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A Necessary Optimality Condition for Optimal Control of Caputo Fractional Evolution Equations ⋆

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Abstract: In this paper, we prove the Pontryagin maximum principle, which constitutes the necessary optimality condition, for the infinite-dimensional optimal control problem of X-valued left Caputo fractional evolution equations, where X is a Banach space. An important step in the proof to obtain the desired Hamiltonian maximization condition is to establish new variational and duality analysis. While the former is characterized by a linear X-valued left Caputo fractional evolution equation via spike variation, the latter requires the adjoint equation characterized by a linear X^*-valued right Riemann-Liouville (RL) fractional evolution equation, where X^* is a dual space of X. We show the variational and duality analysis with the help of the infinite-dimensional fractional version of the technical lemma and the explicit representation of solutions to linear (Caputo and RL) fractional evolution equations using left and right RL state-transition evolution operators.

Keywords: Caputo and RL fractional evolution equations, maximum principle, variational and duality analysis.

1. INTRODUCTION

Fractional derivatives and integrals can be viewed as generalizations of their classical notions to any real arbitrary order. They have been applied to study (finite- and/or infinite-dimensional) *fractional differential equations*, which enables us to analyze more general and extraordinary phenomena observed in real world. Indeed, various types of fractional differential equations and their applications were considered in applied mathematics, science, engineering, and economics; see (Kilbas et al., 2006; Diethelm, 2010; Zhou, 2016) and the references therein.

Infinite-dimensional optimal control has been studied extensively in various settings, which requires dealing with *evolution equations* driven by a (possibly unbounded) linear operator, which could be a generator of the C_0-semigroup of bounded linear operators. In fact, the infinite-dimensional control theory is essential to study several important classes of optimal control problems, including control of partial differential equations (PDEs) and delayed systems, in an abstract way; see (Li and Yong, 1995; Fattorini, 1999; Fabbri et al., 2017) and the references therein.

In the case of finite-dimensions, the maximum principle, which constitutes the necessary optimality condition, for fractional optimal control problems was studied in (Bergounioux and Bourdin, 2020; Ali et al., 2016; Kamocki, 2014; Yildiz et al., 2020; Almeida et al., 2021; Lin and Yong, 2020). On the other hand, the classical maximum principles for nonfractional infinite-dimensional optimal control problems were obtained in (Li and Yong, 1995; Krastanov et al., 2011; Fattorini, 1999; Frankowska et al., 2018; Breitenbach and BorzÍ, 2020; Zhang et al., 2020; Liu et al., 2020). There are some results on existence and uniqueness of (mild) solutions for infinite-dimensional fractional evolution equations (Zhou, 2016; Chen et al., 2014; Sin et al., 2018; Wang and Zhou, 2011; Lizama, 2019). However, it is quite surprising that the maximum principle for infinite-dimensional optimal control problems of fractional evolution equations has not been studied until now, which we address in this paper.

In this paper, we prove the Pontryagin maximum principle for the infinite-dimensional optimal control problem of X-valued left Caputo fractional evolution equations, where X is a (possibly infinite-dimensional) Banach space (see **(P)** in (6) and Theorem 3). An important step in the proof to obtain the desired Hamiltonian maximization condition (see Theorem 3) is to establish variational and duality analysis (see Section 4). While the former is characterized by a linear X-valued left Caputo fractional evolution equation via spike variation, the latter requires the adjoint equation characterized by a linear X^*-valued

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right Riemann-Liouville (RL) fractional evolution equation, where X^* is a dual space of X. We show the variational and duality analysis with the help of the infinite-dimensional fractional version of the technical lemma (see Lemmas 4 and 5) and the explicit characterization of solutions to linear (Caputo and RL) fractional evolution equations using left and right RL state-transition evolution operators (see Lemmas 6 and 7).

Our technique for proving the maximum principle in Theorem 3 (see Section 4) is significantly different from the above-mentioned classical results for finite-dimensional fractional optimal control problems (see (Bergounioux and Bourdin, 2020; Ali et al., 2016; Kamocki, 2014)) and for nonfractional infinite-dimensional optimal control problems (see (Li and Yong, 1995; Krastanov et al., 2011; Fattorini, 1999)). Specifically, in our variational analysis (Section 4.1), we apply a fractional calculus approach to obtain the precise estimates of the variational equation under spike variation (see Lemma 5), in which the key technical lemma (see Lemma 4) can be seen as an extension of the classical nonfractional one in (Li and Yong, 1995, Corollary 3.9, Chapter 4). Moreover, in our duality analysis (see Section 4.2), it is essential to obtain explicit representations of solutions to linear (Caputo and RL) fractional variational and adjoint equations in terms of the associated RL state-transition evolution operators, which has not been reported in the existing literature. Indeed, the maximum principle of this paper (see Theorem 3) is new in the optimal control problem context and its proof requires development of a new technique, both of which are not reported in the existing literature. We also mention that the infinite-dimensional maximum principle of integral-type evolution equations was studied recently in (Ding et al., 2022).

Our paper is organized as follows. We provide the problem statement in Section 2. The statement of the maximum principle is given in Section 3, and its proof is provided in Section 4. An example is given in Section 5. We conclude this paper in Section 6.

2. NOTATION AND PROBLEM STATEMENT

2.1 Notation and Preliminaries on Fractional Calculus

In this subsection, we provide the notation and some preliminary results on fractional calculus. More detailed results on fractional calculus can be found in (Kilbas et al., 2006; Zhou, 2016).

