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Unsteady natural convection flow due to fractional thermal transport and symmetric heat source/sink

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Abstract Unsteady natural convection flow of viscous fluids in a circular cylinder, due to a generalized fractional thermal transport is analytically studied. The considered mathematical model is based on a new fractional differential constitutive equation of the thermal flux suitable to describe the thermal memory effects. To develop the mathematical model, the time-fractional Caputo-Fabrizio derivative is used. The generalized constitutive equation becomes equivalent to the classical Fourier's law for the zero value of the fractional order of derivative. Analytical solutions for the fluid temperature and velocity are determined using the Laplace and finite Hankel transforms. The influence of the memory parameter on heat transfer and fluid motion is discussed by numerical simulations and graphical illustrations.

1. Introduction

The phenomena of free convective fluid flow through cylinders have grabbed the attention of researchers because of their engineering, industrial, and geophysical applications. The applications of the convective flows include thermal insulation, nuclear reactors, drying processes, chemical reactors, heat exchangers, petroleum industries, artificial dialysis, cosmetics, etc.

The influence of the heat source or sink on the convective flow is intensively investigated owing to its utilization in many real-world problems such as the heat transfer control in the internal combustion processes or during exothermic and endothermic chemical reactions.

Dwivedi and Singh [1,2] carried out analytical studies of hydro-magnetic free convective flows of an electrically conducting, viscous fluid through an impermeable vertical cylindrical/rectangular channel with a point/line heat source under the influence of a uniform magnetic field acting perpendicular to the flow direction. The mixed convective steady flows of electrically conducting liquids in a vertical channel with an internal heat source/sink by considering asymmetric

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and symmetric wall heating have been studied by Umavathi and Liu [3]. Natural convective flows of Newtonian fluids inside a circular cylinder under influence of the generalized heat transfer have been investigated by Ahmed et al [4].

Effects of viscous and Ohmic dissipation have been considered in their study in addition to the internal heat generation or absorption. Ramandevi et al. [5] have studied the influence of the non-uniform heat source/sink and viscous dissipation on the non-Newtonian hydromagnetic fluid flow over a stretching surface using the Cattaneo–Christov mathematical model of the thermal flux. A numerical study on natural convective flows in a cylindrical container with different heatings on the surface has been carried out by Nunez [6]. The obtained results show the possibility of obtaining different flow solutions such as steady axisymmetric, steady non-axisymmetric, time-dependent pulsating wave solutions, and other flow states with a variety of spatiotemporal symmetries.

Laidoudi and Ameur [7] studied the natural convective flow of a Newtonian fluid in a circular enclosure that contains three equal-sized cylinders in a tandem arrangement such that the outer cylinder has a cold surface and the enclosure internals have hot surfaces.

During the last period, the methodologies and tools for experimental measuring reached a high level of precision; as a consequence, the experimental measurements revealed that in some thermal processes appear serious anomalies between theoretical and experimental results. For example for the non-Fourier heat conduction phenomena in a porous material heated by a microsecond laser pulse, have been highlighted many disagreements between the experimental results and the theoretical analyses given by Cattaneo-type and Jeffreys-type models [8].

To eliminate some shortcomings of classical mathematical models, the researchers developed mathematical models suitable to describe as accurately as possible various complex transport processes. Thus, mathematical models based on constitutive equations with time-fractional derivatives have been proposed. These models, depending on the used memory kernel, are able to better describe some heat or mass transfer processes [9–11]. Now, it is known that integrals and derivatives non-integer order may have applications in describing complex properties of materials such as non-locality, long-term memory, and fractality. Tarasov and Aifantis [12] elaborated an extension of elasticity theory that describes elasticity of materials with fractional power-law non-locality described by Riesz derivatives of fractional order. The mixed convection heat transfer in nanofluid into fractional calculus approach over an inclined vertical plate has been studied by Saqib et al. [13]. The constitutive equations of the proposed mathematical model are based on the time-fractional Atangana–Baleanu fractional derivatives without a singular kernel with a strong memory. Sarwar et al. [14] studied the transient convective flow of Casson fluids over an oscillating vertical plate using the generalized constitutive equations with time-fractional Prabhakar derivatives. Analytical solutions to the proposed problem are determined using the Laplace transform method. Farooq et al. [15] have determined approximate analytical solutions of fractional-order Navier–Stokes fluid model using the Liouville–Caputo operator.

A generalized thermal and mass transport mathematical model for the unsteady flows of incompressible differential type fluids has been investigated by Razzaque et al. [16], by using time-fractional Caputo–Fabrizio fractional derivative with exponential-decay memory kernel. Baleanu et al. [17] provided an extension for the second-order differential equation of a thermostat model to a fractional hybrid equation. They have demonstrated the existence theorem of solutions for the hybrid fractional thermostat equation and inclusion versions applying the Dhage fixed point theorems.

A computational approach of the heat transfer in convective flows of a power law fluid enclosed in a isosceles triangular cavity has been carried out by Shah et al. [18]. Bilal et al [19] have numerically investigated non-isothermal flows of Williamson fluids over an exponential-stretched surface and thermal process described by the Cattaneo-Christov heat flux theory. Natural convective flows of a power-law liquid in a square enclosure rooted with a *T*-shaped fin have been studied by Bilal et al. [20]. The lower wall of the enclosure along with the fin is uniformly heated, vertical walls are prescribed with cold temperature and the upper boundary is thermally insulated. Numerical solutions are determined using the finite element software COMSOL. A fractional model of Brinkman type fluid holding nanoparticles of titanium dioxide and silver in water base fluid has been investigated by Ikram et al. [21] using the time-fractional Caputo derivative.

Numerical solutions to a class of nonlinear variable-order fractional reaction–diffusion equation with Mittag-Leffler kernel have been obtained by Pandey and Gomez-Aguilar [22], while Dwivedi et al. [23] caried out a numerical study of the variable-order fractional reaction-advection–diffusion equation in a heterogeneous medium. The fractional differential Ambartsumian equation, based on the modified Riemann–Liouville fractional derivative has been studied by El-Zahar et al. [24]. They determined solutions expressed as power series. Other interesting generalized fractional thermal processes have been studied by Abro et al. [25,26]. Kumar et al. [27] developed an operational matrix method based on fractional-order Lagrange polynomials to solve the non-local boundary value problems of fractional order arising in chemical reactor theory.

