


Consistent Sampling Approximations in Abstract Hilbert Spaces

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Abstract: This paper considers generalized consistent sampling and reconstruction processes in an abstract separable Hilbert space. Using an operator-theoretical approach, quasi-consistent and consistent approximations with optimal properties, such as possessing the minimum norm or being closest to the original vector, are derived. The results are illustrated with several examples.

Keywords: consistent sampling; generalized sampling; oblique projection; frame

MSC: 42C15; 47B02

1. Introduction

Sampling is the process of representing a continuous-time signal from a discrete set of measurements, namely the samples. The classical sampling theory focuses mainly on samples that are taken from a signal at some specified instances. A typical example is the Whittaker–Shannon–Kotel’nikov sampling theorem [1] which has been extended in various ways (see [2,3] and references therein).

A more general method of sampling is to consider a signal f in an arbitrary separable Hilbert space \mathcal{H} , and take measurements (i.e., generalized samples) as inner products of f with a set of vectors $\{v_j\}_{j \in J}$, which span a subspace \mathcal{V} called the *sampling space*. With these samples, we reconstruct f using a set of vectors $\{w_k\}_{k \in K}$, which span a subspace \mathcal{W} called the *reconstruction space*. Since any signal lying outside \mathcal{W} cannot be perfectly reconstructed, our goal is to obtain a meaningful approximation for each input signal of \mathcal{H} . A natural approach is to assume the ‘consistency’ which means that an input signal and its approximated signal both yield the same measurements; that is, they look the same to observers through acquisition devices.

The idea of consistent sampling was first introduced by Unser and Aldroubi [4] in a shift-invariant subspace of $L^2(\mathbb{R})$ with single pre- and single post- filters. In [5–8], Eldar et al. studied the consistency in an abstract Hilbert space \mathcal{H} with $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$, under which a unique consistent sampling operator exists. Later, Hirabayashi and Unser [9] studied the consistent sampling in a finite-dimensional Hilbert space $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$ where $\mathcal{W} \cap \mathcal{V}^\perp$ is not necessarily $\{0\}$. Further, Arias and Conde [10] extended the concept of consistency to ‘quasi-consistency’, which requires only that the samples of the approximated signal are as close as possible to the original samples in ℓ_2 sense. Kwon and Lee [11] gave complete characterizations of the quasi-consistency and provided an iterative algorithm to compute the quasi-consistent approximations. Another related work is by Adcock, Hansen, and Poon [12], who analyzed the optimality of consistent sampling using the finite section method [13]. Recently, Arias and Gonzalez [14] studied the problem of reconstructing a vector in a Hilbert space from its samples by means of a weighted least square approximation.



Citation: Kang, S.; Kwon, K.H.; Lee, D.G. Consistent Sampling Approximations in Abstract Hilbert Spaces. *Axioms* **2024**, *13*, 489. <https://doi.org/10.3390/axioms13070489>

Academic Editors: Hadi Roopaei and Feyzi Başar

Received: 21 May 2024

Revised: 6 July 2024

Accepted: 19 July 2024

Published: 21 July 2024



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In this work, we study consistent or quasi-consistent approximations that have optimal properties, such as possessing the minimum norm or being closest to the original signal. We also provide an example to illustrate our results.

2. Preliminaries

For any countable index set I , let $\ell_2(I)$ be the set of all complex-valued sequences $\mathbf{c} = \{c(i)\}_{i \in I}$ with $\|\mathbf{c}\|^2 := \sum_{i \in I} |c(i)|^2 < \infty$. The canonical basis of $\ell_2(I)$ is given by $\{e_i^I\}_{i \in I}$, where $e_i^I(n) = \delta_{i,n}$ for $i, n \in I$.

For any closed subspaces \mathcal{A} and \mathcal{B} of a separable Hilbert space \mathcal{H} , we define the *sum of \mathcal{A} and \mathcal{B}* by

$$\mathcal{A} + \mathcal{B} := \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$$

which may not be closed if \mathcal{H} is infinite-dimensional. If $\mathcal{A} \cap \mathcal{B} = \{0\}$, then $\mathcal{A} + \mathcal{B}$ is also denoted by $\mathcal{A} \oplus \mathcal{B}$ and is referred to as the *direct sum of \mathcal{A} and \mathcal{B}* . In particular, if $\mathcal{H} = \mathcal{A} \oplus \mathcal{B}$ we say that \mathcal{H} is the (*internal*) *direct sum of \mathcal{A} and \mathcal{B}* [15].

For any closed subspaces \mathcal{A} and \mathcal{B} of \mathcal{H} with $\mathcal{H} = \mathcal{A} \oplus \mathcal{B}$, let $P_{\mathcal{A},\mathcal{B}} : \mathcal{H} \rightarrow \mathcal{A}$ be the *oblique projection onto \mathcal{A} along \mathcal{B}* defined by $P_{\mathcal{A},\mathcal{B}}(h) = a$ for $h = a + b$, where $a \in \mathcal{A}$ and $b \in \mathcal{B}$. In particular, $P_{\mathcal{A},\mathcal{A}^\perp} := P_{\mathcal{A}}$ is the orthogonal projection onto \mathcal{A} .

A sequence $\{\phi_n\}_{n \in I}$ in \mathcal{H} is a *frame of \mathcal{H}* if there are constants $B \geq A > 0$, such that we have the following:

$$A\|f\|^2 \leq \sum_{n \in I} |\langle f, \phi_n \rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

Let \mathcal{V} and \mathcal{W} be two closed subspaces of \mathcal{H} . Given a frame $\{w_n\}_{n \in I}$ of \mathcal{W} , a *dual frame of $\{w_n\}_{n \in I}$* is a frame $\{\tilde{w}_n\}_{n \in I}$ of \mathcal{W} satisfying

$$f = \sum_{n \in I} \langle f, \tilde{w}_n \rangle w_n, \quad f \in \mathcal{W}.$$

When $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$, a frame $\{v_n\}_{n \in I}$ of \mathcal{V} is called an *oblique dual frame of $\{w_n\}_{n \in I}$ on \mathcal{V}* if

$$f = \sum_{n \in I} \langle f, v_n \rangle w_n, \quad f \in \mathcal{W},$$

or equivalently (cf. Lemma 3.1 in [16])

$$f = \sum_{n \in I} \langle f, w_n \rangle v_n, \quad f \in \mathcal{V}.$$

For further details on oblique dual frames, see [16,17] and references therein.

For any two Hilbert spaces, \mathcal{H} and \mathcal{K} , let $L(\mathcal{H}, \mathcal{K})$ denote the set of all bounded linear operators from \mathcal{H} into \mathcal{K} , and $L(\mathcal{H}) := L(\mathcal{H}, \mathcal{H})$. For any $T \in L(\mathcal{H}, \mathcal{K})$, let $\text{ran}(T)$ and $\text{ker}(T)$ be the range and the kernel of T , respectively. When $\text{ran}(T)$ is closed, T^\dagger denotes the Moore–Penrose pseudo-inverse of T ([18]).