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R} := \mathbb{R}^1$, where the norm in $\mathbb{R}^n$ is defined by $|\cdot| := |\cdot|_{\mathbb{R}^n}$. Let $\mathbb{1}_A(\cdot)$ be the indicator function of any set A. Let Γ be the Gamma function. Let $[0,T]$ with $T < \infty$ be the (finite) time-horizon. Let $\Delta := \{(s,t) \in [0,T] \times [0,T] \mid 0 \leq s < t \leq T\}$ and $\bar{\Delta}$ be the closure of Δ. In this paper, $M \geq 0$ is a generic constant, whose exact value varies from line to line.

Let X be a Banach space with norm $|\cdot|_{\mathsf{X}}$, and X^* the dual space of X, i.e., the space of all bounded (or continuous) linear functionals on X. The norm on X^* is defined by $|f|_{\mathsf{X}^*} := \sup_{x \in \mathsf{X},\ |x|_{\mathsf{X}} \leq 1} |f(x)|$ for $f \in \mathsf{X}^*$. Let $\langle \cdot, \cdot \rangle_{\mathsf{X}^*,\mathsf{X}}$ be the usual duality paring between X and X^*. We write $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\mathsf{X}^*,\mathsf{X}}$ when there is no confusion. The set of linear bounded operators from X to another Banach space Y is denoted by $\mathcal{L}(\mathsf{X},\mathsf{Y})$. Note that $\mathcal{L}(\mathsf{X}) := \mathcal{L}(\mathsf{X},\mathsf{X})$. Let $|A|_{\mathcal{L}(\mathsf{X},\mathsf{Y})}$ be the (operator) norm of $A \in \mathcal{L}(\mathsf{X},\mathsf{Y})$. Let $\mathrm{I} \in \mathcal{L}(\mathsf{X})$ be an identity operator. Let $A^* \in \mathcal{L}(\mathsf{Y}^*,\mathsf{X}^*)$ be the adjoint operator of $A \in \mathcal{L}(\mathsf{X},\mathsf{Y})$, i.e., $\langle A^* y^*, x \rangle_{\mathsf{X}^*,\mathsf{X}^*} = \langle y^*, Ax \rangle_{\mathsf{Y}^*,\mathsf{Y}}$. Clearly, $A^* \in \mathcal{L}(\mathsf{Y}^*,\mathsf{X}^*)$ is also linear and bounded.

We say that f is a Banach space valued function on $[0,T]$ if $f : [0,T] \to \mathsf{X}$. Note that the integration of Banach spaced valued functions is understood in the Bochner sense (Li and Yong, 1995, page 45). For $1 \leq p \leq \infty$, let $L^p([0,T];\mathsf{X})$ be the usual L^p-space with norm $\|\cdot\|_{L^p}$. Let $C([0,T];\mathsf{X})$ be the space of X-valued continuous functions on $[0,T]$ with norm $\|\cdot\|_\infty$. $\mathrm{AC}([0,T];\mathsf{X})$ denotes the space of absolutely continuous X-valued functions on $[0,T]$.

Definition 1. (i) For $f(\cdot) \in L^1([0,T];\mathsf{X})$ and $t \in [0,T]$, the left Riemann-Liouville (RL) fractional integral $\mathrm{I}^{\alpha}_{0+}[f]$ of order $\alpha > 0$ is defined by

$$\mathrm{I}^{\alpha}_{0+}[f](t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \mathrm{d}s.$$

(ii) For $f(\cdot) \in L^1([0,T];\mathsf{X})$ and $t \in [0,T]$, the right RL fractional integral $\mathrm{I}^{\alpha}_{T-}[f]$ of order $\alpha > 0$ is defined by

$$\mathrm{I}^{\alpha}_{T-}[f](t) := \int_t^T \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} f(s) \mathrm{d}s.$$

(iii) For $\alpha = 0$, we set $\mathrm{I}^{0}_{0+}[f](\cdot) := \mathrm{I}^{0}_{T-}[f](\cdot) := f(\cdot)$. □

Definition 2. (i) For $f(\cdot) \in L^1([0,T];\mathsf{X})$, the left RL fractional derivative $\mathrm{D}^{\alpha}_{0+}[f]$ of order $\alpha \in (0,1)$ is defined by

$${}^{\mathrm{RL}}\mathrm{D}^{\alpha}_{0+}[f](t) := \frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathrm{I}^{1-\alpha}_{0+}[f]\Big](t),$$

provided that $\mathrm{I}^{1-\alpha}_{0+}[f](\cdot) \in \mathrm{AC}([0,T];\mathsf{X})$. In this case, ${}^{\mathrm{RL}}\mathrm{D}^{\alpha}_{0+}[f](\cdot) \in L^1([0,T];\mathsf{X})$.

(ii) For $f(\cdot) \in L^1([0,T];\mathsf{X})$, the right RL fractional derivative $\mathrm{D}^{\alpha}_{T-}[f]$ of order $\alpha \in (0,1)$ is defined by

$${}^{\mathrm{RL}}\mathrm{D}^{\alpha}_{T-}[f](t) := -\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathrm{I}^{1-\alpha}_{T-}[f]\Big](t),$$

provided that $\mathrm{I}^{1-\alpha}_{T-}[f](\cdot) \in \mathrm{AC}([0,T];\mathsf{X})$. In this case, ${}^{\mathrm{RL}}\mathrm{D}^{\alpha}_{T-}[f](\cdot) \in L^1([0,T];\mathsf{X})$.