Ghanbari et al. [28] developed an analytical scheme to determine wave solutions of a partial differential equation involving a local fractional derivative by generalizing the procedure of generalized exponential rational function technique. The physical effects of single-wall carbon nanotube on the free convection slippage flow of fractional viscous fluids under influence of the thermal radiation, heat generation, chemical reaction, and of the Newtonian heating through a porous medium have been investigated by Ahmad et al. [29]. The fractional flow model of viscous fluid with single-wall carbon nanotube was constructed by inserting the non-integer constant proportional Caputo fractional derivative. Analytical solutions of the problem have been determined using the Laplace transform. Tassaddiq et al.[30] formulated and studied a fractional mathematical model for flows of generalized Casson fluids over a vertical plate with Newtonian heating. The fractional equation of flow is based on the time-fractional derivative with Mittag-Leffler kernel. Analytical solutions for

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the tempearture and velocity fields are obtained using Laplace transform. Very interesting problems regarding the psi-Hilfer fractional operator, the discrete proportional fractional operator, Caputo and Caputo-Fabrizio fractional operators have been studied in references [31–37].

In this paper, the unsteady natural convection flow of a viscous fluid in a circular cylinder, due to a generalized fractional thermal transport is analytically studied. The considered mathematical model is based on a fractional constitutive equation of the thermal flux suitable to describe the memory effects. To develop the mathematical model, the time-fractional Caputo-Fabrizio derivative is used. The generalized constitutive equation becomes equivalent to the classical Fourier's law for the zero value of the fractional order of derivative. The fractional mathematical model formulated in this paper is new in the literature. Also, the symmetric form of the heat source considered in the present paper represents a novelty in the convective flows. Analytical solutions for the fluid temperature and velocity are determined using the Laplace and finite Hankel transforms. The influence of the memory parameter on heat transfer and fluid motion is discussed by numerical simulations and graphical illustrations generated by the software Mathcad-15.

We underline the fact that the problem solved by us is interesting and the obtained results could serve to verify the accuracy of numerical schemes developed for this problem. At the same time, since the studied problem is linear, it might not faithfully describe certain flows in which the convective terms have a dominant role. For the future, we propose to study nonlinear convective flows in which the heat transfer is described by fractional differential equations.

The article is structured as follows: The formulation of the classical problem and the generalized problem are presented in Section 2. Section 3 is dedicated to the determination of the analytical solutions of the temperature and velocity fields. The numerical simulations and related discussions are given in Section 4, and the conclusions in Section 5.

2. Statement of the problem

The transient, laminar free convection flow of viscous liquids in an infinite vertical circular cylinder of radius R, in the presence of the axisymmetric heat source/ sink is considered. The flow domain is referred to the cylindrical coordinate system $(O, \vec{e}_r, \vec{e}_\varphi, \vec{e}_z)$ whose z-axis coincides to cylinder's longitudinal axis (Fig. 1). The cylindrical wall is assumed to be heated at the temperature $\tilde{T}_w(t)$, and the velocity and temperature fields are functions of $(\tilde{r}, \tilde{t})$, therefore, $\vec{V} = \left(V_{\tilde{r}} = 0, V_{\tilde{\varphi}} = 0, V_{\tilde{z}} = \tilde{u}(\tilde{r}, \tilde{t})\right)$, $\tilde{T} = \tilde{T}(\tilde{r}, \tilde{t})$. In these assumptions, the continuity equation is identically satisfied. We assume that the pressure gradient in the flow direction is equal to zero and the heat/sink source is given by.

$$\tilde{S}(\tilde{r}, \tilde{t}) = \tilde{S}_0(R^2 - \tilde{r}^2), \tilde{S}_0 = const. \tag{1}$$

The fluid movement is due to the buoyancy forces generated by the variation of the temperature in the mass of liquid. Using the Boussinesq approximation and the above hypotheses, the governing equations of the fluid flow and heat transport are as [1,3,6].

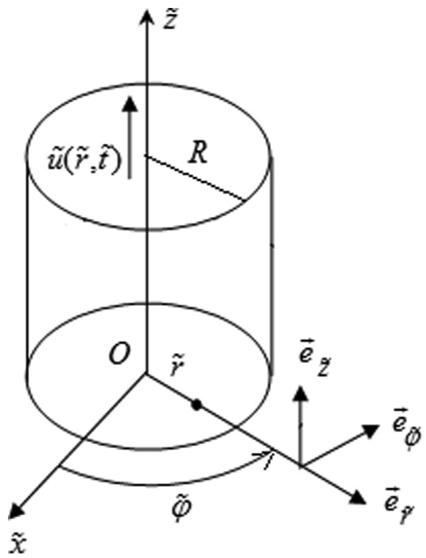

Fig. 1 Flow geometry.

– the balance of linear momentum

$$\rho \frac{\partial \tilde{u}(\tilde{r}, \tilde{t})}{\partial \tilde{t}} = \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r}\, \tilde{\tau}_{z^\sim r^\sim}(\tilde{r}, \tilde{t})\right) + \rho g \beta (\tilde{T}(\tilde{r}, \tilde{t}) - \tilde{T}_0) \tag{2}$$

– the shear stress constitutive equation

$$\tilde{\tau}_{z^\sim r^\sim}(\tilde{r}, \tilde{t}) = \mu \frac{\partial \tilde{u}(\tilde{r}, \tilde{t})}{\partial \tilde{r}} \tag{3}$$

– the equation of energy balance

$$\rho c_p \frac{\partial \tilde{T}(\tilde{r}, \tilde{t})}{\partial \tilde{t}} = -\frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r}\, \tilde{q}_{r^\sim}(\tilde{r}, \tilde{t})\right) + \tilde{S}(\tilde{r}, \tilde{t}) \tag{4}$$

– the constitutive thermal flux equation (Fourier's law)

$$\tilde{q}_{r^\sim}(\tilde{r}, \tilde{t}) = -k \frac{\partial \tilde{T}(\tilde{r}, \tilde{t})}{\partial \tilde{r}} \tag{5}$$

where, ρ is the fluid density, g is the gravitational acceleration, β is the thermal expansion coefficient, $\tilde{T}_0$ is a reference temperature, $\tau_{\tilde{z}\,\tilde{r}}(\tilde{r}, \tilde{t})$ is the shear stress, μ is the fluid viscosity, c_p is the specific heat, $\tilde{q}_{\tilde{r}}(\tilde{r}, \tilde{t}$ is the $\tilde{r}$- component of the thermal flux density vector, and k is the thermal conductivity of the fluid.

