Research Article

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Poisson C^*-algebra derivations in Poisson C^*-algebras

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Abstract: In this study, we introduce the following additive functional equation:

$$g(\lambda u + v + 2y) = \lambda g(u) + g(v) + 2g(y)$$

for all $\lambda \in \mathbb{C}$, all unitary elements u, v in a unital Poisson C^*-algebra P, and all $y \in P$. Using the direct method and the fixed point method, we prove the Hyers-Ulam stability of the aforementioned additive functional equation in unital Poisson C^*-algebras. Furthermore, we apply to study Poisson C^*-algebra homomorphisms and Poisson C^*-algebra derivations in unital Poisson C^*-algebras.

Keywords: Hyers-Ulam stability, fixed point method, additive functional equation, Poisson C^*-algebra derivation in Poisson C^*-algebra, Poisson C^*-algebra homomorphism in Poisson C^*-algebra

MSC 2020: 47B47, 17B63, 17B40, 39B72, 46L05, 47H10, 39B52

1 Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. Rassias [5] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [6] following the same approach as in Rassias [4] gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [6], as well as by Rassias and Šemrl [7] that one cannot prove a Rassias' type theorem when $p = 1$. The counterexamples of Gajda [6], as well as of Rassias and Šemrl [7], have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors [9–18].

We recall a fundamental result in the fixed point theory.

Theorem 1.1. [19,20] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = +\infty$$

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for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1} x) < +\infty, \ \forall n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X | d(J^{n_0} x, y) < +\infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Rassias [21] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [22,23]).

We first recall the definition of a Poisson algebra, given by Bai et al. [24].

Definition 1.2. [24] Let P be a vector space equipped with two bilinear operations $\circ, \{,\} : P \times P \to P$. The triple $(P, \circ, \{,\})$ is called a Poisson algebra if $(P, \circ)$ is a associative algebra and $(P, \{,\})$ is a Lie algebra that satisfy the compatibility condition

$$\{x, y \circ z\} = \{x, y\} \circ z + y \circ \{x, z\} \tag{1.1}$$

for all $x, y, z \in P$.

Equation (1.1) is called the Leibniz rule since the adjoint operators of the Lie algebra are derivations of the commutative associative algebra.

Example 1.3. Let L be a Lie algebra with Lie bracket $[\cdot,\cdot]$, which is a noncommutative (or commutative) algebra. Let $x \circ y$ be $\frac{xy+yx}{2}$ for all $x, y \in L$. Then, $(L, \circ)$ is a commutative algebra. Let $\{\cdot,\cdot\} \coloneqq [\cdot,\cdot]$. Then, we can easily show that $(L, \circ, \{\cdot,\cdot\})$ is a Poisson algebra.

From now on, we denote $x \circ y$ by xy for $x, y \in P$ and $(P, \circ, \{,\})$ by $(P, \{,\})$.

Cho et al. [25] studied homomorphisms and derivations in non-Archimedean Lie C^*-algebras by using the fixed point method. Motivated by their results, we define a unital Poisson C^*-algebra and Poisson C^*-algebra derivations and Poisson C^*-algebra homomorphisms in unital Poisson C^*-algebras as follows.

Definition 1.4. A commutative unital C^*-algebra P is called a *unital Poisson C^*-algebra* if $(P, \{,\})$ is a Poisson algebra.

Definition 1.5. Let P and Q be unital Poisson C^*-algebras.
(i) A $\mathbb{C}$-linear mapping $g : P \to P$ is a *Poisson C^*-algebra derivation* if $g : P \to P$ satisfies

$$\begin{aligned} g(xyz) &= g(x)yz + xg(y)z + xyg(z), \\ g(\{x, yz\}) &= \{g(x), yz\} + \{x, g(y)z\} + \{x, yg(z)\}, \\ g(x^*) &= g(x)^* \end{aligned}$$

for all $x, y, z \in P$.
(ii) A $\mathbb{C}$-linear mapping $h : P \to Q$ is a *Poisson C^*-algebra homomorphism* if $h : P \to Q$ satisfies

$$\begin{aligned} h(xyz) &= h(x)h(y)h(z), \\ h(\{x, yz\}) &= \{h(x), h(y)h(z)\}, \\ h(x^*) &= h(x)^* \end{aligned}$$

for all $x, y, z \in P$.

Throughout this study, assume that P is a unital Poisson C^*-algebra with unit e and unitary group $U(P) \coloneqq \{u \in P | u^*u = uu^* = e\}$ and Q is a unital Poisson C^*-algebra with unit e'.

Remark 1.

(i) If we put $x = y = z = e$ in Definition 1.5 (i), then $g(e) = 0$. If we put $z = e$ in Definition 1.5 (i), then

$$g(xy) = g(x)y + xg(y),$$
$$g(\{x, y\}) = \{g(x), y\} + \{x, g(y)\}$$

for all $x, y \in P$. Thus, $g : P \to P$ satisfies the definition of Poisson C^*-algebra derivation, given in [26, Definition 1.4].
Conversely, assume that $g : P \to P$ satisfies the definition of Poisson C^*-algebra derivation, given in [26, Definition 1.4]. Then,

$$g(xyz) = g(xy)z + xyg(z) = g(x)yz + xg(y)z + xyg(z),$$
$$g(\{x, yz\}) = \{g(x), yz\} + \{x, g(yz)\} = \{g(x), yz\} + \{x, g(y)z\} + \{x, yg(z)\}$$

for all $x, y, z \in P$.

(ii) Assume that $h : P \to Q$ satisfies the definition of Poisson C^*-algebra homomorphism, given in [26, Definition 1.4], i.e., $h : P \to Q$ satisfies

$$h(xy) = h(x)h(y),$$
$$h(\{x, y\}) = \{h(x), h(y)\},$$
$$h(x^*) = h(x)^*$$

for all $x, y \in P$. Then,

$$h(xyz) = h(xy)h(z) = h(x)h(y)h(z),$$
$$h(\{x, yz\}) = \{h(x), h(yz)\} = \{h(x), h(y)h(z)\}$$

for all $x, y, z \in P$.

In general, the converse does not hold. Assume that $h : P \to Q$ satisfies $h(e) = e'$. If we put $z = e$ in Definition 1.5 (ii), then

$$h(xy) = h(xye) = h(x)h(y)h(e) = h(x)h(y)e' = h(x)h(y),$$
$$h(\{x, y\}) = h(\{x, ye\}) = \{h(x), h(y)h(e)\} = \{h(x), h(y)e'\} = \{h(x), h(y)\}$$

for all $x, y \in P$. Thus, $h : P \to Q$ satisfies the definition of Poisson C^*-algebra homomorphism, given in [26, Definition 1.4].

In this study, we solve the additive functional equation (2.1) and prove the Hyers-Ulam stability of the additive functional equation (2.1) in unital Poisson C^*-algebras by using the direct method and by the fixed point method. Furthermore, we investigate Poisson C^*-algebra derivations and Poisson C^*-algebra homomorphisms in unital Poisson C^*-algebras associated with the additive functional equation (2.1).

2 Hyers-Ulam stability of the functional equation (2.1): direct method

In this section, we solve and investigate the functional equation (2.1) in unital C^*-algebras

$$g(\lambda u + v + 2y) = \lambda g(u) + g(v) + 2g(y). \tag{2.1}$$

Lemma 2.1. [27, Lemma 1] *Assume that a mapping $g : P \to Q$ satisfies*

$$g(\lambda u + v + 2y) = \lambda g(u) + g(v) + 2g(y) \tag{2.2}$$

for all $\lambda \in \mathbb{C}$, all $u, v \in U(P)$, and all $y \in P$. Then, the mapping $g : P \to Q$ is $\mathbb{C}$-linear.

Proof. Substituting $\lambda = 1$ and $u = -v$ in (2.2), we obtain $g(2y) = 2g(y)$ for all $y \in P$ and so, we obtain $g(0) = 0$.