Definition 3. (i) For $f(\cdot) \in L^1([0,T];\mathsf{X})$, the left Caputo fractional derivative ${}^{\mathrm{C}}\mathrm{D}^{\alpha}_{0+}[f]$ of order $\alpha \in (0,1)$ is defined by

$${}^{\mathrm{C}}\mathrm{D}^{\alpha}_{0+}[f](t) := \mathrm{D}^{\alpha}_{0+}[f(\cdot) - f(0)](t),$$

where $f(\cdot) - f(0)$ is left RL fractional differentiable in the sense of Definition 2.

(ii) For $f(\cdot) \in L^1([0,T];\mathsf{X})$, the right Caputo fractional derivative ${}^{\mathrm{C}}\mathrm{D}^{\alpha}_{T-}[f]$ of order $\alpha \in (0,1)$ is defined by

$${}^{\mathrm{C}}\mathrm{D}^{\alpha}_{T-}[f](t) := \mathrm{D}^{\alpha}_{T-}[f(\cdot) - f(T)](t),$$

where $f(\cdot) - f(T)$ is right RL fractional differentiable in the sense of Definition 2. □

Lemma 1. (Kilbas et al., 2006, Lemma 2.3) For any $f(\cdot) \in L^1([0,T];\mathsf{X})$ and $\alpha, \beta > 0$, it holds that $\mathrm{I}_{0+}^{\alpha}[\mathrm{I}_{0+}^{\beta}[f]](\cdot) = \mathrm{I}_{0+}^{\alpha+\beta}[f](\cdot) = \mathrm{I}_{0+}^{\beta+\alpha}[f](\cdot) = \mathrm{I}_{0+}^{\beta}[\mathrm{I}_{0+}^{\alpha}[f]](\cdot)$. The same result holds for $\mathrm{I}_{T-}^{\alpha}[f]$. □

2.2 Problem Statement

Let X be a Banach space, which is the state space. Consider the following X-valued left Caputo fractional evolution differential equation on $[0,T]$ with order $\alpha \in (0,1)$:

$$\begin{cases} {}^{\mathrm{C}}\mathrm{D}_{0+}^{\alpha}[X](t) + AX(t) = f(t, X(t), u(t)),\ t \in (0,T], \\ X(0) = X_0 \in \mathsf{X}, \end{cases} \tag{1}$$

where $X(\cdot) \in \mathsf{X}$ is the state, $u : [0,T] \to \mathsf{U}$ is the control input, and $f : [0,T] \times \mathsf{X} \times \mathsf{U} \to \mathsf{X}$ is the driver of the Caputo evolution differential equation in (1). Let $\mathcal{U} := \{u : [0,T] \to \mathsf{U} \mid u \text{ is measurable in } t \in [0,T]\}$ be the set of admissible controls for (1).

Assumption 1. (i) (U, d) is a separable metric space.

(ii) $A : D(A) \subset \mathsf{X} \to \mathsf{X}$, where $D(A)$ is the domain of A, i.e., the subset of X such that A exists, is a linear operator such that $-A$ is the generator of the compact analytic semigroup $(\mathscr{T}(t))_{t\geq 0}$ of uniformly bounded linear operators with $\mathscr{T}(\cdot) : [0,T] \to \mathcal{L}(\mathsf{X})$ (Pazy, 1983).

(iii) $t \mapsto f(t,X,u)$ is strongly measurable with $f(\cdot, X, u) \in L^{\infty}([0,T];\mathsf{X})$, and $(X,u) \mapsto f(t,X,u)$ is Lipschitz continuous, i.e., there is a constant $L \geq 0$ such that for any $t \in [0,T]$ and $(X,u), (X',u') \in \mathsf{X} \times \mathsf{U}$,

$$|f(t,X,u) - f(t,X',u')|_{\mathsf{X}} \leq L(|X - X'|_{\mathsf{X}} + d(u,u'))$$
$$|f(t,0,u)|_{\mathsf{X}} \leq L(1 + |X|_{\mathsf{X}}).$$

(iv) $X \mapsto f(t,X,u)$ is continuously Fréchet differentiable, denoted by $\partial_X f(t,X,u)$, with $(t,X,u) \mapsto \partial_X f(t,X,u)$ being bounded in the L^{∞} sense and $(X,u) \mapsto \partial_X f(t,X,u)$ being Lipschitz continuous. Note that $\partial_X f(t,\cdot,u) : [0,T] \times \mathsf{U} \to \mathcal{L}(\mathsf{X})$. □

Theorem 2. Let Assumption 1 hold. Then (1) admits a unique mild solution. Moreover, the solution of (1) can be expressed by the left RL fractional integral form:

$$X(t) = X_0 - \mathrm{I}_{0+}^{\alpha}[AX(\cdot)](t) + \mathrm{I}_{0+}^{\alpha}[f(\cdot, X(\cdot), u(\cdot))](t). \tag{2}$$

Finally, for any $X_0, X_0' \in \mathsf{X}$ and $u(\cdot), u'(\cdot) \in \mathcal{U}$ (with $X(t) := X(t; X_0, u)$ and $X'(t) := X(t; X_0', u')$), we have

$$\sup_{t\in[0,T]} |X(t;X_0,u) - X(t;X_0',u')|_{\mathsf{X}} \tag{3}$$
$$\leq b(T) + \int_0^T \sum_{k=1}^{\infty} \frac{(M\Gamma(\alpha))^k}{\Gamma(k\alpha)} (T-s)^{k\alpha-1} b(s)\mathrm{d}s$$

$$\sup_{t\in[0,T]} |X(t;X_0,u)|_{\mathsf{X}} \tag{4}$$
$$\leq b'(T) + \int_0^T \sum_{k=1}^{\infty} \frac{(M\Gamma(\alpha))^k}{\Gamma(k\alpha)} (T-s)^{k\alpha-1} b'(s)\mathrm{d}s,$$

where $b(t) = |X_0 - X_0'|_{\mathsf{X}} + M\int_0^t (t-s)^{\alpha-1} d(u(s),u'(s))\mathrm{d}s$ and $b'(t) = |X_0|_{\mathsf{X}} + M\int_0^t (t-s)^{\alpha-1}\mathrm{d}s$.

