f_1\big(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho)\big)\,d\varrho \\ &\quad - \frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_1}^\xi (\xi-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho. \end{aligned}$$

Using the condition, we get

$$\begin{aligned}
x\big(\xi_1^-\big) &= \frac{x_0}{\Gamma(\gamma_1)}\xi_1^{\gamma_1-1} + \int_0^{\xi_1} \frac{(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1\big(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho)\big)\,d\varrho \\
&\quad - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\xi_1}(\xi_\ell-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho, \\
x\big(\xi_1^+\big) &= d_1.
\end{aligned}$$

In view of

$$\Delta x(\xi_1) = x\big(\xi_1^+\big) - x\big(\xi_1^-\big) = I_1\big(x(\xi_1)\big),$$

we get

$$\begin{aligned}
x\big(\xi_1^+\big) - x\big(\xi_1^-\big) &= d_1 - \frac{x_0}{\Gamma(\gamma_1)}\xi_1^{\gamma_1-1} - \int_0^{\xi_1} \frac{(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1\big(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho)\big)\,d\varrho \\
&\quad + \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\xi_1}(\xi_\ell-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho, \\
I_1\big(x(\xi_1)\big) &= d_1 - \frac{x_0}{\Gamma(\gamma_1)}\xi_1^{\gamma_1-1} - \int_0^{\xi_1} \frac{(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1\big(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho)\big)\,d\varrho \\
&\quad + \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\xi_1}(\xi_\ell-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho, \\
d_1 &= \frac{x_0}{\Gamma(\gamma_1)}\xi_1^{\gamma_1-1} + \int_0^{\xi_1} \frac{(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1\big(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho)\big)\,d\varrho \\
&\quad - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\xi_1}(\xi_\ell-\varrho)^{\gamma_1-1}x(\varrho)\,d\varrho + I_1\big(x(\xi_1)\big).
\end{aligned}$$

For $d_1$ value, we have

$$\begin{aligned}
x(\xi) &= \int_{\xi_1}^{\xi} \frac{(\xi-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1\big(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}y(\varrho)\big)\,d\varrho \\
&\quad + \int_0^{\xi_1} \frac{(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1\big(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho)\big)\,d\varrho \\
&\quad - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\xi_1}(\xi_\ell-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho - \frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_1}^{\xi}(\xi-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho + \frac{x_0}{\Gamma(\gamma_1)}\xi_1^{\gamma_1-1} \\
&\quad + I_1\big(x(\xi_1)\big).
\end{aligned}$$

In a similar fashion for $\xi \in J_\ell$, we have

$$\begin{aligned}
x(\xi) &= \frac{x_0}{\Gamma(\gamma_1)}\xi_\ell^{\gamma_1-1} + \sum_{\ell=1}^{m} \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1} f_1\big(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho)\big)\,d\varrho \\
&\quad - \sum_{\ell=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\xi_1}(\xi_\ell-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho + \sum_{\ell=1}^{m} I_\ell\big(x(\xi_\ell)\big).
\end{aligned}$$

Conversely, let $x$ satisfy (3.1). Then it is easy to prove that solution $x(\xi)$ given by (3.1) satisfies (3.2) along with its impulsive and integral boundary conditions. □

From Theorem 3.1 the solution form of (1.1) is given by

$$\begin{cases} x(\xi) = \begin{cases} \frac{x_0}{\Gamma(\gamma_1)}\xi^{\gamma_1-1} + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^\xi (\xi-\varrho)^{\alpha_1+\alpha_2-1} f_1(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho))\,d\varrho \\ \quad - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^\xi (\xi-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho, \quad \xi \in J_0, \\ \frac{x_0}{\Gamma(\gamma_1)}\xi_1^{\gamma_1-1} + \int_{\xi_1}^{\xi} \frac{(\xi-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho))\,d\varrho \\ \quad + \int_0^{\xi_1} \frac{(\xi_1-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho))\,d\varrho \\ \quad - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\xi_1} (\xi_1-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho - \frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_1}^{\xi} (\xi-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho \\ \quad + I_1(x(\xi_1)), \quad \xi \in J_1, \\ \frac{x_0}{\Gamma(\gamma_1)}\xi_\ell^{\gamma_1-1} + \sum_{\ell=1}^m \int_{\xi_{i-1}}^{\xi_\ell} \frac{(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho))\,d\varrho \\ \quad - \sum_{\ell=1}^m \frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho \\ \quad + \sum_{\ell=1}^m I_\ell(x(\xi_\ell)), \quad \xi \in J_\ell, \ell = 1,2,3,4,5,\dots,m, \end{cases} \\ y(\xi) = \begin{cases} \frac{y_0}{\Gamma\gamma_2}\xi^{\gamma_2-1} + \frac{1}{\Gamma(\beta_1+\beta_2)}\int_0^\xi (\xi-\varrho)^{\beta_1+\beta_2-1} f_2(\varrho, x(\varrho), D^{\beta_1,\gamma_2}y(\varrho))\,d\varrho \\ \quad - \frac{\lambda}{\Gamma\beta_1}\int_0^\xi (\xi-\varrho)^{\beta_1-1}y(\varrho)\,d\varrho, \quad \xi \in J_0, \\ \frac{y_0}{\Gamma\gamma_2}\xi_1^{\gamma_2-1} + \int_{\xi_1}^{\xi} \frac{(\xi-\varrho)^{\beta_1+\beta_2-1}}{\Gamma(\beta_1+\beta_2)} f_2(\varrho, x(\varrho), D^{\beta_1,\gamma_2}y(\varrho))\,d\varrho \\ \quad + \int_0^{\xi_1} \frac{(\xi_1-\varrho)^{\beta_1+\beta_2-1}}{\Gamma(\beta_1+\beta_2)} f_2(\varrho, x(\varrho), D^{\beta_1,\gamma_2}y(\varrho))\,d\varrho \\ \quad - \frac{\lambda}{\Gamma\beta_1}\int_0^{\xi_1} (\xi_1-\varrho)^{\beta_1-1}y(\varrho)\,d\varrho - \frac{\lambda}{\Gamma\beta_1}\int_{\xi_1}^{\xi} (\xi-\varrho)^{\beta_1-1}y(\varrho)\,d\varrho \\ \quad + I_1(y(\xi_1)), \quad \xi \in J_1, \\ \frac{y_0}{\Gamma\gamma_2}\xi_\ell^{\gamma_2-1} + \sum_{\ell=1}^m \int_{\xi_{i-1}}^{\xi_\ell} \frac{(\xi_\ell-\varrho)^{\beta_1+\beta_2-1}}{\Gamma(\beta_1+\beta_2)} f_2(\varrho, x(\varrho), D^{\beta_1,\gamma_2}y(\varrho))\,d\varrho \\ \quad - \sum_{\ell=1}^m \frac{\lambda}{\Gamma\beta_1}\int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell-\varrho)^{\beta_1-1}y(\varrho)\,d\varrho \\ \quad + \sum_{\ell=1}^m I_\ell(y(\xi_\ell)), \quad \xi \in J_\ell, \ell = 1,2,3,4,5,\dots,m. \end{cases} \end{cases} \tag{3.4}$$

Consider some assumptions as follows:

$(H_1)$ $f_1 \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is continuous.

$(H_2)$
- $\exists\, 0 < L_{f_1} < 1$ and $0 < L_{g_1} < 1$ such that $|f_1(\omega,\mu,m) - f_1(\omega,\nu,n)| \le L_{f_1}|\mu-\nu| + L_{g_1}|m-n|$, for each $\omega \in J$ and all $\mu,\nu,m,n \in \mathbb{R}$.
- $\exists\, 0 < L_{f_2} < 1$ and $0 < L_{g_2} < 1$ such that $|f_2(\omega,\mu,m) - f_2(\omega,\nu,n)| \le L_{f_2}|\mu-\nu| + L_{g_2}|m-n|$, for each $\omega \in J$ and all $\mu,\nu,m,n \in \mathbb{R}$.