Substituting $\lambda = 0$ in (2.2), we obtain

$$g(v + 2y) = g(v) + 2g(y)$$

and so,

$$g(\lambda u + v + 2y) = \lambda g(u) + g(v) + 2g(y) = \lambda g(u) + g(v + 2y) \tag{2.3}$$

for all $\lambda \in \mathbb{C}$, all $u, v \in U(P)$, and all $y \in P$. Substituting $z = v + 2y$ in (2.3), we obtain

$$g(\lambda u + z) = \lambda g(u) + g(z) \tag{2.4}$$

for all $\lambda \in \mathbb{C}$, all $u \in U(P)$, and all $z \in P$.

Substituting $z = 0$ in (2.4), we obtain $g(\lambda u) = \lambda g(u)$ for all $\lambda \in \mathbb{C}$ and all $u \in U(P)$.

Since each $x \in P$ is a finite linear combination of unitary elements [28], i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in U(P))$, it follows from (2.4) that

$$\begin{aligned}
g(\lambda x + z) &= g\left(\lambda \sum_{j=1}^{m} \lambda_j u_j + z\right) = g\left(\sum_{j=1}^{m} \lambda\lambda_j u_j + z\right) = g\left(\lambda\lambda_1 u_1 + \sum_{j=2}^{m} \lambda\lambda_j u_j + z\right) \\
&= \lambda\lambda_1 g(u_1) + g\left(\sum_{j=2}^{m} \lambda\lambda_j u_j + z\right) \\
&\vdots \\
&= \lambda\lambda_1 g(u_1) + \lambda\lambda_2 g(u_2) + \ldots + g(\lambda\lambda_m u_m + z) \\
&= \lambda\lambda_1 g(u_1) + \lambda\lambda_2 g(u_2) + \ldots + \lambda\lambda_m g(u_m) + g(z) \\
&= \lambda(\lambda_1 g(u_1) + \lambda_2 g(u_2) + \ldots + \lambda_m g(u_m)) + g(z) \\
&= \lambda(\lambda_1 g(u_1) + \lambda_2 g(u_2) + \ldots + \lambda_{m-1} g(u_{m-1}) + g(\lambda_m u_m)) + g(z) \\
&= \lambda(\lambda_1 g(u_1) + \lambda_2 g(u_2) + \ldots + g(\lambda_{m-1} u_{m-1} + \lambda_m u_m)) + g(z) \\
&\vdots \\
&= \lambda(\lambda_1 g(u_1) + g(\lambda_2 u_2 + \ldots + \lambda_m u_m)) + g(z) \\
&= \lambda(g(\lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_{m-1} u_{m-1} + \lambda_m u_m)) + g(z) \\
&= \lambda g(x) + g(z)
\end{aligned}$$

for all $\lambda \in \mathbb{C}$, and all $z \in P$. So the mapping $g : P \to Q$ is $\mathbb{C}$-linear. □

Now, we prove the Hyers-Ulam stability of the additive functional equation (2.1) in unital C^*-algebras.

Theorem 2.2. [27, Theorem 2] *Let $\varphi : P^3 \to [0, \infty)$ be a function such that*

$$\Phi(u, v, y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{u}{2^j}, \frac{v}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.5}$$

for all $u, v \in U(P)$, and all $y \in P$. Let $g : P \to Q$ be a mapping satisfying

$$\|g(\lambda(2^s u) + 2^s v + 2y) - \lambda g(2^s u) - g(2^s v) - 2g(y)\| \le \varphi(2^s u, 2^s v, y) \tag{2.6}$$

for all $\lambda \in \mathbb{C}$, all $u, v \in U(P)$, all $s \in \mathbb{Z}$ with $s \le 0$, and all $y \in P$. Then, there exists a unique $\mathbb{C}$-linear mapping $G : P \to Q$ such that

$$\|g(y) - G(y)\| \le \frac{1}{2}\Phi(u, u, y), \tag{2.7}$$

for all $u \in U(P)$ and all $y \in P$.

Proof. Letting $\lambda = -1$ and replacing u by $\frac{u}{2}$ and v by $\frac{u}{2}$ in (2.6), we obtain

$$\|g(2y) - 2g(y)\| \le \varphi\left(\frac{u}{2}, \frac{u}{2}, y\right)$$

and so,

$$\left\|g(y) - 2g\left(\frac{y}{2}\right)\right\| \le \varphi\left(\frac{u}{2}, \frac{u}{2}, \frac{y}{2}\right)$$

for all $u \in U(P)$ and all $y \in P$.

Similarly, we can show that

$$\left\|2^j g\left(\frac{y}{2^j}\right) - 2^{j+1} g\left(\frac{y}{2^{j+1}}\right)\right\| \le 2^j \varphi\left(\frac{u}{2^{j+1}}, \frac{u}{2^{j+1}}, \frac{y}{2^{j+1}}\right)$$

for all $u \in U(P)$, all $y \in P$, and each positive integer j. Thus,

$$\left\|2^l g\left(\frac{y}{2^l}\right) - 2^m g\left(\frac{y}{2^m}\right)\right\| \le \sum_{j=l}^{m-1} \left\|2^j g\left(\frac{y}{2^j}\right) - 2^{j+1} g\left(\frac{y}{2^{j+1}}\right)\right\| \le \frac{1}{2} \sum_{j=l+1}^{m} 2^j \varphi\left(\frac{u}{2^j}, \frac{u}{2^j}, \frac{y}{2^j}\right) \tag{2.8}$$

for all nonnegative integers m and l with $m > l$, all $u \in U(P)$ and all $y \in P$. It follows from (2.8) that the sequence $\{2^k g(\frac{y}{2^k})\}$ is Cauchy for all $y \in P$. Since Q is complete, the sequence $\{2^k g(\frac{y}{2^k})\}$ converges. So one can define the mapping $G : P \to Q$ by

$$G(y) := \lim_{k\to\infty} 2^k g\left(\frac{y}{2^k}\right)$$

for all $y \in P$. Moreover, substituting $l = 0$ and passing to the limit $m \to \infty$ in (2.8), we obtain (2.7).

It follows from (2.5) and (2.6) that

$$\begin{aligned}
&\|G(\lambda u + v + 2y) - \lambda G(u) - G(v) - 2G(y)\| \\
&= \lim_{n\to\infty} 2^n \left\|g\left(\frac{\lambda u + v + 2y}{2^n}\right) - \lambda g\left(\frac{u}{2^n}\right) - g\left(\frac{v}{2^n}\right) - 2g\left(\frac{y}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty} 2^n \varphi\left(\frac{u}{2^n}, \frac{v}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $\lambda \in \mathbb{C}$, all $u, v \in U(P)$, and all $y \in P$. So,

$$G(\lambda u + v + 2y) = \lambda G(u) + G(v) + 2G(y)$$

for all $\lambda \in \mathbb{C}$, all $u, v \in U(P)$, and all $y \in P$. By Lemma 2.1, the mapping $G : P \to Q$ is $\mathbb{C}$-linear.

The proof of the uniqueness of the mapping G is similar to the proof of [29, Theorem 2.3]. □

Theorem 2.3. [27, Theorem 3] *Let $\varphi : P^3 \to [0, \infty)$ be a function such that*

$$\Psi(u, v, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j u, 2^j v, 2^j y) < \infty \tag{2.9}$$

for all $u, v \in U(P)$, and all $y \in P$. Let $g : P \to Q$ be a mapping satisfying (2.6) *for all $s \in \mathbb{Z}$ with $s \ge 0$. Then, there exists a unique $\mathbb{C}$-linear mapping $G : P \to Q$ such that*

$$\|g(y) - G(y)\| \le \frac{1}{2} \Psi(u, u, y) \tag{2.10}$$

for all $u \in U(P)$ and all $y \in P$.