The objective functional is given by the following left RL fractional integral with order $\beta \geq 1$:

$$J(X_0; u(\cdot)) = I_{0+}^{\beta}[l(\cdot, X(\cdot), u(\cdot)](T) + m(X(T)). \tag{5}$$

The fractional optimal control problem of this paper is

$$\textbf{(P)} \quad \inf_{u(\cdot)\in\mathcal{U}} J(X_0; u(\cdot)),\ \text{s.t. (1)}. \tag{6}$$

Note that **(P)** can be regarded as the optimal control problem for Caputo fractional evolution equations in infinite dimensions. The aim of this paper is to derive the Pontryagin maximum principle for **(P)**.

Assumption 2. (i) $l : [0,T] \times \mathsf{X} \times \mathsf{U} \to \mathbb{R}$ is the running cost, where $t \mapsto l(t,X,u)$ is strongly measurable, $l(\cdot,X,u) \in L^{\infty}([0,T];\mathbb{R})$, and $(X,u) \mapsto l(t,X,u)$ is Lipschitz continuous. $m : \mathsf{X} \to \mathbb{R}$ is the terminal cost, where $X \mapsto m(X)$ is Lipschitz continuous.

(ii) $X \mapsto l(t,X,u)$ is continuously Fréchet differentiable, where $(t,X,u) \mapsto \partial_X l(t,X,u)$ is continuous and bounded, and $(X,u) \mapsto \partial_X l(t,X,u)$ is Lipschitz continuous.$X \mapsto m(X)$ is continuously Fréchet differentiable, where $X \mapsto \partial_X m(X)$ is bounded and Lipschitz continuous. Note that $\partial_X l(t,\cdot,u) : [0,T] \times \mathsf{U} \to \mathcal{L}(\mathsf{X},\mathbb{R}) = \mathsf{X}^*$ and $\partial_X m(\cdot) \in \mathcal{L}(\mathsf{X},\mathbb{R}) = \mathsf{X}^*$. □

3. STATEMENT OF MAIN RESULT

Assume that $(\overline{u}(\cdot), \overline{X}(\cdot)) \in \mathcal{U} \times C([0,T];\mathsf{X})$ is the optimal solution of **(P)**, i.e., $\overline{u}(\cdot) \in \mathcal{U}$ is the optimal control of **(P)** and $\overline{X}(\cdot) := \overline{X}(\cdot; \overline{X}_0, \overline{u}) \in C([0,T];\mathsf{X})$ is the optimal state trajectory of (1) controlled by $\overline{u}(\cdot) \in \mathcal{U}$. We let

$$\overline{f}(t) := f(t, \overline{X}(t), \overline{u}(t)),\ \partial_X \overline{f}(t) := \partial_X f(t, \overline{X}(t), \overline{u}(t))$$
$$\overline{l}(t) := l(\cdot, \overline{X}(t), \overline{u}(t)),\ \partial_X \overline{l}(t) := \partial_X l(t, \overline{X}(t), \overline{u}(t))$$
$$\partial_X \overline{m}(T) := \partial_X m(\overline{X}(T)).$$

We state the main result of this paper:

Theorem 3. Suppose that Assumptions 1-2 hold. Assume that $(\overline{u}(\cdot), \overline{X}(\cdot)) \in \mathcal{U} \times C([0,T];\mathsf{X})$ is the optimal solution of **(P)**. Then the following conditions hold:

(i) There is a nontrivial $P(\cdot) \in L^p([0,T];\mathsf{X}^*)$ $(1 \leq p < \infty)$ such that $P(\cdot) \in L^p([0,T];\mathsf{X}^*)$ is a unique (mild) solution to the following adjoint equation:

$$\begin{cases} {}^{\mathrm{RL}}\mathrm{D}_{T-}^{\alpha}[P](t) = -A^*P(t) + \partial_X \overline{f}(t)^* P(t) \\ \qquad - \dfrac{(T-t)^{\beta-1}}{\Gamma(\beta)} \partial_X \overline{l}(t),\ t \in [0,T), \\ \mathrm{I}_{T-}^{1-\alpha}[P](T) = -\partial_X \overline{m}(T). \end{cases} \tag{7}$$

(ii) $\overline{u}(\cdot) \in \mathcal{U}$ satisfies the following Hamiltonian maximization condition:

$$H(t, \overline{X}(t), P(t), \overline{u}(t)) \tag{8}$$
$$= \max_{u\in\mathsf{U}} H(t, \overline{X}(t), P(t), u), \text{a.e. } t \in [0,T],$$

where $H : [0,T] \times \mathsf{X} \times \mathsf{X}^* \times \mathsf{U} \to \mathbb{R}$ is the Hamiltonian defined by

$$H(t,X,P,u) \tag{9}$$
$$:= \langle P, f(t,X,u)\rangle_{\mathsf{X}^*,\mathsf{X}} - \frac{(T-t)^{\beta-1}}{\Gamma(\beta)} l(t,X,u). \qquad \square$$

Remark 1. (i) Theorem 3 is different from the nonfractional infinite-dimensional maximum principles in (Li and Yong, 1995; Krastanov et al., 2011; Fattorini, 1999), where Theorem 3 involves the RL fractional adjoint equation and the fractional Hamiltonian.
(ii) If $\mathsf{X} = \mathbb{R}^n$ and $\mathsf{U} \subset \mathbb{R}^p$ in **(P)**, then Theorem 3 is reduced to the maximum principle for the finite-dimensional case studied in (Bergounioux and Bourdin, 2020; Ali et al., 2016; Kamocki, 2014).