Along with Eqs. (2)–(5), we consider the initial and boundary conditions.

$$\tilde{u}(\tilde{r},0)=0,\ \tilde{T}(\tilde{r},0)=\tilde{T}_0,\ \tilde{r}\in[0,R] \tag{6}$$

$$\tilde{u}(R,\tilde{t})=0,\ \tilde{t}>0 \tag{7}$$

$$\tilde{T}_w(R,\tilde{t})=\tilde{T}_0+(\tilde{T}_w-\tilde{T}_0)\exp(-k_0t),\ \tilde{t}>0 \tag{8}$$

Using the following dimensionless parameters and functions.

$$\begin{gathered} r=\tfrac{\tilde{r}}{R},t=\tfrac{\nu\tilde{t}}{R^2},u=\tfrac{R\tilde{u}}{\nu},\tau=\tfrac{R^2\tilde{\tau}_{\tilde{z}\tilde{r}}}{\mu\nu},\theta=\tfrac{\tilde{T}-\tilde{T}_0}{\tilde{T}_w-\tilde{T}_0},q=\tfrac{R\tilde{q}_{\tilde{r}}}{k(\tilde{T}_w-\tilde{T}_0)},\\ \mathrm{Pr}=\tfrac{\mu c_p}{k},Gr=\tfrac{gR^3\beta(\tilde{T}_w-\tilde{T}_0)}{\nu^2},k_0=\tfrac{\tilde{k}_0R^2}{\nu},S_0=\tfrac{\tilde{S}_0R^4}{k(\tilde{T}_w-\tilde{T}_0)}, \end{gathered} \tag{9}$$

Eqs. (2)–(8) become

$$\frac{\partial u(r,t)}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}(r\,\tau(r,t))+Gr\theta(r,t) \tag{10}$$

$$\tau(r,t)=\frac{\partial u(r,t)}{\partial r} \tag{11}$$

$$\mathrm{Pr}\frac{\partial\theta(r,t)}{\partial t}=-\frac{1}{r}\frac{\partial}{\partial r}(rq(r,t))+S_0(1-r^2) \tag{12}$$

$$q(r,t)=-\frac{\partial\theta(r,t)}{\partial r} \tag{13}$$

The non-dimensional initial and boundary condition are as follows

$$u(r,0)=0,\ \ \theta(r,0)=0,\ \ r\in[0,1] \tag{14}$$

$$u(1,t)=0,\ \ \theta(1,t)=\exp(-k_0t),\ \ t>0 \tag{15}$$

2.1. The generalized model of the thermal process

In the present paper we consider a generalized model for the thermal flux considering a constitutive equation with the time-fractional derivative proposed by Caputo and Fabrizio [35,38]. First, let's present the basic theoretical elements related to the Caputo-Fabrizio derivative.

Let $T>0$, $I=[0,T]$, and $\chi:I\rightarrow\mathbb{R}$. The function space $H^1(I)$ is defined as $H^1(I)=\left\{\chi|\chi(t)\in L_2(I),\frac{d\chi(t)}{dt}=\dot{\chi}(t)\in L_2(I),\ t\in I\right\}$, where $L_2(I)$ ia the space of square integrable functions on the interval I.

Let ${}^{CF}h(t,\alpha)=\frac{1}{1-\alpha}\exp\left(\frac{-\alpha t}{1-\alpha}\right)$, $t\geqslant0$, $\alpha\in[0,1)$ be the exponential kernel.

Definition 2.1.1. If $\chi(t)\in H^1(I)$ the Caputo-Fabrizio derivative of function $\chi(t)$ is defined as ${}^{CF}D\,t\alpha\chi(t)={}^{CF}h(t,\alpha)*\dot{\chi}(t)$, where the notation "$*$" denotes the convolution product.

The above relation can be written in the following equivalent forms:

$${}^{CF}D_t^{a}\chi(t)=\frac{1}{1-\alpha}\int_0^t\exp\left(\frac{-\alpha(t-)}{1-\alpha}\right)\dot{\chi}()d$$

$$\begin{aligned} {}^{CF}D_t^{\alpha}\chi(t)=\frac{\chi(t)}{1-\alpha}-\frac{\chi(0)}{1-\alpha}\exp\left(\frac{-\alpha t}{1-\alpha}\right)-\frac{\alpha}{(1-\alpha)^2}\int_0^t\\ \times\exp\left(\frac{-\alpha(t-)}{1-\alpha}\right)\chi()d \end{aligned}$$

The last relation highlights the non-local character of the Caputo-Fabrizio derivative. Indeed, the value of the Caputo-Fabrizio derivative of the function $\chi(t)$ at the moment $t>0$ is determined not only by the value of the function of the function χ at that moment but also by all the values of the function χ at the points of the interval $(0,t)$. So, the history of the function χ influences the values of the Caputo-Fabrizio derivative; in other words, the memory of the function χ influences its derivative. Also, it is important to note that unlike other fractional derivatives such as the Riemann-Liouville derivative and the Caputo derivative, the kernel of the Caputo-Fabrizio derivative has no singularities.

Remark. For $\alpha=0$, the Caputo-Fabrizio derivative of function $\chi(t)$ becomes ${}^{CF}D_t^0\chi(t)=\int_0^t\dot{\chi}()d=\chi(t)-\chi(0)$. If $\chi(0)=0$, we get ${}^{CF}D_t^0\chi(t)=\int_0^t\dot{\chi}()d=\chi(t)$.

The Laplace transform of Caputo-Fabrizio derivative is given by [39].

$$L\{{}^{CF}D_t^0\chi(t)\}=L\{{}^{CF}h(t,\alpha)*\dot{\chi}(t)\}=L\{{}^{CF}h(t,\alpha)\}L\{\dot{\chi}(t)\}=\tfrac{sL\{\chi(t)\}-\chi(0)}{(1-\alpha)s+\alpha},$$

where the Laplace transform of function $\chi(t)$ is defined by $L\{\chi(t)\}=\int_0^\infty\chi(t)\exp(-st)dt$.

In the following, we consider a generalized thermal transport described by a new fractional constitutive equation of the thermal flux, namely [9].

$$q(r,t)=-{}^{CF}D_t^{\alpha}\left(\frac{\partial\theta(r,t)}{\partial r}\right),\alpha\in[0,1) \tag{16}$$

where

$$\begin{aligned} D_t^{\alpha}\left(\tfrac{\partial\theta(r,t)}{\partial r}\right)&=\tfrac{1}{1-\alpha}\int_0^t\exp\left(\tfrac{-\alpha(t-s)}{1-\alpha}\right)\tfrac{\partial^2\theta(r,s)}{\partial s\partial r}\\ ds&=\left(\tfrac{1}{1-\alpha}\exp\left(\tfrac{-\alpha t)}{1-\alpha}\right)\right)*\tfrac{\partial^2\theta(r,t)}{\partial t\partial r}. \end{aligned} \tag{17}$$

The Laplace transform of (17) is given by.

$$L\left\{{}^{CF}D_t^{\alpha}\left(\frac{\partial\theta(r,t)}{\partial r}\right)\right\}=\frac{1}{(1-\alpha)s+\alpha}\frac{\partial}{\partial r}\left[s\bar{\theta}(r,s)-\theta(r,0)\right], \tag{18}$$

It is easily observed that for $\alpha=0$, from the generalized equation (16) we recover classical Fourier's law (13).

3. Exact analytical solution for the thermal transport

In this section we shall determine the analytical solution for the energy equations for both fractional and classical constitutive equation of the thermal flux.