$(H_3)$ $\exists Ł_k > 0$ such that $|I_\ell(\mu) - I_\ell(\nu)| \le Ł_k|\mu-\nu|$, for each $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$, and for all $\mu,\nu \in \mathbb{R}$.

$(H_4)$ For $\varphi_x \in C(J, \mathbb{R}_+)$, there exists $c_\varphi > 0$ such that

$$\int_0^\xi \big(\varphi(s)\big)\,ds \le c_\varphi \varphi_x(\xi) \quad \text{for each } \xi \in J.$$

**Theorem 3.2** *Let* $(H_1)$–$(H_3)$ *hold. If*

$$\begin{aligned} &\frac{mL_{f_1}(T)^{\alpha_1+\alpha_2}}{(1-L_{g_1})\Gamma(\alpha_1+\alpha_2+1)} - \frac{m\lambda L_{f_1}(T)^{\alpha_1}}{(1-L_{g_1})\Gamma\alpha_1+1} + \frac{mL_{f_2}(T)^{\beta_1+\beta_2}}{(1-L_{g_2})\Gamma(\beta_1+\beta_2+1)} \\ &\quad - \frac{m\lambda L_{f_2}(T)^{\beta_1}}{(1-L_{g_2})\Gamma\beta_1+1} + 2m\text{Ł}_k < 1 \end{aligned} \tag{3.5}$$

*then* (1.1) *has a unique solution* $x$ *in* $C_{1-\eta_1}[a,b]$.

*Proof* Define $N : C_{1-\eta_1}[a,b] \to C_{1-\eta_1}[a,b]$ by

$$\begin{cases} (Nx)(\xi) = \frac{x_0}{\Gamma(\gamma_1)}\xi^{\gamma_1-1} + \int_0^\xi \frac{(\xi-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho))\,d\varrho \\ \qquad - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^\xi (\xi-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho, \quad \xi \in J_0, \\ (Nx)(\xi) = \frac{x_0}{\Gamma(\gamma_1)}\xi_m^{\gamma_1-1} + \sum_{\ell=1}^m \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1} f_1(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho))\,d\varrho \\ \qquad - \sum_{\ell=1}^m \frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho \\ \qquad + \sum_{\ell=1}^m I_\ell(x(\xi_\ell)), \quad \xi \in J_m, m = 1,2,\ldots,q. \end{cases}$$

For any $x, y \in C_{1-\eta_1}[a,b]$ and $\xi \in J_\ell$, consider

$$\begin{aligned} &\bigl|(Nx)(\xi) - (N\bar{x})(\xi)\bigr| \\ &\quad \le \sum_{\ell=1}^m \int_{\xi_{i-1}}^{\xi_\ell} \frac{(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}\bigl|x(\varrho)-\bar{x}(\varrho)\bigr|\,d\varrho \\ &\qquad - \sum_{\ell=1}^m \frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}\bigl|x(\varrho)-\bar{x}(\varrho)\bigr|\,d\varrho + \sum_{\ell=1}^m \bigl|I_\ell\bigl(x(\xi_\ell)\bigr) - I_\ell\bigl(\bar{x}(\xi_\ell)\bigr)\bigr| \\ &\quad \le \sum_{\ell=1}^m \int_{\xi_{i-1}}^{\xi_\ell} \frac{(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}\bigl|x(\varrho)-\bar{x}(\varrho)\bigr|\,d\varrho \\ &\qquad - \sum_{\ell=1}^m \frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}\bigl|x(\varrho)-\bar{x}(\varrho)\bigr|\,d\varrho + \text{Ł}_k\sum_{\ell=1}^m \bigl|x(\xi)-\bar{x}(\xi)\bigr|, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} x(\xi) &:= f_1\bigl(\xi, y(\xi), D^{\alpha_1,\gamma_1}x(\xi)\bigr) = f_1\bigl(\xi, y(\xi), x(\xi)\bigr), \\ \bar{x}(\xi) &:= f_1\bigl(\xi, \bar{y}(\xi), D^{\alpha_1,\gamma_1}\bar{x}(\xi)\bigr) = f_1\bigl(\xi, \bar{y}(\xi), \bar{x}(\xi)\bigr), \\ \bigl|x(\xi) - \bar{x}(\xi)\bigr| &= \bigl|f_1\bigl(\xi, y(\xi), x(\xi)\bigr) - f_1\bigl(\xi, \bar{y}(\xi), \bar{x}(\xi)\bigr)\bigr| \\ &\le L_{f_1}\bigl|y(\xi) - \bar{y}(\xi)\bigr| + L_{g_1}\bigl|x(\xi) - \bar{x}(\xi)\bigr|. \end{aligned}$$

This further gives

$$\bigl|x(\xi) - \bar{x}(\xi)\bigr| \le \frac{L_{f_1}}{1-L_{g_1}}\bigl|y(\xi) - \bar{y}(\xi)\bigr|, \tag{3.7}$$

and similarly

$$\bigl|y(\xi) - \bar{y}(\xi)\bigr| \le \frac{L_{f_2}}{1-L_{g_2}}\bigl|x(\xi) - \bar{x}(\xi)\bigr|.$$

Putting (3.7) in (3.6), we obtain

$$\begin{aligned}
&\left|(Nx)(\xi)-(N\bar{x})(\xi)\right| \\
&\leq \sum_{\ell=1}^{m}\frac{L_{f_1}}{(1-L_{g_1})\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}\left|y(\varrho)-\bar{y}(\varrho)\right|d\varrho \\
&\quad-\sum_{\ell=1}^{m}\frac{\lambda L_{f_1}}{(1-L_{g_1})\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}\left|y(\varrho)-\bar{y}(\varrho)\right|d\varrho+\frac{L_{f_1}Ł_k}{1-L_{g_1}}\sum_{\ell=1}^{m}\left|y(\varrho)-\bar{y}(\varrho)\right| \\
&\leq\left(\frac{mL_{f_1}(\xi_\ell-\xi_{i-1})^{\alpha_1+\alpha_2}}{(1-L_{g_1})\Gamma(\alpha_1+\alpha_2+1)}-\frac{m\lambda L_{f_1}}{(1-L_{g_1})\Gamma\alpha_1+1}(\xi_\ell-\xi_{i-1})^{\alpha_1}+mŁ_k\right)\left|y(\xi)-\bar{y}(\xi)\right| \\
&\leq\left(\frac{mL_{f_1}(T)^{\alpha_1+\alpha_2}}{(1-L_{g_1})\Gamma(\alpha_1+\alpha_2+1)}-\frac{m\lambda L_{f_1}}{(1-L_{g_1})\Gamma\alpha_1+1}(T)^{\alpha_1}+mŁ_k\right)\left|y(\xi)-\bar{y}(\xi)\right|.
\end{aligned} \tag{3.8}$$

In a similar fashion, we can get

$$\begin{aligned}
&\left|(Ny)(\xi)-(N\bar{y})(\xi)\right| \\
&\leq\left(\frac{mL_{f_2}(T)^{\beta_1+\beta_2}}{(1-L_{g_2})\Gamma(\beta_1+\beta_2+1)}-\frac{m\lambda L_{f_2}}{(1-L_{g_2})\Gamma\beta_1+1}(T)^{\beta_1}+mŁ_k\right)\left|x(\xi)-\bar{x}(\xi)\right|.
\end{aligned} \tag{3.9}$$