3. Generalized Consistent Sampling

We consider a generalized consistent sampling problem in a separable Hilbert space \mathcal{H} . Let $\{v_j \neq 0\}_{j \in J}$ be a set of sampling vectors in \mathcal{H} , which forms a frame of the sampling space $\mathcal{V} := \overline{\text{span}}\{v_j\}_{j \in J}$ with synthesis operator $S : \ell_2(J) \rightarrow \mathcal{V}$ given by $S(\mathbf{c}) = \sum_{j \in J} c_j v_j$ for $\mathbf{c} = \{c_j\}_{j \in J} \in \ell_2(J)$. Similarly, let $\{w_k \neq 0\}_{k \in K}$ be a set of reconstruction vectors in \mathcal{H} , which forms a frame of the reconstruction space $\mathcal{W} := \overline{\text{span}}\{w_k\}_{k \in K}$ with synthesis operator $T : \ell_2(K) \rightarrow \mathcal{W}$ given by $T(\mathbf{d}) = \sum_{k \in K} d_k w_k$ for $\mathbf{d} = \{d_k\}_{k \in K} \in \ell_2(K)$. For any signal f in \mathcal{H} , we take generalized samples $S^*(f) = \{\langle f, v_j \rangle\}_{j \in J}$ of f and we seek

its approximation $\tilde{P}(f)$ of f . Specifically, we seek an operator $\tilde{P} : \mathcal{H} \rightarrow \mathcal{W}$ satisfying the following:

$$(stability) \quad \tilde{P} \in L(\mathcal{H}, \mathcal{W}); \tag{1}$$

$$\begin{aligned} \tilde{P}(f) &= 0 \text{ if } S^*(f) = 0, \\ \text{i.e., } \ker(S^*) &\subseteq \ker(\tilde{P}); \end{aligned} \tag{2}$$

$$\begin{aligned} (consistency) \quad \tilde{P} &\text{ is consistent,} \\ \text{i.e., } S^*(\tilde{P}f) &= S^*(f) \text{ for any } f \in \mathcal{H}. \end{aligned} \tag{3}$$

We call \tilde{P} satisfying (1)–(3) a *consistent sampling operator*, and denote the set of all such operators by $\mathcal{C}(\mathcal{W}, \mathcal{V})$. It follows from ([11], Lemma 2.1) (see also [18,19]) that (1) and (2) hold if and only if $\tilde{P} = TQS^*$ for some $Q \in L(\ell_2(J), \ell_2(K))$. We call Q a *consistent filter* if $\tilde{P} = TQS^*$ satisfies (3), and denote the set of all such filters by $\mathcal{F}(\mathcal{W}, \mathcal{V})$. Then

$$\mathcal{F}(\mathcal{W}, \mathcal{V}) = \{Q \in L(\ell_2(J), \ell_2(K)) \mid S^*TQS^* = S^*\}$$

and

$$\mathcal{C}(\mathcal{W}, \mathcal{V}) = \{TQS^* \mid S^*TQS^* = S^*\} = T\mathcal{F}(\mathcal{W}, \mathcal{V})S^*.$$

Throughout the paper, we will always assume that $\mathcal{W} + \mathcal{V}^\perp$ is closed (equivalently, and $\text{ran}(S^*T)$ is closed so that $(S^*T)^\dagger$ exists). Let $\tilde{U} := T(S^*T)^\dagger S^* \in L(\mathcal{H}, \mathcal{W})$.

Theorem 1. $\mathcal{C}(\mathcal{W}, \mathcal{V}) \neq \emptyset$ if and only if $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$. In this case,

$$\begin{aligned} \mathcal{C}(\mathcal{W}, \mathcal{V}) &= \{\tilde{U} + TP_{\ker(S^*T)}YS^* \mid Y \in L(\ell_2(J), \ell_2(K))\} \\ &= \{P_{L, \mathcal{V}^\perp} \mid L \in \mathcal{L}\}, \end{aligned}$$

where $\mathcal{L} := \{L \mid L \text{ is a closed complementary subspace of } \mathcal{W} \cap \mathcal{V}^\perp \text{ in } \mathcal{W}\}$.

Proof. See Theorem 3.1 in [10] and Theorem 3.2 in [11]. \square

When $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$, the consistent approximation $P_{L, \mathcal{V}^\perp}(f)$ of f , with some $L \in \mathcal{L}$, can be expressed using oblique dual frames as follows:

Proposition 1. Assume that $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$ and let $L \in \mathcal{L}$ and $\{u_n\}_{n \in I}$ a frame of L with synthesis operator U . Then $P_{L, \mathcal{V}^\perp} = U(S^*U)^\dagger S^*$ and, moreover, we have the following.

- (a) $\{\tilde{v}_n := S(U^*S)^\dagger(e_n^I) \mid n \in I\}$ is an oblique dual frame of $\{u_n\}_{n \in I}$ on \mathcal{V} (with synthesis operator $S(U^*S)^\dagger$), where $\{e_n^I\}_{n \in I}$ denotes the canonical basis for $\ell_2(I)$.
- (b) $\{\tilde{u}_j := U(S^*U)^\dagger(e_j^J) \mid j \in J\}$ is an oblique dual frame of $\{v_j\}_{j \in J}$ on L (with synthesis operator $U(S^*U)^\dagger$), where $\{e_j^J\}_{j \in J}$ denotes the canonical basis for $\ell_2(J)$.
- (c) For any $f \in \mathcal{H}$,

$$P_{L, \mathcal{V}^\perp}(f) = \sum_{n \in I} \langle f, \tilde{v}_n \rangle u_n = \sum_{j \in J} \langle f, v_j \rangle \tilde{u}_j$$

where $\mathbf{b} = \{\langle f, \tilde{v}_n \rangle\}_{n \in I}$ and $\mathbf{c} = S^*(f) = \{\langle f, v_j \rangle\}_{j \in J}$ have the minimum norm properties:

$$\begin{aligned} \|\mathbf{b}\| &\leq \|\tilde{\mathbf{b}}\| \quad \text{for any } \tilde{\mathbf{b}} = \{\tilde{b}(n)\}_{n \in I} \text{ with } f = \sum_{n \in I} \tilde{b}(n) u_n, \\ \|\mathbf{c}\| &\leq \|\tilde{\mathbf{c}}\| \quad \text{for any } \tilde{\mathbf{c}} = \{\tilde{c}(j)\}_{j \in J} \text{ with } f = \sum_{j \in J} \tilde{c}(j) \tilde{u}_j. \end{aligned}$$