3.1. Fluid temperature for the fractional constitutive equation

To obtain the analytical solution of the problem given by equations (12) and (16) along with the initial condition $(14)_2$ and the boundary condition $(15)_2$, the Laplace transform and finite Hankel transform are used. Applying the Laplace transform to Eqs. (12) and (16), using Eq. (18) and the initial condition $(14)_2$, we obtain the transformed equations.

$$\mathrm{Pr}s\bar{\theta}(r,s)=-\frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{q}(r,s)\right)+\frac{1}{s}S_0(1-r^2) \tag{19}$$

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$$\bar{q}(r,s) = -\frac{s}{(1-\alpha)s+\alpha}\,\frac{\partial\bar{\theta}(r,s)}{\partial r} \tag{20}$$

Replacing (20) in Eq. (19), we find that the Laplace transform $\bar{\theta}(r,s)$ of the temperature $\theta(r,t)$ has to satisfy the differential equation.

$$w_1(s)\,\bar{\theta}(r,s) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{\theta}(r,s)}{\partial r}\right) + w_2(s)(1-r^2) \tag{21}$$

where

$$w_1(s) = \mathrm{Pr}[(1-\alpha)s+\alpha],\ w_2(s) = S_0\frac{(1-\alpha)s+\alpha}{s^2} \tag{22}$$

Applying the Laplace transform of the boundary condition $(15)_2$, the following boundary condition for the transformed function is obtained:

$$\bar{\theta}(1,s) = \frac{1}{s+k_0} \tag{23}$$

The finite Hankel transform of function $\bar{\theta}(r,s)$ and its inverse transform are defined as [40].

$$\begin{aligned}&\bar{\theta}_H(n,s) = \int_0^1 \bar{\theta}(r,s)r\,J_0(r\,r_n)dr,\\&\bar{\theta}(r,s) = 2\sum_{n=1}^{\infty}\tfrac{J_0(r\,r_n)}{J_1^2(r_n)}\bar{\theta}_H(n,s).\end{aligned} \tag{24}$$

where, r_n, $n = 1,2,\dots$ are the positive roots of the equation $J_0(r) = 0$, $J_n(\cdot)$ being the Bessel function of the first kind and order n.

A straightforward calculus leads to the following relations:

$$\int_0^1 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{\theta}(r,s)}{\partial r}\right)r\,J_0(r\,r_n)dr = r_nJ_1(r_n)\,\bar{\theta}(1,s) - r_n^2\bar{\theta}_H(n,s) \tag{25}$$

$$\int_0^1 (1-r^2)r\,J_0(r\,r_n)dr = \frac{4}{r_n^3}J_1(r_n) \tag{26}$$

To obtain equations (25) and (26), the following properties of the Bessel functions were used [41]:

$$\begin{aligned}&\int z^nJ_{n-1}(z) = z^nJ_n(z)+C,\\&\int z^3J_0(z)dz = z^3J_1(z) - 2z^2J_2(z)+C,\\&J_{n-1}(z)+J_{n+1}(z) = \tfrac{2n}{z}J_n(z),\\&J_{n-1}(z)-J_{n+1}(z) = 2J_n'(z),\ n\in\mathbb{N}.\end{aligned} \tag{27}$$

Apllying the finite Hankel transform to Eq. (21), using Eqs. (23), (25) and (26) we obtain.

$$\bar{\theta}_H(n,s) = \frac{r_n^4+4w_2(s)(s+k_0)}{r_n^3(s+k_0)(w_1(s)+r_n^2)}J_1(r_n) \tag{28}$$

Now, we consider the auxiliary function $f(r) = r^2$ whose finite Hankel transform is given by $f_H(n) = \frac{r_n^2-4}{r_n^3}J_1(r_n),\ n = 1,2,\dots$.

The finite Hankel transform (28) is written in the following equivalent form.

$$\begin{aligned}\bar{\theta}_H(n,s) = {}&\frac{1}{s+k_0}\,\frac{r_n^2-4}{r_n^3}J_1(r_n)\\&+\frac{J_1(r_n)}{r\,n3}\left(\frac{A_n}{s}+\frac{B_n}{s^2}+\frac{C_n}{s+k_0}+\frac{D_n}{r_n^2+\alpha\mathrm{Pr}+(1-\alpha)s\mathrm{Pr}}\right)\end{aligned} \tag{29}$$

where

$$\begin{aligned}&A_n = \tfrac{4S_0(1-\alpha)r_n^2}{\left(r\,n2+\alpha\mathrm{Pr}\right)^2},\ B_n = \tfrac{4\alpha S_0}{r\,n2+\alpha\mathrm{Pr}},\ C_n = \tfrac{(1-\alpha)k_0\mathrm{Pr}(r_n^2-4)+4(r_n^2+\alpha\mathrm{Pr})-\alpha\mathrm{Pr}r_n^2}{r_n^2+\alpha\mathrm{Pr}-(1-\alpha)k_0\mathrm{Pr}},\\&D_n = -\tfrac{r_n^2(1-\alpha)\mathrm{Pr}[r_n^2\left(r\,n2+\alpha\mathrm{Pr}\right)^2+4S_0(1-\alpha)(r_n^2+\alpha\mathrm{Pr}-(1-\alpha)k_0\mathrm{Pr})]}{\left(r\,n2+\alpha\mathrm{Pr}\right)^2\left[r_n^2+\alpha\mathrm{Pr}-(1-\alpha)k_0\mathrm{Pr}\right]}.\end{aligned} \tag{30}$$

Applying the inverse Laplace transform and the inverse Hankel transform to Eq. (29) we obtain the following expression for the temperature field:

$$\begin{aligned}&\theta(r,t) = r^2\exp(-k_0t)+2\sum_{n=1}^{\infty}\tfrac{J_0(r\,r_n)}{r_n^3J_1(r_n)}\\&\left[A_n+B_nt+C_n\exp(-k_0t)+\tfrac{D_n}{(1-\alpha)\mathrm{Pr}}\exp\left(-\tfrac{r_n^2+\alpha\mathrm{Pr}}{(1-\alpha)\mathrm{Pr}}t\right)\right].\end{aligned} \tag{31}$$

3.2. Fluid temperature for the classical law of Fourier

As mentioned in Section 2.1, the classical case corresponding to Fourier's law for heat flux is obtained from the generalized model for the zero value of the fractional parameter α. For $\alpha = 0$, coefficients in Eq. (30) becomes.