Therefore from (3.8) and (3.9), we get

$$\begin{aligned}
&\left|N(x,y)-N(\bar{x},\bar{y})\right| \\
&\leq\Bigg(\frac{mL_{f_1}(T)^{\alpha_1+\alpha_2}}{(1-L_{g_1})\Gamma(\alpha_1+\alpha_2+1)}-\frac{m\lambda L_{f_1}}{(1-L_{g_1})\Gamma\alpha_1+1}(T)^{\alpha_1}+2mŁ_k \\
&\quad+\frac{mL_{f_2}(T)^{\beta_1+\beta_2}}{(1-L_{g_2})\Gamma(\beta_1+\beta_2+1)}-\frac{m\lambda L_{f_2}}{(1-L_{g_2})\Gamma\beta_1+1}(T)^{\beta_1}\Bigg)\left|(x,y)-(\bar{x},\bar{y})\right|.
\end{aligned}$$

Now since

$$\begin{aligned}
&\frac{mL_{f_1}(T)^{\alpha_1+\alpha_2}}{(1-L_{g_1})\Gamma(\alpha_1+\alpha_2+1)}-\frac{m\lambda L_{f_1}(T)^{\alpha_1}}{(1-L_{g_1})\Gamma\alpha_1+1}+\frac{mL_{f_2}(T)^{\beta_1+\beta_2}}{(1-L_{g_2})\Gamma(\beta_1+\beta_2+1)} \\
&\quad-\frac{m\lambda L_{f_2}(T)^{\beta_1}}{(1-L_{g_2})\Gamma\beta_1+1}+2mŁ_k<1,
\end{aligned}$$

$(x,y)$ is a contraction and according to Banach's contraction theorem its only fixed point is the only solution of (1.1). □

## 4 UH stability analysis

Let $\varepsilon_x,\varepsilon_y>0$, $\psi_x,\psi_y\geq 0$ and $\varphi:J\to\mathbb{R}^+$ be a continuous function. Consider

$$\left\{\begin{cases}
|D^{\alpha_1,\gamma_1}(D^{\alpha_2,\gamma_1}+\lambda)x(\xi)-f(\xi,y(\xi),x(\xi))|\leq\varepsilon_x, & \xi\in J_\ell,\ell=1,2,3,4,5,\dots,q, \\
|\Delta x(\xi_\ell)-I_\ell(x(\xi_\ell))|\leq\varepsilon_x, & \ell=1,2,3,4,5,\dots,m, \\
|D^{\beta_1,\gamma_2}(D^{\beta_2,\gamma_2}+\lambda)y(\xi)-f(\xi,x(\xi),y(\xi))|\leq\varepsilon_y, & \xi\in J_\ell,\ell=1,2,3,4,5,\dots,q, \\
|\Delta y(\xi_\ell)-I_\ell(y(\xi_\ell))|\leq\varepsilon_y, & \ell=1,2,3,4,5,\dots,m,
\end{cases}\right. \tag{4.1}$$

$$\begin{cases} \begin{cases} |D^{\alpha_1,\gamma_1}(D^{\alpha_2,\gamma_1}+\lambda)x(\xi)-f(\xi,y(\xi),x(\xi))| \le \varphi_x(\xi), \\ \quad \xi \in J_\ell, \ell = 1,2,3,4,5,\dots,q, \\ |\Delta x(\xi_\ell) - I_\ell(x(\xi_\ell))| \le \psi_x, \quad \ell = 1,2,3,4,5,\dots,m, \end{cases} \\ \begin{cases} |D^{\beta_1,\gamma_2}(D^{\beta_2,\gamma_2}+\lambda)y(\xi)-f(\xi,x(\xi),y(\xi))| \le \varphi_y(\xi), \\ \quad \xi \in J_\ell, \ell = 1,2,3,4,5,\dots,q, \\ |\Delta y(\xi_\ell) - I_\ell(y(\xi_\ell))| \le \psi_y, \quad \ell = 1,2,3,4,5,\dots,m, \end{cases} \end{cases} \tag{4.2}$$

and

$$\begin{cases} \begin{cases} |D^{\alpha_1,\gamma_1}(D^{\alpha_2,\gamma_1}+\lambda)x(\xi)-f(\xi,y(\xi),x(\xi))| \le \varepsilon_x\varphi_x(\xi), \\ \quad \xi \in J_\ell, \quad \ell = 1,2,3,4,5,\dots,q, \\ |\Delta x(\xi_\ell) - I_\ell(x(\xi_\ell))| \le \varepsilon_x\psi_x, \ell = 1,2,3,4,5,\dots,m, \end{cases} \\ \begin{cases} |D^{\beta_1,\gamma_2}(D^{\beta_2,\gamma_2}+\lambda)y(\xi)-f(\xi,x(\xi),y(\xi))| \le \varepsilon_y\varphi_y(\xi), \\ \quad \xi \in J_\ell, \ell = 1,2,3,4,5,\dots,q, \\ |\Delta y(\xi_\ell) - I_\ell(y(\xi_\ell))| \le \varepsilon_y\psi_y, \quad \ell = 1,2,3,4,5,\dots,m. \end{cases} \end{cases} \tag{4.3}$$

**Definition 4.1** (1.1) is called UH stable if there exists $C_{f,i,q,\sigma} > 0$ such that for each $\varepsilon > 0$ and for each solution $(x,y) \in C_{1-\eta_1}[0,T] \times C_{1-\eta_1}[0,T]$ of (4.1), there exists a solution $(\bar{x},\bar{y}) \in C_{1-\eta_1}[0,T] \times C_{1-\eta_1}[0,T]$ of (1.1) such that

$$\bigl\|(x,y)(\xi) - (\bar{x},\bar{y})(\xi)\bigr\| \le C_{f,i,q,\sigma}\varepsilon, \quad \xi \in J. \tag{4.4}$$

**Definition 4.2** (1.1) is called generalized UH stable if there exists $\phi_{f,i,q,\sigma} \in C_{1-\eta_1}[0,T] \times C_{1-\eta_1}[0,T]$, $\phi_{f,i,q,\sigma}(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $(x,y) \in C_{1-\gamma[0,T]}$ of (4.1), there exists a solution $(\bar{x},\bar{y}) \in C_{1-\eta_1}[0,T] \times C_{1-\eta_1}[0,T]$ of (1.1) such that

$$\bigl\|(x,y)(\xi) - (\bar{x},\bar{y})(\xi)\bigr\| \le \phi_{f,i,q,\sigma}\varepsilon, \quad \xi \in J. \tag{4.5}$$

*Remark* 4.3 Definition 4.1 implies Definition 4.2.

**Definition 4.4** (1.1) is called UHR stable with respect to $(\varphi,\psi)$ if there exists $C_{f,i,q,\sigma,\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $(x,y) \in C_{1-\eta_1}[0,T] \times C_{1-\eta_1}[0,T]$ of (4.3), there is a solution $(\bar{x},\bar{y}) \in C_{1-\eta_1}[0,T] \times C_{1-\eta_1}[0,T]$ of (1.1) such that

$$\bigl\|(x,y)(\xi) - (\bar{x},\bar{y})(\xi)\bigr\| \le C_{f,i,q,\sigma,\varphi}\varepsilon\bigl(\varphi(\xi) + \psi\bigr)\varepsilon, \quad \xi \in J. \tag{4.6}$$

**Definition 4.5** (1.1) is called generalized UHR stable with respect to $(\varphi,\psi)$ if there exists $C_{f,i,q,\sigma,\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $(x,y) \in C_{1-\eta_1}[0,T] \times C_{1-\eta_1}[0,T]$ of (4.2), there is a solution $(\bar{x},\bar{y}) \in C_{1-\eta_1}[0,T] \times C_{1-\eta_1}[0,T]$ of (1.1) such that

$$\bigl\|(x,y)(\xi) - (\bar{x},\bar{y})(\xi)\bigr\| \le C_{f,i,q,\sigma,\varphi}\bigl(\varphi(\xi) + \psi\bigr)\varepsilon, \quad \xi \in J. \tag{4.7}$$

*Remark* 4.6 Definition 4.4 implies Definition 4.5.