Proof. See Proposition 3.2 in [7] and Proposition 5.1 in [8]. \square

A generalization of the consistency is the ‘quasi-consistency’ introduced by Arias and Conde [10]. Recall that an operator \tilde{P} satisfies (1) and (2) if and only if $\tilde{P} = TQS^*$ for some $Q \in L(\ell_2(J), \ell_2(K))$. An operator $\tilde{P} = TQS^*$ with $Q \in L(\ell_2(J), \ell_2(K))$ is called a

quasi-consistent sampling operator if $\|S^*\tilde{P}(f) - S^*(f)\|$ is as small as possible for every $f \in \mathcal{H}$; that is, for all $Q_1 \in L(\ell_2(J), \ell_2(K))$ and all $f \in \mathcal{H}$,

$$\|S^*TQS^*(f) - S^*(f)\| \leq \|S^*TQ_1S^*(f) - S^*(f)\|. \tag{4}$$

When $\tilde{P} = TQS^*$ is a quasi-consistent sampling operator, we call Q a quasi-consistent filter. We denote by $\mathcal{QC}(\mathcal{W}, \mathcal{V})$ the set of all quasi-consistent sampling operators and by $\mathcal{QF}(\mathcal{W}, \mathcal{V})$ the set of all quasi-consistent filters, so that $\mathcal{QC}(\mathcal{W}, \mathcal{V}) = T\mathcal{QF}(\mathcal{W}, \mathcal{V})S^*$. It is easily seen that $\mathcal{QF}(\mathcal{W}, \mathcal{V}) = \mathcal{F}(\mathcal{W}, \mathcal{V}) \Leftrightarrow \mathcal{QC}(\mathcal{W}, \mathcal{V}) = \mathcal{C}(\mathcal{W}, \mathcal{V})$ if and only if $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$.

Note that $S^*T = 0$ implies $\mathcal{QC}(\mathcal{W}, \mathcal{V}) = \{TQS^* \mid Q \in L(\ell_2(J), \ell_2(K))\}$, a situation that is not interesting. Therefore, we will assume that $S^*T \neq 0$. Then we have the following:

$$\begin{aligned} \mathcal{QC}(\mathcal{W}, \mathcal{V}) &= \{TQS^* \mid Q \in L(\ell_2(J), \ell_2(K)), S^*TQS^* = P_{\text{ran}(S^*T)}S^*\} \\ &= \{\tilde{U} + TP_{\text{ker}(S^*T)}YS^* \mid Y \in L(\ell_2(J), \ell_2(K))\}, \end{aligned}$$

where $\tilde{U} := T(S^*T)^\dagger S^* \in L(\mathcal{H}, \mathcal{W})$ (see Theorem 5.1 in [10], Proposition 4.2 in [11]).

Proposition 2 (Theorem 4.10 in [11]). *There exists a one-to-one correspondence between $\mathcal{QC}(\mathcal{W}, \mathcal{V})$ and $\mathcal{L} \times L(\mathcal{H}_0^\perp, \mathcal{W} \cap \mathcal{V}^\perp)$, where $\mathcal{H}_0 := \mathcal{W} + \mathcal{V}^\perp$.*

Now we consider the sets of consistent or quasi-consistent approximations of f . For any $f \in \mathcal{H}$, we define

$$\begin{aligned} \mathcal{C}(\mathcal{W}, \mathcal{V})(f) &:= \{\tilde{P}(f) \mid \tilde{P} \in \mathcal{C}(\mathcal{W}, \mathcal{V})\}, \\ \mathcal{QC}(\mathcal{W}, \mathcal{V})(f) &:= \{\tilde{P}(f) \mid \tilde{P} \in \mathcal{QC}(\mathcal{W}, \mathcal{V})\}, \\ \mathcal{C}(f) &:= \{\tilde{f} \in \mathcal{W} \mid S^*(\tilde{f}) = S^*(f)\}, \\ \mathcal{QC}(f) &:= \{\tilde{f} \in \mathcal{W} \mid S^*(\tilde{f}) = P_{\text{ran}(S^*T)}S^*(f)\}. \end{aligned}$$

Clearly, we have $\mathcal{C}(\mathcal{W}, \mathcal{V})(f) \subseteq \mathcal{C}(f)$, and $\tilde{U}(f) \in \mathcal{QC}(\mathcal{W}, \mathcal{V})(f) \subseteq \mathcal{QC}(f)$. Note that $\mathcal{C}(\mathcal{W}, \mathcal{V})(f) \neq \emptyset$ if and only if $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$, in which case, $\mathcal{C}(\mathcal{W}, \mathcal{V})(f) = \mathcal{QC}(\mathcal{W}, \mathcal{V})(f) \subseteq \mathcal{C}(f) = \mathcal{QC}(f)$.

Proposition 3. *Let $f \in \mathcal{H}$.*

- (a) *If $\mathcal{C}(f)$ is nonempty, then it is a closed affine subspace of \mathcal{H} . Moreover, $\mathcal{C}(f) = \tilde{f}_\mathcal{C} + \mathcal{W} \cap \mathcal{V}^\perp$ for any $\tilde{f}_\mathcal{C} \in \mathcal{C}(f)$.*
- (b) *The set $\mathcal{QC}(f) = \tilde{U}(f) + \mathcal{W} \cap \mathcal{V}^\perp$ is a closed affine subspaces of \mathcal{H} . Moreover, we have $\mathcal{QC}(f) = \tilde{f}_{\mathcal{QC}} + \mathcal{W} \cap \mathcal{V}^\perp$ for any $\tilde{f}_{\mathcal{QC}} \in \mathcal{QC}(f)$.*

Proof. (a) It suffices to show that if $\tilde{f}_\mathcal{C} \in \mathcal{C}(f)$, then $\mathcal{C}(f) = \tilde{f}_\mathcal{C} + \mathcal{W} \cap \mathcal{V}^\perp$. Assume that $\tilde{f}_\mathcal{C} \in \mathcal{C}(f)$. Then $S^*(\tilde{f}_\mathcal{C}) = S^*(f)$ so that $\mathcal{C}(f) = \{\tilde{f} \in \mathcal{W} \mid S^*(\tilde{f}) = S^*(\tilde{f}_\mathcal{C})\} = \mathcal{C}(\tilde{f}_\mathcal{C})$. If $g \in \mathcal{C}(f) = \mathcal{C}(\tilde{f}_\mathcal{C})$, then $h := g - \tilde{f}_\mathcal{C} \in \mathcal{W} \cap \mathcal{V}^\perp$ since both g and $\tilde{f}_\mathcal{C}$ belong in \mathcal{W} and $S^*(g - \tilde{f}_\mathcal{C}) = 0$. Then, $g = \tilde{f}_\mathcal{C} + h \in (\tilde{f}_\mathcal{C} + \mathcal{W} \cap \mathcal{V}^\perp)$, which shows that $\mathcal{C}(f) \subseteq \tilde{f}_\mathcal{C} + \mathcal{W} \cap \mathcal{V}^\perp$. Conversely, if $g = \tilde{f}_\mathcal{C} + h$ for some $h \in \mathcal{W} \cap \mathcal{V}^\perp$, then $g \in \mathcal{W}$ and $S^*(g) = S^*(\tilde{f}_\mathcal{C} + h) = S^*(\tilde{f}_\mathcal{C})$ so that $g \in \mathcal{C}(\tilde{f}_\mathcal{C}) = \mathcal{C}(f)$. Therefore, we conclude that $\mathcal{C}(f) = \tilde{f}_\mathcal{C} + \mathcal{W} \cap \mathcal{V}^\perp$.