Proof. Letting $\lambda = -1$ and replacing v by u in (2.6), we obtain

$$\|g(2y) - 2g(y)\| \le \varphi(u, u, y)$$

and so,

$$\left\|g(y) - \frac{1}{2}g(2y)\right\| \le \frac{1}{2}\varphi(u, u, y)$$

for all $u \in U(P)$ and all $y \in P$.

Similarly, we can show that

$$\left\|\frac{1}{2^j}g(2^j y) - \frac{1}{2^{j+1}}g(2^{j+1}y)\right\| \le \frac{1}{2^{j+1}}\varphi(2^j u, 2^j u, 2^j y)$$

for all $u \in U(P)$, all $y \in P$, and each positive integer j. Thus,

$$\left\|\frac{1}{2^l}g(2^l y) - \frac{1}{2^m}g(2^m y)\right\| \le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}g(2^j y) - \frac{1}{2^{j+1}}g(2^{j+1}y)\right\| \le \frac{1}{2}\sum_{j=l}^{m-1}\frac{1}{2^j}\varphi(2^j u, 2^j u, 2^j y) \tag{2.11}$$

for all nonnegative integers m and l with $m > l$, all $u \in U(P)$, and all $y \in P$. It follows from (2.11) that the sequence $\{\frac{1}{2^k}g(2^k y)\}$ is Cauchy for all $y \in P$. Since Q is complete, the sequence $\{\frac{1}{2^k}g(2^k y)\}$ converges. So one can define the mapping $G : P \to Q$ by

$$G(y) \coloneqq \lim_{k\to\infty}\frac{1}{2^k}g(2^k y)$$

for all $y \in P$. Moreover, substituting $l = 0$ and passing to the limit $m \to \infty$ in (2.11), we obtain (2.10).

It follows from (2.6) and (2.9) that

$$\begin{aligned} &\|G(\lambda u + v + 2y) - \lambda G(u) - G(v) - 2G(y)\| \\ &= \lim_{n\to\infty}\frac{1}{2^n}\|g(2^n(\lambda u + v + 2y)) - \lambda g(2^n u) - g(2^n v) - 2g(2^n y)\| \\ &\le \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n u, 2^n v, 2^n y) = 0 \end{aligned}$$

for all $\lambda \in \mathbb{C}$, all $u, v \in U(P)$, and all $y \in P$. So,

$$G(\lambda u + v + 2y) = \lambda G(u) + G(v) + 2G(y)$$

for all $\lambda \in \mathbb{C}$, all $u, v \in U(P)$, and all $y \in P$. By Lemma 2.1, the mapping $G : P \to Q$ is $\mathbb{C}$-linear.

The proof of the uniqueness of the mapping G is similar to the proof of [29, Theorem 2.3]. □

3 Hyers-Ulam stability of Poisson C^*-algebra derivations and Poisson C^*-algebra homomorphisms in Poisson C^*-algebras: Direct method

Using the direct method, we prove the Hyers-Ulam stability of Poisson C^*-algebra homomorphisms in unital Poisson C^*-algebras associated with the additive functional equation (2.2).

Theorem 3.1. *Let $\varphi : P^3 \to [0, \infty)$ be a function such that*

$$\sum_{j=0}^{\infty} 8^j \varphi\left(\frac{u}{2^j}, \frac{v}{2^j}, \frac{y}{2^j}\right) < \infty \tag{3.1}$$

for all $u, v \in U(P)$ and all $y \in P$. Let $g : P \to Q$ be a mapping satisfying (2.6) for all $s \in \mathbb{Z}$ with $s \le 0$. If the mapping $g : P \to Q$ satisfies

$$\|g(8^s uvw) - g(2^s u)g(2^s v)g(2^s w)\| \le \varphi(2^s u, 2^s v, 2^s w), \tag{3.2}$$

$$\|g(2^s u^*) - g(2^s u)^*\| \le \varphi(2^s u, 2^s u, 2^s u), \tag{3.3}$$

$$\|g(\{2^s u, 4^s vw\}) - \{g(2^s u), g(2^s v)g(2^s w)\}\| \le \varphi(2^s u, 2^s v, 2^s w), \tag{3.4}$$

for all $u, v, w \in U(P)$ *and all* $s \in \mathbb{Z}$ *with* $s \le 0$, *then there exists a unique Poisson C*-algebra homomorphism* $H : P \to Q$ *such that*

$$\|g(y) - H(y)\| \le \frac{1}{2}\sum_{j=1}^{\infty} 2^j \varphi\left(\frac{u}{2^j}, \frac{u}{2^j}, \frac{y}{2^j}\right) \tag{3.5}$$

for all $u \in U(P)$, *and all* $y \in P$.

Proof. By Theorem 2.2, there exists a unique $\mathbb{C}$-linear mapping $H : P \to Q$ satisfying (3.5). The mapping $H : P \to Q$ is given by

$$H(y) := \lim_{k\to\infty} 2^k g\left(\frac{y}{2^k}\right)$$

for all $y \in P$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}\|H(uvw) - H(u)H(v)H(w)\| &= \lim_{n\to\infty} 8^n \left\| g\left(\frac{uvw}{8^n}\right) - g\left(\frac{u}{2^n}\right) g\left(\frac{v}{2^n}\right) g\left(\frac{w}{2^n}\right) \right\| \\ &\le \lim_{n\to\infty} 8^n \varphi\left(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n}\right) = 0\end{aligned}$$

and so,

$$H(uvw) = H(u)H(v)H(w)$$

for all $u, v, w \in U(P)$.

Since $H : P \to Q$ is $\mathbb{C}$-linear and each $x, y, z \in P$ is a finite linear combination of unitary elements [28], i.e., $x = \sum_{j=1}^m \lambda_j u_j$, $y = \sum_{k=1}^l \mu_k v_k$, and $z = \sum_{s=1}^t \nu_s w_s (\lambda_j, \mu_k, \nu_s \in \mathbb{C}, u_j, v_k, w_s \in U(P))$,

$$\begin{aligned}H(xyz) &= H\left(\sum_{j=1}^m \sum_{k=1}^l \sum_{s=1}^t \lambda_j \mu_k \nu_s u_j v_k w_s\right) = \sum_{j=1}^m \sum_{k=1}^l \sum_{s=1}^t \lambda_j \mu_k \nu_s H(u_j v_k w_s) \\ &= \sum_{j=1}^m \sum_{k=1}^l \sum_{s=1}^t \lambda_j \mu_k \nu_s H(u_j)H(v_k)H(w_s) \\ &= \left(\sum_{j=1}^m \lambda_j H(u_j)\right)\left(\sum_{k=1}^l \mu_k H(v_k)\right)\left(\sum_{s=1}^t \nu_s H(w_s)\right) \\ &= H\left(\sum_{j=1}^m \lambda_j u_j\right) H\left(\sum_{k=1}^l \mu_k v_k\right) H\left(\sum_{s=1}^t \nu_s w_s\right) = H(x)H(y)H(z)\end{aligned}$$

for all $x, y, z \in P$. So, the $\mathbb{C}$-linear mapping $H : P \to Q$ satisfies $H(xyz) = H(x)H(y)H(z)$ for all $x, y, z \in P$.

It follows from (3.1) and (3.3) that

$$\begin{aligned}\|H(u^*) - H(u)^*\| &= \lim_{n\to\infty} 2^n \left\| g\left(\frac{u^*}{2^n}\right) - g\left(\frac{u}{2^n}\right)^* \right\| \\ &\le \lim_{n\to\infty} 2^n \varphi\left(\frac{u}{2^n}, \frac{u}{2^n}, \frac{u}{2^n}\right) \le \lim_{n\to\infty} 8^n \varphi\left(\frac{u}{2^n}, \frac{u}{2^n}, \frac{u}{2^n}\right) = 0\end{aligned}$$

and so,

$$H(u^*) = H(u)^*$$

for all $u \in U(P)$.