*Remark* 4.7 $x, y \in C_{1-\eta_1}[0,T]$ satisfy (4.1) if and only if there exist $g \in C_{1-\eta_1}[0,T]$ and a sequence $g_\ell$, $\ell = 1,2,3,4,5,\dots,m$, depending on $g$, such that

(a) • $|g_x(\xi)| \le \epsilon_x$, $|g_\ell| \le \epsilon_x$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,
• $|g_y(\xi)| \le \epsilon_y(\xi)$, $|g_\ell| \le \epsilon_y$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,

(b) • $D^{\alpha_1,\gamma_1}(D^{\alpha_2,\gamma_1} + \lambda)x(\xi) = f(\xi, y(\xi), x(\xi)) + g_x(\xi)$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,
• $D^{\beta_1,\gamma_2}(D^{\beta_2,\gamma_2} + \lambda)y(\xi) = f(\xi, x(\xi), y(\xi)) + g_y(\xi)$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,

(c) • $\Delta x(\xi_\ell) = I_\ell(x(\xi_\ell)) + g_\ell$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,
• $\Delta y(\xi_\ell) = I_\ell(y(\xi_\ell)) + g_\ell$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$.

*Remark* 4.8 $x, y \in C_{1-\eta_1}[0,T]$ satisfy (4.2) if and only if there exist $g \in C_{1-\eta_1}[0,T]$ and a sequence $g_\ell$, $\ell = 1,2,3,4,5,\dots,m$, depending on $g$, such that

(a) • $|g_x(\xi)| \le \varphi_x(\xi)$, $|g_\ell| \le \psi_x$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,
• $|g_y(\xi)| \le \varphi_y(\xi)$, $|g_\ell| \le \psi_y$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,

(b) • $D^{\alpha_1,\gamma_1}(D^{\alpha_2,\gamma_1} + \lambda)x(\xi) = f(\xi, y(\xi), x(\xi)) + g_x(\xi)$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,
• $D^{\beta_1,\gamma_2}(D^{\beta_2,\gamma_2} + \lambda)y(\xi) = f(\xi, x(\xi), y(\xi)) + g_y(\xi)$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,

(c) • $\Delta x(\xi_\ell) = I_\ell(x(\xi_\ell)) + g_\ell$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,
• $\Delta y(\xi_\ell) = I_\ell(y(\xi_\ell)) + g_\ell$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$.

*Remark* 4.9 $x, y \in C_{1-\eta_1}[0,T]$ satisfy (4.2) if and only if there exist $g \in C_{1-\eta_1}[0,T]$ and a sequence $g_\ell$, $\ell = 1,2,3,4,5,\dots,m$, depending on $g$, such that

(a) • $|g_x(\xi)| \le \epsilon_x\varphi_x(\xi)$, $|g_\ell| \le \epsilon_x\psi_x$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,
• $|g_y(\xi)| \le \epsilon_y\varphi_y(\xi)$, $|g_\ell| \le \epsilon_y\psi_y$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,

(b) • $D^{\alpha_1,\gamma_1}(D^{\alpha_2,\gamma_1} + \lambda)x(\xi) = f(\xi, y(\xi), x(\xi)) + g_x(\xi)$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,
• $D^{\beta_1,\gamma_2}(D^{\beta_2,\gamma_2} + \lambda)y(\xi) = f(\xi, x(\xi), y(\xi)) + g_y(\xi)$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,

(c) • $\Delta x(\xi_\ell) = I_\ell(x(\xi_\ell)) + g_\ell$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$,
• $\Delta y(\xi_\ell) = I_\ell(y(\xi_\ell)) + g_\ell$, $\xi \in J_\ell$, $\ell = 1,2,3,4,5,\dots,m$.

**Theorem 4.10** *If* $(H_1)$*–*$(H_3)$ *and* (3.5) *hold, then* (1.1) *is UH stable and consequently generalized UH stable.*

*Proof* Let $\bar{x} \in C_{1-\eta_1}[a,b]$ satisfy (4.1) and let $x$ be the only solution of

$$\begin{cases} D^{\alpha_1,\gamma_1}(D^{\alpha_2,\gamma_1} + \lambda)x(\xi) = f_1(\xi, y(\xi), D^{\alpha_1,\gamma_1}x(\xi)), \\ \quad \xi \in J = [0,T], 0 < \alpha_1, \alpha_2 < 1, 0 \le \gamma_1 \le 1, \\ \Delta x(\xi_\ell) = I_\ell(x(\xi_\ell)), \quad \ell = 1,2,3,4,5,\dots,m, \\ I^{1-\gamma_1}x(0) = x_0, \qquad \eta_1 = (\alpha_1 + \alpha_2)(1-\gamma_1) + \gamma_1. \end{cases}$$

By Theorem 3.1, we have for each $\xi \in J_\ell$

$$\begin{aligned} x(\xi) = {} & \frac{x_0}{\Gamma\gamma_1}\xi_m^{\gamma_1 - 1} + \sum_{\ell=1}^{m} \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1+\alpha_2-1} f_1\big(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho)\big)\, d\varrho \\ & - \sum_{\ell=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1 - 1} x(\varrho)\, d\varrho + \sum_{\ell=1}^{m} I_\ell\big(x(\xi_\ell)\big), \\ & \xi \in J_\ell, \ell = 1,2,3,4,5,\dots,m. \end{aligned}$$

Since $\bar{x}$ satisfies (4.1), by Remark 4.7, we get

$$\begin{cases} D^{\alpha_1,\gamma_1}(D^{\alpha_2,\gamma_1}+\lambda)\bar{x}(\xi)=f_1(\xi,\bar{y}(\xi),D^{\alpha_1,\gamma_1}\bar{x}(\xi))+g_\ell, \\ \quad \xi\in J=[0,T], 0<\alpha_1,\alpha_2<1, 0\le\gamma_1\le 1, \\ \Delta\bar{x}(\xi_\ell)=I_\ell(\bar{x}(\xi_\ell))+g_\ell, \quad \ell=1,2,3,4,5,\dots,m, \\ I^{1-\gamma_1}\bar{x}(0)=\bar{x}_0, \qquad \eta_1=(\alpha_1+\alpha_2)(1-\gamma_1)+\gamma_1. \end{cases} \tag{4.8}$$

Obviously the solution of (4.8) is

$$\bar{x}(\xi)=\begin{cases} \frac{\bar{x}_0}{\Gamma\gamma_1}\xi^{\gamma_1-1}+\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^\xi(\xi-\varrho)^{\alpha_1+\alpha_2-1}f_1(\varrho,\bar{y}(\varrho),D^{\alpha_1,\gamma_1}\bar{x}(\varrho))\,d\varrho \\ \quad -\frac{\lambda}{\Gamma(\alpha_1)}\int_0^\xi(\xi-\varrho)^{\alpha_1-1}\bar{x}(\varrho)\,d\varrho \\ \quad +\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^\xi(\xi-\varrho)^{\alpha_1+\alpha_2-1}g_\ell(\varrho)\,d\varrho-\frac{\lambda}{\Gamma(\alpha_1)}\int_0^\xi(\xi-\varrho)^{\alpha_1-1}g_\ell(\varrho)\,d\varrho, \quad \xi\in J_0, \\ \frac{\bar{x}_0}{\Gamma\gamma_1}\xi_m^{\gamma_1-1}+\sum_{\ell=1}^m\int_{\xi_{i-1}}^{\xi_\ell}\frac{(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}f_1(\varrho,\bar{y}(\varrho),D^{\alpha_1,\gamma_1}\bar{x}(\varrho))\,d\varrho \\ \quad -\sum_{\ell=1}^m\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}\bar{x}(\varrho)\,d\varrho \\ \quad +\sum_{\ell=1}^m\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}g_\ell(\varrho)\,d\varrho \\ \quad -\sum_{\ell=1}^m\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}g_\ell(\varrho)\,d\varrho \\ \quad +\sum_{\ell=1}^m I_\ell(\bar{x}(\xi_\ell))+\sum_{\ell=1}^m g_\ell, \quad \xi\in J_\ell, \ell=1,2,3,4,5,\dots,m. \end{cases}$$