4. PROOF OF THEOREM 3

This section proves Theorem 3.

Recall that the pair $(\overline{u}(\cdot), \overline{X}(\cdot)) \in \mathcal{U} \times C([0,T];\mathsf{X})$ is the optimal solution of **(P)**.

Let $\epsilon \in (0,1)$ and $E_\epsilon \subset [0,T]$ a measurable set, whose Lebesgue measure is $|E_\epsilon| = \epsilon T$. Consider the following *spike variation*:

$$u^\epsilon(t) := \begin{cases} \overline{u}(t), & t \in [0,T] \setminus E_\epsilon, \\ u(t), & t \in E_\epsilon, \end{cases} \tag{10}$$

where $u(\cdot) \in \mathcal{U}$. Clearly, $u^\epsilon(\cdot) \in \mathcal{U}$. We introduce the following *Ekeland metric:*

$$\tilde{d}(u(\cdot), u'(\cdot)) := \{t \in [0,T] \mid u(t) \neq u'(t)\}.$$

Then by definition, it follows that

$$\tilde{d}(u^\epsilon(\cdot), \overline{u}(\cdot)) \leq |E_\epsilon| = \epsilon T. \tag{11}$$

Let $X^\epsilon(\cdot) := X^\epsilon(\cdot; X_0, u^\epsilon)$ be the state trajectory controlled by u^ϵ in (10). Note that $\overline{X}(0) = X^\epsilon(0) = X_0$.

4.1 Variational Analysis

We begin this subsection by the following technical lemma. The proof is omitted.

Lemma 4. For any $\alpha, \epsilon \in (0,1)$ and $g \in L^1([0,T];\mathsf{X}) \cap C([0,T];\mathsf{X})$, there is an E_ϵ with $|E_\epsilon| = \epsilon T$ such that

$$\sup_{t\in[0,T]} \Big| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big(\frac{\mathbb{1}_{E_\epsilon}(s)}{\epsilon} - 1 \Big) g(s) \mathrm{d}s \Big|_{\mathsf{X}} = o(1). \tag{12}$$

We define (see the notation in Section 3): $\widehat{f}(t) := f(t, \overline{X}(t), u(t)) - \overline{f}(t)$ and $\widehat{l}(t) := l(t, \overline{X}(t), u(t)) - \overline{l}(t)$. Below, we state the variational analysis. The proof is omitted.

Lemma 5. (Variational Analysis). It holds that

$$\sup_{t\in[0,T]} \left| \frac{X^\epsilon(t) - \overline{X}(t)}{\epsilon} - Z(t) \right|_{\mathsf{X}} = o(1), \tag{13}$$

where Z is the variational equation given by

$$\begin{cases} {}^{\mathrm{C}}\mathrm{D}^\alpha_{0+}[Z](t) + AZ(t) = \partial_X \overline{f}(t) Z(t) + \widehat{f}(t), \ t \in (0,T], \\ Z(0) = 0. \qquad \square \end{cases} \tag{14}$$

Remark 2. The variational equation in (14) can be viewed as a linear X-valued left Caputo fractional evolution equation. Its explicit expression of the solution is given in Lemma 6 in terms of the RL state-transition operator. □

In addition to Lemma 5, we have

$$\left| \frac{J(X_0; u^\epsilon(\cdot)) - J(X_0; \overline{u}(\cdot))}{\epsilon} - \overline{Z}(T) \right| = o(1), \tag{15}$$

where

$$\overline{Z}(T) = \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} \Big(\partial_X \overline{l}(s) Z(s) + \widehat{l}(s) \Big) \mathrm{d}s + \partial_X \overline{m}(T) Z(T). \tag{16}$$

To show (15), note that

$$\begin{aligned} &\frac{J(X_0; u^\epsilon(\cdot)) - J(X_0; \overline{u}(\cdot))}{\epsilon} - \overline{Z}(T) \\ &= \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} l^\epsilon_X(s) \Big(Z^\epsilon(s) - Z(s) \Big) \mathrm{d}s \\ &\quad + \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} \Big(l^\epsilon_X(s) - \partial_X \overline{l}(s) \Big) Z(s) \mathrm{d}s \\ &\quad + \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} \Big(\frac{\mathbb{1}_{E_\epsilon}}{\epsilon} - 1 \Big) \widehat{l}(s) \mathrm{d}s \\ &\quad + m^\epsilon_X(T) \Big(Z^\epsilon(T) - Z(T) \Big) + \Big(m^\epsilon_X(T) - \partial_X \overline{m}(T) \Big) Z(T). \end{aligned}$$

Then using the similar approach of the proof for Lemma 5, we are able to show (15).