(b) The proof is similar to (a), except that $\mathcal{QC}(f)$ is always nonempty. This is because $\tilde{U}(f) \in \mathcal{QC}(\mathcal{W}, \mathcal{V})(f) \subseteq \mathcal{QC}(f)$. \square

Our first main result is the following.

Theorem 2. *The following are equivalent.*

- (a) $f \in \mathcal{W} + \mathcal{V}^\perp$;
- (b) $\mathcal{C}(f) \neq \emptyset$.

(c) $\mathcal{C}(f) = \mathcal{QC}(f)$;

Moreover, if $f \in \mathcal{V}^\perp$, then $\mathcal{C}(\mathcal{W}, \mathcal{V})(f)$ is either \emptyset or $\{0\}$, $\mathcal{QC}(\mathcal{W}, \mathcal{V})(f) = \{0\}$, and $\mathcal{C}(f) = \mathcal{QC}(f) = \mathcal{W} \cap \mathcal{V}^\perp$; if $f \in (\mathcal{W} + \mathcal{V}^\perp) \setminus \mathcal{V}^\perp$, then $\mathcal{C}(\mathcal{W}, \mathcal{V})(f) \subseteq \mathcal{C}(f) = \mathcal{QC}(f) = \mathcal{QC}(\mathcal{W}, \mathcal{V})(f)$.

Proof. (a) \Rightarrow (c): Let $f = g + h \in \mathcal{W} + \mathcal{V}^\perp$, where $g = T(\mathbf{d}) \in \mathcal{W}$, $\mathbf{d} \in \ell_2(K)$, and $h \in \mathcal{V}^\perp$. Then $P_{\text{ran}(S^*T)}S^*(f) = P_{\text{ran}(S^*T)}S^*(T(\mathbf{d}) + h) = S^*T(\mathbf{d}) = S^*(T(\mathbf{d}) + h) = S^*(f)$, which yields that $\mathcal{C}(f) = \mathcal{QC}(f)$.

(c) \Rightarrow (b): Assume that $\mathcal{C}(f) = \mathcal{QC}(f)$. Then $\tilde{U}(f) \in \mathcal{QC}(f) = \mathcal{C}(f)$ and, therefore, $\mathcal{C}(f) \neq \emptyset$.

(b) \Rightarrow (a): Assume that $\tilde{f} \in \mathcal{C}(f)$. Then since $S^*(\tilde{f}) = S^*(f)$, we have $h := f - \tilde{f} \in \ker(S^*) = \mathcal{V}^\perp$. Therefore, $f = \tilde{f} + h \in \mathcal{W} + \mathcal{V}^\perp$.

Now, let $f \in \mathcal{W} + \mathcal{V}^\perp$, so that $\mathcal{C}(\mathcal{W}, \mathcal{V})(f)$, $\mathcal{QC}(\mathcal{W}, \mathcal{V})(f) \subseteq \mathcal{C}(f) = \mathcal{QC}(f)$ by (c). First, assume that $f \in \mathcal{V}^\perp = \ker(S^*)$. Then, $\mathcal{C}(f) = \mathcal{QC}(f) = \{\tilde{f} \in \mathcal{W} \mid S^*(\tilde{f}) = 0\} = \mathcal{W} \cap \mathcal{V}^\perp$. Note that $\tilde{P}(f) = 0$ for any $\tilde{P} = TQS^*$ with $Q \in L(\ell_2(J), \ell_2(K))$. Therefore, if $\mathcal{C}(\mathcal{W}, \mathcal{V})$ is nonempty, i.e., $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$, then $\mathcal{C}(\mathcal{W}, \mathcal{V})(f) = \{0\}$; if $\mathcal{C}(\mathcal{W}, \mathcal{V})$ is empty, then by definition $\mathcal{C}(\mathcal{W}, \mathcal{V})(f) = \emptyset$. Since $\mathcal{QC}(\mathcal{W}, \mathcal{V}) (\ni \tilde{U})$ is always nonempty, we have $\mathcal{QC}(\mathcal{W}, \mathcal{V})(f) = \{0\}$.

Finally, assume that $f \in (\mathcal{W} + \mathcal{V}^\perp) \setminus \mathcal{V}^\perp$. Then $S^*\tilde{U}(f) = P_{\text{ran}(S^*T)}S^*(f) = S^*(f) \neq 0$ shows that $\tilde{U}(f) \notin \mathcal{V}^\perp$. Let $\tilde{f} \in \mathcal{QC}(f)$, which can be expressed as $\tilde{f} = \tilde{U}(f) + h$ with $h \in \mathcal{W} \cap \mathcal{V}^\perp$, due to the fact that $\mathcal{QC}(f) = \tilde{U}(f) + \mathcal{W} \cap \mathcal{V}^\perp$. Then $\tilde{U}(f) \notin \mathcal{V}^\perp$ yields $\tilde{f} = \tilde{U}(f) + h \in \mathcal{W} \setminus (\mathcal{W} \cap \mathcal{V}^\perp)$ and, therefore, exists $L \in \mathcal{L}$ containing \tilde{f} . Noticing that

$$\tilde{P} := \begin{cases} P_{L, \mathcal{V}^\perp} & \text{on } \mathcal{W} + \mathcal{V}^\perp \\ \tilde{U} & \text{on } (\mathcal{W} + \mathcal{V}^\perp)^\perp \end{cases}$$

belongs in $\mathcal{QC}(\mathcal{W}, \mathcal{V})$ (see Proposition 2 and its proof given in [11]), we write $\tilde{P} = TQS^*$ and compute $\tilde{f} = P_{L, \mathcal{V}^\perp}(\tilde{f}) = \tilde{P}(\tilde{f}) = \tilde{P}(\tilde{U}(f) + h) = \tilde{P}\tilde{U}(f) = TQS^*\tilde{U}(f) = TQS^*(f) = \tilde{P}(f) \in \mathcal{QC}(\mathcal{W}, \mathcal{V})(f)$. Therefore, $\mathcal{QC}(f) \subseteq \mathcal{QC}(\mathcal{W}, \mathcal{V})(f)$. Since $\mathcal{QC}(\mathcal{W}, \mathcal{V})(f) \subseteq \mathcal{QC}(f)$ by definition, we conclude that $\mathcal{QC}(f) = \mathcal{QC}(\mathcal{W}, \mathcal{V})(f)$. This completes the proof. \square

Let $L_0 := \mathcal{W} \cap (\mathcal{W} \cap \mathcal{V}^\perp)^\perp (\in \mathcal{L})$ be the orthogonal complementary subspace of $\mathcal{W} \cap \mathcal{V}^\perp$ in \mathcal{W} . Note that if $\mathcal{W} \cap \mathcal{V}^\perp = \{0\}$, then $L_0 = \mathcal{W}$ is the unique element of \mathcal{L} .