Therefore, for each $\xi\in J_\ell$, we have the following

$$\begin{aligned} &\left|x(\xi)-\bar{x}(\xi)\right| \\ &\le\sum_{\ell=1}^m\int_{\xi_{i-1}}^{\xi_\ell}\frac{(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}\left|f_1\left(\varrho,y(\varrho),D^{\alpha_1,\gamma_1}x(\varrho)\right)-f_1\left(\varrho,\bar{y}(\varrho),D^{\alpha_1,\gamma_1}\bar{x}(\varrho)\right)\right|d\varrho \\ &\quad -\sum_{\ell=1}^m\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}\left|x(\varrho)-\bar{x}(\varrho)\right|d\varrho \\ &\quad +\sum_{\ell=1}^m\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}g_\ell(\varrho)\,d\varrho \\ &\quad -\sum_{\ell=1}^m\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}g_\ell(\varrho)\,d\varrho+\sum_{\ell=1}^m\left|I_\ell\left(x(\xi_\ell)\right)-I_\ell\left(\bar{x}(\xi_\ell)\right)\right|+\sum_{\ell=1}^m g_\ell \\ &\le\sum_{\ell=1}^m\int_{\xi_{i-1}}^{\xi_\ell}\frac{(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}\left|\left(x(\varrho)-\bar{x}(\varrho)\right)\right|d\varrho \\ &\quad -\sum_{\ell=1}^m\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}\left|x(\varrho)-\bar{x}(\varrho)\right|d\varrho \\ &\quad +\sum_{\ell=1}^m\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}g_\ell(\varrho)\,d\varrho \\ &\quad -\sum_{\ell=1}^m\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}g_\ell(\varrho)\,d\varrho \\ &\quad +Ł_k\sum_{\ell=1}^m\left|x(\xi)-y(\xi)\right|+\sum_{\ell=1}^m g_\ell, \end{aligned}$$

where

$$\begin{aligned}
x(\xi) &:= f_1\big(\xi, y(\xi), D^{\alpha_1,\gamma_1}x(\xi)\big) = f_1\big(\xi, y(\xi), x(\xi)\big),\\
\bar{x}(\xi) &:= f_1\big(\xi, \bar{y}(\xi), D^{\alpha_1,\gamma_1}\bar{x}(\xi)\big) = f_1\big(\xi, \bar{y}(\xi), \bar{x}(\xi)\big),\\
\big|x(\xi) - \bar{x}(\xi)\big| &= \big|f_1\big(\xi, y(\xi), x(\xi)\big) - f_1\big(\xi, \bar{y}(\xi), \bar{x}(\xi)\big)\big|\\
&\leq L_{f_1}\big|y(\xi) - \bar{y}(\xi)\big| + L_{g_1}\big|x(\xi) - \bar{x}(\xi)\big|.
\end{aligned}$$

This further gives

$$\big|x(\xi) - \bar{x}(\xi)\big| \leq \frac{L_{f_1}}{1 - L_{g_1}}\big|y(\xi) - \bar{y}(\xi)\big|, \tag{4.9}$$

and similarly

$$\begin{aligned}
\big|y(\xi) - \bar{y}(\xi)\big| &\leq \frac{L_{f_2}}{1 - L_{g_2}}\big|x(\xi) - \bar{x}(\xi)\big|,\\
\big|x(\xi) - \bar{x}(\xi)\big| &\leq \sum_{\ell=1}^{m}\int_{\xi_{i-1}}^{\xi_\ell}\frac{(\xi_\ell - \varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}\big|\big(x(\varrho) - \bar{x}(\varrho)\big)\big|\,d\varrho\\
&\quad - \sum_{\ell=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell - \varrho)^{\alpha_1-1}\big|x(\varrho) - \bar{x}(\varrho)\big|\,d\varrho\\
&\quad + \sum_{\ell=1}^{m}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell - \varrho)^{\alpha_1+\alpha_2-1}g_\ell(\varrho)\,d\varrho\\
&\quad - \sum_{\ell=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell - \varrho)^{\alpha_1-1}g_\ell(\varrho)\,d\varrho\\
&\quad + Ł_k\sum_{\ell=1}^{m}\big|x(\xi) - \bar{x}(\xi)\big| + \sum_{\ell=1}^{m}g_\ell.
\end{aligned} \tag{4.10}$$

Putting (4.9) in (4.10), we obtain

$$\begin{aligned}
&\big|x(\xi) - \bar{x}(\xi)\big|\\
&\quad\leq \sum_{\ell=1}^{m}\frac{L_{f_1}}{(1 - L_{g_1})\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell - \varrho)^{\alpha_1+\alpha_2-1}\big|y(\varrho) - \bar{y}(\varrho)\big|\,d\varrho\\
&\qquad - \sum_{\ell=1}^{m}\frac{\lambda L_{f_1}}{\Gamma(\alpha_1)(1 - L_{g_1})}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell - \varrho)^{\alpha_1-1}\big|y(\varrho) - \bar{y}(\varrho)\big|\,d\varrho + \frac{mŁ_k L_{f_1}}{1 - L_{g_1}}\big|y(\varrho) - \bar{y}(\varrho)\big|\\
&\qquad + \sum_{\ell=1}^{m}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell - \varrho)^{\alpha_1+\alpha_2-1}g_\ell(\varrho)\,d\varrho\\
&\qquad - \sum_{\ell=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell - \varrho)^{\alpha_1-1}g_\ell(\varrho)\,d\varrho + \sum_{\ell=1}^{m}g_\ell\\
&\quad\leq \sum_{\ell=1}^{m}\frac{L_{f_1}C_x}{(1 - L_{g_1})\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell - \varrho)^{\alpha_1+\alpha_2-1}\big|\varphi_x(\varrho)\big|\,d\varrho
\end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{\ell=1}^{m} \frac{C_x \lambda L_{f_1}}{\Gamma(\alpha_1)(1-L_{g_1})} \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1 - 1} \left|\varphi_x(\varrho)\right| d\varrho \\
&\quad + \frac{m Ł_k L_{f_1} \psi}{1 - L_{g_1}} + \sum_{\ell=1}^{m} \frac{\epsilon}{\Gamma(\alpha_1 + \alpha_2)} \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1 + \alpha_2 - 1} \, d\varrho \\
&\quad - \sum_{\ell=1}^{m} \frac{\lambda \epsilon}{\Gamma(\alpha_1)} \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1 - 1} \, d\varrho + m\epsilon \\
&\leq \sum_{\ell=1}^{m} \frac{L_{f_1} C_x}{(1 - L_{g_1})\Gamma(\alpha_1 + \alpha_2)} \left( \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1 + \alpha_2 - 1} \, d\varrho \right) \left( \int_{\xi_{i-1}}^{\xi_\ell} \left|\varphi_x(\varrho)\right| d\varrho \right) \\
&\quad - \sum_{\ell=1}^{m} \frac{C_x \lambda L_{f_1}}{\Gamma(\alpha_1)(1 - L_{g_1})} \left( \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1 - 1} \, d\varrho \right) \left( \int_{\xi_{i-1}}^{\xi_\ell} \left|\varphi_x(\varrho)\right| d\varrho \right) + \frac{m Ł_k L_{f_1} \psi}{1 - L_{g_1}} \\
&\quad + \sum_{\ell=1}^{m} \frac{\epsilon}{\Gamma(\alpha_1 + \alpha_2)} \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1 + \alpha_2 - 1} \, d\varrho - \sum_{\ell=1}^{m} \frac{\lambda \epsilon}{\Gamma(\alpha_1)} \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1 - 1} \, d\varrho + m\epsilon \\
&\leq \frac{m L_{f_1} C_x (T)^{\alpha_1 + \alpha_2 + 1}}{(1 - L_{g_1})\Gamma(\alpha_1 + \alpha_2 + 1)} - \frac{m C_x \lambda L_{f_1} (T)^{\alpha_1 + 1}}{(1 - L_{g_1})\Gamma\alpha_1 + 1} + \frac{m Ł_k L_{f_1} \psi}{1 - L_{g_1}} \\
&\quad + \frac{m\varepsilon (T)^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1 + \alpha_2 + 1)} - \frac{m\varepsilon\lambda (T)^{\alpha_1}}{\Gamma\alpha_1 + 1} + m\varepsilon,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\left|x(\xi) - \bar{x}(\xi)\right| \leq \varepsilon \Bigg( &\frac{m L_{f_1} C_x (T)^{\alpha_1 + \alpha_2 + 1}}{(1 - L_{g_1})\Gamma(\alpha_1 + \alpha_2 + 1)} - \frac{m C_x \lambda L_{f_1} (T)^{\alpha_1 + 1}}{(1 - L_{g_1})\Gamma\alpha_1 + 1} + \frac{m Ł_k L_{f_1} \psi}{1 - L_{g_1}} \\
&+ \frac{m}{\Gamma(\alpha_1 + \alpha_2 + 1)} (T)^{\alpha_1 + \alpha_2} - \frac{m\lambda}{\Gamma\alpha_1 + 1} (T)^{\alpha_1} + m \Bigg).
\end{aligned}
$$