4.2 Proof of (i): Variational and Adjoint Equations

This section provides the representation results of linear fractioanl evolution equations. The case of finite dimensions was reported in (Bourdin, 2017; Gomoyunov, 2020).

Lemma 6. The (mild) solution of the variational equation in (14) is can be written as follows:

$$Z(t) = \int_0^t \Pi(t,s) \widehat{f}(s) \mathrm{d}s, \tag{17}$$

where Π is the X-valued left RL fractional state-transition evolution operator $\Pi : \Delta \to \mathcal{L}(\mathsf{X})$ given by

$$\Pi(t,s)x = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mathrm{I}x + \int_s^t \frac{(t-r)^{\alpha-1}}{\Gamma(\alpha)} \Big(-A + \partial_X \overline{f}(r) \Big) \Pi(r,s) x \mathrm{d}r. \qquad \square \tag{18}$$

In fact, by Fubini's formula (see (Bogachev, 2000, Theorem 3.4.4)), we can show that (18) can be rewritten as:

$$\Pi(t,s)x = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mathrm{I}x + \int_s^t \frac{(r-s)^{\alpha-1}}{\Gamma(\alpha)} \Pi(t,r) \Big(-A + \partial_X \overline{f}(r) \Big) x \mathrm{d}r. \tag{19}$$

As (19) is equivalent to (18), which is expressed in a backward manner, by Definition 1, (19) is the X-valued right RL fractional state-transition evolution operator.

Recall the adjoint equation $P(\cdot) \in L^p([0,T];\mathsf{X}^*)$ in (7):

$$\begin{cases} {}^{\mathrm{RL}}\mathrm{D}^\alpha_{T-}[P](t) = -A^* P(t) + \partial_X \overline{f}(t)^* P(t) \\ \qquad - \dfrac{(T-t)^{\beta-1}}{\Gamma(\beta)} \partial_X \overline{l}(t), \ t \in [0,T), \\ \mathrm{I}^{1-\alpha}_{T-}[P](T) = -\partial_X \overline{m}(T). \end{cases} \tag{20}$$

Remark 3. The adjoint equation in (20) (equivalently (7)) is the X^*-valued right RL fractional evolution equation with the terminal condition. Its explicit expression of the solution is given in Lemma 7 in terms of the RL state-transition operator. □

Lemma 7. (Proof for (i) of Theorem 3). The (mild) solution of the adjoint equation in (20) can be written as follows:

$$P(t) = -\Pi(T,t)^* \partial_X \overline{m}(T) - \int_t^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} \Pi(s,t)^* \partial_X \bar{l}(s) \mathrm{d}s, \tag{21}$$

where Π is the X-valued right RL fractional state-transition evolution operator $\Pi : \Delta \to \mathcal{L}(\mathsf{X})$ in (19). □

4.3 Proof of (ii): Duality Analysis and Hamiltonian Maximization Condition

To prove (ii) in Theorem 3, we obtain the duality analysis between variational and adjoint equations using Lemmas 6 and 7.

Note that (15) and (16) imply that

$$0 \leq \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} \Big(\langle \partial_X \bar{l}(s), Z(s) \rangle + \widehat{l}(s) \Big) \mathrm{d}s + \partial_X \overline{m}(T) Z(T) + o(1).$$

As $\epsilon \downarrow 0$, using (17), we have

$$0 \leq \int_0^T \Big\langle \int_s^T \frac{(T-r)^{\beta-1}}{\Gamma(\beta)} \Pi(r,s)^* \partial_X \bar{l}(r) \mathrm{d}r + \Pi(T,s)^* \partial_X \overline{m}(T), \widehat{f}(s) \Big\rangle \mathrm{d}s + \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} \widehat{l}(s) \mathrm{d}s.$$

Here, to get the last equality, we have used the definition of the duality paring between X and X^*, the definition of the adjoint operator (see Section 2.1), and the Fubini's formula (see (Bogachev, 2000, Theorem 3.4.4)).

By (21) and (9), (with $\overline{H}(s) := H(s, \overline{X}(s), P(s), \overline{u}(s))$)

$$\int_0^T H(s, \overline{X}(s), P(s), u(s)) \mathrm{d}s \leq \int_0^T \overline{H}(s) \mathrm{d}s.$$

Since U is separable, there exists a countable dense set $\mathsf{U}_i = \{u_i, \ i \geq 1\} \subset \mathsf{U}$. Moreover, there exists a measurable set $S_i \subset [0,T]$ such that $|S_i| = T$ and any $t \in S_i$ is the Lebesgue point of $H(t, \overline{X}(t), P(t), u(t))$, i.e., we have $\lim_{\tau \downarrow 0} \frac{1}{2\tau} \int_{t-\tau}^{t+\tau} H(s, \overline{X}(s), P(s), u(s)) \mathrm{d}s = H(t, \overline{X}(t), P(t), u(t))$ (Bogachev, 2000, Theorem 5.6.2). We fix $u_i \in \mathsf{U}_i$. For any $t \in S_i$, define

$$u(s) := \begin{cases} \overline{u}(s), & s \in [0,T] \setminus (t-\tau, t+\tau), \\ u_i, & s \in (t-\tau, t+\tau). \end{cases}$$

It then follows that

$$H(t, \overline{X}(t), P(t), u(t)) \leq H(t, \overline{X}(t), P(t), \overline{u}(t)).$$

Since $\cap_{i \geq 1} S_i = [0,T]$, H is continuous in $u \in \mathsf{U}$, and U is separable, for any $u \in \mathsf{U}$ and a.e. $t \in [0,T]$,

$$H(t, \overline{X}(t), P(t), u) \leq H(t, \overline{X}(t), P(t), \overline{u}(t)),$$

which shows the Hamiltonian maximization condition (see (8)). This is the end of the proof for Theorem 3.