Since $H: P \to Q$ is $\mathbb{C}$-linear and each $x \in P$ is a finite linear combination of unitary elements [28], i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in U(P))$,

$$H(x^*) = H\left(\sum_{j=1}^{m} \overline{\lambda_j} u_j^*\right) = \sum_{j=1}^{m} \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^{m} \overline{\lambda_j} H(u_j)^* = H\left(\sum_{j=1}^{m} \lambda_j u_j\right)^* = H(x)^*$$

for all $x \in P$. So, the $\mathbb{C}$-linear mapping $H: P \to Q$ is involutive.

It follows from (3.1) and (3.4) that

$$\begin{aligned}\|H(\{u, vw\}) - \{H(u), H(v)H(w)\}\| &= \lim_{n\to\infty} 8^n \left\| g\left(\frac{\{u, vw\}}{8^n}\right) - \left\{ g\left(\frac{u}{2^n}\right), g\left(\frac{v}{2^n}\right) g\left(\frac{w}{2^n}\right)\right\} \right\| \\ &\le \lim_{n\to\infty} 8^n \varphi\left(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n}\right) = 0\end{aligned}$$

and so,

$$H(\{u, vw\}) = \{H(u), H(v)H(w)\}$$

for all $u, v, w \in U(P)$.

Since $H: P \to Q$ is $\mathbb{C}$-linear and each $x, y, z \in P$ is a finite linear combination of unitary elements [28], i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$, $y = \sum_{k=1}^{l} \mu_k v_k$, and $z = \sum_{s=1}^{t} \nu_s w_s (\lambda_j, \mu_k, \nu_s \in \mathbb{C}, u_j, v_k, w_s \in U(P))$,

$$\begin{aligned}H(\{x, yz\}) &= H\left(\left\{\sum_{j=1}^{m} \lambda_j u_j, \left(\sum_{k=1}^{l} \mu_k v_k\right)\left(\sum_{s=1}^{t} \nu_s w_s\right)\right\}\right) \\ &= \sum_{j=1}^{m}\sum_{k=1}^{l}\sum_{s=1}^{t} \lambda_j \mu_k \nu_s H(\{u_j, v_k w_s\}) \\ &= \sum_{j=1}^{m}\sum_{k=1}^{l}\sum_{s=1}^{t} \lambda_j \mu_k \nu_s \{H(u_j), H(v_k)H(w_s)\} \\ &= \left\{\sum_{j=1}^{m} \lambda_j H(u_j), \left(\sum_{k=1}^{l} \mu_k H(v_k)\right)\left(\sum_{s=1}^{t} \nu_s H(w_s)\right)\right\} \\ &= \left\{H\left(\sum_{j=1}^{m} \lambda_j u_j\right), H\left(\sum_{k=1}^{l} \mu_k v_k\right) H\left(\sum_{s=1}^{t} \nu_s w_s\right)\right\} = \{H(x), H(y)H(z)\}\end{aligned}$$

for all $x, y, z \in P$. So, the mapping $H: P \to Q$ satisfies $H(\{x, yz\}) = \{H(x), H(y)H(z)\}$ for all $x, y, z \in P$. Thus, the $\mathbb{C}$-linear mapping $H: P \to Q$ is a Poisson C^*-algebra homomorphism. □

Corollary 3.2. *Let $r > 3$ and θ be nonnegative real numbers and $g: P \to Q$ be a mapping satisfying*

$$\|g(\lambda(2^s u) + 2^s v + 2y) - \lambda g(2^s u) - g(2^s v) - 2g(y)\| \le \theta(2^{s+1} + \|y\|^r) \tag{3.6}$$

for all $\lambda \in \mathbb{C}$, all $u, v \in U(P)$, all $s \in \mathbb{Z}$ with $s \le 0$, and all $y \in P$. If the mapping $g: P \to Q$ satisfies

$$\|g(8^s uvw) - g(2^s u)g(2^s v)g(2^s w)\| \le 3 \cdot 2^{rs}\theta, \tag{3.7}$$

$$\|g(2^s u^*) - g(2^s u)^*\| \le 3 \cdot 2^{rs}\theta, \tag{3.8}$$

$$\|g(\{2^s u, 4^s vw\}) - \{g(2^s u), g(2^s v)g(2^s w)\}\| \le 3 \cdot 2^{rs}\theta \tag{3.9}$$

for all $u, v, w \in U(P)$ and all $s \in \mathbb{Z}$ with $s \le 0$, then there exists a unique Poisson C^*-algebra homomorphism $H: P \to Q$ such that

$$\|g(y) - H(y)\| \le \frac{\theta}{2^r - 2}(2 + \|y\|^r) \tag{3.10}$$

for all $y \in P$.

Proof. The result follows from Theorem 3.1 by taking $\varphi(2^s u, 2^s v, y) = \theta(2^{rs+1} + \|y\|^r)$ for all $u, v \in U(P)$, all $s \in \mathbb{Z}$ with $s \leq 0$, and all $y \in P$. □

Theorem 3.3. *Let $\varphi : P^3 \to [0, \infty)$ be a function and $g : P \to Q$ be a mapping satisfying* (2.9), (2.6), (3.2), (3.3) *and* (3.4) *for all $s \in \mathbb{Z}$ with $s \geq 0$. Then, there exists a unique Poisson C^*-algebra homomorphism $H : P \to Q$ such that*

$$\|g(y) - H(y)\| \leq \frac{1}{2}\sum_{j=0}^{\infty}\frac{1}{2^j}\varphi(2^j u, 2^j u, 2^j y) \tag{3.11}$$

for all $u \in U(P)$ and all $y \in P$.

Proof. By Theorem 2.3, there exists a unique $\mathbb{C}$-linear mapping $H : P \to Q$ satisfying (3.11). The mapping $H : P \to Q$ is given by

$$H(y) := \lim_{k\to\infty}\frac{1}{2^k}g(2^k y)$$

for all $y \in P$.

It follows from (2.9) and (3.2) that

$$\begin{aligned}\|H(uvw) - H(u)H(v)H(w)\| &= \lim_{n\to\infty}\frac{1}{8^n}\|g(8^n uvw) - g(2^n u)g(2^n v)g(2^n w)\| \\ &\leq \lim_{n\to\infty}\frac{1}{8^n}\varphi(2^n u, 2^n v, 2^n w) \\ &\leq \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n u, 2^n v, 2^n w) = 0\end{aligned}$$

and so $H(uvw) = H(u)H(v)H(w)$ for all $u, v, w \in U(P)$.

It follows from (2.9) and (3.4) that

$$\begin{aligned}&\|H(\{u, vw\}) - \{H(u), H(v)H(w)\}\| \\ &= \lim_{n\to\infty}\frac{1}{8^n}\|g(8^n\{u, vw\}) - \{g(2^n u), g(2^n v)g(2^n w)\}\| \\ &\leq \lim_{n\to\infty}\frac{1}{8^n}\varphi(2^n u, 2^n v, 2^n w) \leq \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n u, 2^n v, 2^n w) = 0\end{aligned}$$

and so $H(\{u, vw\}) = \{H(u), H(v)H(w)\}$ for all $u, v, w \in U(P)$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let $r < 1$ and θ be nonnegative real numbers and $g : P \to Q$ be a mapping satisfying* (3.6), (3.7), (3.8), *and* (3.9) *for all $s \in \mathbb{Z}$ with $s \geq 0$. Then, there exists a unique Poisson C^*-algebra homomorphism $H : P \to Q$ such that*

$$\|g(x) - H(x)\| \leq \frac{\theta}{2 - 2^r}(2 + \|y\|^r) \tag{3.12}$$

for all $y \in P$.