Thus

$$
\left|x(\xi) - \bar{x}(\xi)\right| \leq \varepsilon C_{f_1, g_1, \alpha_1, \alpha_2}, \tag{4.11}
$$

where

$$
\begin{aligned}
C_{f_1, g_1, \alpha_1, \alpha_2} = &\frac{m L_{f_1} C_x (T)^{\alpha_1 + \alpha_2 + 1}}{(1 - L_{g_1})\Gamma(\alpha_1 + \alpha_2 + 1)} - \frac{m C_x \lambda L_{f_1} (T)^{\alpha_1 + 1}}{(1 - L_{g_1})\Gamma\alpha_1 + 1} + \frac{m Ł_k L_{f_1} \psi}{1 - L_{g_1}} \\
&+ \frac{m (T)^{\alpha_1 + \alpha_2}}{\Gamma(\alpha_1 + \alpha_2 + 1)} - \frac{m\lambda (T)^{\alpha_1}}{\Gamma\alpha_1 + 1} + m.
\end{aligned}
$$

In a similar fashion, we can get

$$
\left|y(\xi) - \bar{y}(\xi)\right| \leq \varepsilon C_{f_2, g_2, \beta_1, \beta_2}, \tag{4.12}
$$

where

$$
\begin{aligned}
C_{f_2, g_2, \beta_1, \beta_2} = &\frac{m L_{f_2} C_y (T)^{\beta_1 + \beta_2 + 1}}{(1 - L_{g_2})\Gamma(\beta_1 + \beta_2 + 1)} - \frac{m C_y \lambda L_{f_2} (T)^{\beta_1 + 1}}{(1 - L_{g_2})\Gamma\beta_1 + 1} + \frac{m Ł_k L_{f_2} \psi}{1 - L_{g_2}} \\
&+ \frac{m (T)^{\beta_1 + \beta_2}}{\Gamma(\beta_1 + \beta_2 + 1)} - \frac{m\lambda (T)^{\beta_1}}{\Gamma\beta_1 + 1} + m.
\end{aligned}
$$

Therefore, from (4.12) and (4.11), we get

$$\begin{aligned}\left|(x,y)-(\bar{x},\bar{y})\right| &\leq \varepsilon C_{f_1,g_1,\alpha_1,\alpha_2}+\varepsilon C_{f_2,g_2,\beta_1,\beta_2}\\ &\leq \varepsilon C_{f,g,\alpha_1,\alpha_2,\beta_1,\beta_2}.\end{aligned}$$

So Eq. (1.1) is UH stable and if we set $\phi(\varepsilon)=\varepsilon C_{f,g,\beta_1,\beta_2,\alpha_1,\alpha_2}$, $\phi(0)=0$, then Eq. (1.1) is generalized UH stable. □

**Theorem 4.11** *If* $(H_1)$–$(H_4)$ *and* (3.5) *hold, then* (1.1) *is UHR stable with respect to* $(\varphi,\psi)$, *consequently, generalized UHR stable.*

*Proof* Let $\bar{x}\in C_{1-\gamma_1}[a,b]$ satisfy (4.1) and let $x$ be the only solution of

$$\begin{cases} D^{\alpha_1,\gamma_1}(D^{\alpha_2,\gamma_1}+\lambda)x(\xi)=f_1(\xi,y(\xi),D^{\alpha_1,\gamma_1}x(\xi)),\\ \quad \xi\in J=[0,T], 0<\alpha_1,\alpha_2<1, 0\leq\gamma_1\leq 1,\\ \Delta x(\xi_\ell)=I_\ell(x(\xi_\ell)), \quad \ell=1,2,3,4,5,\dots,m,\\ I^{1-\gamma_1}x(0)=x_0, \qquad \eta_1=(\alpha_1+\alpha_2)(1-\gamma_1)+\gamma_1.\end{cases}$$

By Theorem 3.1, we have for each $\xi\in J_\ell$

$$\begin{aligned} x(\xi)={}&\frac{x_0}{\Gamma\gamma_1}\xi_m^{\gamma_1-1}+\sum_{\ell=1}^{m}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}f_1\big(\varrho,y(\varrho),D^{\alpha_1,\gamma_1}x(\varrho)\big)\,d\varrho\\ &-\sum_{\ell=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}x(\varrho)\,d\varrho+\sum_{\ell=1}^{m}I_\ell\big(x(\xi_\ell)\big),\\ &\xi\in J_\ell, \ell=1,2,3,4,5,\dots,m.\end{aligned}$$

Since $\bar{x}$ satisfies (4.1), by Remark 4.7, we get

$$\begin{cases} D^{\alpha_1,\gamma_1}(D^{\alpha_2,\gamma_1}+\lambda)\bar{x}(\xi)=f_1(\xi,\bar{y}(\xi),D^{\alpha_1,\gamma_1}\bar{x}(\xi))+g_\ell,\\ \quad \xi\in J=[0,T], 0<\alpha_1,\alpha_2<1, 0\leq\gamma_1\leq 1,\\ \Delta \bar{x}(\xi_\ell)=I_\ell(\bar{x}(\xi_\ell))+g_\ell, \quad \ell=1,2,3,4,5,\dots,m,\\ I^{1-\gamma_1}\bar{x}(0)=\bar{x}_0, \qquad \eta_1=(\alpha_1+\alpha_2)(1-\gamma_1)+\gamma_1.\end{cases}\tag{4.13}$$