$$\begin{aligned}&A_n = \tfrac{4S_0}{r\,n2},\ B_n = 0,\ C_n = \tfrac{k_0r_n^2\mathrm{Pr}+4(r_n^2-k_0\mathrm{Pr})}{r_n^2-k_0\mathrm{Pr}},\\&D_n = -\tfrac{\left[r_n^6+4S_0(r_n^2-k_0\mathrm{Pr})\right]\mathrm{Pr}}{r_n^2(r_n^2-k_0\mathrm{Pr})},\end{aligned} \tag{32}$$

and the temperature field corresponding to classical Fourier's law is given by

$$\begin{aligned}&\theta(r,t) = r^2\exp(-k_0t)+2\sum_{n=1}^{\infty}\tfrac{J_0(r\,r_n)}{r_n^3J_1(r_n)}\\&\left[\tfrac{4S_0}{r\,n2}+\tfrac{k_0\mathrm{Pr}r_n^2+4(r_n^2-k_0\mathrm{Pr})}{r_n^2-k_0\mathrm{Pr}}\exp(-k_0t)-\tfrac{\left[r_n^6+4S_0(r_n^2-k_0\mathrm{Pr})\right]}{r_n^2(r_n^2-k_0\mathrm{Pr})}\exp\left(-\tfrac{r_n^2}{\mathrm{Pr}}t\right)\right].\end{aligned} \tag{33}$$

3.3. Velocity field

This sections deals with finding the solutions of the fluid motion equation along with the initial condition $(14)_1$ and the boundary condition $(15)_1$. Eliminating the shear stress $\tau(r,t)$ between Eqs. (10) and (11) we find that the velocity $u(r,t)$ has to satisfy the following partial differential equation:

$$\frac{\partial u(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u(r,t)}{\partial r}\right)+Gr\theta(r,t) \tag{34}$$

Applying the Laplace transform and finite Hankel transform to Eq. (34), using $(14)_1$, $(15)_1$ and (25) we obtain the double transformed velocity field.

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$$\bar{u}_H(n,s)=\frac{Gr}{s+r_n^2}\bar{\theta}_H(n,s)=\frac{GrJ_1(r_n)}{r_n^3(s+r_n^2)(s+k_0)}\frac{r_n^4+4w_2(s)(s+k_0)}{r_n^2+w_1(s)}$$
$$=\frac{GrJ_1(r_n)}{r_n^3}\frac{[r_n^4+4(1-\alpha)S_0]s^2+4S_0[(1-\alpha)k_0+\alpha]+4\alpha k_0S_0}{s^2(s+r_n^2)(s+k_0)[(1-\alpha)\Pr s+\alpha\Pr+r_n^2]} \tag{35}$$
$$=\frac{GrJ_1(r_n)}{r_n^3}\left[\frac{M_n}{s}+\frac{N_n}{s^2}+\frac{P_n}{s+k_0}+\frac{Q_n}{s+r_n^2}+\frac{R_n}{(1-\alpha)\Pr s+\alpha\Pr+r_n^2}\right],$$

where,

$$M_n=-\left(P_n+Q_n+\frac{R_n}{(1-\alpha)\Pr}\right),N_n=\frac{4\alpha S_0}{r_n^2(r_n^2+\alpha\Pr)},$$
$$P_n=\frac{r_n^4}{(r_n^2-k_0)[r_n^2+\alpha\Pr-(1-\alpha)k_0\Pr]},$$
$$Q_n=\frac{r_n^8+4S_0[(1-\alpha)r_n^2-\alpha](r_n^2-k_0)}{r\,n4(k_0-r_n^2)[r_n^2+\alpha\Pr-(1-\alpha)r_n^2Pr]},$$
$$R_n=\frac{(1-\alpha)^2\Pr^2\left\{r_n^4+4S_0(1-\alpha)-\frac{4(1-\alpha)S_0\Pr[\alpha+(1-\alpha)k_0]}{r_n^2+\alpha\Pr}+\frac{4\alpha k_0S_0(1-\alpha)^2\Pr^2}{(r_n^2+\alpha\Pr)^2}\right\}}{[r_n^2+\alpha\Pr-(1-\alpha)r_n^2\Pr][r_n^2+\alpha\Pr-(1-\alpha)k_0Pr]} \tag{36}$$

By inverting the integral transforms, we obtain the following expression of the fluid velocity for the case of fractional thermal constitutive equation:

$$u(r,t)=2\sum_{n=1}^{\infty}\frac{GrJ_0(r\,r_n)}{r_n^3J_1(r_n)}\left[M_n+N_nt+P_n\exp(-k_0t)+Q_n\exp(-r_n^2t)+\right.$$
$$\left.\frac{R_n}{(1-\alpha)\Pr}\exp\left(-\frac{r_n^2+\alpha\Pr}{(1-\alpha)\Pr}t\right)\right]. \tag{37}$$

In the case $\alpha=0$, corresponding to the classical Fourier law, Eq. (37) becomees:

For Pr≠1.

$$u(r,t)=2\sum_{n=1}^{\infty}\times\frac{GrJ_0(r\,r_n)}{r_n^3J_1(r_n)}\left[M_{0n}+P_{0n}\exp(-k_0t)+Q_{0n}\exp(-r_n^2t)+R_{0n}\exp\left(-\frac{r_n^2}{\Pr}t\right)\right], \tag{38}$$

where,

$$M_{0n}=-\left(P_{0n}+Q_{0n}+\frac{R_{0n}}{\Pr}\right),\ P_{0n}=\frac{r_n^4}{(r_n^2-k_0)(r_n^2-k_0\Pr)},$$
$$Q_{0n}=\frac{r_n^6+4S_0(r_n^2-k_0)}{(1-\Pr)r\,n4(k_0-r_n^2)},R_{0n}=\frac{\Pr^2\left\{r_n^2(r_n^4+4S_0)-4k_0S_0\Pr\right\}}{(1-\Pr)r_n^4(r_n^2-k_0Pr)}. \tag{39}$$

For Pr = 1.

$$u(r,t)=2\sum_{n=1}^{\infty}\frac{GrJ_0(r\,r_n)}{r_n^3J_1(r_n)}\left[\frac{4S_0}{r_n^4}+\frac{r_n^4}{(r_n^2-k_0)^2}\exp(-k_0t)-\right.$$
$$\left.\frac{r_n^8+4S_0(r_n^2-k_0)^2}{r_n^4(r_n^2-k_0)^2}\exp(-r_n^2t)-\frac{r_n^6+4S_0(r_n^2-k_0)}{r_n^2(r_n^2-k_0)}t\exp\left(-r_n^2t\right)\right]. \tag{40}$$

4. Numerical results and discussion

Natural convection flow of a viscous fluid, due to a generalized fractional thermal transport, has been analytically studied. The fluid flows within a circular cylinder whose surface has a temperature that varies exponentially over time, namely $\theta(1,t)=\exp(-k_0t)$, $k_0=0.25$. In the considered mathematical model, the constitutive equation of the thermal flux is a fractional differential equation based on the time-fractional Caputo-Fabrizio derivative of the order $\alpha\in[0,1)$. Note that for $\alpha=0$, the generalized constitutive equation is equivalent to the classical Fourier's law.