Proof. The result follows from Theorem 3.3 by taking $\varphi(2^s u, 2^s v, 2^j y) = \theta(2^{rs+1} + \|y\|^r)$ for all $u, v \in U(P)$, all $s \in \mathbb{Z}$ with $s \geq 0$, and all $y \in P$. □

Now, we prove the Hyers-Ulam stability of Poisson C^*-algebra derivations in unital Poisson C^*-algebras associated with the additive functional equation (2.2) by using the direct method.

Theorem 3.5. *Let $\varphi : P^3 \to [0, \infty)$ be a function satisfying* (3.1). *If a mapping $g : P \to P$ satisfies* (3.3), (2.6), *and*

$$\|g(8^s uvw) - g(2^s u)4^s vw - 4^s ug(2^s v)w - 4^s uvg(2^s w)\| \leq \varphi(2^s u, 2^s v, 2^s w), \tag{3.13}$$

$$\|g(\{2^s u, 4^s vw\}) - \{g(2^s u), 4^s vw\} - \{2^s u, g(2^s v)2^s w\} - \{2^s u, 2^s vg(2^s w)\}\| \leq \varphi(2^s u, 2^s v, 2^s w) \tag{3.14}$$

for all $u, v, w \in U(P)$ and all $s \in \mathbb{Z}$ with $s \leq 0$, then there exists a unique Poisson C^*-algebra derivation $D : P \to P$ such that

$$\|g(y) - D(y)\| \leq \frac{1}{2}\sum_{j=1}^{\infty} 2^j \varphi\left(\frac{u}{2^j}, \frac{u}{2^j}, \frac{y}{2^j}\right) \tag{3.15}$$

for all $u \in U(P)$ and all $y \in P$.

Proof. By Theorem 2.2, there exists a unique $\mathbb{C}$-linear mapping $D : P \to P$ satisfying (3.15). The mapping $D : P \to P$ is given by

$$D(y) \coloneqq \lim_{k\to\infty} 2^k g\left(\frac{y}{2^k}\right)$$

for all $y \in P$.

It follows from (3.1) and (3.13) that

$$\begin{aligned} &\|D(uvw) - D(u)vw - uD(v)w - uvD(w)\| \\ &= \lim_{n\to\infty} 8^n \left\| g\left(\frac{uvw}{8^n}\right) - g\left(\frac{u}{2^n}\right)\frac{vw}{4^n} - \frac{u}{2^n} g\left(\frac{v}{2^n}\right)\frac{w}{2^n} - \frac{uv}{4^n} g\left(\frac{w}{2^n}\right) \right\| \\ &\leq \lim_{n\to\infty} 8^n \varphi\left(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n}\right) = 0 \end{aligned}$$

for all $u, v, w \in U(P)$. So,

$$D(uvw) = D(u)vw + uD(v)w + uvD(w)$$

for all $u, v, w \in U(P)$.

Since $D : P \to P$ is $\mathbb{C}$-linear and each $x, y \in P$ is a finite linear combination of unitary elements [28], i.e., $x = \sum_{j=1}^m \lambda_j u_j$, $y = \sum_{k=1}^l \mu_k v_k$, and $z = \sum_{s=1}^t \nu_s w_s$ ($\lambda_j, \mu_k, \nu_s \in \mathbb{C}$, $u_j, v_k, w_s \in U(P)$),

$$\begin{aligned} D(xyz) &= D\left(\sum_{j=1}^m \sum_{k=1}^l \sum_{s=1}^t \lambda_j \mu_k u_j v_k w_s\right) \\ &= \sum_{j=1}^m \sum_{k=1}^l \sum_{s=1}^t \lambda_j \mu_k \nu_s D(u_j v_k w_s) \\ &= \sum_{j=1}^m \sum_{k=1}^l \sum_{s=1}^t \lambda_j \mu_k \nu_s (D(u_j) v_k w_s + u_j D(v_k) w_s + u_j v_k D(w_s)) \\ &= \left(\sum_{j=1}^m \lambda_j D(u_j)\right)\left(\sum_{k=1}^l \mu_k v_k\right)\left(\sum_{s=1}^t \nu_s w_s\right) + \left(\sum_{j=1}^m \lambda_j u_j\right)\left(\sum_{k=1}^l \mu_k D(v_k)\right)\left(\sum_{s=1}^t \nu_s w_s\right) + \left(\sum_{j=1}^m \lambda_j u_j\right)\left(\sum_{k=1}^l \mu_k v_k\right)\left(\sum_{s=1}^t \nu_s D(w_s)\right) \\ &= D\left(\sum_{j=1}^m \lambda_j u_j\right) yz + xD\left(\sum_{k=1}^l \mu_k v_k\right) z + xyD\left(\sum_{s=1}^t \nu_s w_s\right) = D(x)yz + xD(y)z + xyD(z) \end{aligned}$$

for all $x, y, z \in P$. So, the $\mathbb{C}$-linear mapping $D : P \to P$ satisfies

$$D(xyz) = D(x)yz + xD(y)z + xyD(z)$$

for all $x, y, z \in P$.

It follows from (3.1) and (3.3) that

$$\|D(u^*) - D(u)^*\| = \lim_{n\to\infty} 2^n \left\| g\left(\frac{u^*}{2^n}\right) - g\left(\frac{u}{2^n}\right)^* \right\| \leq \lim_{n\to\infty} 2^n \varphi\left(\frac{u}{2^n}, \frac{u}{2^n}, \frac{u}{2^n}\right) \leq \lim_{n\to\infty} 8^n \varphi\left(\frac{u}{2^n}, \frac{u}{2^n}, \frac{u}{2^n}\right) = 0$$

and so,

$$D(u^*) = D(u)^*$$

for all $u \in U(P)$.

Since $D : P \to P$ is $\mathbb{C}$-linear and each $x \in P$ is a finite linear combination of unitary elements [28], i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in U(P)$),

$$D(x^*) = D\left(\sum_{j=1}^{m} \overline{\lambda_j} u_j^*\right) = \sum_{j=1}^{m} \overline{\lambda_j} D(u_j^*) = \sum_{j=1}^{m} \overline{\lambda_j} D(u_j)^* = D\left(\sum_{j=1}^{m} \lambda_j u_j\right)^* = D(x)^*$$

for all $x \in P$. So, the $\mathbb{C}$-linear mapping $D : P \to P$ is involutive.

It follows from (3.1) and (3.14) that

$$\begin{aligned}
&\|D(\{u, vw\}) - \{D(u), vw\} - \{u, D(v)w\} - \{u, vD(w)\}\| \\
&= \lim_{n\to\infty} 8^n \left\| g\left(\frac{\{u, vw\}}{8^n}\right) - \left\{g\left(\frac{u}{2^n}\right), \frac{vw}{4^n}\right\} - \left\{\frac{u}{2^n}, g\left(\frac{v}{2^n}\right)\frac{w}{2^n}\right\} - \left\{\frac{uv}{4^n}, g\left(\frac{w}{2^n}\right)\right\} \right\| \\
&\le \lim_{n\to\infty} 8^n \varphi(2^n u, 2^n v, 2^n w) = 0
\end{aligned}$$

and so, $D(\{u, vw\}) = \{D(u), vw\} + \{u, D(v)w\} + \{u, vD(w)\}$ for all $u, v, w \in U(P)$.