Obviously the solution of (4.13) is

$$\bar{x}(\xi)=\begin{cases} \frac{\bar{x}_0}{\Gamma\gamma_1}\xi^{\gamma_1-1}+\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^\xi(\xi-\varrho)^{\alpha_1+\alpha_2-1}f_1(\varrho,\bar{y}(\varrho),D^{\alpha_1,\gamma_1}\bar{x}(\varrho))\,d\varrho\\ \quad -\frac{\lambda}{\Gamma(\alpha_1)}\int_0^\xi(\xi-\varrho)^{\alpha_1-1}\bar{x}(\varrho)\,d\varrho\\ \quad +\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^\xi(\xi-\varrho)^{\alpha_1+\alpha_2-1}g_\ell(\varrho)\,d\varrho-\frac{\lambda}{\Gamma(\alpha_1)}\int_0^\xi(\xi-\varrho)^{\alpha_1-1}g_\ell(\varrho)\,d\varrho, \quad \xi\in J_0,\\ \frac{\bar{x}_0}{\Gamma\gamma_1}\xi_m^{\gamma_1-1}+\sum_{\ell=1}^m\int_{\xi_{i-1}}^{\xi_\ell}\frac{(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}f_1(\varrho,\bar{y}(\varrho),D^{\alpha_1,\gamma_1}\bar{x}(\varrho))\,d\varrho\\ \quad -\sum_{\ell=1}^m\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}\bar{x}(\varrho)\,d\varrho\\ \quad +\sum_{\ell=1}^m\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}g_\ell(\varrho)\,d\varrho\\ \quad -\sum_{\ell=1}^m\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}g_\ell(\varrho)\,d\varrho\\ \quad +\sum_{\ell=1}^m I_\ell(\bar{x}(\xi_\ell))+\sum_{\ell=1}^m g_\ell, \quad \xi\in J_\ell, \ell=1,2,3,4,5,\dots,m.\end{cases}$$

Therefore, for each $\xi \in J_\ell$, we have the following

$$\begin{aligned}
&\left|x(\xi) - \bar{x}(\xi)\right| \\
&\quad \leq \sum_{\ell=1}^{m} \int_{\xi_{i-1}}^{\xi_\ell} \frac{(\xi_\ell - \varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \left|f_1\big(\varrho, y(\varrho), D^{\alpha_1,\gamma_1}x(\varrho)\big) - f_1\big(\varrho, \bar{y}(\varrho), D^{\alpha_1,\gamma_1}\bar{x}(\varrho)\big)\right| d\varrho \\
&\qquad - \sum_{\ell=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1-1} \left|x(\varrho) - \bar{x}(\varrho)\right| d\varrho \\
&\qquad + \sum_{\ell=1}^{m} \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1+\alpha_2-1} g_\ell(\varrho)\, d\varrho \\
&\qquad - \sum_{\ell=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1-1} g_\ell(\varrho)\, d\varrho + \sum_{\ell=1}^{m} \left|I_\ell\big(x(\xi_\ell)\big) - I_\ell\big(\bar{x}(\xi_\ell)\big)\right| + \sum_{\ell=1}^{m} g_\ell \\
&\quad \leq \sum_{\ell=1}^{m} \int_{\xi_{i-1}}^{\xi_\ell} \frac{(\xi_\ell - \varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \left|\big(x(\varrho) - \bar{x}(\varrho)\big)\right| d\varrho \\
&\qquad - \sum_{\ell=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1-1} \left|x(\varrho) - \bar{x}(\varrho)\right| d\varrho \\
&\qquad + \sum_{\ell=1}^{m} \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1+\alpha_2-1} g_\ell(\varrho)\, d\varrho \\
&\qquad - \sum_{\ell=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\xi_{i-1}}^{\xi_\ell} (\xi_\ell - \varrho)^{\alpha_1-1} g_\ell(\varrho)\, d\varrho \\
&\qquad + \text{Ł}_k \sum_{\ell=1}^{m} \left|x(\xi) - y(\xi)\right| + \sum_{\ell=1}^{m} g_\ell,
\end{aligned}$$

where

$$\begin{aligned}
x(\xi) &:= f_1\big(\xi, y(\xi), D^{\alpha_1,\gamma_1}x(\xi)\big) = f_1\big(\xi, y(\xi), x(\xi)\big), \\
\bar{x}(\xi) &:= f_1\big(\xi, \bar{y}(\xi), D^{\alpha_1,\gamma_1}\bar{x}(\xi)\big) = f_1\big(\xi, \bar{y}(\xi), \bar{x}(\xi)\big), \\
\left|x(\xi) - \bar{x}(\xi)\right| &= \left|f_1\big(\xi, y(\xi), x(\xi)\big) - f_1\big(\xi, \bar{y}(\xi), \bar{x}(\xi)\big)\right| \\
&\leq L_{f_1}\left|y(\xi) - \bar{y}(\xi)\right| + L_{g_1}\left|x(\xi) - \bar{x}(\xi)\right|.
\end{aligned}$$

This further gives

$$\left|x(\xi) - \bar{x}(\xi)\right| \leq \frac{L_{f_1}}{1 - L_{g_1}} \left|y(\xi) - \bar{y}(\xi)\right| \tag{4.14}$$

and similarly

$$\left|y(\xi) - \bar{y}(\xi)\right| \leq \frac{L_{f_2}}{1 - L_{g_2}} \left|x(\xi) - \bar{x}(\xi)\right|,$$

$$\begin{aligned}
\left|x(\xi)-\bar{x}(\xi)\right| \leq{}& \sum_{\ell=1}^{m}\int_{\xi_{i-1}}^{\xi_\ell}\frac{(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}\left|\left(x(\varrho)-\bar{x}(\varrho)\right)\right|d\varrho \\
&-\sum_{\ell=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}\left|x(\varrho)-\bar{x}(\varrho)\right|d\varrho \\
&+\sum_{\ell=1}^{m}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}g_\ell(\varrho)\,d\varrho \\
&-\sum_{\ell=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}g_\ell(\varrho)\,d\varrho \\
&+Ł_k\sum_{\ell=1}^{m}\left|x(\xi)-\bar{x}(\xi)\right|+\sum_{\ell=1}^{m}g_\ell.
\end{aligned} \tag{4.15}$$

Putting (4.14) in (4.15), we obtain

$$\begin{aligned}
&\left|x(\xi)-\bar{x}(\xi)\right| \\
&\quad\leq \sum_{\ell=1}^{m}\frac{L_{f_1}}{(1-L_{g_1})\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}\left|y(\varrho)-\bar{y}(\varrho)\right|d\varrho \\
&\qquad-\sum_{\ell=1}^{m}\frac{\lambda L_{f_1}}{\Gamma(\alpha_1)(1-L_{g_1})}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}\left|y(\varrho)-\bar{y}(\varrho)\right|d\varrho+\frac{mŁ_kL_{f_1}}{1-L_{g_1}}\left|y(\varrho)-\bar{y}(\varrho)\right| \\
&\qquad+\sum_{\ell=1}^{m}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}g_\ell(\varrho)\,d\varrho \\
&\qquad-\sum_{\ell=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}g_\ell(\varrho)\,d\varrho+\sum_{\ell=1}^{m}g_\ell \\
&\quad\leq \sum_{\ell=1}^{m}\frac{L_{f_1}C_x}{(1-L_{g_1})\Gamma(\alpha_1+\alpha_2)}\left(\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}\,d\varrho\right)\left(\int_{\xi_{i-1}}^{\xi_\ell}\left|\varphi_x(\varrho)\right|d\varrho\right) \\
&\qquad-\sum_{\ell=1}^{m}\frac{C_x\lambda L_{f_1}}{\Gamma(\alpha_1)(1-L_{g_1})}\left(\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}\,d\varrho\right)\left(\int_{\xi_{i-1}}^{\xi_\ell}\left|\varphi_x(\varrho)\right|d\varrho\right)+\frac{mŁ_kL_{f_1}\psi}{1-L_{g_1}} \\
&\qquad+\sum_{\ell=1}^{m}\frac{\varphi_x(\xi)}{\Gamma(\alpha_1+\alpha_2)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1+\alpha_2-1}\,d\varrho \\
&\qquad-\sum_{\ell=1}^{m}\frac{\lambda\varphi_x(\xi)}{\Gamma(\alpha_1)}\int_{\xi_{i-1}}^{\xi_\ell}(\xi_\ell-\varrho)^{\alpha_1-1}\,d\varrho+m\varphi_x(\xi) \\
&\quad\leq \frac{mL_{f_1}C_xC_\varphi(T)^{\alpha_1+\alpha_2}\varphi_x(\xi)}{(1-L_{g_1})\Gamma(\alpha_1+\alpha_2+1)}-\frac{mC_xC_\varphi\lambda L_{f_1}(T)^{\alpha_1}\varphi_x(\xi)}{(1-L_{g_1})\Gamma\alpha_1+1}+\frac{mŁ_kL_{f_1}\psi}{1-L_{g_1}} \\
&\qquad+\frac{m\varphi_x(\xi)}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\varphi_x(\xi)\lambda}{\Gamma\alpha_1+1}(T)^{\alpha_1}+m\varphi_x(\xi),
\end{aligned}$$