The generalized constitutive equation (16) highlights that the history of the temperature gradient influences the thermal

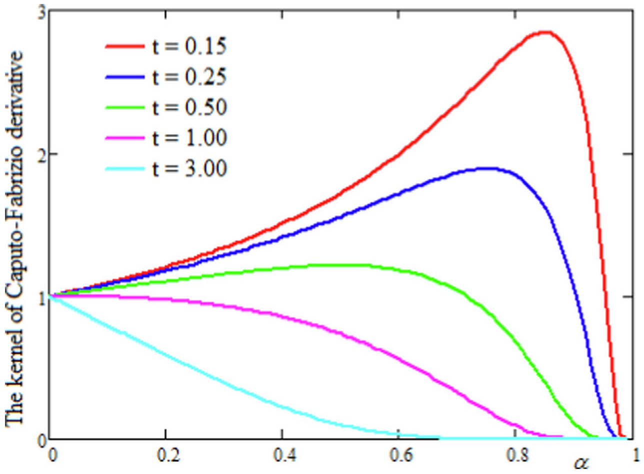

Fig. 2 The profiles of the Caputo-Fabrizio kernel $h_{CF}(\alpha,t)$ versus fractional parameter $\alpha\in[0,1)$, for different values of time t.

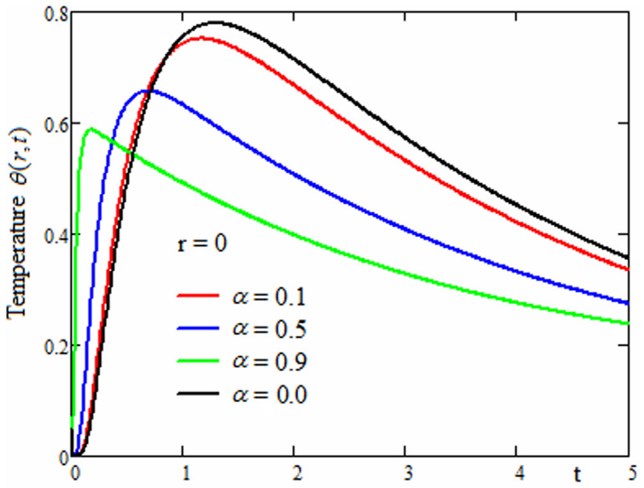

Fig. 3 Time-variation of the non-dimensional temperature $\theta(r,t)$.

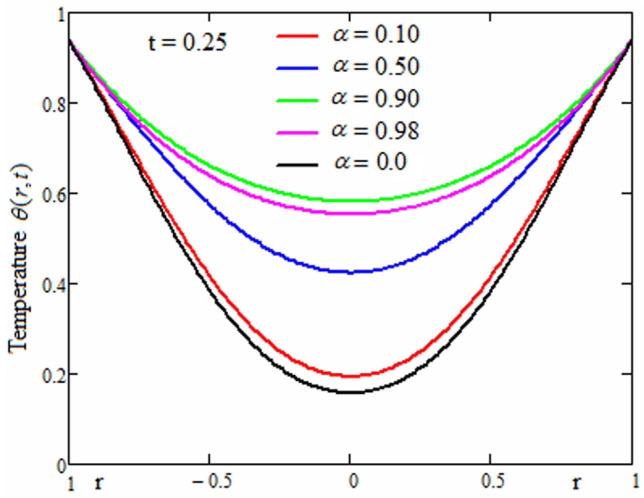

Fig. 4 Profiles of the non-dimensional temperature $\theta(r,t)$ versus radial coordinate.

flux, and therefore the heat transfer in the mass of fluid. From definition of Caputo-Fabrizio fractional derivative, it is clearly observed that history of the function $\chi(\tau)$ on the interval $[0, t]]$ influences the value of the fractional derivative at time t. Also, the kernel of the fractional derivative has a significant influence, being the function that weights the values of the function $\chi(\tau)$. Also, let's note that $\lim_{\substack{\alpha\to 1\\ \alpha<1}} \frac{1}{1-\alpha}\exp\left(\frac{-\alpha t}{1-\alpha}\right) = 0, t \geqslant 0$, and for large values of time t ${}^{CF}h(t,\alpha) \cong 0, \alpha \in [0,1)$. These properties of the kernel ${}^{CF}h(t,\alpha)$ of Caputo-Fabrizio derivative are highlighted by curves in Fig. 2. These curves show the variation of the Caputo-Fabrizio kernel $h_{CF}(\alpha, t)$ with the fractional parameter $\alpha \in [0, 1)$, for different values of time t. It is observed in Fig. 2 that the fractional derivative kernel has significant influences for small values of the time t. In this case, the Caputo-Fabrizio kernel attains a maximum value for a large value of the fractional parameter. For larger values of the time t, the influence of the weight function is significant only for low values of the.

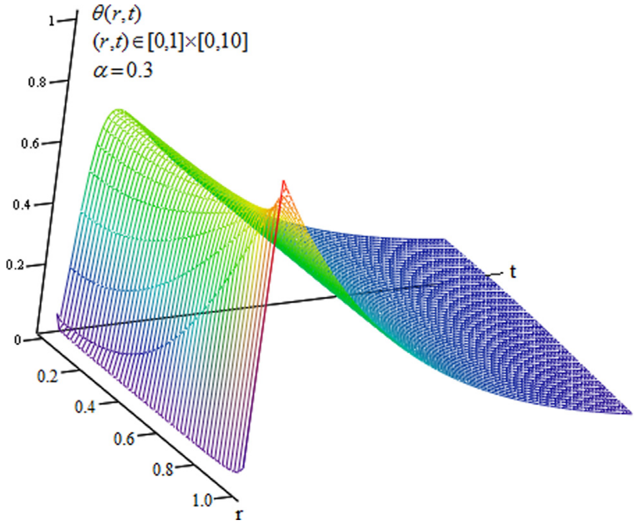

Fig. 5 the non-dimensional temperature $\theta(r,t)$ for $\theta(r,t)$ for $(r,t) \in [0,1] \times [0,10]$ and $\alpha = 0.3$.

fractional parameter. An important conclusion derived from this discussion is that the considered model offers an optimal heat transfer only for small values of the fractional parameter.