Since $D : P \to P$ is $\mathbb{C}$-linear and each $x, y, z \in P$ is a finite linear combination of unitary elements [28], i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$, $y = \sum_{k=1}^{l} \mu_k v_k$, and $z = \sum_{s=1}^{t} \nu_s w_s (\lambda_j, \mu_k, \nu_s \in \mathbb{C}, u_j, v_k, w_s \in U(P))$,

$$\begin{aligned}
D(\{x, yz\}) &= D\left(\left\{\sum_{j=1}^{m} \lambda_j u_j, \left(\sum_{k=1}^{l} \mu_k v_k\right)\left(\sum_{s=1}^{t} \nu_s w_s\right)\right\}\right) \\
&= \sum_{j=1}^{m}\sum_{k=1}^{l}\sum_{s=1}^{t} \lambda_j \mu_k \nu_s D(\{u_j, v_k w_s\}) \\
&= \sum_{j=1}^{m}\sum_{k=1}^{l}\sum_{s=1}^{t} \lambda_j \mu_k \nu_s (\{D(u_j), v_k w_s\} + \{u_j, D(v_k) w_s\} + \{u_j v_k, D(w_s)\}) \\
&= \left\{\sum_{j=1}^{m} \lambda_j D(u_j), \left(\sum_{k=1}^{l} \mu_k v_k\right)\left(\sum_{s=1}^{t} \nu_s w_s\right)\right\} + \left\{\sum_{j=1}^{m} \lambda_j u_j, \sum_{k=1}^{l} \mu_k D(v_k) \sum_{s=1}^{t} \nu_s w_s\right\} \\
&\quad + \left\{\left(\sum_{j=1}^{m} \lambda_j u_j\right)\left(\sum_{k=1}^{l} \mu_k v_k\right), \sum_{s=1}^{t} \nu_s D(w_s)\right\} \\
&= \left\{D\left(\sum_{j=1}^{m} \lambda_j u_j\right), yz\right\} + \left\{x, D\left(\sum_{k=1}^{l} \mu_k v_k\right) z\right\} + \left\{xy, D\left(\sum_{s=1}^{t} \nu_s w_s\right)\right\} \\
&= \{D(x), yz\} + \{x, D(y)z\} + \{xy, D(z)\}
\end{aligned}$$

for all $x, y, z \in P$. Thus, the mapping $D : P \to P$ is a Poisson C^*-algebra derivation. □

Corollary 3.6. *Let $r > 3$ and θ be nonnegative real numbers and $g : P \to P$ be a mapping satisfying* (3.6), (3.8), *and*

$$\|g(8^s uvw) - g(2^s u)4^s vw - 4^s u g(2^s v) w - 4^s uv g(2^s w)\| \le 3 \cdot 2^{rs}\theta, \tag{3.16}$$

$$\|g(\{2^s u, 4^s vw\}) - \{g(2^s u), 4^s vw\} - \{2^s u, g(2^s v) 2^s w\} - \{4^s uv, g(2^s w)\}\| \le 3 \cdot 2^{rs}\theta \tag{3.17}$$

for all $u, v, w \in U(P)$ and all $s \in \mathbb{Z}$ with $s \le 0$. Then, there exists a unique Poisson C^*-algebra derivation $D : P \to P$ such that

$$\|g(y) - D(y)\| \le \frac{\theta}{2^r - 2}(2 + \|y\|^r) \tag{3.18}$$

for all $y \in P$.

Proof. The result follows from Theorem 3.5 by taking $\varphi(2^s u, 2^s v, y) = \theta(2^{rs+1} + \|y\|^r)$ for all $u, v \in U(P)$, all $s \in \mathbb{Z}$ with $s \le 0$ and all $y \in P$. □

Theorem 3.7. *Let $\varphi : P^3 \to [0, \infty)$ be a function and $g : P \to P$ be a mapping satisfying* (2.9), (2.6), (3.13), (3.14), *and* (3.3) *for all $s \in \mathbb{Z}$ with $s \geq 0$. Then, there exists a unique Poisson C^*-algebra derivation $D : P \to P$ satisfying*

$$\|g(y) - D(y)\| \leq \frac{1}{2}\Psi(u, u, y) \tag{3.19}$$

for all $u \in U(P)$ and all $y \in P$, where $\Psi(u, v, y)$ is given in the statement of Theorem 2.3.

Proof. By Theorem 2.3, there exists a unique $\mathbb{C}$-linear mapping $D : P \to P$ satisfying (3.19). The mapping $D : P \to P$ is given by

$$D(y) := \lim_{k\to\infty} \frac{1}{2^k} g(2^k y)$$

for all $y \in P$.

It follows from (2.9) and (3.13) that

$$\begin{aligned}
&\|D(uvw) - D(u)vw - uD(v)w - uvD(w)\| \\
&= \lim_{n\to\infty} \frac{1}{8^n}\|g(8^n uvw) - g(2^n u)(4^n vw) - (2^n u)g(2^n v)(2^n w) - (4^n uv)g(2^n w)\| \\
&\leq \lim_{n\to\infty} \frac{1}{8^n}\varphi(2^n u, 2^n v, 2^n w) \leq \frac{1}{2^n}\varphi(2^n u, 2^n v, 2^n w) = 0
\end{aligned}$$

and so, $D(uvw) = D(u)vw + uD(v)w + uvD(w)$ for all $u, v, w \in U(P)$.

It follows from (2.9) and (3.14) that

$$\begin{aligned}
&\|D(\{u, vw\}) - \{D(u), vw\} - \{u, D(v)w\} - \{u, vD(w)\}\| \\
&= \lim_{n\to\infty} \frac{1}{8^n}\|g(8^n\{u, vw\}) - \{g(2^n u), 4^n vw\} - \{2^n u, g(2^n v)2^n w\} - \{2^n u, 2^n v g(2^n w)\}\| \\
&\leq \lim_{n\to\infty} \frac{1}{8^n}\varphi(2^n u, 2^n v, 2^n w) \leq \frac{1}{2^n}\varphi(2^n u, 2^n v, 2^n w) = 0
\end{aligned}$$

and so, $D(\{u, vw\}) = \{D(u), vw\} + \{u, D(v)w\} + \{u, vD(w)\}$ for all $u, v, w \in U(P)$.

The rest of the proof is similar to the proof of Theorem 3.5. □

Corollary 3.8. *Let $r < 1$ and θ be nonnegative real numbers and $g : P \to P$ be a mapping satisfying* (3.6), (3.16), (3.17), *and* (3.8) *for all $s \in \mathbb{Z}$ with $s \geq 0$. Then, there exists a unique Poisson C^*-algebra derivation $D : P \to P$ such that*

$$\|g(x) - D(x)\| \leq \frac{\theta}{2 - 2^r}(2 + \|y\|^r) \tag{3.20}$$

for all $y \in P$.

Proof. The result follows from Theorem 3.7 by taking $\varphi(2^s u, 2^s v, y) = \theta(2^{rs+1} + \|y\|^r)$ for all $u, v \in U(P)$, all $s \in \mathbb{Z}$ with $s \geq 0$, and all $y \in P$. □

4 Hyers-Ulam stability of the functional equation (2.1): fixed point method

Using the fixed point method, we prove the Hyers-Ulam stability of the additive functional equation (2.1) in unital C^*-algebras.

Theorem 4.1. [27, Theorem 8] *Let $\varphi : P^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{u}{2}, \frac{v}{2}, \frac{y}{2}\right) \leq \frac{L}{2}\varphi(u, v, y)$$

for all $u, v \in U(P)$ and all $y \in P$. Let $g : P \to Q$ be a mapping satisfying (2.6) *for all $s \in \mathbb{Z}$ with $s \le 0$. Then, there exists a unique $\mathbb{C}$-linear mapping $H : P \to Q$ such that*

$$\|g(y) - H(y)\| \le \frac{L}{2(1-L)}\varphi(u, u, y) \tag{4.1}$$

for all $u \in U(P)$ and all $y \in P$.

Proof. Consider the set

$$S := \{g : P \to Q\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(y) - h(y)\| \le \mu\varphi(u, u, y),\ \forall u \in U(P), \forall y \in P\},$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that (S, d) is complete [30].

Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(y) := 2g\left(\frac{y}{2}\right)$$

for all $y \in P$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then,

$$\|g(y) - h(y)\| \le \varepsilon\varphi(u, u, y)$$

for all $u \in U(P)$ and all $y \in P$. Since

$$\left\|2g\left(\frac{y}{2}\right) - 2h\left(\frac{y}{2}\right)\right\| \le 2\varepsilon\varphi\left(\frac{u}{2}, \frac{u}{2}, \frac{y}{2}\right) \le 2\varepsilon\frac{L}{2}\varphi(u, u, y) = L\varepsilon\varphi(u, u, y)$$

for all $u \in U(P)$ and all $y \in P$, $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$.

It follows from (2) that

$$\left\|g(y) - 2g\left(\frac{y}{2}\right)\right\| \le \varphi\left(\frac{u}{2}, \frac{u}{2}, \frac{y}{2}\right) \le \frac{L}{2}\varphi(u, u, y)$$

for all $u \in U(P)$ and all $y \in P$. So $d(g, Jg) \le \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $H : P \to P$ satisfying the following:

(1) H is a fixed point of J, i.e.,

$$H(y) = 2H\left(\frac{y}{2}\right) \tag{4.2}$$

for all $y \in P$. The mapping H is a unique fixed point of J. This implies that H is a unique mapping satisfying (4.2) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|g(y) - H(y)\| \le \mu\varphi(u, u, y)$$

for all $u \in U(P)$ and all $y \in P$;

(2) $d(J^l g, H) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^l g\left(\frac{y}{2^l}\right) = H(y)$$

for all $y \in P$;

(3) $d(g, H) \leq \frac{1}{1-L} d(g, Jg)$, which implies

$$\|g(x) - H(x)\| \leq \frac{L}{2(1-L)} \varphi(u, u, y)$$

for all $u \in U(P)$ and all $y \in P$. Thus, we obtain the inequality (4.1).

The rest of the proof is the same as in the proof of Theorem 2.2. □

Theorem 4.2. [27, Theorem 8] *Let $\varphi : P^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(u, v, y) \leq 2L\varphi\left(\frac{u}{2}, \frac{v}{2}, \frac{y}{2}\right) \tag{4.3}$$

for all $u, v \in U(P)$ and all $y \in P$. Let $g : P \to Q$ be a mapping satisfying (2.6) *for all $s \in \mathbb{Z}$ with $s \geq 0$. Then, there exists a unique $\mathbb{C}$-linear mapping $H : P \to Q$ such that*

$$\|g(y) - H(y)\| \leq \frac{1}{2(1-L)} \varphi(u, u, y) \tag{4.4}$$

for all $u \in U(P)$ and all $y \in P$.

Proof. Let S and d be given in the proof of Theorem 4.1.

Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(y) := \frac{1}{2} g(2y)$$

for all $y \in P$.

Letting $\lambda = -1$ and replacing v by u in (2.6), we obtain

$$\left\| g(y) - \frac{1}{2} g(2y) \right\| \leq \frac{1}{2} \varphi(u, u, y)$$

for all $u \in U(P)$ and all $y \in P$. So $d(g, Jg) \leq \frac{1}{2}$.

The rest of the proof is similar to the proofs of Theorems 2.3 and 4.1. □

5 Hyers-Ulam stability of Poisson C^*-algebra derivations and Poisson C^*-algebra homomorphisms in Poisson C^*-algebras: Fixed point method

Using the fixed point method, we prove the Hyers-Ulam stability of Poisson C^*-algebra homomorphisms in unital Poisson C^*-algebras associated with the additive functional equation (2.2).

Theorem 5.1. *Let $\varphi : P^3 \to [0, \infty)$ be a function such that*

$$\varphi\left(\frac{u}{2}, \frac{v}{2}, \frac{y}{2}\right) \leq \frac{L}{8} \varphi(u, v, y) \leq \frac{L}{2} \varphi(u, v, y) \tag{5.1}$$

for all $u, v \in U(P)$ and all $y \in P$. Let $g : P \to Q$ be a mapping satisfying (2.6), (3.2), (3.3), *and* (3.4) *for all $s \in \mathbb{Z}$ with $s \leq 0$. Then, there exists a unique Poisson C^*-algebra homomorphism $H : P \to Q$ satisfying* (4.1).

Proof. It follows from (5.1) and (3.2) that

$$\begin{aligned}\|H(uvw) - H(u)H(v)H(w)\| &= \lim_{n\to\infty} 8^n \left\| g\left(\frac{uvw}{8^n}\right) - g\left(\frac{u}{2^n}\right)g\left(\frac{v}{2^n}\right)g\left(\frac{w}{2^n}\right) \right\| \\ &\le \lim_{n\to\infty} 8^n \varphi\left(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n}\right) \le \lim_{n\to\infty} 8^n \frac{L^n}{8^n}\varphi(u, v, w) = 0\end{aligned}$$

and so,

$$H(uvw) = H(u)H(v)H(w)$$

for all $u, v, w \in U(P)$.

It follows from (5.1) and (3.4) that

$$\begin{aligned}\|H(\{u, vw\}) - \{H(u), H(v)H(w)\}\| &= \lim_{n\to\infty} 8^n \left\| g\left(\frac{\{u, vw\}}{8^n}\right) - \left\{g\left(\frac{u}{2^n}\right), g\left(\frac{v}{2^n}\right)g\left(\frac{w}{2^n}\right)\right\} \right\| \\ &\le \lim_{n\to\infty} 8^n \varphi\left(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n}\right) \le \lim_{n\to\infty} 8^n \frac{L^n}{8^n}\varphi(u, v, w) = 0\end{aligned}$$

and so,

$$H(\{u, vw\}) = \{H(u), H(v)H(w)\}$$

for all $u, v, w \in U(P)$.

The rest of the proof is similar to the proofs of Theorems 2.2, 3.1, and 4.1. □

Corollary 5.2. *Let $r > 3$ and θ be nonnegative real numbers and $g : P \to Q$ be a mapping satisfying* (3.6), (3.7), (3.8), *and* (3.9) *for all $s \in \mathbb{Z}$ with $s \le 0$. Then, there exists a unique Poisson C^*-algebra homomorphism $H : P \to Q$ satisfying* (3.10).

Proof. The result follows from Theorem 5.1 by taking $L = 2^{1-r}$ and $\varphi(2^s u, 2^s v, y) = \theta(2^{rs+1} + \|y\|^r)$ for all $u, v \in U(P)$, all $s \in \mathbb{Z}$ with $s \le 0$, and all $y \in P$. □

Theorem 5.3. *Let $\varphi : P^3 \to [0, \infty)$ be a function and $g : P \to Q$ be a mapping satisfying* (4.3), (2.6), (3.2), (3.3), *and* (3.4) *for all $s \in \mathbb{Z}$ with $s \ge 0$. Then, there exists a unique Poisson C^*-algebra homomorphism $H : P \to Q$ satisfying* (4.4).

Proof. It follows from (2.9) and (3.2) that

$$\begin{aligned}\|H(uvw) - H(u)H(v)H(w)\| &= \lim_{n\to\infty} \frac{1}{8^n}\|g(8^n uvw) - g(2^n u)g(2^n v)g(2^n w)\| \\ &\le \lim_{n\to\infty} \frac{1}{8^n}\varphi(2^n u, 2^n v, 2^n w) \le \lim_{n\to\infty} \frac{2^n L^n}{8^n}\varphi(u, v, w) = 0\end{aligned}$$

and so, $H(uvw) = H(u)H(v)H(w)$ for all $u, v, w \in U(P)$.

It follows from (2.9) and (3.4) that

$$\begin{aligned}\|H(\{u, vw\}) - \{H(u), H(v)H(w)\}\| &= \lim_{n\to\infty} \frac{1}{8^n}\|g(8^n\{u, vw\}) - \{g(2^n u), g(2^n v)g(2^n w)\}\| \\ &\le \lim_{n\to\infty} \frac{1}{8^n}\varphi(2^n u, 2^n v, 2^n w) \le \lim_{n\to\infty} \frac{2^n L^n}{8^n}\varphi(u, v, w) = 0\end{aligned}$$

and so, $H(\{u, vw\}) = \{H(u), H(v)H(w)\}$ for all $u, v, w \in U(P)$.