Proposition 4. We have $P_{L_0}\tilde{U} = P_{L_0, \ker(\tilde{U})} \in \mathcal{QC}(\mathcal{W}, \mathcal{V})$ and $P_{L_0}\tilde{U}|_{\mathcal{W} + \mathcal{V}^\perp} = P_{L_0, \mathcal{V}^\perp}$. If $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$, then $P_{L_0}\tilde{U} = F^\dagger P_{\mathcal{V}} = P_{L_0, \mathcal{V}^\perp} \in \mathcal{C}(\mathcal{W}, \mathcal{V})$ where $F := I - P_{\mathcal{V}^\perp}P_{\mathcal{W}}$.

Proof. See Proposition 3.8, Theorem 3.10, and Lemma 4.11 in [11]. \square

Among the quasi-consistent approximations of f , we can identify some special ones that have optimal properties.

Theorem 3. Let $f \in \mathcal{H}$. Then,

$$\operatorname{argmin}_{\tilde{f} \in \mathcal{QC}(f)} \|f - \tilde{f}\| = P_{L_0}\tilde{U}(f) + P_{\mathcal{W} \cap \mathcal{V}^\perp}(f),$$

$$\operatorname{argmin}_{\tilde{f} \in \mathcal{QC}(f)} \|\tilde{f}\| = P_{L_0}\tilde{U}(f).$$

Proof. Note that since $\mathcal{QC}(f) = \tilde{U}(f) + \mathcal{W} \cap \mathcal{V}^\perp$, every element \tilde{f} in $\mathcal{QC}(f)$ can be written as $\tilde{f} = \tilde{U}(f) + h$ for some $h \in \mathcal{W} \cap \mathcal{V}^\perp$. Observe that

$$\begin{aligned} \|f - \tilde{f}\|^2 &= \|f - \tilde{U}(f) - h\|^2 \\ &= \|P_{\mathcal{W} \cap \mathcal{V}^\perp}(f - \tilde{U}(f)) + P_{(\mathcal{W} \cap \mathcal{V}^\perp)^\perp}(f - \tilde{U}(f)) - h\|^2 \\ &= \|P_{\mathcal{W} \cap \mathcal{V}^\perp}(f - \tilde{U}(f)) - h\|^2 + \|P_{(\mathcal{W} \cap \mathcal{V}^\perp)^\perp}(f - \tilde{U}(f))\|^2 \\ &\geq \|P_{(\mathcal{W} \cap \mathcal{V}^\perp)^\perp}(f - \tilde{U}(f))\|^2, \end{aligned}$$

where equality is achieved if and only if $h = P_{\mathcal{W} \cap \mathcal{V}^\perp}(f - \tilde{U}(f))$, i.e., $\tilde{f} = \tilde{U}(f) + P_{\mathcal{W} \cap \mathcal{V}^\perp}(f - \tilde{U}(f))$. Since $P_{(\mathcal{W} \cap \mathcal{V}^\perp)^\perp} \tilde{U} = P_{L_0} \tilde{U}$, we obtain that $\operatorname{argmin}_{\tilde{f} \in \mathcal{QC}(f)} \|f - \tilde{f}\| = P_{L_0} \tilde{U}(f) + P_{\mathcal{W} \cap \mathcal{V}^\perp}(f)$. Similarly, observe that

$$\begin{aligned} \|\tilde{f}\|^2 &= \|\tilde{U}(f) + h\|^2 \\ &= \|P_{\mathcal{W} \cap \mathcal{V}^\perp} \tilde{U}(f) + P_{(\mathcal{W} \cap \mathcal{V}^\perp)^\perp} \tilde{U}(f) + h\|^2 \\ &= \|P_{\mathcal{W} \cap \mathcal{V}^\perp} \tilde{U}(f) + h\|^2 + \|P_{(\mathcal{W} \cap \mathcal{V}^\perp)^\perp} \tilde{U}(f)\|^2 \\ &\geq \|P_{(\mathcal{W} \cap \mathcal{V}^\perp)^\perp} \tilde{U}(f)\|^2, \end{aligned}$$

where equality is achieved if and only if $h = -P_{\mathcal{W} \cap \mathcal{V}^\perp} \tilde{U}(f)$, i.e., $\tilde{f} = \tilde{U}(f) - P_{\mathcal{W} \cap \mathcal{V}^\perp} \tilde{U}(f) = P_{L_0} \tilde{U}(f)$. Therefore, $\operatorname{argmin}_{\tilde{f} \in \mathcal{QC}(f)} \|\tilde{f}\| = \tilde{U}(f) = P_{L_0} \tilde{U}(f)$. \square

Note that if $f \in \mathcal{W} + \mathcal{V}^\perp$, then the set $\mathcal{QC}(f)$ coincides with $\mathcal{C}(f)$ by Theorem 2 and moreover, $P_{L_0} \tilde{U}(f) = P_{L_0, \mathcal{V}^\perp}(f)$ by Proposition 4. As a consequence, we obtain the following.

Corollary 1 (cf. Proposition 3.1 in [10]). *Let $f \in \mathcal{W} + \mathcal{V}^\perp$. Then*

$$\begin{aligned} \operatorname{argmin}_{\tilde{f} \in \mathcal{C}(f)} \|f - \tilde{f}\| &= P_{L_0, \mathcal{V}^\perp}(f) + P_{\mathcal{W} \cap \mathcal{V}^\perp}(f), \\ \operatorname{argmin}_{\tilde{f} \in \mathcal{C}(f)} \|\tilde{f}\| &= P_{L_0, \mathcal{V}^\perp}(f). \end{aligned}$$

Remark 1. (i) It should be noted that $\mathcal{C}(\mathcal{W}, \mathcal{V})(f) \neq \mathcal{C}(f)$ for a generic $f \in \mathcal{H}$. Theorem 3.2 in [10] asserts that if $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$, then $\mathcal{C}(\mathcal{W}, \mathcal{V})(f) = F^\dagger P_{\mathcal{V}}(f) + \mathcal{W} \cap \mathcal{V}^\perp$ with $F := I - P_{\mathcal{V}^\perp} P_{\mathcal{W}}$, which is not exactly accurate. A correct statement is that if $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$, then $\mathcal{C}(\mathcal{W}, \mathcal{V})(f) \subseteq \mathcal{C}(f) = F^\dagger P_{\mathcal{V}}(f) + \mathcal{W} \cap \mathcal{V}^\perp$.