Figs. 3-5 show the influence of the memory parameter α on the temperature variation in the fluid mass. The time-variation of the nondimensional temperature $\theta(r, t)$ is presented in Fig. 3 in the central pozition of cylinder $r = 0$, for different values of the fractional parameter α. In Fig. 4 are plotted the profiles of

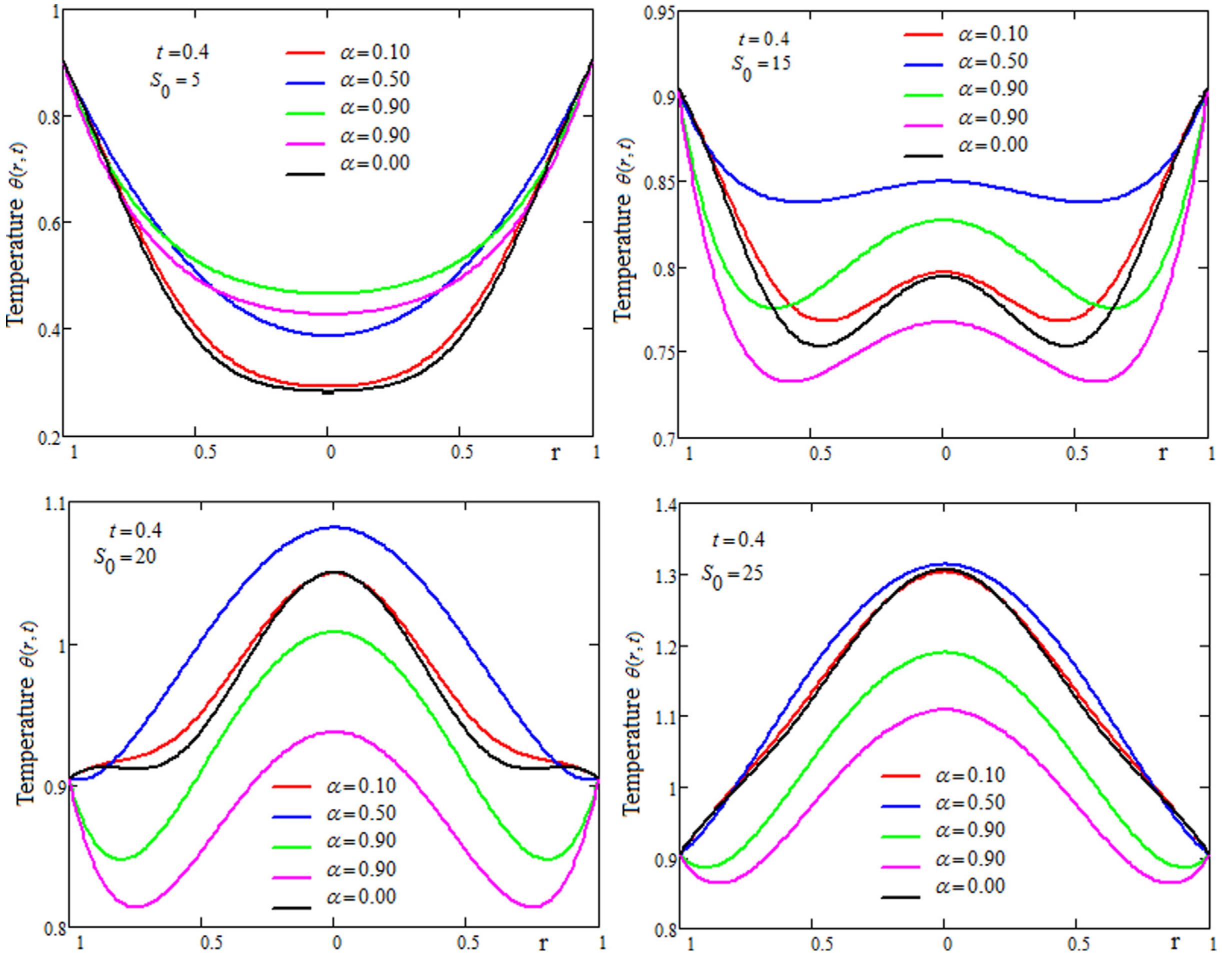

Fig. 6 The influence of heat source intensity S_0 on the fluid temperature at small value of the time t.

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the non-dimensional temperature $\theta(r,t)$ in the inner of the cylindrical pipe. It is observed in these figures that the fractional derivative has a significant influence on the fluid temperature for small values of the memory parameter α. It is also observed that the influence of the history of the temperature gradient on the temperature is more accentuated for small values of time t. This behavior is in accordance with the evolution in time of the Caputo-Fabrizio's kernel discussed with the help of Fig. 2. As expected, the temperature has symmetric distribution with respect to the longitudinal axis of the cylinder. Fig. 5 shows the time-spatial variation of the non-dimensional temperature for $(r,t) \in [0,1] \times [0,10]$. In the previous discussions, we showed that the thermal memory has a significant influence on the thermal process only for small values of the fractional parameter. For this reason, in the numerical data related to Fig. 5, we would use the value $\alpha = 0.3$ for the fractional parameter. The previously discussed properties of the temperature are easily observed in this figure.

The simultaneous influence of the heat source and the thermal memory on the fluid temperature is analyzed in the graphs in Fig. 6 for a small value of time t, and in Fig. 7 for a large value of time t. As expected, the increase in the values of the intensity of the source of heat S_0 has the effect of increasing the temperature of the fluid. For low values of the intensity of the heat source, the external temperature of the cylinder is dominant and as a result the temperature has higher values near the cylindrical surface and lower values in the central area of the cylinder. The temperature distribution changes its character for high values of the intensity of the heat source S_0.

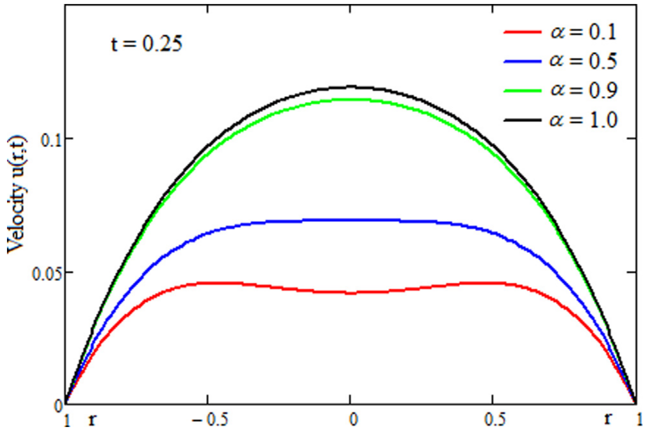

Fig. 8 The profiles of fluid velocity $u(r,t)$ for different values of the fractional parameter α and $t = 0.25$.