The rest of the proof is similar to the proofs of Theorems 3.1, 3.3, and 5.1. □

Corollary 5.4. *Let $r < 1$ and θ be nonnegative real numbers and $g : P \to Q$ be a mapping satisfying* (3.6), (3.7), (3.8), *and* (3.9) *for all $s \in \mathbb{Z}$ with $s \geq 0$. Then, there exists a unique Poisson C^*-algebra homomorphism $H : P \to Q$ satisfying* (3.12).

Proof. The result follows from Theorem 5.3 by taking $L = 2^{r-1}$ and $\varphi(2^s u, 2^s v, y) = \theta(2^{rs+1} + \|y\|^r)$ for all $u, v \in U(P)$, all $s \in \mathbb{Z}$ with $s \geq 0$ and all $y \in P$. □

Now, we prove the Hyers-Ulam stability of Poisson C^*-algebra derivations in unital Poisson C^*-algebras associated with the additive functional equation (2.2) by using the fixed point method.

Theorem 5.5. *Let $\varphi : P^3 \to [0, \infty)$ be a function satisfying* (5.1). *If the mapping $g : P \to P$ satisfies* (2.6), (3.3), (3.13), *and* (3.14) *for all $s \in \mathbb{Z}$ with $s \leq 0$, then there exists a unique Poisson C^*-algebra derivation $D : P \to P$ such that*

$$\|g(y) - D(y)\| \leq \frac{L}{2(1-L)}\varphi(u, u, y)$$

for all $u \in U(P)$ and all $y \in P$.

Proof. It follows from (5.1) and (3.13) that

$$\begin{aligned}
&\|D(uvw) - D(u)vw - uD(v)w - uvD(w)\| \\
&= \lim_{n\to\infty} 8^n \left\| g\left(\frac{uvw}{8^n}\right) - g\left(\frac{u}{2^n}\right)\frac{vw}{4^n} - \frac{u}{2^n}g\left(\frac{v}{2^n}\right)\frac{w}{2^n} - \frac{uv}{4^n}g\left(\frac{w}{2^n}\right)\right\| \\
&\leq \lim_{n\to\infty} 8^n \varphi\left(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n}\right) \leq \lim_{n\to\infty} 8^n \frac{L^n}{8^n}\varphi(u, v, w) = 0
\end{aligned}$$

and so,

$$D(uvw) = D(u)vw + uD(v)w + uvD(w)$$

for all $u, v, w \in U(P)$.

It follows from (5.1) and (3.14) that

$$\begin{aligned}
&\|D(\{u, vw\}) - \{D(u), vw\} - \{u, D(v)w\} - \{u, vD(w)\}\| \\
&= \lim_{n\to\infty} 8^n \left\| g\left(\frac{\{u, vw\}}{8^n}\right) - \left\{g\left(\frac{u}{2^n}\right), \frac{vw}{4^n}\right\} - \left\{\frac{u}{2^n}, g\left(\frac{v}{2^n}\right)\frac{w}{2^n}\right\} - \left\{\frac{u}{2^n}, \frac{v}{2^n}g\left(\frac{w}{2^n}\right)\right\}\right\| \\
&\leq \lim_{n\to\infty} 8^n \varphi\left(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n}\right) \leq \lim_{n\to\infty} 8^n \frac{L^n}{8^n}\varphi(u, v, w) = 0
\end{aligned}$$

and so,

$$D(\{u, vw\}) = \{D(u), vw\} + \{u, D(v)w\} + \{u, vD(w)\}$$

for all $u, v, w \in U(P)$.

The rest of the proof is similar to the proofs of Theorems 3.5 and 5.1. □

Corollary 5.6. *Let $r > 3$ and θ be nonnegative real numbers and $g : P \to P$ be a mapping satisfying* (3.6), (3.8), (3.16), *and* (3.17) *for all $s \in \mathbb{Z}$ with $s \leq 0$. Then, there exists a unique Poisson C^*-algebra derivation $D : P \to P$ satisfying* (3.18).

Proof. The result follows from Theorem 5.5 by taking $L = 2^{1-r}$ and $\varphi(2^s u, 2^s v, y) = \theta(2^{rs+1} + \|y\|^r)$ for all $u, v \in U(P)$, all $s \in \mathbb{Z}$ with $s \leq 0$ and all $y \in P$. □

Theorem 5.7. *Let $\varphi : P^3 \to [0, \infty)$ be a function and $g : P \to P$ be a mapping satisfying* (2.6), (4.3), (3.3), (3.13), *and* (3.14) *for all $s \in \mathbb{Z}$ with $s \geq 0$. Then, there exists a unique Poisson C^*-algebra derivation $D : P \to P$ such that*

$$\|g(y) - D(y)\| \leq \frac{1}{2(1-L)}\varphi(u, u, y)$$

for all $u \in U(P)$ and all $y \in P$.

Proof. It follows from (4.3) and (3.2) that

$$\begin{aligned}
&\|D(uvw) - D(u)vw - uD(v)w - uvD(w)\| \\
&= \lim_{n\to\infty} \frac{1}{8^n}\|g(8^n uvw) - g(2^n u)(4^n vw) - (2^n u)g(2^n v)(2^n w) - (4^n uv)g(2^n w)\| \\
&\leq \lim_{n\to\infty} \frac{1}{8^n}\varphi(2^n u, 2^n v, 2^n w) \leq \lim_{n\to\infty} \frac{2^n L^n}{8^n}\varphi(u, v, w) = 0
\end{aligned}$$

and so, $D(uvw) = D(u)vw + uD(v)w + uvD(w)$ for all $u, v, w \in U(P)$.

It follows from (4.3) and (3.14) that

$$\begin{aligned}
&\|D(\{u, vw\}) - \{D(u), vw\} - \{u, D(v)w\} - \{u, vD(w)\}\| \\
&= \lim_{n\to\infty} \frac{1}{8^n}\|g(8^n\{u, vw\}) - \{g(2^n u), 4^n w\} - \{2^n u, g(2^n v)2^n w\} - \{2^n u, 2^n v g(2^n w)\}\| \\
&\leq \lim_{n\to\infty} \frac{1}{8^n}\varphi(2^n u, 2^n v, 2^n w) \leq \lim_{n\to\infty} \frac{2^n L^n}{8^n}\varphi(u, v, w) = 0
\end{aligned}$$

and so, $D(\{u, vw\}) = \{D(u), vw\} + \{u, D(v)w\} + \{u, vD(w)\}$ for all $u, v, w \in U(P)$.

The rest of the proof is similar to the proofs of Theorems 3.5 and 5.1. □

Corollary 5.8. *Let $r < 1$ and θ be nonnegative real numbers and $g : P \to P$ be a mapping satisfying* (3.6), (3.8), (3.16), *and* (3.17) *for all $s \in \mathbb{Z}$ with $s \geq 0$. Then, there exists a unique Poisson C^*-algebra derivation $D : P \to P$ satisfying* (3.20).

Proof. The result follows from Theorem 5.7 by taking $L = 2^{r-1}$ and $\varphi(2^s u, 2^s v, y) = \theta(2^{rs+1} + \|y\|^r)$ for all $u, v \in U(P)$, all $s \in \mathbb{Z}$ with $s \geq 0$, and all $y \in P$. □

6 Conclusion

We introduced the additive functional equation (2.1) in a unital Poisson C^*-algebra P. Using the direct method and the fixed point method, we proved the Hyers-Ulam stability of the additive functional equation (2.1) in unital Poisson C^*-algebras. Furthermore, we applied to study Poisson C^*-algebra homomorphisms and Poisson C^*-algebra derivations in unital Poisson C^*-algebras.

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