(ii) It was shown in ([5], Theorem 1) that if $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$ is of finite dimension, then $\mathcal{C}(f) = F^\dagger P_{\mathcal{V}}(f) + \mathcal{W} \cap \mathcal{V}^\perp$ with $F = I - P_{\mathcal{V}^\perp} P_{\mathcal{W}}$. The authors of [10] noticed that this is true even if \mathcal{H} is infinite dimensional (see Theorem 3.2 in [10]), and showed that if $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$, then $\operatorname{argmin}_{\tilde{f} \in \mathcal{C}(f)} \|f - \tilde{f}\| = F^\dagger P_{\mathcal{V}}(f) + P_{\mathcal{W} \cap \mathcal{V}^\perp}(f)$ (see Proposition 3.1 in [10]). Since $F^\dagger P_{\mathcal{V}}(f) = P_{L_0, \mathcal{V}^\perp}(f)$ by Proposition 4, this follows from the first part of Corollary 1. Note that we have replaced $\mathcal{C}(\mathcal{W}, \mathcal{V})(f)$ with $\mathcal{C}(f)$ in the original statements of [10], as discussed in (i).

Let us now illustrate our results with some examples. In the finite-dimensional case, we consider the band-limited sampling of time-limited vectors (cf. [6,11]).

Example 1. Let J_0, J, K , and N be positive integers such that $J = 2J_0 + 1 < N$ and $K < N$, and let $\mathcal{H} = \mathbb{C}^N$ be the space of N -dimensional vectors $\mathbf{x} = \{x(n)\}_{n=0}^{N-1}$ with $x(n) \in \mathbb{C}$ for all

$n = 0, \dots, N-1$. Define the sampling vectors $\{v_j\}_{j=0}^{J-1}$ by $v_j(n) = e^{2\pi i(j-J_0)n/N}$ for $0 \leq n < N$, and the reconstruction vectors $\{w_k\}_{k=0}^{K-1}$ by $w_k(n) = \delta_{k,n}$ for $0 \leq n < N$. Then it is easily seen that

$$\mathcal{V}_J := \overline{\text{span}}\{v_j\}_{j=0}^{J-1} = \{\mathbf{x} \in \mathbb{C}^N \mid \hat{x}(n) = 0 \text{ for } J_0 < n < N - J_0\},$$

$$\mathcal{W}_K := \overline{\text{span}}\{w_k\}_{k=0}^{K-1} = \{\mathbf{x} \in \mathbb{C}^N \mid x(n) = 0 \text{ for } n \geq K\} \ (\cong \mathbb{C}^K),$$

where $\hat{x}(\cdot)$ denotes the N point DFT (discrete Fourier transform) of \mathbf{x} . Note that \mathcal{W}_K consists of time-limited sequences while \mathcal{V}_J consists of band-limited sequences. The synthesis operators $S : \mathbb{C}^J \rightarrow \mathcal{V}_J$ of $\{v_j\}_{j=0}^{J-1}$ and $T : \mathbb{C}^K \rightarrow \mathcal{W}_K$ of $\{w_k\}_{k=0}^{K-1}$ are given by $S(\mathbf{c}) = \sum_{j=0}^{J-1} c_j v_j$ and $T(\mathbf{d}) = \sum_{k=0}^{K-1} d_k w_k$ respectively.

For any $\mathbf{x} \in \mathcal{H}$, its measurements $\mathbf{c} = \{c_j\}_{j=0}^{J-1} = S^*(\mathbf{x})$ are given by

$$c_j = \langle \mathbf{x}, v_j \rangle = \sum_{n=0}^{N-1} x(n) \overline{v_j(n)} = \sum_{n=0}^{N-1} x(n) e^{-2\pi i(j-J_0)n/N}$$

$$= \hat{x}[(j - J_0)_N], \quad 0 \leq j < J,$$

where $(\cdot)_N$ denotes the residue modulo N . That is, the measurements $\mathbf{c} = S^*(\mathbf{x})$ are precisely the J low-pass DFT coefficients of the N point DFT of \mathbf{x} . Therefore, a consistent approximation $\tilde{\mathbf{x}}$ of \mathbf{x} in \mathcal{W}_K has the same low-pass DFT coefficients as \mathbf{x} .

Note that for $0 \leq j < J$ and $0 \leq k < K$, we have the following:

$$\langle w_k, v_j \rangle = \sum_{n=0}^{N-1} w_k(n) \overline{v_j(n)} = \overline{v_j(k)} = (e^{-2\pi i/N})^{jk} (e^{2\pi i J_0/N})^k.$$

Therefore, the input-output cross-correlation matrix (or the generalized Gram matrix) $B = [\langle w_k, v_j \rangle]_{0 \leq j < J, 0 \leq k < K} \in \mathbb{C}^{J \times K}$ is given by

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & z & z^2 & \cdots & z^{K-1} \\ 1 & z^2 & z^4 & \cdots & z^{2(K-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z^{J-1} & z^{2(J-1)} & \cdots & z^{(J-1)(K-1)} \end{bmatrix} D,$$

where $z := e^{-2\pi i/N}$, $d := e^{2\pi i J_0/N}$ and $D = \text{diag}(1, d^1, \dots, d^{K-1})$. Note that $B \in \mathbb{C}^{J \times K}$ always has full rank. In general, we have from ([11], Lemma 2.4) that

$$B \text{ is injective} \Leftrightarrow \mathcal{W}_K \cap \mathcal{V}_J^\perp = \{0\},$$

$$B \text{ is surjective} \Leftrightarrow \mathcal{H} = \mathcal{W}_K + \mathcal{V}_J^\perp,$$

but since B has full rank, the size of B immediately determines the injectivity/surjectivity of B and the corresponding conditions.

We will focus on the under-determined case ($J < K$), where the number of measurements is strictly less than the number of reconstruction vectors. In this case, B is surjective but not injective; correspondingly, we have $\mathcal{H} = \mathcal{W}_K + \mathcal{V}_J^\perp$ and $\mathcal{W}_K \cap \mathcal{V}_J^\perp \neq \{0\}$. Then, $\mathcal{L} = \{L \mid L \text{ is a closed complementary subspace of } \mathcal{W}_K \cap \mathcal{V}_J^\perp \text{ in } \mathcal{W}_K\}$ has infinite cardinality, so there exist infinitely many consistent sampling operators $\{P_{L, \mathcal{V}_J^\perp} \mid L \in \mathcal{L}\}$. Note that if we fix a subspace $L \in \mathcal{L}$, then every element \mathbf{x} in L ($\subsetneq \mathcal{W}_K$) can be exactly recovered from its measurements $\mathbf{c} = S^*(\mathbf{x})$. Unfortunately, this does not apply to every element \mathbf{x} in \mathcal{H} . For a generic element \mathbf{x} in \mathcal{H} , we seek its (quasi-) consistent approximations in the subspace \mathcal{W}_K based on its