5. EXAMPLE: FRACTIONAL HEAT EQUATION

In this section, we provide an example of **(P)**. Let $T = 1$. Let $\Omega = (0,1) \times (0,1) \times (0,1) \subset \mathbb{R}^3$ and $\bar{\Omega} = \Omega \cup \partial\Omega$, where $\partial\Omega \subset \mathbb{R}^3$ is a (smooth) boundary. Recall $L^2(\Omega) := L^2(\Omega; \mathbb{R})$. We denote $H^k(\Omega)$ by the usual Sobolev space of real-valued functions on Ω, whose distributional derivatives, up to the order k, are square-integrable, and $H_0^k(\Omega)$ by the closure of $C_c^\infty(\Omega)$ in $H^k(\Omega)$, where $C_c^\infty(\Omega)$ is the set of continuously differentiable real-valued functions on Ω having the support $\bar{\Omega}$.

With $\omega = (\omega_1, \omega_2, \omega_3) \in \bar{\Omega}$, consider the following *fractional heat equation*:

$$\begin{cases} {}^{\mathrm{C}}\mathrm{D}_{0+}^{\alpha}[x(\cdot,\omega)](t) \\ \quad - \Big(\frac{\partial^2}{\partial \omega_1^2} x(t,\omega) + \frac{\partial^2}{\partial \omega_2^2} x(t,\omega) + \frac{\partial^2}{\partial \omega_1^3} x(t,\omega) \Big) \\ \quad = b_1(t, x(t,\omega)) + b_2(t) u(t,\omega), \ (t,\omega) \in (0,1] \times \Omega, \\ x(t,\omega) = 0, \ (t,\omega) \in (0,1] \times \partial\Omega, \\ x(0,\omega) = x_0(\omega), \ \omega \in \Omega. \end{cases} \tag{22}$$

The objective functional is given by

$$(\mathbf{E}_1) \ J'(x_0; u(\cdot)) = \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} \int_\Omega \Big(\gamma_1(x(s,\omega)) + \gamma_2(u(s,\omega)) \Big) \mathrm{d}\omega \mathrm{d}s + \int_\Omega \gamma_3(x(T,\omega)) \mathrm{d}\omega. \tag{23}$$

Let $\mathsf{X} = \mathsf{U} = L^2(\Omega)$. Let $Ax := -(\frac{\partial^2}{\partial \omega_1^2} x + \frac{\partial^2}{\partial \omega_2^2} x + \frac{\partial^2}{\partial \omega_1^3} x)$, where $x \in D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then by (Pazy, 1983, Theorem 2.7, Chapter 7) and the embedding theorem of Sobolev spaces, it follows that $-A$ is the generator of the compact analytic semigroup of uniformly bounded linear operators on X. Let $X(t) := x(t,\cdot) \in \mathsf{X}$ with $X_0 := x_0(\cdot) \in \mathsf{X}$ and $u(t) := u(t,\cdot) \in \mathsf{U}$. By defining the Nemytskii operator $b_1(t, X(t))(\omega) := b_1(t, x(t,\omega))$, let $f(t, X(t), u(t)) := b_1(t, X(t)) + b_2(t) u(t)$. Then (22) can be converted into the following abstract form of the X-valued Caputo fractional evolution equation:

$$\begin{cases} {}^{\mathrm{C}}\mathrm{D}_{0+}^{\alpha}[X](t) + AX(t) \\ \qquad = b_1(t, X(t)) + b_2(t) u(t), \ t \in (0,T], \\ X(0) = X_0 \in \mathsf{X}. \end{cases} \tag{24}$$

Let $l(s, X(s), u(s)) := \int_\Omega (\gamma_1(x(s,\omega)) + \gamma_2(u(s,\omega))) \mathrm{d}\omega$ and $m(X(T)) := \int_\Omega \gamma_3(x(T,\omega)) \mathrm{d}\omega$. Then the objective functional in (23) is equivalent to

$$J'(X_0; u(\cdot)) = \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} l(s, X(s), u(s)) \mathrm{d}s + m(x(T)).$$

We can easily observe that the minimization problem $(\mathbf{E}_1)$ is equivalent to the following abstract problem:

$$(\mathbf{E}_2) \ \inf_{u(\cdot) \in \mathcal{U}} J'(X_0; u(\cdot)), \ \text{s.t. (24)}.$$

Notice that $(\mathbf{E}_2)$ is a special case of **(P)** of this paper.

Let $b_1(t, X(t)) = b_1(t) X(t)$, $l(s, X, u) = q|X|_{\mathsf{X}}^2 + r|u|_{\mathsf{U}}^2$ and $m(X) = m|X|_{\mathsf{X}}^2$, where q, r, m are positive constants. Then we can easily see that $(\mathbf{E}_2)$ holds Assumptions 1-2. Hence, Theorem 3 can be applied to solve $(\mathbf{E}_2)$.