Profiles of the non-dimensional fluid velocity $u(r,t)$ versus the radial coordinate r have been plotted in Figs. 8 and 9 for

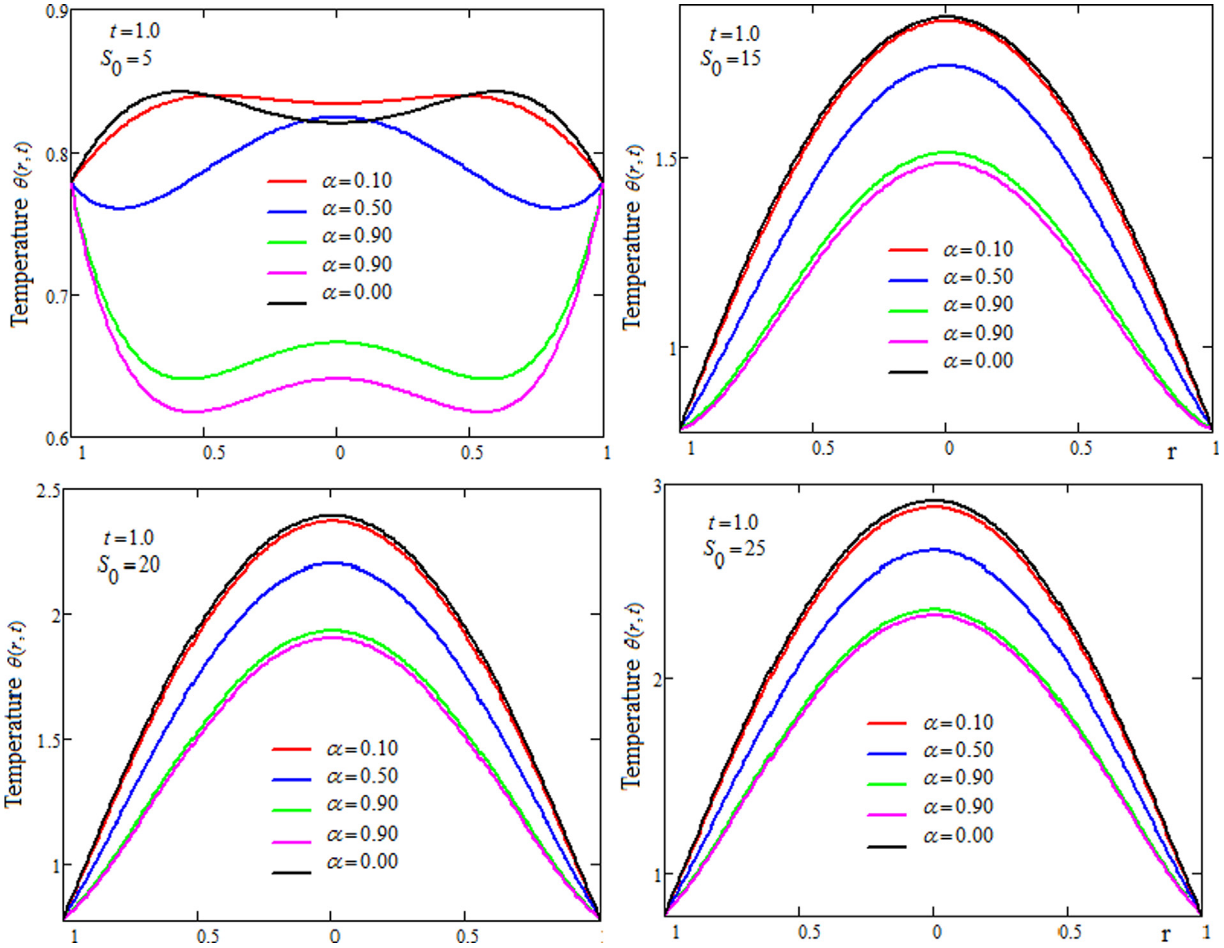

Fig. 7 The influence of heat source intensity S_0 on the fluid temperature at small value of the time t.

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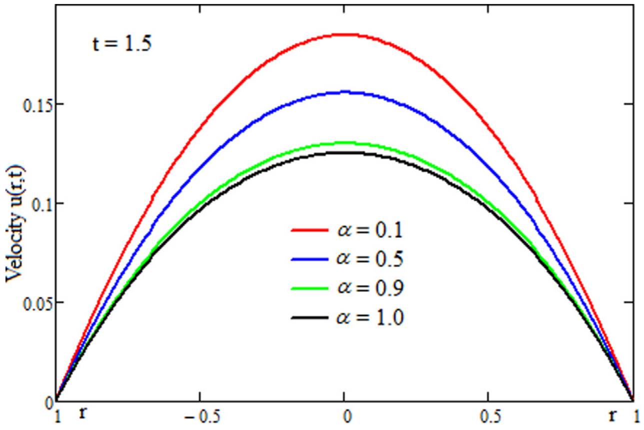

Fig. 9 The profiles of fluid velocity $u(r,t)$ for different values of the fractional parameter α and $t = 1.50$.

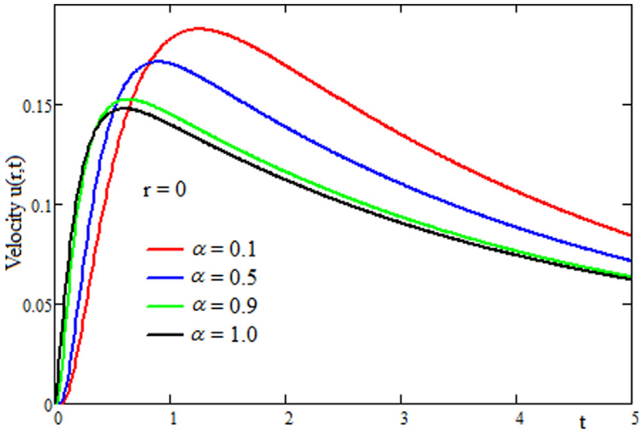

Fig. 10 Time-variation of fluid velocity $u(r,t)$ in the middle of the pipe for different values of the fractional parameter α.

different values of the fractional parameter α. It is observed in these figures that the fluid motion is symmetric with respect to the longitudinal axis of the cylinder. The time-variation of the fluid velocity is sketched in Fig. 10. It is observed that for small values of the time t, the velocity is increasing with the fractional parameter and has an opposite behavior for large values of the time. These properties also are clearly observed in Figs. 8 and 9.

5. Conclusions

The non-integer order differential and integral operators with memory have a great importance for the modelling of problems of flow and heat transfer.

Present investigation explores the consequences of the fractional Caputo-Fabrizio operator to analyse the implications of exponential heating and heat source on convective flow of a Newtonian fluid in a circular cylinder.

The flow and heat transfer problem is formulated with fractional partial differential equations. Laplace and finite Hankel transforms are used to acquire the fluid temperature and velocity. The influence of different involved parameters on the heat transfer and fluid motion are studied by numerical simulations and graphics.

The fractional mathematical model formulated in this paper is new in the literature. Also, the symmetric form of the heat source considered in the present paper represents a novelty in the convective flows.

Our study highlights that at small values of time t, the heat transfer in the fractional model is faster to the classical model described by Fourier's law. For large values of time t, in the fractional model the heat transfer is slower than in the classical case of Fourier's law.

It is important to note that the influence of the thermal memory on the heat transfer becomes significant for small values of the fractional parameter. This influence becomes negligible once the memory parameter tends to 1. This result could provide optimal choices of the fractional model depending on the goal pursued in the studied problem.

For the future we propose to study nonlinear models of the problem studied in this article. We believe that the comparative results of the classical nonlinear model with those of the fractional nonlinear model could highlight important properties of the generalized models.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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