measurements $\mathbf{c} = S^*(\mathbf{x})$. In fact, all such approximations are collected in the sets $\mathcal{C}(\mathcal{W}_K, \mathcal{V}_J)(\mathbf{x})$, $\mathcal{QC}(\mathcal{W}_K, \mathcal{V}_J)(\mathbf{x})$, $\mathcal{C}(\mathbf{x})$, and $\mathcal{QC}(\mathbf{x})$. Since $\mathcal{H} = \mathcal{W}_K + \mathcal{V}_J^\perp$, we have from Theorem 2 that

$$\begin{aligned} \mathcal{C}(\mathcal{W}_K, \mathcal{V}_J)(\mathbf{x}) &= \mathcal{QC}(\mathcal{W}_K, \mathcal{V}_J)(\mathbf{x}) \\ &= \begin{cases} \{0\} & \text{if } \mathbf{x} \in \mathcal{V}_J^\perp \\ P_{L_0, \mathcal{V}_J^\perp}(\mathbf{x}) + \mathcal{W}_K \cap \mathcal{V}_J^\perp & \text{if } \mathbf{x} \notin \mathcal{V}_J^\perp, \end{cases} \end{aligned}$$

and

$$\mathcal{C}(\mathbf{x}) = \mathcal{QC}(\mathbf{x}) = P_{L_0, \mathcal{V}_J^\perp}(\mathbf{x}) + \mathcal{W}_K \cap \mathcal{V}_J^\perp,$$

where $L_0 := \mathcal{W}_K \cap (\mathcal{W}_K \cap \mathcal{V}_J^\perp)^\perp \in \mathcal{L}$. In particular, the set $\mathcal{C}(\mathbf{x}) = P_{L_0, \mathcal{V}_J^\perp}(\mathbf{x}) + \mathcal{W}_K \cap \mathcal{V}_J^\perp$ contains all possible candidates for consistent approximations of \mathbf{x} in \mathcal{W}_K . Among these candidates, Corollary 1 identifies those with optimal properties:

$$\begin{aligned} \operatorname{argmin}_{\tilde{\mathbf{x}} \in \mathcal{C}(\mathbf{x})} \|\mathbf{x} - \tilde{\mathbf{x}}\| &= P_{L_0, \mathcal{V}_J^\perp}(\mathbf{x}) + P_{\mathcal{W}_K \cap \mathcal{V}_J^\perp}(\mathbf{x}), \\ \operatorname{argmin}_{\tilde{\mathbf{x}} \in \mathcal{C}(\mathbf{x})} \|\tilde{\mathbf{x}}\| &= P_{L_0, \mathcal{V}_J^\perp}(\mathbf{x}). \end{aligned}$$

The determined case ($J = K$) and over-determined case ($J > K$) are rather obvious, so we refer interested readers to ([11], Example 4.19) for further details.

For an example in the infinite dimensional case, we consider complex exponential systems. For a discrete set $\Lambda \subset \mathbb{R}$, we define $\mathcal{E}(\Lambda) := \{e^{2\pi i \lambda \cdot} : \lambda \in \Lambda\}$, which consists of complex exponential functions with frequencies from Λ .

Example 2. It is well known that $\mathcal{E}(\mathbb{Z})$ is an orthonormal basis for $\mathcal{H} := L^2[0, 1]$. Let $\{v_j\}_{j \in J} = \mathcal{E}(2\mathbb{Z}+1) \cup \mathcal{E}(4\mathbb{Z}+2)$ be the sampling vectors and let $\{w_k\}_{k \in K} = \mathcal{E}(2\mathbb{Z})$ be the reconstruction vectors, so that

$$\begin{aligned} \mathcal{V} &:= \overline{\operatorname{span}}\{v_j\}_{j \in J} = \overline{\operatorname{span}}(\mathcal{E}(2\mathbb{Z}+1) \cup \mathcal{E}(8\mathbb{Z}+4)), \\ \mathcal{W} &:= \overline{\operatorname{span}}\{w_k\}_{k \in K} = \overline{\operatorname{span}}(\mathcal{E}(4\mathbb{Z})). \end{aligned}$$

For any $f \in \mathcal{H} = L^2[0, 1]$, its measurements are given by

$$\begin{aligned} S^*(f) &= \left\{ \langle f, e^{2\pi i n \cdot} \rangle_{L^2[0,1]} \right\}_{n \in (2\mathbb{Z}+1) \cup (8\mathbb{Z}+4)} \\ &= \left\{ \int_0^1 f(x) e^{-2\pi i n x} dx \mid n \in (2\mathbb{Z}+1) \cup (8\mathbb{Z}+4) \right\} \end{aligned}$$

which are in fact the Fourier coefficients of f for $n \in (2\mathbb{Z}+1) \cup (8\mathbb{Z}+4)$. Clearly, these coefficients are not sufficient for the exact recovery of $f \in \mathcal{H}$ in general, and we aim to approximate f in the reconstruction space \mathcal{W} using the coefficients. That is, we seek some approximations of f in \mathcal{W} based on the measurements $S^*(f)$, namely the consistent approximations, which produce the same measurements or the quasi-consistent approximations, which minimize the measurement error in the sense of (4). All such approximations are collected in the sets $\mathcal{C}(\mathcal{W}, \mathcal{V})(f)$, $\mathcal{QC}(\mathcal{W}, \mathcal{V})(f)$, $\mathcal{C}(f)$, and $\mathcal{QC}(f)$.

Note that

$$\begin{aligned} \mathcal{V}^\perp &= \left(\overline{\operatorname{span}}(\mathcal{E}(2\mathbb{Z}+1) \cup \mathcal{E}(8\mathbb{Z}+4)) \right)^\perp = \overline{\operatorname{span}}(\mathcal{E}(4\mathbb{Z}+2) \cup \mathcal{E}(8\mathbb{Z})), \\ \mathcal{W} \cap \mathcal{V}^\perp &= \overline{\operatorname{span}}(\mathcal{E}(4\mathbb{Z})) \cap \overline{\operatorname{span}}(\mathcal{E}(4\mathbb{Z}+2) \cup \mathcal{E}(8\mathbb{Z})) = \overline{\operatorname{span}}(\mathcal{E}(8\mathbb{Z})), \\ \mathcal{W} + \mathcal{V}^\perp &= \overline{\operatorname{span}}(\mathcal{E}(4\mathbb{Z})) + \overline{\operatorname{span}}(\mathcal{E}(4\mathbb{Z}+2) \cup \mathcal{E}(8\mathbb{Z})) = \overline{\operatorname{span}}(\mathcal{E}(2\mathbb{Z})), \\ \mathcal{H} &= L^2[0, 1] = \overline{\operatorname{span}}(\mathcal{E}(\mathbb{Z})). \end{aligned}$$

Since $\mathcal{H} \neq \mathcal{W} + \mathcal{V}^\perp$, we have $\mathcal{C}(\mathcal{W}, \mathcal{V}) = \emptyset$ and, thus, $\mathcal{C}(\mathcal{W}, \mathcal{V})(f) = \emptyset$ for all $f \in \mathcal{H}$. Moreover, Theorem 2 shows the following:

- if $f \in \mathcal{W} + \mathcal{V}^\perp = \overline{\text{span}}(\mathcal{E}(2\mathbb{Z}))$, then $\emptyset \neq \mathcal{C}(f) = \mathcal{QC}(f)$;
- if $f \in \mathcal{V}^\perp = \overline{\text{span}}(\mathcal{E}(4\mathbb{Z}+2) \cup \mathcal{E}(8\mathbb{Z}))$, then $\mathcal{QC}(\mathcal{W}, \mathcal{V})(f) = \{0\}$, and $\mathcal{C}(f) = \mathcal{QC}(f) = \mathcal{W} \cap \mathcal{V}^\perp = \overline{\text{span}}(\mathcal{E}(8\mathbb{Z}))$;
- if $f \in (\mathcal{W} + \mathcal{V}^\perp) \setminus \mathcal{V}^\perp = \overline{\text{span}}(\mathcal{E}(2\mathbb{Z})) \setminus \overline{\text{span}}(\mathcal{E}(4\mathbb{Z}+2) \cup \mathcal{E}(8\mathbb{Z}))$, in particular, if $f \in \overline{\text{span}}(\mathcal{E}(8\mathbb{Z}+4))$, then $\mathcal{C}(f) = \mathcal{QC}(f) = \mathcal{QC}(\mathcal{W}, \mathcal{V})(f)$.

Further, Corollary 1 shows that if $f \in \mathcal{W} + \mathcal{V}^\perp = \overline{\text{span}}(\mathcal{E}(2\mathbb{Z}))$, then

$$\begin{aligned} \operatorname{argmin}_{\tilde{f} \in \mathcal{C}(f)} \|f - \tilde{f}\| &= P_{L_0, \mathcal{V}^\perp}(f) + P_{\mathcal{W} \cap \mathcal{V}^\perp}(f), \\ \operatorname{argmin}_{\tilde{f} \in \mathcal{C}(f)} \|\tilde{f}\| &= P_{L_0, \mathcal{V}^\perp}(f), \end{aligned}$$

where $L_0 := \mathcal{W} \cap (\mathcal{W} \cap \mathcal{V}^\perp)^\perp = \overline{\text{span}}(\mathcal{E}(4\mathbb{Z})) \cap (\overline{\text{span}}(\mathcal{E}(8\mathbb{Z})))^\perp = \overline{\text{span}}(\mathcal{E}(8\mathbb{Z}+4))$. This means that if

$$f(x) = \sum_{k \in 2\mathbb{Z}} c_k e^{2\pi i k x} \quad \text{for some } \{c_k\}_{k \in 2\mathbb{Z}} \in \ell_2(2\mathbb{Z}), \tag{5}$$

then

$$P_{L_0, \mathcal{V}^\perp}(f) = \sum_{k \in 8\mathbb{Z}+4} c_k e^{2\pi i k x} \quad \text{and} \quad P_{\mathcal{W} \cap \mathcal{V}^\perp}(f) = \sum_{k \in 8\mathbb{Z}} c_k e^{2\pi i k x},$$

and therefore,

$$\begin{aligned} \operatorname{argmin}_{\tilde{f} \in \mathcal{C}(f)} \|f - \tilde{f}\| &= \sum_{k \in 4\mathbb{Z}} c_k e^{2\pi i k (\cdot)}, \\ \operatorname{argmin}_{\tilde{f} \in \mathcal{C}(f)} \|\tilde{f}\| &= \sum_{k \in 8\mathbb{Z}+4} c_k e^{2\pi i k (\cdot)}. \end{aligned}$$

Comparison with Related Work

In the early papers on consistent sampling, the authors studied the consistency under the assumption $\mathcal{H} = \mathcal{W} \oplus \mathcal{V}^\perp$, meaning that $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$ and $\mathcal{W} \cap \mathcal{V}^\perp = 0$ [4,6,7]. In this setting, there exists a unique consistent sampling operator $P_{\mathcal{W}, \mathcal{V}^\perp}$, which is the oblique projection onto \mathcal{W} along \mathcal{V}^\perp . Motivated by applications in wavelets, the cases where \mathcal{W} is a shift-invariant space of $\mathcal{H} = L^2(\mathbb{R})$ were extensively studied in [20–23].

Hirabayashi and Unser ([9]) investigated the consistent sampling in a finite-dimensional Hilbert space \mathcal{H} where $\mathcal{H} = \mathcal{W} + \mathcal{V}^\perp$ but $\mathcal{W} \cap \mathcal{V}^\perp$ is not necessarily trivial. They showed that if $\mathcal{W} \cap \mathcal{V}^\perp \neq \{0\}$, then there exist infinitely many consistent sampling operators, which are the oblique projections P_{L, \mathcal{V}^\perp} , with L belonging to $\mathcal{L} = \{L \mid L \text{ is a closed complementary subspace of } \mathcal{W} \cap \mathcal{V}^\perp \text{ in } \mathcal{W}\}$.

Arias and Conde ([10]) extended the problem to the general situation where $\mathcal{H} \supsetneq \mathcal{W} + \mathcal{V}^\perp$ in which case consistent sampling operators do not exist. Generalizing the consistency, they formalized the concept of quasi-consistency and obtained some characterizations for $\mathcal{QC}(\mathcal{W}, \mathcal{V})$ and $\mathcal{QC}(f)$. The quasi-consistency was then studied extensively by Kwon and Lee [11]. They obtained complete characterizations of the quasi-consistency (see Theorems 4.5 and 4.10 in [11]). They also showed that the quasi-consistency can be interpreted geometrically in terms of oblique projections and provided an iterative algorithm to compute the quasi-consistent approximations (see Theorem 4.16 and Corollary 4.17 in [11]).

This paper extends and builds upon the results of [11]. Proposition 3 shows that $\mathcal{C}(f)$ and $\mathcal{QC}(f)$ are certain closed affine subspaces of \mathcal{H} . Theorem 2 shows that $f \in \mathcal{W} + \mathcal{V}^\perp$ if and only if $\mathcal{C}(f) \neq \emptyset$ if and only if $\mathcal{C}(f) = \mathcal{QC}(f)$. Additionally, Theorem 3 and Corollary 1 identify some quasi-consistent approximations of f that possess optimal properties.

4. Conclusions

In this paper, we studied generalized consistent sampling and reconstruction processes in an abstract separable Hilbert space. Using an operator-theoretical approach, we derived quasi-consistent and consistent approximations with optimal properties. In particular, we identified those that have the minimum norm and those that are closest to the original vector. The obtained results are illustrated with several examples.

Author Contributions: Conceptualization, S.K., K.H.K. and D.G.L.; investigation, S.K., K.H.K. and D.G.L.; writing—original draft preparation, S.K., K.H.K. and D.G.L.; writing—review and editing, S.K., K.H.K. and D.G.L.; supervision, D.G.L. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (RS-2023-00275360).

Data Availability Statement: All data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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