By applying Theorem 3, the (candidate) optimal solution is one that maximizes the Hamiltonian, which is given by

$$\overline{u}(t) = \frac{\Gamma(\beta)}{2r(T-t)^{\beta-1}} b_2(t)P(t), \ t \in [0,T), \tag{25}$$

where $P(\cdot) \in L^p([0,1];\mathsf{X})$ (here $\mathsf{X}^* = \mathsf{X}$) is the adjoint equation satisfying

$$\begin{cases} {}^{\mathrm{RL}}\mathrm{D}^{\alpha}_{T-}[P](t) = -A^* P(t) + b_1(t)P(t) \\ \qquad - \dfrac{2q(T-t)^{\beta-1}}{\Gamma(\beta)} \overline{X}(t), \ t \in [0,T), \\ \mathrm{I}^{1-\alpha}_{T-}[P](T) = -2m\overline{X}(T). \end{cases} \tag{26}$$

6. CONCLUSIONS

We have shown the maximum principle for the optimal control problem of Caputo fractional evolution equations. In the proof, due to the inherent complex nature of infinite-dimensional fractional control, we have to establish new variational and duality analysis to get the desired Hamiltonian maximization condition. One important future research direction is the infinite-dimensional fractional control problem with terminal and running state constraints.

REFERENCES

Ali, H.M., Pereira, F.L., and Gamma, S.M.A. (2016). A new approach to the Pontryagin maximum principle for nonlinear fractional optimal control problems. *Mathematical Methods in the Applied Sciences*, 39(13), 3640–3649.

Almeida, R., Kamocki, R., Malinowska, A.B., and Odzijewicz, T. (2021). On the necessary optimality conditions for the fractional Cucker–Smale optimal control problem. *Communications in Nonlinear Science and Numerical Simulation*, 96, 105678.

Bergounioux, M. and Bourdin, L. (2020). Pontryagin maximum principle for general Caputo fractional optimal control problems with Bolza cost and terminal constraints. *ESAIM Control Optimisation and Calculus of Variations*, 26(35), 1–38.

Bogachev, V.I. (2000). *Measure Theory*. Springer.

Bourdin, L. (2017). Cauchy-Lipschitz theory for fractional multi-order dynamics–state-transition matrices, Duhamel formulas and duality theorems. *Differential and Integral Equations*, 31(7-8), 559–594.

Breitenbach, T. and BorzÍ, A. (2020). The Pontryagin maximum principle for solving Fokker-Planck optimal control problems. *Computational Optimization and Applications*, 76, 499–533.

Chen, P., Li, Y., Chen, Q., and Feng, B. (2014). On the initial value problem of fractional evolution equations with noncompact semigroup. *Computers and Mathematics with Applications*, 67, 1108–1115.

Diethelm, K. (2010). *The Analysis of Fractional Differential Equations*. Springer.

Ding, X.L., Area, I., and Nieto, J. (2022). Controlled singular evolution equations and Pontryagin type maximum principle with applications. *Evolution Equations and Control Theory*, 11(5), 1655–1679.

Fabbri, G., Gozzi, F., and Swiech, A. (2017). *Stochastic Optimal Control in Infinite Dimension*. Springer.

Fattorini, H.O. (1999). *Infinite Dimensional Optimization and Control Theory*. Cambridge University Press.

Frankowska, H., Marchini, E., and Mazzola, M. (2018). Necessary optimality conditions for infinite dimensional state constrained control problems. *Journal of Differential Equations*, 264, 7294–7327.

Gomoyunov, M. (2020). On representation formulas for solutions of linear differential equations with Caputo fractional derivatives. *Fractional Calculus and Applied Analysis*, 23, 1141–1160.

Kamocki, R. (2014). Pontryagin maximum principle for fractional ordinary optimal control problems. *Mathematical Methods in the Applied Sciences*, 37(11), 1668–1686.

Kilbas, A.A., Srivastava, H.M., and Trujillo, J.J. (2006). *Theory and Applications of Fractional Differential Equations*. Elsevier.

Krastanov, M.I., Ribarska, N.K., and Tsachev, T.Y. (2011). A Pontryagin maximum principle for infinite dimensional problems. *SIAM Journal on Control and Optimization*, 49(5), 2155–2182.

Li, X. and Yong, J. (1995). *Optimal Control Theory for Infinite Dimensional Systems*. Birkhauser.

Lin, P. and Yong, J. (2020). Controlled singular Volterra integral equations and Pontryagin maximum principle. *SIAM Journal on Control and Optimization*, 58(1), 136–164.

Liu, X., Lü, Q., and Zhang, X. (2020). Finite codimensional controllability and optimal control problems with endpoint state constraints. *Journal de Mathématiques Pures et Appliquées*, 138, 164–203.

Lizama, C. (2019). *Abstract nonlinear fractional evolution equations*, 499–514. De Gruyter.

Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, 2nd edition.

Sin, C.S., In, H.C., and Kim, K.C. (2018). Existence and uniqueness of mild solutions to initial value problems for fractional evolution equations. *Advances in Difference Equations*, 61, 1–13.

Wang, J. and Zhou, Y. (2011). A class of fractional evolution equations and optimal controls. *Nonlinear Analysis: Real World Applications*, 12, 262–272.

Yildiz, T.A., Jajarmi, A., Yildiz, B., and Baleanu, D. (2020). New aspects of time fractional optimal control problems within operators with nonsingular kernel. *Discrete and Continuous Dynamical Systems (Series S)*, 13(3), 407–428.

Zhang, X., Li, H., and Liu, C. (2020). Optimal control problem for the Cahn–Hilliard/Allen–Cahn equation with state constraint. *Applied Mathematics and Optimization*, 82, 721–754.

Zhou, Y. (2016). *Fractional Evolution Equations and Inclusions: Analysis and Control*. Academic Press.