which implies that

$$\begin{aligned}
\left|x(\xi)-\bar{x}(\xi)\right| \leq{}& \Bigg(\frac{mL_{f_1}C_xC_\varphi(T)^{\alpha_1+\alpha_2}}{(1-L_{g_1})\Gamma(\alpha_1+\alpha_2+1)}-\frac{mC_xC_\varphi\lambda L_{f_1}(T)^{\alpha_1}}{(1-L_{g_1})\Gamma\alpha_1+1}+\frac{mŁ_kL_{f_1}}{1-L_{g_1}} \\
&+\frac{m}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma\alpha_1+1}(T)^{\alpha_1}+m\Bigg)\left(\varphi_x(\xi)+\psi\right).
\end{aligned}$$

Thus

$$\left|x(\xi)-\bar{x}(\xi)\right| \leq C_{f_1,g_2,\alpha_1,\alpha_2}\left(\varphi_x(\xi)+\psi\right), \tag{4.16}$$

where

$$\begin{aligned}C_{f,g,\alpha_1,\alpha_2} &= \frac{mL_{f_1}C_xC_\varphi(T)^{\alpha_1+\alpha_2}}{(1-L_{g_1})\Gamma(\alpha_1+\alpha_2+1)} - \frac{mC_xC_\varphi\lambda L_{f_1}(T)^{\alpha_1}}{(1-L_{g_1})\Gamma\alpha_1+1} + \frac{m\text{Ł}_kL_{f_1}}{1-L_{g_1}}\\ &\quad + \frac{m(T)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} - \frac{m\lambda(T)^{\alpha_1}}{\Gamma\alpha_1+1} + m.\end{aligned}$$

In a similar fashion, we can get

$$\left|y(\xi)-\bar{y}(\xi)\right| \leq C_{f_2,g_2,\alpha_1,\beta_2}\left(\varphi_x(\xi)+\psi\right), \tag{4.17}$$

where

$$\begin{aligned}C_{f_2,g_2,\beta_1,\beta_2} &= \frac{mL_{f_1}C_xC_\varphi(T)^{\beta_1+\beta_2}}{(1-L_{g_1})\Gamma(\beta_1+\beta_2+1)} - \frac{mC_xC_\varphi\lambda L_{f_1}(T)^{\beta_1}}{(1-L_{g_1})\Gamma\beta_1+1} + \frac{m\text{Ł}_kL_{f_1}}{1-L_{g_1}}\\ &\quad + \frac{m(T)^{\beta_1+\beta_2}}{\Gamma(\beta_1+\beta_2+1)} - \frac{m\lambda(T)^{\beta_1}}{\Gamma\beta_1+1} + m.\end{aligned}$$

Therefore from (4.12) and (4.11), we get

$$\begin{aligned}\left|(x,y)-(\bar{x},\bar{y})\right| &\leq C_{f_1,g_1,\alpha_1,\alpha_2}\left(\varphi_x(\xi)+\psi\right) + C_{f_2,g_2,\beta_1,\beta_2}\left(\varphi_x(\xi)+\psi\right)\\ &\leq \varepsilon C_{f,g,\alpha_1,\alpha_2,\beta_1,\beta_2}\left(\varphi_x(\xi)+\psi\right).\end{aligned}$$

Hence (1.1) is UHR stable and is obviously generalized UHR stable. □

*Example* 4.12

$$\begin{cases}\begin{cases}D^{(\frac{1}{2},\frac{1}{2})}(D^{(\frac{1}{3},\frac{1}{2})}+\frac{1}{2})x(\xi) = \frac{|y(\xi)+D^{(\frac{1}{2},\frac{1}{2})}x(\xi)|}{8+e^{\xi}+\xi^2},\\ \quad \xi\in J=[0,1], 0<\frac{1}{2},\frac{1}{3}<1, 0\leq\frac{1}{2}\leq 1,\\ I_\ell x(\frac{1}{2}) = \frac{x|(\frac{1}{2})|}{70+|x(\frac{1}{2})|},\\ I^{1-\gamma}x(0)=0, \qquad \eta_1=(\alpha_1+\alpha_2)(1-\gamma_1)+\gamma_1,\end{cases}\\ \begin{cases}D^{(\frac{1}{3},\frac{1}{3})}(D^{(\frac{1}{4},\frac{1}{2})}+\frac{1}{2})y(\xi) = \frac{|x(\xi)+D^{(\frac{1}{2},\frac{1}{2})}y(\xi)|}{8+e^{\xi}+\xi^2},\\ \quad \xi\in J=[0,1], 0<\frac{1}{3},\frac{1}{4}<1, 0\leq\frac{1}{3}\leq 1,\\ I_\ell y(\frac{1}{2}) = \frac{y|(\frac{1}{2})|}{70+|y(\frac{1}{2})|},\\ I^{1-\gamma_1}y(0)=0, \qquad \eta_1=(\beta_1+\beta_2)(1-\gamma_2)+\gamma_2.\end{cases}\end{cases} \tag{4.18}$$

Let $J_0 = [0, \frac{1}{2}]$, $J_1 = [\frac{1}{2}, 1]$ $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3}$, $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{1}{3}$, $\lambda = \lambda_\varphi = \frac{1}{2}$, $L_{f_1} = L_{f_2} = Ł_k = \frac{1}{90e^2}$, $L_{g_1} = L_{g_2} = \frac{1}{9.0e^{-2}}$ and $m = T = 1$. Then, obviously

$$\begin{aligned}&\frac{mL_{f_1}(T)^{\alpha_1+\alpha_2}}{(1-L_{g_1})\Gamma(\alpha_1+\alpha_2+1)} - \frac{m\lambda L_{f_1}(T)^{\alpha_1}}{(1-L_{g_1})\Gamma\alpha_1+1} + \frac{mL_{f_2}(T)^{\beta_1+\beta_2}}{(1-L_{g_2})\Gamma(\beta_1+\beta_2+1)}\\ &\quad - \frac{m\lambda L_{f_2}(T)^{\beta_1}}{(1-L_{g_2})\Gamma\beta_1+1} + 2mŁ_k < 1.\end{aligned}$$

Thus, by Theorem 3.2, (4.18) has a unique solution. Furthermore, the conditions of Theorem 4.10 are satisfied, so the solution of (4.18) is UH stable and generalized UH stable. Furthermore, it can be easily verified that the conditions of Theorem 4.11 hold and thus (4.18) is UHR stable and consequently generalized UHR stable.

## 5 Conclusion

In this article, switched coupled system of implicit impulsive LEs with four HFDs is considered. Some assumptions are made to avoid the hurdles, to examine the existence and uniqueness, and to discuss different types of UH stability of our considered model, using Banach fixed-point theorem. The main aim of the authors is that these qualitative properties can be examined and established in the future on some impulsive real-world systems arising in mathematical models of the brain.

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## Declarations

**Competing interests**
The authors declare no competing interests.

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